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THE MASSERA-SCHÄFFER PROBLEM FOR A FIRST ORDER LINEAR DIFFERENTIAL EQUATION

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Abstract. We consider the Massera-Schäffer problem for the equation

$$-y'(x)+q(x)y(x)=f(x),\quad x\in\mathbb{R},$$

where $f\in L_p^{\rm loc}(\mathbb{R})$, $p\in[1,\infty)$ and $0\leqslant q\in L_1^{\rm loc}(\mathbb{R})$. By a solution of the problem we mean any function y, absolutely continuous and satisfying the above equation almost everywhere in $\mathbb{R}$. Let positive and continuous functions $\mu(x)$ and $\theta(x)$ for $x\in\mathbb{R}$ be given. Let us introduce the spaces

$$L_p(\mathbb{R},\mu)=\Big\{f\in L_p^{\rm loc}(\mathbb{R})\colon\ \|f\|_{L_p(\mathbb{R},\mu)}^p=\int_{-\infty}^{\infty}|\mu(x)f(x)|^p{\rm d}x<\infty\Big\},$$
$$L_p(\mathbb{R},\theta)=\Big\{f\in L_p^{\rm loc}(\mathbb{R})\colon\ \|f\|_{L_p(\mathbb{R},\theta)}^p=\int_{-\infty}^{\infty}|\theta(x)f(x)|^p{\rm d}x<\infty\Big\}.$$

We obtain requirements to the functions μ, θ and q under which (1) for every function $f\in L_p(\mathbb{R},\theta)$ there exists a unique solution $y\in L_p(\mathbb{R},\mu)$ of the above equation; (2) there is an absolute constant $c(p)\in(0,\infty)$ such that regardless of the choice of a function $f\in L_p(\mathbb{R},\theta)$ the solution of the above equation satisfies the inequality

$$\|y\|_{L_p(\mathbb{R},\mu)}\leqslant c(p)\|f\|_{L_p(\mathbb{R},\theta)}.$$

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1. Introduction

In the present paper, we consider the equation

$$-y'(x) + q(x)y(x) = f(x), \quad x \in \mathbb{R}, \tag{1.1}$$

where $f \in L_p^{\mathrm{loc}}(\mathbb{R})$, $p \in (1,\infty)$ and

$$0 \leqslant q \in L_1^{\mathrm{loc}}(\mathbb{R}). \tag{1.2}$$

We continue the research started in [6], where a criterion has been obtained for the correct solvability of equation (1.1) in the space $L_p(\mathbb{R})$, see Theorem 1.2 below. However, in this paper we do not restrict our considerations to the space $L_p(\mathbb{R})$, $p \in (1,\infty)$, but also investigate a more general problem to find a pair of spaces which is admissible for equation (1.1), i.e., the Massera-Schäffer problem, see [7], Chapter 5, Sections 50–51 and Definition 1.1 below. Thus, our goal is to determine the space frame within which equation (1.1) always has a unique bounded solution, see (1.5) below.

Let us now go to precise formulations. Throughout the sequel, $\mu(x)$ and $\theta(x)$ stand for positive functions continuous on $x \in \mathbb{R}$ (weights), and $L_{p,\mu}(\mathbb{R})$, $L_{p,\theta}(\mathbb{R})$, $p \in (1,\infty)$, denote the corresponding weight spaces

$$L_{p,\mu}(\mathbb{R}) = \left\{ f \in L_p^{\mathrm{loc}}(\mathbb{R}) \colon \|f\|_{L_{p,\mu}(\mathbb{R})}^p = \int_{-\infty}^{\infty} |\mu(x)f(x)|^p \,\mathrm{d}x < \infty \right\}, \tag{1.3}$$

$$L_{p,\theta}(\mathbb{R}) = \left\{ f \in L_p^{\mathrm{loc}}(\mathbb{R}) \colon \|f\|_{L_{p,\theta}(\mathbb{R})}^p = \int_{-\infty}^{\infty} |\theta(x)f(x)|^p \,\mathrm{d}x < \infty \right\}. \tag{1.4}$$

For brevity, we write $L_{p,\mu}$, $L_{p,\theta}$, $\|\cdot\|_{p,\mu}$, $\|\cdot\|_{p,\theta}$ instead of $L_{p,\mu}(\mathbb{R})$, $L_{p,\theta}(\mathbb{R})$, $\|\cdot\|_{L_{p,\mu}(\mathbb{R})}$, $\|\cdot\|_{L_{p,\theta}(\mathbb{R})}$, respectively. In the case $\mu \equiv 1$ ($\theta \equiv 1$), we write $L_p(L_p)$ and $\|\cdot\|_p(\|\cdot\|_p)$, respectively. By a solution of (1.1), we understand any absolutely continuous function $y(x)$, $x \in \mathbb{R}$, satisfying equality (1.1) almost everywhere on $\mathbb{R}$.

Definition 1.1. We say that the spaces $L_{p,\mu}$ and $L_{p,\theta}$ constitute a pair $\{L_{p,\mu}; L_{p,\theta}\}$ admissible for equation (1.1) if requirements (I)–(II) hold:

(I) for every function $f \in L_{p,\theta}$, there exists a unique solution $y \in L_{p,\mu}$ of (1.1),

(II) there exists a constant $c \in (0,\infty)$ such that regardless of the choice of $f \in L_{p,\theta}$, the solution $y \in L_{p,\mu}$ of (1.1) satisfies the inequality

$$\|y\|_{p,\mu} \leqslant c\|f\|_{p,\theta}. \tag{1.5}$$

Note that Definition 1.1 is a restriction of a general definition given in [7], Chapter 5, Sections 50–51. Therefore, the question on the validity of requirements (I)–(II) for equation (1.1) is called the *Massera-Schäffer problem*.

Below we use the following convention: for brevity, we say "problem (I)–(II)" or "question on (I)–(II)" instead of "problem (or question) on the requirements to the functions $q(\cdot)$, $\mu(\cdot)$ and $\theta(\cdot)$ under which requirements (I)–(II) of Definition 1.1 hold"; we say "a pair $\{L_{p,\mu}; L_{p,\theta}\}$ admissible for (1.1)" instead of "a pair of spaces $\{L_{p,\mu}; L_{p,\theta}\}$ admissible for equation (1.1)"; we often omit the word "equation" before (1.1); by the letter c we denote absolute positive constants which are not essential for exposition; requirement (1.2) is our standing assumption, i.e., throughout the sequel we assume it holds and do not mention this in the statements.

In [6], in the case that the pair $\{L_p; L_p\}$ is admissible for (1.1), (1.1) is said to be correctly solvable in L_p. We keep this terminology in the present paper. Let us quote the main result of [6].

Theorem 1.2 ([6]). *For $p \in [1, \infty)$, equation (1.1) is correctly solvable in L_p if and only if there exists $a \in (0, \infty)$ such that $q_0(a) > 0$. Here*

$$q_0(a) = \inf_{x\in\mathbb{R}} \int_{x-a}^{x+a} q(t)\,\mathrm{d}t. \tag{1.6}$$

A new proof of Theorem 1.2 can be found in Section 5 below.

We continue the research of [6] with the following goal: given equation (1.1), determine the requirements to the weights $\mu(\cdot)$ and $\theta(\cdot)$ under which the pair $\{L_{p,\mu}; L_{p,\theta}\}$ is admissible for (1.1) also in the case, where $q_0(a) = 0$ for all $a \in (0, \infty)$. Such an approach allows one to continue being interested also in equations (1.1) to which Theorem 1.2 is not applicable. For example:

(1) $q_0(a) > 0$ for some $a \in (0, \infty)$ but $f \notin L_p(\mathbb{R})$,
(2) $q_0(a) = 0$ for all $a \in (0, \infty)$, $f \in L_p(\mathbb{R})$,
(3) $q_0(a) = 0$ for all $a \in (0, \infty)$, $f \notin L_p(\mathbb{R})$.

Note that for a second order linear differential equation, an analogue of problem (I)–(II) was studied in [3]. As in [3], our main result (see Theorem 3.3) reduces the question on (I)–(II) to the problem on the boundedness of a certain integral operator $S\colon L_p \to L_p$. From this criterion, under additional requirements to the functions $q(\cdot)$, $\mu(\cdot)$ and $\theta(\cdot)$, one can deduce various particular conditions controlling the solution of problem (I)–(II), see Sections 3 and 5.

To conclude this section, for the reader's convenience we describe the structure of the present paper. In Section 2 we collect preliminaries, Section 3 contains the statements and comments, all the proofs are given in Section 4, see Section 5 for examples of the solution of problem (I)–(II).

2. Preliminaries

Below we give a summary of the main results used in our proofs.

Theorem 2.1 ([5]). *Let $\mu(x)$ and $\theta(x)$ be continuous, positive functions for $x \in \mathbb{R}$, and let H be the integral operator*

$$(Hf)(x) = \mu(x) \int_x^\infty \theta(t) f(t)\,\mathrm{d}t, \quad x \in \mathbb{R}. \tag{2.1}$$

For $p \in (1, \infty)$ the operator $H\colon L_p \to L_p$ is bounded if and only if $H_p < \infty$. Here $H_p = \sup_{x \in \mathbb{R}} H_p(x)$,

$$H_p(x) = \left(\int_{-\infty}^x \mu(t)^p\,\mathrm{d}t\right)^{1/p} \left(\int_x^\infty \theta(t)^{p'}\,\mathrm{d}t\right)^{1/p'}, \quad x \in \mathbb{R},\ p' = \frac{p}{p-1}. \tag{2.2}$$

In addition,

$$H_p \leqslant \|H\|_{p\to p} \leqslant (p)^{1/p} (p')^{1/p'} H_p. \tag{2.3}$$

Suppose that together with (1.2) the following condition holds:

$$\int_{-\infty}^0 q(t)\,\mathrm{d}t = \int_0^\infty q(t)\,\mathrm{d}t = \infty. \tag{2.4}$$

Define the function $d(x)$, $x \in \mathbb{R}$, by the equality (see [1])

$$d(x) = \inf_{d>0} \left\{ d\colon \int_{x-d}^{x+d} q(t)\,\mathrm{d}t = 2 \right\}. \tag{2.5}$$

From (2.4) it follows that the function $d(x)$, $x \in \mathbb{R}$, is well-defined, it is positive and finite for all $x \in \mathbb{R}$, and we have

$$\int_{x-d(x)}^{x+d(x)} q(t)\,\mathrm{d}t = 2, \quad x \in \mathbb{R}. \tag{2.6}$$

Lemma 2.2 ([2]). *Suppose that* (2.4) *holds, $x \in \mathbb{R}$, and $|h| \leqslant d(x)$. Then we have the inequalities*

$$|d(x+h) - d(x)| \leqslant |h|, \quad d(x+h) \leqslant 2d(x). \tag{2.7}$$

Remark 2.3. The function $d(\cdot)$ was introduced in [1]. Such functions were first used by Otelbaev and therefore all similar functions are called *Otelbaev functions*, see [8].

Definition 2.4 ([2]). Let there be given a positive continuous function $\sigma(t)$ for $t \in \mathbb{R}$, a point x on the real axis, and a sequence of points $\{x_n\}_{n\in N'}$, $N' = \pm1, \pm2, \dots$ Consider the segments

$$\sigma_n = [\sigma_n^{(-)}, \sigma_n^{(+)}], \quad \sigma_n^{(\pm)} = x_n \pm \sigma(x_n), \quad n \in N'.$$

We say that the segments $\{\sigma_n\}_{n=1}^{\infty}$ (or $\{\sigma_n\}_{n=-\infty}^{-1}$) form an $\mathbb{R}(x, \sigma(\cdot))$-covering of the half-axis $[x, \infty)$ (or an $\mathbb{R}(x, \sigma(\cdot))$-covering of the half axis $(-\infty, x]$)), respectively, if the following conditions hold:

(1) $\sigma_n^{(+)} = \sigma_{n+1}^{(-)}$ for $n \geqslant 1$ and $\sigma_{n-1}^{(+)} = \sigma_n^{(-)}$ for $n \leqslant -1$,

(2) $\sigma_1^{(-)} = \sigma_{-1}^{(+)} = x$; $\bigcup\limits_{n\geqslant1} \sigma_n = [x, \infty)$; $\bigcup\limits_{n\leqslant-1} \sigma_n = (-\infty, x]$.

The system of segments $\{\sigma_n\}_{n\in N'}$ is called an $\mathbb{R}(x, \sigma(\cdot))$-*covering* of the real axis, or just an $\mathbb{R}(x, \sigma(\cdot))$-covering. In the latter case, we also say that the covering of $\mathbb{R}$ is generated by the function $\sigma(\cdot)$ (the generating function of the covering) and the point x (the initial point of the covering).

Lemma 2.5 ([2]). *Suppose that a positive, continuous function $\sigma(t)$ satisfies for $t \in \mathbb{R}$ the conditions*

$$\lim_{t\to-\infty}(t + \sigma(t)) = -\infty, \quad \lim_{t\to\infty}(t - \sigma(t)) = \infty. \tag{2.8}$$

Then for every $x \in \mathbb{R}$ there exists an $\mathbb{R}(x, \sigma(\cdot))$-covering of $(-\infty, x]$ and an $\mathbb{R}(x, \sigma(\cdot))$-covering of $[x, \infty)$.

Lemma 2.6 ([2]). *Under criterion* (2.4) *for every $x \in \mathbb{R}$ there exist $\mathbb{R}(x, d(\cdot))$-coverings of $(-\infty, x]$ and of (x, ∞) (here $d(\cdot)$ denotes Otelbaev's function* (2.5)*).*

Remark 2.7. Since in the case of $\mathbb{R}(x, d(\cdot))$-coverings we have chosen the particular function $d(\cdot)$ (see (2.5)) as the generating function of the covering, we denote the segments $\{\sigma_n\}_{n\in\mathbb{N}'}$ by the symbol Δ. Therefore, according to Definition 2.4, we have the relations

$$\Delta(x) = [x - d(x), x + d(x)], \quad x \in \mathbb{R}, \tag{1}$$

$$\Delta_n = [\Delta_n^{(-)}, \Delta_n^{(+)}], \quad \Delta_n^{(\pm)} = x_n \pm d(x_n), \quad n \in \mathbb{N}'. \tag{2}$$

(1′) $\Delta_n^{(+)} = \Delta_{n+1}^{(-)}$ for $n \geqslant 1$ and $\Delta_{n-1}^{(+)} = \Delta_n^{(-)}$ for $n \leqslant -1$,

(2′) $\Delta_1^{(-)} = \Delta_{-1}^{(+)} = x$; $\bigcup\limits_{n\geqslant1} \Delta_n = [x, \infty)$ and $\bigcup\limits_{n\leqslant-1} \Delta_n = (-\infty, x]$.

Note that Definition 2.4 and Lemmas 2.5 and 2.6 can be viewed as modifications of the notions and statements due to Otelbaev, see [8].

3. Results

Our main results are based on Lemmas 3.1 and 3.2.

Lemma 3.1. *Suppose that*

$$\int_0^\infty \mu(t)^p \,\mathrm{d}t = \infty \tag{3.1}$$

and let $f(\cdot)$ be a function with compact support from the class $L_p(\mathbb{R})$. Then the solution $y \in L_{p,\mu}$ of (1.1) *admits the representation*

$$y(x) = \int_x^\infty f(t)\exp\Big(-\int_x^t q(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}t, \quad x \in \mathbb{R}. \tag{3.2}$$

Let us introduce the integral operator

$$(Sf)(x) = \mu(x)\int_x^\infty \frac{1}{\theta(t)}\exp\Big(-\int_x^t q(\xi)\,\mathrm{d}\xi\Big)f(t)\,\mathrm{d}t, \quad x \in \mathbb{R}. \tag{3.3}$$

Lemma 3.2. *We have the following estimates for the norm of the operator $S\colon L_p \to L_p$*

$$S_p \leqslant \|S\|_{p\to p} \leqslant (p)^{1/p}(p')^{1/p'}S_p, \quad p' = \frac{p}{p-1}. \tag{3.4}$$

Here

$$S_p = \sup_{x\in\mathbb{R}} S_p(x), \quad S_p(x) = (J_p^{(-)}(x))^{1/p}(J_{p'}^{(+)}(x))^{1/p'}, \quad x \in \mathbb{R}, \tag{3.5}$$

$$J_p^{(-)}(x) = \int_{-\infty}^x \mu(t)^p \exp\Big(-p\int_t^x q(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}t, \quad x \in \mathbb{R}, \tag{3.6}$$

$$J_{p'}^{(+)}(x) = \int_x^\infty \frac{1}{\theta(t)^{p'}}\exp\Big(-p'\int_x^t q(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}t, \quad x \in \mathbb{R}. \tag{3.7}$$

Our main result is Theorem 3.3.

Theorem 3.3. *Suppose condition* (3.1) *holds and let $p \in (1,\infty)$. Then the pair $\{L_{p,\mu}; L_{p,\theta}\}$ is admissible for* (1.1) *if and only if the operator $S\colon L_p \to L_p$ is bounded.*

Corollary 3.4. *Suppose condition* (3.1) *holds and let $q \in L_1(\mathbb{R})$. Then the pair $\{L_{p,\mu}; L_{p,\theta}\}$ is admissible for* (1.1) *if and only if $A_p < \infty$. Here*

$$A_p = \sup_{x\in\mathbb{R}} A_p(x), \quad A_p(x) = \Big(\int_{-\infty}^x \mu(t)^p\,\mathrm{d}t\Big)^{1/p}\Big(\int_x^\infty \frac{1}{\theta(t)^{p'}}\,\mathrm{d}t\Big)^{1/p'}, \quad x \in \mathbb{R}. \tag{3.8}$$

Corollary 3.5. *Suppose condition* (3.1) *holds and the function* $q(\cdot)$ *from* (1.1) *can be written in the form*

$$q(x) = q_1(x) + q_2(x), \quad x \in \mathbb{R}. \tag{3.9}$$

Suppose that the functions $q_1(\cdot)$ *and* $q_2(\cdot)$ *satisfy the conditions*

$$0 \leqslant q_1(\cdot) \in L_1^{\text{loc}}(\mathbb{R}), \quad \mathbb{P} = \sup_{x,t\in\mathbb{R}} \left| \int_x^t q_2(\xi)\,\mathrm{d}\xi \right| < \infty. \tag{3.10}$$

Then the pair $\{L_{p,\mu}; L_{p,\theta}\}$ *is admissible for* (1.1) *if and only if it is admissible for the equation*

$$-y'(x) + q_1(x)y(x) = f(x), \quad x \in \mathbb{R}. \tag{3.11}$$

Corollary 3.6. *Suppose that the function* $q(\cdot)$ *can be written in the form* (3.9), *where the function* $q_1(x)$ *is positive for* $x \in \mathbb{R}$ *and condition* (3.10) *holds. In addition, suppose that for* $q(\cdot) \equiv q_1(\cdot)$ *equalities* (2.4) *hold. Set*

$$\mu(x) = q_1(x)^{1/p}, \quad \theta(x) = q_1(x)^{-1/p'}, \quad x \in \mathbb{R}. \tag{3.12}$$

Then equality (3.1) *holds and the pair* $\{L_{p,\mu}; L_{p,\theta}\}$ *is admissible for equation* (1.1).

From Theorem 3.3 it follows that its criterion does not give a straightforward description of the set of solutions of the Massera-Schäffer problem. However, for the question on (I)–(II), Theorem 3.3 plays the role of Cauchy's criterion from the theory of number series — any solution of problem (I)–(II) is based on this theorem, in an explicit or implicit way. For example, this is the case for the particular solution of the question on (I)–(II) given in Corollary 3.6, see also the proof of Theorem 1.2 in Section 5. On the other hand, in order to find a general relation between the functions $q(\cdot)$, $\mu(\cdot)$ and $\theta(\cdot)$ which controls the solution of problem (I)–(II), we change the terminology and instead of speaking about the properties of the operator $S\colon L_p \to L_p$ from Theorem 3.3, we study the properties of the above mentioned functions. Note that to achieve this goal, we had to restrict the class of equations (1.1) under consideration as well as the choice of the spaces $\{L_{p,\mu}; L_{p,\theta}\}$.

Let us now turn to precise statements.

Definition 3.7. We say that (1.1) is *standard* if the function $q(x)$ is positive and continuously differentiable for $x \in \mathbb{R}$ and, in addition, satisfies the condition

$$\lim_{|x|\to\infty} \frac{q'(x)}{q^2(x)} = 0. \tag{3.13}$$

We call the function $q(\cdot)$ the *coefficient of a standard equation* (1.1) (or just a standard coefficient) and denote by K the set of all standard coefficients.

Definition 3.8. We say that a *pair* $\{L_{p,\mu}; L_{p,\theta}\}$ *agrees with the standard equation* (1.1) if for every $\alpha > 0$ there exists a constant $c(\alpha) \in [1,\infty)$ such that for all $t, x \in \mathbb{R}$ the functions $q(\cdot)$, $\mu(\cdot)$ and $\theta(\cdot)$ satisfy the inequalities

$$(3.14)\qquad c(\alpha)^{-1}\exp\Big(-\alpha\Big|\int_x^t q(\xi)\,\mathrm{d}\xi\Big|\Big) \leqslant f_k(x,t) \leqslant c(\alpha)\exp\Big(\alpha\Big|\int_x^t q(\xi)\,\mathrm{d}\xi\Big|\Big), \quad k=1,2,$$

where

$$f_1(x,t) = \frac{q(x)}{q(t)}\Big(\frac{\mu(t)}{\mu(x)}\Big)^p, \quad f_2(x,t) = \frac{q(x)}{q(t)}\Big(\frac{\theta(x)}{\theta(t)}\Big)^{p'}, \quad t,x\in\mathbb{R}.$$

To check inequalities (3.14), the following lemma is useful.

Lemma 3.9. *Let* $q(\cdot) \in K$*, suppose that the weights* $\mu(\cdot)$ *and* $\theta(\cdot)$ *are continuously differentiable, and*

$$(3.15)\qquad \lim_{|x|\to\infty}\frac{\mu'(x)}{\mu(x)}\frac{1}{q(x)} = \lim_{|x|\to\infty}\frac{\theta'(x)}{\theta(x)}\frac{1}{q(x)} = 0.$$

Then for $p \in (1,\infty)$ *the pair* $\{L_{p,\mu}; L_{p,\theta}\}$ *agrees with* (1.1)*. In particular, for* $p\in(1,\infty)$ *the pair* $\{L_p; L_p\}$ *agrees with any standard equation* (1.1)*.*

Definition 3.10. We say that *equation* (1.1) is *quasi-standard* if the function $q(\cdot)$ can be written in the form (3.9), where $q_1(\cdot) \in K$ and the function $q_2(\cdot) \in L_1^{\mathrm{loc}}(\mathbb{R})$ satisfies the condition

$$(3.16)\qquad \lim_{|x|\to\infty} \varkappa(x) = 0.$$

Here

$$(3.17)\qquad \varkappa(x) = \sup_{|s|\leqslant 2/q_1(x)}\Big|\int_{x-s}^{x+s} q_2(t)\,\mathrm{d}t\Big|, \quad x\in\mathbb{R}.$$

We call $q_1(\cdot)$ the *principal part* of $q(\cdot)$ and $q_2(\cdot)$ the *perturbation* of $q_1(\cdot)$, or the non-principal part of $q(\cdot)$.

Note that a quasi-standard equation is standard if $q_2(\cdot) = 0$ almost everywhere on $\mathbb{R}$.

Theorem 3.11. *Suppose* (3.1) *holds and* (1.1) *is a quasi-standard equation. Let* $q_1(\cdot)$ *be the principal part of* $q(\cdot)$ *and suppose that the pair* $\{L_{p,\mu}; L_{p,\theta}\}$*,* $p\in(1,\infty)$*, agrees with equation* (3.11)*. Then this pair is admissible for* (1.1) *if and only if* $m(q_1(\cdot),\mu(\cdot),\theta(\cdot)) < \infty$*. Here*

$$(3.18)\qquad m(q_1(\cdot),\mu(\cdot),\theta(\cdot)) = \sup_{x\in\mathbb{R}}\Big(\frac{\mu(x)}{\theta(x)}\frac{1}{q_1(x)}\Big).$$

4. Proofs

P r o o f of Lemma 3.1. Let $f \in L_p$, and suppose that $\operatorname{supp} f = [x_1, x_2]$, $-\infty < x_1 < x_2 < \infty$. Then from (1.1) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\Big[y(\xi)\exp\Big(-\int_0^\xi q(s)\,\mathrm{d}s\Big)\Big] = -f(\xi)\exp\Big(-\int_0^\xi q(s)\,\mathrm{d}s\Big), \quad \xi \in \mathbb{R}.$$

For $x \in \mathbb{R}$ and $t > x$, we get

$$\begin{aligned} y(t)\exp\Big(-\int_0^t q(s)\,\mathrm{d}s\Big) - y(x)\exp\Big(-\int_0^x q(s)\,\mathrm{d}s\Big) \\ = -\int_x^t f(\xi)\exp\Big(-\int_0^\xi q(s)\,\mathrm{d}s\Big)\,\mathrm{d}\xi. \end{aligned} \tag{4.1}$$

In (4.1), set $x = x_2$. Then

$$y(t) = y(x_2)\exp\Big(\int_{x_2}^t q(s)\,\mathrm{d}s\Big), \quad t \geqslant x_2. \tag{4.2}$$

Assume that $y(x_2) \neq 0$. Then (4.2) and the inclusion $y \in L_{p,\mu}$ imply that the relations

$$\infty > \|y\|_{p,\mu}^p \geqslant |y(x_2)|^p \int_{x_2}^\infty \mu(t)^p \exp\Big(p\int_{x_2}^t q(s)\,\mathrm{d}s\Big)\,\mathrm{d}t \geqslant |y(x_2)|^p \int_{x_2}^\infty \mu(t)^p\,\mathrm{d}t = \infty$$

hold and we arrive at a contradiction. Hence, $y(x_2) = 0$ and therefore $y(t) = 0$ for $t \geqslant x_2$, see (4.2). Thus from (4.1) we obtain, as $t \to \infty$:

$$\begin{aligned} y(x)\exp\Big(-\int_0^x q(s)\,\mathrm{d}s\Big) &= \lim_{t\to\infty}\int_x^t f(\xi)\exp\Big(-\int_0^\xi q(s)\,\mathrm{d}s\Big)\,\mathrm{d}\xi \\ &= \int_x^\infty f(\xi)\exp\Big(-\int_0^\xi q(s)\,\mathrm{d}s\Big)\,\mathrm{d}\xi. \end{aligned}$$

The latter equality gives (3.2). □

P r o o f of Lemma 3.2. The operator S (see (3.3)) can be written in a different way:

$$(Sf)(x) = \mu_1(x)\int_x^\infty \theta_1(t)f(t)\,\mathrm{d}t, \quad x \in \mathbb{R}, \tag{4.3}$$

where

$$\mu_1(x) = \mu(x)\exp\Big(\int_0^x q(\xi)\,\mathrm{d}\xi\Big), \quad \theta_1(x) = \frac{1}{\theta(x)}\exp\Big(-\int_0^x q(\xi)\,\mathrm{d}\xi\Big), \quad x \in \mathbb{R}.$$

By Theorem 2.1, this gives the estimates

$$\widetilde{H}_p \leqslant \|S\|_{p\to p} \leqslant (p)^{1/p}(p')^{1/p'}\widetilde{H}_p, \quad \widetilde{H}_p = \sup_{x\in\mathbb{R}} \widetilde{H}_p(x).$$

Here

$$\widetilde{H}_p(x) = \left(\int_{-\infty}^{x} \mu_1^p(t)\,\mathrm{d}t\right)^{1/p}\left(\int_{x}^{\infty} \theta_1^{p'}(t)\,\mathrm{d}t\right)^{1/p'}, \quad x \in \mathbb{R}. \tag{4.4}$$

From (4.4) and the obvious relations

$$\int_0^t q(\xi)\,\mathrm{d}\xi = \int_0^x q(\xi)\,\mathrm{d}\xi - \int_t^x q(\xi)\,\mathrm{d}\xi \quad \text{for } t \leqslant x,$$
$$\int_0^t q(\xi)\,\mathrm{d}\xi = \int_0^x q(\xi)\,\mathrm{d}\xi + \int_x^t q(\xi)\,\mathrm{d}\xi \quad \text{for } t \geqslant x$$

together with (3.6) and (3.7), we get the equalities

$$\widetilde{H}_p(x) = S_p(x), \quad x \in \mathbb{R}, \quad \widetilde{H}_p = S_p,$$

which proves the lemma. □

P r o o f of Theorem 3.3. *Necessity.* Let x_1 and x_2 be arbitrary numbers ($x_1 < x_2$). Set

$$f(t) = \begin{cases} 0 & \text{if } t \notin [x_1, x_2], \\ \theta(t)^{-p'} \exp\left(-(p'-1)\int_0^t q(s)\,\mathrm{d}s\right) & \text{if } t \in [x_1, x_2]. \end{cases} \tag{4.5}$$

Then we have the obvious relations

$$\begin{aligned} \|f\|_{p,\theta}^p &= \int_{-\infty}^{\infty} |\theta(t)f(t)|^p\,\mathrm{d}t \\ &= \int_{x_1}^{x_2} \frac{\theta(t)^p}{\theta(t)^{pp'}} \exp\left(-p(p'-1)\int_0^t q(s)\,\mathrm{d}s\right)\mathrm{d}t \\ &= \int_{x_1}^{x_2} \frac{1}{\theta(t)^{p'}} \exp\left(-p'\int_0^t q(s)\,\mathrm{d}s\right)\mathrm{d}t < \infty. \end{aligned} \tag{4.6}$$

Since the pair $\{L_{p,\mu}; L_{p,\theta}\}$ is admissible for (1.1), (1.1) with $f(\cdot)$ from (4.5) has a unique solution $y \in L_{p,\mu}$ in view of (4.6). By Lemma 3.1, this solution is given by formula (3.2). This gives the estimate

$$
\begin{aligned}
\|y\|_{p,\mu}^p &= \int_{-\infty}^{\infty} \Big|\mu(x) \int_x^{\infty} f(t) \exp\Big(-\int_x^t q(s)\,\mathrm{d}s\Big)\,\mathrm{d}t\Big|^p \mathrm{d}x \\
&\geqslant \int_{-\infty}^{x_1} \mu(x)^p \exp\Big(p\int_0^x q(s)\,\mathrm{d}s\Big) \\
&\quad\times \Big[\int_x^{\infty} f(t) \exp\Big(-\int_0^t q(s)\,\mathrm{d}s\Big)\,\mathrm{d}t\Big]^p \mathrm{d}x \\
&= \int_{-\infty}^{x_1} \mu(x)^p \exp\Big(p\int_0^x q(s)\,\mathrm{d}s\Big) \\
&\quad\times \Big[\int_x^{x_2} f(t) \exp\Big(-\int_0^t q(s)\,\mathrm{d}s\Big)\,\mathrm{d}t\Big]^p \mathrm{d}x \\
&\geqslant \Big[\int_{-\infty}^{x_1} \mu(x)^p \exp\Big(p\int_0^x q(s)\,\mathrm{d}s\Big)\,\mathrm{d}x\Big] \\
&\quad\times \Big[\int_{x_1}^{x_2} \frac{1}{\theta(t)^{p'}} \exp\Big(-p'\int_0^t q(s)\,\mathrm{d}s\Big)\,\mathrm{d}t\Big]^p.
\end{aligned} \tag{4.7}
$$

On the other hand, the solution (3.2) satisfies inequality (1.5), in which the constant c does not depend on the choice of $f \in L_{p,\theta}$. Therefore, from (4.6) and (4.7) we get

$$
\begin{aligned}
&\Big[\int_{-\infty}^{x_1} \mu(x)^p \exp\Big(p\int_0^x q(s)\,\mathrm{d}s\Big)\,\mathrm{d}x\Big] \\
&\qquad\times \Big[\int_{x_1}^{x_2} \frac{1}{\theta(x)^{p'}} \exp\Big(-p'\int_0^x q(s)\,\mathrm{d}s\Big)\,\mathrm{d}x\Big]^p \\
&\quad\leqslant \|y\|_{p,\mu}^p \leqslant c^p \|f\|_{p,\theta}^p \\
&\quad= c^p \int_{x_1}^{x_2} \frac{1}{\theta(x)^{p'}} \exp\Big(-p'\int_0^x q(s)\,\mathrm{d}s\Big)\,\mathrm{d}x \\
&\Rightarrow \Big[\int_{-\infty}^{x_1} \mu(x)^p \exp\Big(\int_0^x q(s)\,\mathrm{d}s\Big)\,\mathrm{d}x\Big]^{1/p} \\
&\qquad\times \Big[\int_{x_1}^{x_2} \frac{1}{\theta(x)^{p'}} \exp\Big(-p'\int_0^x q(s)\,\mathrm{d}s\Big)\,\mathrm{d}x\Big]^{1/p'} \leqslant c, \quad x_1, x_2 \in \mathbb{R}.
\end{aligned} \tag{4.8}
$$

In (4.8), the numbers x_1 and x_2 ($x_1 < x_2$) are arbitrary. Therefore, from (4.8) it follows that $\widetilde{H}_p(x) \leqslant c$, $x \in \mathbb{R}$, see (4.4). Hence, (see the proof of Lemma 3.2), we obtain $S_p < \infty$. It remains to refer to (3.4).

Sufficiency. Let us show that under the assumptions of the theorem, the requirements of Definition 1.1 hold. Since $\|S\|_{p\to p} < \infty$, we have also $S_p < \infty$, see (3.5).

Therefore, the integrals $J_p^{(-)}(x)$ and $J_p^{(+)}(x)$ (see (3.6), (3.7)) converge for all $x \in \mathbb{R}$. We introduce the function $y(x)$, $x \in \mathbb{R}$, of (3.2). This function is well-defined because by Hölder's inequality the integral in (3.2) converges:

$$\begin{aligned} |y(x)| &\leqslant \left[\int_x^\infty \frac{1}{\theta(t)^{p'}} \exp\left(-p' \int_x^t q(\xi)\,\mathrm{d}\xi\right) \mathrm{d}t\right]^{1/p'} \|f\|_{p,\theta} \\ &= (J_{p'}^{(+)}(x))^{1/p'} \|f\|_{p,\theta}, \quad x \in \mathbb{R}. \end{aligned}$$

From (3.2), we deduce the obvious formula

$$y(x) = \exp\left(\int_0^x q(\xi)\,\mathrm{d}\xi\right)\left(\int_x^\infty \exp\left(-\int_0^t q(\xi)\,\mathrm{d}\xi\right) f(t)\,\mathrm{d}t\right), \quad x \in \mathbb{R}. \tag{4.9}$$

We then compute the derivative of $y(\cdot)$ written in the form (4.9) and obtain that $y(\cdot)$ is a solution of (1.1). In addition, $y \in L_{p,\mu}$ because $\|S\|_{p\to p} < \infty$:

$$\|y\|_{p,\mu} = \|S(\theta f)\|_p \leqslant \|S\|_{p\to p}\|f\|_{p,\theta} < \infty.$$

Let us check that $y(\cdot)$ is a unique solution of (1.1) in the class $L_{p,\mu}$. Indeed, the general solution $Y(x)$, $x \in \mathbb{R}$, of equation (1.1) is of the form

$$Y(x) = cz(x) + y(x), \quad x \in \mathbb{R},\ c = \text{const.},$$

where the function $y(\cdot)$ is defined in (3.2) and

$$z(x) = \exp\left(\int_0^x q(\xi)\,\mathrm{d}\xi\right), \quad x \in \mathbb{R}.$$

Assume that $Y(\cdot) \in L_{p,\mu}$. Then also $cz(\cdot) \in L_{p,\mu}$ because $y \in L_{p,\mu}$. Hence,

$$\begin{aligned} \infty > |c|^p \|z\|_{p,\mu}^p &= |c|^p \int_{-\infty}^\infty |\mu(x) z(x)|^p \,\mathrm{d}x \\ &\geqslant |c|^p \int_0^\infty \mu(x)^p \exp\left(p \int_0^x q(\xi)\,\mathrm{d}\xi\right) \mathrm{d}x \\ &\geqslant |c|^p \int_0^\infty \mu(x)^p \,\mathrm{d}x = \infty \quad \text{if } c \neq 0. \end{aligned}$$

Therefore, $c = 0$, as required. □

P r o o f of Corollary 3.4. *Necessity.* Put $Q = \|q\|_1$. If the pair $\{L_{p,\mu}; L_{p,\theta}\}$ is admissible for (1.1), then $\|S\|_{p\to p} < \infty$ by Theorem 3.3, and therefore $S_p < \infty$ by Lemma 3.2. This implies (see (3.5), (3.6), (3.7) and (3.8)) that

$$\begin{aligned}\infty > S_p &= \sup_{x\in\mathbb{R}}\biggl[\int_{-\infty}^{x}\mu(t)^p\exp\biggl(-p\int_t^x q(\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}t\biggr]^{1/p}\\ &\quad\times\biggl[\int_x^{\infty}\frac{1}{\theta(t)^{p'}}\exp\biggl(-p'\int_x^t q(\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}t\biggr]^{1/p'}\\ &\geqslant \sup_{x\in\mathbb{R}}\biggl[\int_{-\infty}^{x}\mu(t)^p\exp(-pQ)\,\mathrm{d}t\biggr]^{1/p}\biggl[\int_x^{\infty}\frac{1}{\theta(t)^{p'}}\exp(-p'Q)\,\mathrm{d}t\biggr]^{1/p'}\\ &= \exp(-2Q)A_p.\end{aligned}$$

Sufficiency. Let $A_p < \infty$. Then (see (3.5), (3.6), (3.7) and (3.8))

$$\begin{aligned}\infty > A_p &= \sup_{x\in\mathbb{R}}\biggl[\int_{-\infty}^{x}\mu(t)^p\,\mathrm{d}t\biggr]^{1/p}\biggl[\int_x^{\infty}\frac{\mathrm{d}t}{\theta(t)^{p'}}\biggr]^{1/p'}\\ &\geqslant \sup_{x\in\mathbb{R}}\biggl[\int_{-\infty}^{x}\mu(t)^p\exp\biggl(-p\int_t^x q(\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}t\biggr]^{1/p}\\ &\quad\times\biggl[\int_x^{\infty}\frac{1}{\theta(t)^{p'}}\exp\biggl(-p'\int_x^t q(\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}t\biggr]^{1/p'}\\ &= S_p.\end{aligned}$$

Hence, $S_p < \infty$, and therefore $\|S\|_{p\to p} < \infty$, see (3.4). It remains to refer to Theorem 3.3. □

P r o o f of Corollary 3.5. To prove this statement, we have to use the line of reasoning of the proof of Corollary 3.4 and the obvious inequalities (see (3.4), (3.5), (3.6), (3.7) and (3.10)):

$$\begin{aligned}&\exp(-p\mathbb{P})\int_{-\infty}^{x}\mu(t)^p\biggl(-p\int_t^x q_1(\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}t\\ &\qquad\leqslant J_p^{(-)}(x)\leqslant \exp(p\mathbb{P})\int_{-\infty}^{x}\mu(t)^p\exp\biggl(-p\int_t^x q_1(\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}t;\\ &\exp(-p'\mathbb{P})\int_x^{\infty}\frac{1}{\theta(t)^{p'}}\biggl(-p\int_x^t q_1(\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}t\\ &\qquad\leqslant J_p^{(+)}(x)\leqslant \exp(p'\mathbb{P})\int_x^{\infty}\frac{1}{\theta(t)^{p'}}\exp\biggl(-p\int_x^t q_1(\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}t.\end{aligned}$$

□

Proof of Corollary 3.6. Equality (3.1) in case (3.12) can be checked in a straightforward way. Let us show that the following estimates hold (see (3.6), (3.7) and (3.10)):

$$J_p^{(-)}(x) \leqslant \frac{\exp(p\mathbb{P})}{p}, \quad J_{p'}^{(+)}(x) \leqslant \frac{\exp(p'\mathbb{P})}{p'}, \quad x \in \mathbb{R}. \tag{4.10}$$

Both estimates in (4.10) are checked in the same way. Consider, say, the second one:

$$\begin{aligned}
J_{p'}^{(+)}(x) &= \int_x^\infty \frac{1}{\theta(t)^{p'}} \exp\Big(-p' \int_x^t q(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}t \\
&= \int_x^\infty \frac{1}{\theta(t)^{p'}} \exp\Big(-p' \int_x^t q_1(\xi)\,\mathrm{d}\xi - p' \int_x^t q_2(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}t \\
&\leqslant \int_x^\infty q_1(t) \exp\Big(-p' \int_x^t q_1(\xi)\,\mathrm{d}\xi + p' \Big|\int_x^t q_2(\xi)\,\mathrm{d}\xi\Big|\Big)\,\mathrm{d}t \\
&\leqslant \int_x^\infty q_1(t) \exp\Big(-p' \int_x^t q_1(\xi)\,\mathrm{d}\xi\Big) \exp\Big(p' \sup_{x,t\in\mathbb{R}} \Big|\int_x^t q_2(\xi)\,\mathrm{d}\xi\Big|\Big)\,\mathrm{d}t \\
&\leqslant \exp(p'\mathbb{P}) \int_x^\infty q_1(t) \exp\Big(-p' \int_x^t q_1(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}t \\
&= \exp(p'\mathbb{P}) \Big[-\frac{1}{p'} \exp\Big(-p' \int_x^t q_1(\xi)\Big)\Big|_{t=x}^{t=\infty}\Big] \\
&= \frac{\exp(p'\mathbb{P})}{p'} \Big(1 - \exp\Big(-p' \int_x^\infty q_1(\xi)\,\mathrm{d}\xi\Big)\Big) \\
&= \frac{\exp(p'\mathbb{P})}{p'}.
\end{aligned}$$

Note that here we used the equality $\|q_1(\cdot)\|_{L_1(0,\infty)} = \infty$ which holds by the asumption of Corollary 3.6. Now from (3.4), (3.5), and (4.10), it follows that $\|S\|_{p\to p} < \infty$. It remains to refer to Theorem 3.3. □

Proof of Lemma 3.9. The assertions of the lemma for the functions $\mu(\cdot)$ and $\theta(\cdot)$ are checked in the same way; therefore, we only consider the case of the function $\mu(\cdot)$. By (3.13) and (3.15), for a given $\beta > 0$, there exists $x_0(\beta)$ such that

$$\frac{|\mu'(t)|}{\mu(t)} \frac{1}{q(t)} \leqslant \frac{\beta}{2}, \quad \frac{|q'(t)|}{q^2(t)} \leqslant \frac{\beta}{2} \quad \text{for } |t| \geqslant x_0(\beta). \tag{4.11}$$

Then from the obvious equality

$$\frac{1}{\mu(t)} \Big(\frac{\mu(t)}{q(t)}\Big)' = \frac{\mu'(t)}{\mu(t)} \frac{1}{q(t)} - \frac{q'(t)}{q^2(t)}, \quad t \in \mathbb{R},$$

and (4.11), we obtain the estimates

$$(4.12)\qquad -\beta \leqslant \frac{1}{\mu(t)}\Big(\frac{\mu(t)}{q(t)}\Big)' \leqslant \beta \quad \text{for } |t| \geqslant x_0(\beta)$$
$$\Rightarrow -\beta q(t) \leqslant \Big(\frac{\mu(t)}{q(t)}\Big)'\Big(\frac{\mu(t)}{q(t)}\Big)^{-1} \leqslant \beta q(t) \quad \text{for } |t| \geqslant x_0(\beta).$$

Put

$$(4.13)\qquad \begin{aligned} m(\beta) &= \min_{t\in[-x_0(\beta),x_0(\beta)]} \frac{\mu(t)}{q(t)},\\ M(\beta) &= \max_{t\in[-x_0(\beta),x_0(\beta)]} \frac{\mu(t)}{q(t)},\\ c(\beta) &= \max\Big\{\frac{1}{m(\beta)}, M(\beta)\Big\}. \end{aligned}$$

In the following table, we present all possible locations of the points $x_0(\beta)$, t and x on $\mathbb{R}$.

Case 1.1	Case 1.2	Case 1.3
$t \leqslant -x_0(\beta)$ $x \leqslant -x_0(\beta)$	$-x_0(\beta) \leqslant t \leqslant x_0(\beta)$ $x \leqslant -x_0(\beta)$	$t \geqslant x_0(\beta)$ $x \leqslant -x_0(\beta)$
Case 2.1	Case 2.2	Case 2.3
$t \leqslant -x_0(\beta)$ $-x_0(\beta) \leqslant x \leqslant x_0(\beta)$	$-x_0(\beta) \leqslant t \leqslant x_0(\beta)$ $-x_0(\beta) \leqslant x \leqslant x_0(\beta)$	$t \geqslant x_0(\beta)$ $-x_0(\beta) \leqslant x \leqslant x_0(\beta)$
Case 3.1	Case 3.2	Case 3.3
$t \leqslant -x_0(\beta)$ $x \geqslant x_0(\beta)$	$-x_0(\beta) \leqslant t \leqslant x_0(\beta)$ $x \geqslant x_0(\beta)$	$t \geqslant x_0(\beta)$ $x \geqslant x_0(\beta)$

Table 1.

Let us check that in all the cases of Table 1 we have the inequalities (see (4.13))

$$(4.14)\qquad c(\beta)^{-2}\exp\Big(-\beta\Big|\int_x^t q(\xi)\,\mathrm{d}\xi\Big|\Big) \leqslant \frac{\mu(t)}{q(t)}\frac{q(x)}{\mu(x)} \leqslant c(\beta)^2\exp\Big(\beta\Big|\int_x^t q(\xi)\,\mathrm{d}\xi\Big|\Big).$$

Case 1.1 and *Case 3.3*: Both the cases are treated in the same way. For example, in Case 3.3 for $t \geqslant x \geqslant x_0(\beta)$ (and similarly for $x \geqslant t \geqslant x_0(\beta)$) from (4.12) we get

$$(4.15)\qquad \exp\Big(-\beta\Big|\int_x^t q(\xi)\,\mathrm{d}\xi\Big|\Big) \leqslant \exp\Big(-\beta\int_x^t q(\xi)\,\mathrm{d}\xi\Big) \leqslant \frac{\mu(t)}{q(t)}\frac{q(x)}{\mu(x)}$$
$$\leqslant \exp\Big(\beta\int_x^t q(\xi)\,\mathrm{d}\xi\Big) \leqslant \exp\Big(\beta\Big|\int_x^t q(\xi)\,\mathrm{d}\xi\Big|\Big).$$

Case 1.2 and *Case 2.1*: Both the cases are treated in the same way using (4.15). For example, in Case 1.2, we have

$$\begin{aligned}
\frac{\mu(t)}{q(t)}\Big(\frac{\mu(x)}{q(x)}\Big)^{-1} &= \Big[\frac{\mu(t)}{q(t)}\Big(\frac{\mu(-x_0(\beta))}{q(-x_0(\beta))}\Big)^{-1}\Big]\Big[\frac{\mu(-x_0(\beta))}{q(-x_0(\beta))}\Big(\frac{\mu(x)}{q(x)}\Big)^{-1}\Big] \\
&\leqslant \frac{M(\beta)}{m(\beta)}\exp\Big(\beta\Big|\int_x^{-x_0(\beta)} q(\xi)\,\mathrm{d}\xi\Big|\Big) \\
&\leqslant c(\beta)^2\exp\Big(\beta\Big|\int_x^{-x_0(\beta)} q(\xi)\,\mathrm{d}\xi + \int_{-x_0(\beta)}^t q(\xi)\,\mathrm{d}\xi\Big|\Big) \\
&= c(\beta)^2\exp\Big(\beta\Big|\int_x^t q(\xi)\,\mathrm{d}\xi\Big|\Big), \\
\frac{\mu(t)}{q(t)}\Big(\frac{\mu(x)}{q(x)}\Big)^{-1} &= \Big[\frac{\mu(t)}{q(t)}\Big(\frac{\mu(-x_0(\beta))}{q(-x_0(\beta))}\Big)^{-1}\Big]\Big[\frac{\mu(-x_0(\beta))}{q(-x_0(\beta))}\Big(\frac{\mu(x)}{q(x)}\Big)^{-1}\Big] \\
&\geqslant \frac{m(\beta)}{M(\beta)}\exp\Big(-\beta\Big|\int_x^{-x_0(\beta)} q(\xi)\,\mathrm{d}\xi\Big|\Big) \\
&\geqslant c(\beta)^{-2}\exp\Big(-\beta\Big|\int_x^{-x_0(\beta)} q(\xi)\,\mathrm{d}\xi + \int_{-x_0(\beta)}^t q(\xi)\,\mathrm{d}\xi\Big|\Big) \\
&\geqslant c(\beta)^{-2}\exp\Big(-\beta\Big|\int_x^t q(\xi)\,\mathrm{d}\xi\Big|\Big).
\end{aligned}$$

Case 1.3 and *Case 3.1*: Both the cases are treated in the same way using (4.15). For example, in Case 1.3, we have

$$\begin{aligned}
&\frac{\mu(t)}{q(t)}\Big(\frac{\mu(x)}{q(x)}\Big)^{-1} \\
&\quad= \Big[\frac{\mu(-x_0(\beta))}{q(-x_0(\beta))}\Big(\frac{\mu(x)}{q(x)}\Big)^{-1}\Big]\Big[\Big(\frac{\mu(-x_0(\beta))}{q(-x_0(\beta))}\Big)^{-1}\frac{\mu(x_0(\beta))}{q(x_0(\beta))}\Big]\Big[\Big(\frac{\mu(x_0(\beta))}{q(x_0(\beta))}\Big)^{-1}\frac{\mu(t)}{q(t)}\Big] \\
&\quad\leqslant c(\beta)^2\exp\Big(\beta\Big|\int_x^{-x_0(\beta)} q(\xi)\,\mathrm{d}\xi\Big| + \beta\Big|\int_{x_0(\beta)}^t q(\xi)\,\mathrm{d}\xi\Big|\Big) \\
&\quad\leqslant c(\beta)^2\exp\Big(\beta\Big|\int_x^{-x_0(\beta)} q(\xi)\,\mathrm{d}\xi + \int_{-x_0(\beta)}^{x_0(\beta)} q(\xi)\,\mathrm{d}\xi + \int_{x_0(\beta)}^t q(\xi)\,\mathrm{d}\xi\Big|\Big) \\
&\quad\leqslant c(\beta)^2\exp\Big(\beta\Big|\int_x^t q(\xi)\,\mathrm{d}\xi\Big|\Big), \\
&\frac{\mu(t)}{q(t)}\Big(\frac{\mu(x)}{q(x)}\Big)^{-1} \\
&\quad= \Big[\frac{\mu(-x_0(\beta))}{q(-x_0(\beta))}\Big(\frac{\mu(x)}{q(x)}\Big)^{-1}\Big]\Big[\Big(\frac{\mu(-x_0(\beta))}{q(-x_0(\beta))}\Big)^{-1}\frac{\mu(x_0(\beta))}{q(x_0(\beta))}\Big]\Big[\Big(\frac{\mu(x_0(\beta))}{q(x_0(\beta))}\Big)^{-1}\frac{\mu(t)}{q(t)}\Big] \\
&\quad\geqslant c(\beta)^{-2}\exp\Big(-\beta\Big|\int_x^{-x_0(\beta)} q(\xi)\,\mathrm{d}\xi\Big| - \beta\Big|\int_{x_0(\beta)}^t q(\xi)\,\mathrm{d}\xi\Big|\Big)
\end{aligned}$$

$$\geqslant c(\beta)^{-2}\exp\Big(-\beta\Big|\int_x^{-x_0(\beta)} q(\xi)\,\mathrm{d}\xi+\int_{-x_0(\beta)}^{x_0(\beta)} q(\xi)\,\mathrm{d}\xi+\int_{x_0(\beta)}^{t} q(\xi)\,\mathrm{d}\xi\Big|\Big)$$
$$= \ \leqslant c(\beta)^{-2}\exp\Big(-\beta\Big|\int_x^t q(\xi)\,\mathrm{d}\xi\Big|\Big).$$

Case 2.2: Here inequalities (4.14) follow from (4.13)

$$\frac{\mu(t)}{q(t)}\Big(\frac{\mu(x)}{q(x)}\Big)^{-1}\leqslant\frac{M(\beta)}{m(\beta)}\leqslant c(\beta)^2\exp\Big(\beta\Big|\int_x^t q(\xi)\,\mathrm{d}\xi\Big|\Big),$$
$$\frac{\mu(t)}{q(t)}\Big(\frac{\mu(x)}{q(x)}\Big)^{-1}\geqslant c(\beta)^{-2}\exp\Big(-\beta\Big|\int_x^t q(\xi)\,\mathrm{d}\xi\Big|\Big).$$

Case 2.3 and *Case 3.2*: Both the cases are treated in the same way using (4.15). For example, in Case 2.3, we have

$$\begin{aligned}\frac{\mu(t)}{q(t)}\Big(\frac{\mu(x)}{q(x)}\Big)^{-1}&=\Big[\frac{\mu(t)}{q(t)}\Big(\frac{\mu(x_0(\beta))}{q(x_0(\beta))}\Big)^{-1}\Big]\Big[\frac{\mu(x_0(\beta))}{q(x_0(\beta))}\Big(\frac{\mu(x)}{q(x)}\Big)^{-1}\Big]\\&\leqslant c^2(\beta)\exp\Big(\beta\Big|\int_{x_0(\beta)}^t q(\xi)\,\mathrm{d}\xi\Big|+\beta\Big|\int_x^{x_0(\beta)} q(\xi)\,\mathrm{d}\xi\Big|\Big)\\&\leqslant c^2(\beta)\exp\Big(\beta\Big|\int_x^t q(\xi)\,\mathrm{d}\xi\Big|\Big),\\\frac{\mu(t)}{q(t)}\Big(\frac{\mu(x)}{q(x)}\Big)^{-1}&=\Big[\frac{\mu(t)}{q(t)}\Big(\frac{\mu(x_0(\beta))}{q(x_0(\beta))}\Big)^{-1}\Big]\Big[\frac{\mu(x_0(\beta))}{q(x_0(\beta))}\Big(\frac{\mu(x)}{q(x)}\Big)\Big]\\&\geqslant c(\beta)^{-2}\exp\Big(-\beta\Big|\int_{x_0(\beta)}^t q(\xi)\,\mathrm{d}\xi\Big|-\beta\Big|\int_x^{x_0(\beta)} q(\xi)\,\mathrm{d}\xi\Big|\Big)\\&\geqslant c(\beta)^{-2}\exp\Big(-\beta\Big|\int_x^t q(\xi)\,\mathrm{d}\xi\Big|\Big).\end{aligned}$$

Thus, inequalities (4.14) are proven.

Let us emphasize that in (4.14) t and x are arbitrary points from $\mathbb{R}$; in particular, this implies that for any $t,x\in\mathbb{R}$ the inequalities (4.14) remain true after switching x for t and t for x. In (4.14), set $\mu(x)\equiv 1$, $x\in\mathbb{R}$. Then we obtain that if $q(\cdot)\in K$, then for all x and t from $\mathbb{R}$ we have the estimates

$$c(\beta)^{-2}\exp\Big(-\beta\Big|\int_x^t q(\xi)\,\mathrm{d}\xi\Big|\Big)\leqslant\frac{q(t)}{q(x)}\leqslant c(\beta)^2\exp\Big(\beta\Big|\int_x^t q(\xi)\,\mathrm{d}\xi\Big|\Big).\tag{4.16}$$

In particular, from (4.16) it follows that for $p\in(1,\infty)$ the pair $\{L_p;L_p\}$ agrees with (1.1) if $q(\cdot)\in K$. Further, write (4.14) in a different way:

$$\begin{aligned}c(\beta)^{-2}\Big(\frac{q(t)}{q(x)}\Big)^{1/p'}&\exp\Big(-\beta\Big|\int_x^t q(\xi)\,\mathrm{d}\xi\Big|\Big)\\&\leqslant\frac{\mu(t)}{\mu(x)}\Big(\frac{q(x)}{q(t)}\Big)^{1/p}\leqslant c(\beta)^2\Big(\frac{q(t)}{q(x)}\Big)^{1/p'}\exp\Big(\beta\Big|\int_x^t q(\xi)\,\mathrm{d}\xi\Big|\Big),\quad x,t\in\mathbb{R}.\end{aligned}\tag{4.17}$$

Then from (4.16) and (4.17), we obtain the relations

$$(4.18)\quad \Big(\frac{\mu(t)}{\mu(x)}\Big)^p \frac{q(x)}{q(t)} \leqslant c(\beta)^{4p-2} \exp\biggl((2p-1)\beta\biggl|\int_x^t q(\xi)\,\mathrm{d}\xi\biggr|\biggr), \quad t, x \in \mathbb{R},$$

$$(4.19)\quad \Big(\frac{\mu(t)}{\mu(x)}\Big)^p \frac{q(x)}{q(t)} \geqslant c(\beta)^{-(4p-2)} \exp\biggl(-(2p-1)\beta\biggl|\int_x^t q(\xi)\,\mathrm{d}\xi\biggr|\biggr), \quad t, x \in \mathbb{R}.$$

Now for a given $\alpha > 0$, set

$$\beta = \frac{\alpha}{2p-1}, \quad c(\alpha) := c(\beta)\big|_{\beta=\alpha/(2p-1)},$$

and inequalities (4.18) and (4.19) take the form (3.14). □

Let us go over to some auxiliary assertions needed for the proof of Theorem 3.11.

Lemma 4.1. *If* $q(\cdot) \in K$, *then equalities* (2.4) *hold.*

Proof. Both equalities in (2.4) are checked in the same way. Therefore, we only check the second one. Put $B = \|q\|_{L_1(\mathbb{R}_+)}$ and assume that $B < \infty$. Then for $\beta = 1$ from (4.16) we obtain

$$\begin{aligned}\infty > B &= \int_0^\infty q(t)\,\mathrm{d}t = q(0)\int_0^\infty \frac{q(t)}{q(0)}\,\mathrm{d}t \geqslant c(1)^{-2} q(0) \int_0^\infty \exp\biggl(-\biggl|\int_0^t q(\xi)\,\mathrm{d}\xi\biggr|\biggr) \\ &\geqslant c(1)^{-2} q(0)\int_0^\infty \exp(-B)\,\mathrm{d}t \\ &= \infty.\end{aligned}$$

We get a contradiction. □

Lemma 4.2. *Let* $q(\cdot) \in K$ *and let* $d(\cdot)$ *be the Otelbaev function, see Definition* 3.7 *and* (2.5), (2.6). *Then*

$$(4.20)\qquad \lim_{x\to-\infty}(x + d(x)) = -\infty; \quad \lim_{x\to\infty}(x - d(x)) = \infty.$$

Proof. The existence of the function $d(\cdot)$ follows from (2.4), see [1]. Further, both equalities (4.20) are checked in the same way; consider, say, the second one. Assume the contrary. Then there exist a constant $c \in [1, \infty)$ and a sequence $\{x_n\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty} x_n = \infty, \quad x_n - d(x_n) \leqslant c < \infty, \quad x_n \geqslant c,\ n = 1, 2, \ldots$$

By (2.4) and (2.6), this implies that

$$2 = \int_{x_n - d(x_n)}^{x_n + d(x_n)} q(\xi)\,\mathrm{d}\xi \geqslant \int_c^{x_n} q(\xi)\,\mathrm{d}\xi \to \infty \quad \text{as } n \to \infty.$$

We get a contradiction. Hence, equalities (4.20) hold. □

Lemma 4.3. *Let $q(\cdot) \in K$. Then*

$$q(x)d(x) \leqslant 2 \quad \textit{if } |x| \gg 1. \tag{4.21}$$

P r o o f. Put (see Remark 2.7)

$$\varepsilon(x) = 2 \sup_{t \in \Delta(x)} \frac{|q'(t)|}{q(t)^2}, \quad x \in \mathbb{R}. \tag{4.22}$$

Clearly, (4.20) and (3.13) imply that

$$\lim_{|x| \to \infty} \varepsilon(x) = 0. \tag{4.23}$$

Furthermore, the cases $x \ll -1$ and $x \gg 1$ in (4.21) are treated in the same way. Therefore, below we only consider the case $x \gg 1$.

From (2.6), the Schwarz inequality and (4.23), we obtain

$$\begin{aligned}
2d(x) &= \int_{x-d(x)}^{x+d(x)} \frac{\sqrt{q(t)}}{\sqrt{q(t)}} \,\mathrm{d}t \leqslant \Big[\int_{\Delta(x)} q(t)\,\mathrm{d}t\Big]^{1/2} \Big[\int_{\Delta(x)} \frac{\mathrm{d}t}{q(t)}\Big]^{1/2} \\
&\leqslant \sqrt{2}\Big[\frac{2d(x)}{q(x)} + \int_{\Delta(x)} \Big(\frac{1}{q(t)} - \frac{1}{q(x)}\Big)\,\mathrm{d}t\Big]^{1/2} \\
&\leqslant \sqrt{2}\Big[\frac{2d(x)}{q(x)} + \int_{\Delta(x)} \Big|\int_x^t \frac{|q'(\xi)|}{q(\xi)^2}\,\mathrm{d}\xi\Big|\,\mathrm{d}t\Big]^{1/2} \\
&\leqslant \sqrt{2}\Big[\frac{2d(x)}{q(x)} + \frac{\varepsilon(x)}{2}\int_{\Delta(x)} |t-x|\,\mathrm{d}t\Big]^{1/2} \\
&= \sqrt{2}\Big[\frac{2d(x)}{q(x)} + \frac{\varepsilon(x)}{2} d^2(x)\Big]^{1/2} \\
\Rightarrow d^2(x)\Big(2 - \frac{\varepsilon(x)}{2}\Big) &\leqslant \frac{2d(x)}{q(x)} \Rightarrow d(x)q(x) \leqslant \frac{2}{2 - \frac{1}{2}\varepsilon(x)} \leqslant 2 \quad \text{for } x \gg 1.
\end{aligned}$$

□

Lemma 4.4. *Let $q(\cdot) \in K$, $|x| \gg 1$, $t \in \Delta(x)$. Then we have the estimates, see* (4.22)

$$1 - \varepsilon(x) \leqslant \frac{1}{1+\varepsilon(x)} \leqslant \frac{q(t)}{q(x)} \leqslant \frac{1}{1-\varepsilon(x)}. \tag{4.24}$$

P r o o f. Let $|x| \gg 1$, $t \in \Delta(x)$. We now use (4.22) and (4.21) to get

$$\Big|\frac{1}{q(t)} - \frac{1}{q(x)}\Big| \leqslant \Big|\int_x^t \frac{|q'(\xi)|}{q(\xi)^2}\,\mathrm{d}\xi\Big| \leqslant \frac{\varepsilon(x)}{2}|t-x| \leqslant \frac{\varepsilon(x)d(x)}{2} \leqslant \frac{\varepsilon(x)}{q(x)}.$$

The obtained estimate implies inequalities (4.24). □

Lemma 4.5. *Let $q(\cdot) \in K$ and $|x| \gg 1$. Then*

$$1 - \varepsilon(x) \leqslant q(x)d(x) \leqslant 1 + \varepsilon(x). \tag{4.25}$$

P r o o f. We now use relations (2.6) and (4.24):

$$2 = \int_{x-d(x)}^{x+d(x)} q(t)\,\mathrm{d}t = \int_{x-d(x)}^{x+d(x)} \frac{q(t)}{q(x)} q(x)\,\mathrm{d}t \leqslant \frac{2d(x)}{1-\varepsilon(x)} q(x),$$
$$2 = \int_{x-d(x)}^{x+d(x)} q(t)\,\mathrm{d}t = \int_{x-d(x)}^{x+d(x)} \frac{q(t)}{q(x)} q(x)\,\mathrm{d}t \geqslant \frac{2d(x)}{1+\varepsilon(x)} q(x).$$

The inequalities obtained imply estimates (4.25). □

Lemma 4.6. *If $q(\cdot) \in K$, then the Otelbaev function $d(x)$, $x \in \mathbb{R}$, corresponding to $q(\cdot)$ is differentiable for all $x \in \mathbb{R}$ and*

$$\lim_{|x|\to\infty} d'(x) = 0. \tag{4.26}$$

P r o o f. Since $q(\cdot) \in K$, then the function $q(\xi)$ is continuous for all $\xi \in \mathbb{R}$ and therefore by the theorem of implicit functions (see [4], Chapter X, Section 5), the function $d(x)$, $x \in \mathbb{R}$, is everywhere differentiable. Therefore, from (2.6) we get

$$\begin{aligned} 0 &= \Big[\int_{x-d(x)}^{x+d(x)} q(t)\,\mathrm{d}t\Big]' \\ &= (1+d'(x))q(x+d(x)) - (1-d'(x))q(x-d(x)), \quad x \in \mathbb{R} \\ \Rightarrow d'(x) &= -\frac{q(x+d(x)) - q(x-d(x))}{q(x+d(x)) + q(x-d(x))}, \quad x \in \mathbb{R}. \end{aligned} \tag{4.27}$$

Note that from (4.21), (4.22) and (4.24), it follows that

$$\int_{\Delta(x)} |q'(x)|\,\mathrm{d}t = \int_{\Delta(x)} \frac{|q'(t)|}{q(t)^2} \Big(\frac{q(t)}{q(x)}\Big)^2 q(x)^2\,\mathrm{d}t \leqslant \frac{2\varepsilon(x)}{(1-\varepsilon(x))^2} q(x). \tag{4.28}$$

Therefore, by (4.28) we get

$$|q(x+d(x)) - q(x-d(x))| \leqslant \int_{\Delta(x)} |q'(t)|\,\mathrm{d}t \leqslant \frac{2\varepsilon(x)}{(1-\varepsilon(x))^2} q(x), \tag{4.29}$$

$$\begin{aligned} q(x+d(x)) + q(x-d(x)) &= 2q(x) + \int_x^{x+d(x)} q'(t)\,\mathrm{d}t - \int_{x-d(x)}^{x} q'(t)\,\mathrm{d}t \\ &\geqslant 2q(x) - \int_{\Delta(x)} |q'(t)| \geqslant 2q(x) - \frac{2\varepsilon(x)}{(1-\varepsilon(x))^2} q(x). \end{aligned} \tag{4.30}$$

Now from (4.23), (4.27), (4.29) and (4.30), we get (4.26)

$$|d'(x)| \leqslant \frac{\delta(x)}{1-\delta(x)} \leqslant 2\delta(x), \quad \delta(x) = \frac{\varepsilon(x)}{(1-\varepsilon(x))^2} \to 0 \quad \text{as } |x| \to \infty.$$

□

Lemma 4.7. *If* $q(\cdot) \in K$, *then*

$$\lim_{|x|\to\infty} |x|q(x) = \infty. \tag{4.31}$$

P r o o f. Equality (4.31) is considered in the same way for the cases $x \to -\infty$ and $x \to \infty$, and below we only treat the second case. By (4.20), (4.26) and l'Hôpital's rule, we have

$$\lim_{x\to\infty} \frac{x-d(x)}{x} = \lim_{x\to\infty} \frac{1-d'(x)}{1} = 1.$$

Hence, we have the asymptotic equality

$$x - d(x) = x(1-\nu(x)), \ \lim_{x\to\infty} \nu(x) = 0 \quad \Rightarrow \quad x\nu(x) = d(x), \ \lim_{x\to\infty} \nu(x) = 0. \tag{4.32}$$

Since according to (4.25) we have the inequalities

$$2^{-1} \leqslant q(x)d(x) \leqslant 2 \quad \text{for all } |x| \gg 1, \tag{4.33}$$

by (4.32) and (4.33) we get

$$\frac{1}{2\nu(x)} \leqslant xq(x) \leqslant \frac{2}{\nu(x)}, \quad \lim_{|x|\to\infty} \nu(x) = 0.$$

These relations imply equality (4.31). □

Corollary 4.8. *Let* $f(x)$ *be defined, positive, and continuously differentiable for all* $x \in \mathbb{R}$. *Suppose, in addition, that* $\lim_{|x|\to\infty} f'(x) = 0$. *Then*

$$\lim_{|x|\to\infty} \frac{f(x)}{x} = 0.$$

P r o o f. Set

$$q(x) = \frac{1}{f(x)}, \quad x \in \mathbb{R}.$$

The assertion then follows from Lemma 4.7. □

Lemma 4.9. *If* (1.1) *is a quasi-standard equation, then equalities* (2.4) *hold, the function* $d(x)$, $x \in \mathbb{R}$, *see* (2.5), *is well-defined, and for every* $x \in \mathbb{R}$ *there exist* $\mathbb{R}(x, d(\cdot))$*-coverings of the half-axes* $(-\infty, x]$ *and* $[x, \infty)$, *see Remark* 2.7.

P r o o f. Equalities (2.4) are checked in the same way, so we only consider the second one. Since $q_1 \in K$ and (3.16) holds, there exists $x_0 \gg 1$ such that $\varkappa(x) \leqslant 1$ and the estimate (4.21) holds for all $|x| \geqslant x_0$. Put

$$\tau(x) = \frac{2}{q_1(x)}, \quad x \in \mathbb{R}, \quad \omega(x) = [\omega^{(-)}(x), \omega^{(+)}(x)], \quad \omega^{(\pm)}(x) = x \pm \tau(x). \tag{4.34}$$

Then by (4.31) we have

$$\lim_{x\to\infty} (x - \tau(x)) = \lim_{x\to\infty} x\Big(1 - \frac{2}{xq_1(x)}\Big) = \infty. \tag{4.35}$$

Therefore, by Lemma 2.5 there exists an $\mathbb{R}(x_0, \tau(\cdot))$-covering of the half axis $[x_0, \infty)$ by segments $\{\omega_n\}_{n=1}^{\infty}$, where (see Definition 2.4)

$$\omega_n = [x_n - \tau(x_n), x_n + \tau(x_n)] = \Big[x_n - \frac{2}{q_1(x_n)}, x_n + \frac{2}{q_1(x_n)}\Big], \quad n \geqslant 1. \tag{4.36}$$

On the other hand, since $q_1(\cdot) \in K$, by Lemma 4.1 equalities (2.4) hold for the function $q_1(\cdot)$. Hence, one can define the function

$$d_1(x) = \inf_{d>0}\Big\{d\colon \int_{x-d}^{x+d} q_1(t)\,\mathrm{d}t = 2\Big\} \tag{4.37}$$

for which we have the equality (see (2.5) and (2.6))

$$\int_{x-d_1(x)}^{x+d_1(x)} q_1(t)\,\mathrm{d}t = 2, \quad x \in \mathbb{R}. \tag{4.38}$$

By (4.36), Lemma 4.3 and the choice of x_0, we get the inclusions

$$\Delta_n = [x_n - d_1(x_n), x_n + d_1(x_n)] \subseteq \Big[x_n - \frac{2}{q_1(x_n)}, x_n + \frac{2}{q_1(x_n)}\Big] = \omega_n, \quad n \geqslant 1.$$

Then by Definition 2.4 and the choice of x_0,

$$\begin{aligned}\int_{x_0}^{\infty} q(t)\,\mathrm{d}t &= \sum_{n=1}^{\infty}\int_{\omega_n} q(t)\,\mathrm{d}t = \sum_{n=1}^{\infty}\Big[\int_{\omega_n} q_1(t)\,\mathrm{d}t + \int_{\omega_n} q_2(t)\,\mathrm{d}t\Big] \\ &\geqslant \sum_{n=1}^{\infty}\Big[\int_{\Delta_n} q_1(t)\,\mathrm{d}t - \Big|\int_{\omega_n} q_2(t)\,\mathrm{d}t\Big|\Big] \geqslant \sum_{n=1}^{\infty}(2 - \varkappa(x_n)) \geqslant \sum_{n=1}^{\infty}(2-1) = \infty.\end{aligned}$$

The remaining assertions of the lemma now follow from (2.5) and Lemma 2.6. □

Lemma 4.10. *Let* (1.1) *be a quasi-standard equation. For a fixed* $x \in \mathbb{R}$, *consider the segments* $\{\Delta_n\}_{n=-\infty}^{-1}$ *and* $\{\Delta_n\}_{n=1}^{\infty}$ *from* $\mathbb{R}(x, d(\cdot))$*-coverings of the half-axes* $(-\infty, x]$ *and* $[x, \infty)$, *respectively. Then we have the equalities*

$$\int_{\Delta_1^{(-)}}^{\Delta_n^{(+)}} q(\xi)\,\mathrm{d}\xi = 2n, \quad n \geqslant 1, \qquad \int_{\Delta_n^{(-)}}^{\Delta_{-1}^{(+)}} q(\xi)\,\mathrm{d}\xi = 2|n|, \quad n \leqslant -1. \tag{4.39}$$

If, in addition, there exists $\delta \in (0, 2]$ *such that for every* $t \in \mathbb{R}$, *we have the relations*

$$\delta d(t) \leqslant d(s) \leqslant 2d(t) \quad \text{if } s \in \Delta(t) = [\Delta^{(-)}(t), \Delta^{(+)}(t)], \ \ \Delta^{(\pm)}(t) = t \pm d(t), \tag{4.40}$$

and the inequalities

$$\delta \int_{\Delta_1^{(-)}}^{\Delta_n^{(+)}} \frac{\mathrm{d}\xi}{d(\xi)} \leqslant \int_{\Delta_1^{(-)}}^{\Delta_n^{(+)}} q(\xi)\,\mathrm{d}\xi \leqslant 2 \int_{\Delta_1^{(-)}}^{\Delta_n^{(+)}} \frac{\mathrm{d}\xi}{d(\xi)}, \quad n \geqslant 1, \tag{4.41}$$

$$\delta \int_{\Delta_n^{(-)}}^{\Delta_{-1}^{(+)}} \frac{\mathrm{d}\xi}{d(\xi)} \leqslant \int_{\Delta_n^{(-)}}^{\Delta_{-1}^{(+)}} q(\xi)\,\mathrm{d}\xi \leqslant 2 \int_{\Delta_n^{(-)}}^{\Delta_{-1}^{(+)}} \frac{\mathrm{d}\xi}{d(\xi)}, \quad n \leqslant -1, \tag{4.42}$$

$$1 \leqslant \int_{t-d(t)}^{t+d(t)} \frac{\mathrm{d}\xi}{d(\xi)} \leqslant \frac{2}{\delta}. \tag{4.43}$$

P r o o f. Note that the upper estimate in (4.40) is an a priori one, see (2.7). Let us check the first equality in (4.39) (the second one can be established in a similar way). Now we use Definition 2.4 and (2.6) to get

$$\int_{\Delta_1^{(-)}}^{\Delta_n^{(+)}} q(\xi)\,\mathrm{d}\xi = \sum_{k=1}^{n} \int_{\Delta_k} q(\xi)\,\mathrm{d}\xi = \sum_{k=1}^{n} \int_{x_k - d(x_k)}^{x_k + d(x_k)} q(\xi)\,\mathrm{d}\xi = \sum_{k=1}^{n} 2 = 2n. \tag{4.44}$$

In the proof of estimates (4.41) (inequalities (4.42) are checked in a similar way), we use relations (4.40) and (4.44):

$$\int_{\Delta_1^{(-)}}^{\Delta_n^{(+)}} q(\xi)\,\mathrm{d}\xi = \sum_{k=1}^{n} 2 = \sum_{k=1}^{n} \int_{\Delta_k^{(-)}}^{\Delta_k^{(+)}} \frac{d(\xi)}{d(x_k)} \frac{\mathrm{d}\xi}{d(\xi)} \geqslant \sum_{k=1}^{n} \int_{\Delta_k^{(-)}}^{\Delta_k^{(+)}} \delta \frac{\mathrm{d}\xi}{d(\xi)} = \delta \int_{\Delta_1^{(-)}}^{\Delta_n^{(+)}} \frac{\mathrm{d}\xi}{d(\xi)},$$

$$\begin{aligned} \int_{\Delta_1^{(-)}}^{\Delta_n^{(+)}} q(\xi)\,\mathrm{d}\xi &= \sum_{k=1}^{n} \int_{\Delta_k^{(-)}}^{\Delta_k^{(+)}} \frac{d(\xi)}{d(x_k)} \frac{\mathrm{d}\xi}{d(\xi)} \leqslant \sum_{k=1}^{n} \int_{\Delta_n^{(-)}}^{\Delta_k^{(+)}} 2 \frac{\mathrm{d}\xi}{d(\xi)} \\ &= 2 \sum_{k=1}^{n} \int_{\Delta_k} \frac{\mathrm{d}\xi}{d(\xi)} = 2 \int_{\Delta_1^{(-)}}^{\Delta_n^{(+)}} \frac{\mathrm{d}\xi}{d(\xi)}. \end{aligned}$$

In the proof of estimates (4.43) we use (4.40) to obtain

$$\int_{t-d(t)}^{t+d(t)} \frac{\mathrm{d}s}{d(s)} = \int_{t-d(t)}^{t+d(t)} \frac{d(t)}{d(s)} \frac{\mathrm{d}s}{d(t)} \leqslant \int_{t-d(t)}^{t+d(t)} \frac{1}{\delta} \frac{\mathrm{d}s}{d(t)} \leqslant \frac{2}{\delta},$$
$$\int_{t-d(t)}^{t+d(t)} \frac{\mathrm{d}s}{d(s)} = \int_{t-d(t)}^{t+d(t)} \frac{d(t)}{d(s)} \frac{\mathrm{d}s}{d(t)} \geqslant \int_{t-d(t)}^{t+d(t)} \frac{1}{2d(t)} \, \mathrm{d}t = 1.$$

□

Lemma 4.11. *Under the assumptions of Lemma* 4.10, *we have the inequalities*

$$(4.45)\quad c^{-1} \int_{-\infty}^{x} \mu(t)^p \exp\Big(-2p \int_t^x \frac{\mathrm{d}\xi}{d(\xi)}\Big) \, \mathrm{d}t \leqslant J_p^{(-)}(x) \leqslant c \int_{-\infty}^{x} \mu(t)^p \exp\Big(-\delta p \int_t^x \frac{\mathrm{d}\xi}{d(\xi)}\Big) \, \mathrm{d}t, \quad x \in \mathbb{R},\ p \in (1,\infty);$$

$$(4.46)\quad c^{-1} \int_{x}^{\infty} \frac{1}{\theta(t)^{p'}} \exp\Big(-2p' \int_x^t \frac{\mathrm{d}\xi}{d(\xi)}\Big) \, \mathrm{d}t \leqslant J_{p'}^{(+)}(x) \leqslant c \int_{x}^{\infty} \frac{1}{\theta(t)^{p'}} \exp\Big(-\delta p' \int_x^t \frac{\mathrm{d}\xi}{d(\xi)}\Big) \, \mathrm{d}t, \quad x \in \mathbb{R},\ p' = \frac{p}{p-1}.$$

P r o o f. Inequalities (4.45) and (4.46) are checked in the same way. Therefore, we only consider (4.46). Moreover, the proof of (4.46) differs from the proof of (4.41) only in applying, in a an obvious way, (4.41) and (4.43), so that no additional comments are needed.

Thus we arrive at the lower estimate in (4.46):

$$(4.47)\qquad \begin{aligned} J_{p'}^{(+)}(x) &= \int_x^\infty \frac{1}{\theta(t)^{p'}} \exp\Big(-p' \int_x^t q(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}t \\ &= \sum_{n=1}^\infty \int_{\Delta_n} \frac{1}{\theta(t)^{p'}} \exp\Big(-p' \int_{\Delta_1^{(-)}}^t q(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}t \\ &= \sum_{n=1}^\infty \int_{\Delta_n} \frac{1}{\theta(t)^{p'}} \exp\Big(-p' \Big(\int_{\Delta_1^{(-)}}^{\Delta_n^{(+)}} q(\xi)\,\mathrm{d}\xi - \int_t^{\Delta_n^{(+)}} q(\xi)\,\mathrm{d}\xi\Big)\Big)\,\mathrm{d}t \\ &\geqslant \sum_{n=1}^\infty \int_{\Delta_n} \frac{1}{\theta(t)^{p'}} \exp\Big(-p' \int_{\Delta_1^{(-)}}^{\Delta_n^{(+)}} q(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}t \\ &\geqslant \sum_{n=1}^\infty \int_{\Delta_n} \frac{1}{\theta(t)^{p'}} \exp\Big(-2p' \int_{\Delta_1^{(-)}}^{\Delta_n^{(+)}} \frac{\mathrm{d}\xi}{d(\xi)}\Big)\,\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \int_{\Delta_n} \frac{1}{\theta(t)^{p'}} \exp\Big(-2p'\Big(\int_{\Delta_1^{(-)}}^{t} \frac{\mathrm{d}\xi}{d(\xi)} + \int_t^{\Delta_n^{(+)}} \frac{\mathrm{d}\xi}{d(\xi)}\Big)\Big)\,\mathrm{d}t \\
&\geqslant \sum_{n=1}^{\infty} \int_{\Delta_n} \frac{1}{\theta(t)^{p'}} \exp\Big(-2p'\Big(\int_{\Delta_1^{(-)}}^{t} \frac{\mathrm{d}\xi}{d(\xi)} + \int_{\Delta_n^{(-)}}^{\Delta_n^{(+)}} \frac{\mathrm{d}\xi}{d(\xi)}\Big)\Big)\,\mathrm{d}t \\
&\geqslant \sum_{n=1}^{\infty} \int_{\Delta_n} \frac{1}{\theta(t)^{p'}} \exp\Big(-2p'\Big(\int_{\Delta_1^{(-)}}^{t} \frac{\mathrm{d}\xi}{d(\xi)} + \frac{2}{\delta}\Big)\Big)\,\mathrm{d}t \\
&= \exp\Big(-\frac{4p'}{\delta}\Big) \sum_{n=1}^{\infty} \int_{\Delta_n} \frac{1}{\theta(t)^{p'}} \exp\Big(-2p'\int_x^t \frac{\mathrm{d}\xi}{d(\xi)}\Big)\,\mathrm{d}t \\
&= \exp\Big(-\frac{4p'}{\delta}\Big) \int_x^{\infty} \frac{1}{\theta(t)^{p'}} \exp\Big(-2p'\int_x^t \frac{\mathrm{d}\xi}{d(\xi)}\Big)\,\mathrm{d}t.
\end{aligned}
$$

In a similar way, we obtain the upper estimate in (4.46):

$$
\begin{aligned}
(4.48)\quad J_{p'}^{(+)}(x) &= \int_x^{\infty} \frac{1}{\theta(t)^{p'}} \exp\Big(-p'\int_x^t q(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}t \\
&= \sum_{n=1}^{\infty} \int_{\Delta_n} \frac{1}{\theta(t)^{p'}} \exp\Big(-p'\int_{\Delta_1^{(-)}}^t q(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}t \\
&= \sum_{n=1}^{\infty} \int_{\Delta_n} \frac{1}{\theta(t)^{p'}} \exp\Big(-p'\Big(\int_{\Delta_1^{(-)}}^{\Delta_n^{(+)}} q(\xi)\,\mathrm{d}\xi - \int_t^{\Delta_n^{(+)}} q(\xi)\,\mathrm{d}\xi\Big)\Big)\,\mathrm{d}t \\
&\leqslant \sum_{n=1}^{\infty} \int_{\Delta_n} \frac{1}{\theta(t)^{p'}} \exp\Big(-p'\delta\int_{\Delta_1^{(-)}}^{\Delta_n^{(+)}} \frac{\mathrm{d}\xi}{d(\xi)} + p'\int_{\Delta_n^{(-)}}^{\Delta_n^{(+)}} q(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}t \\
&= \sum_{n=1}^{\infty} \int_{\Delta_n} \frac{1}{\theta(t)^{p'}} \exp\Big(-p'\delta\int_{\Delta_1^{(-)}}^{\Delta_n^{(+)}} \frac{\mathrm{d}\xi}{d(\xi)} + 2p'\Big)\,\mathrm{d}t \\
&\leqslant \exp(2p') \sum_{n=1}^{\infty} \int_{\Delta_n} \frac{1}{\theta(t)^{p'}} \exp\Big(-p'\delta\Big(\int_{\Delta_1^{(-)}}^{t} \frac{\mathrm{d}\xi}{d(\xi)} + \int_t^{\Delta_n^{(+)}} \frac{\mathrm{d}\xi}{d(\xi)}\Big)\Big)\,\mathrm{d}t \\
&\leqslant \exp(2p') \sum_{n=1}^{\infty} \int_{\Delta_n} \frac{1}{\theta(t)^{p'}} \exp\Big(-p'\delta\int_{\Delta_1^{(-)}}^{t} \frac{\mathrm{d}\xi}{d(\xi)}\Big)\,\mathrm{d}t \\
&= \exp(2p') \int_x^{\infty} \frac{1}{\theta(t)^{p'}} \exp\Big(-p'\delta\int_x^t \frac{\mathrm{d}\xi}{d(\xi)}\Big)\,\mathrm{d}t.
\end{aligned}
$$

□

In the next lemmas we show that under the assumptions of Theorem 3.11, there exists $\delta \in (0,2]$ such that inequalities (4.40) hold. (Recall that under condition (2.4), the upper estimate in (4.40) is an a priori one, see (2.7).) The requirements of Theorem 3.11 are assumed to hold below and are not mentioned in the statements.

Below we also need one simple assertion from [2]. To state it, for a fixed $x \in \mathbb{R}$, let us introduce the function $F(x,\eta)$ with $\eta \geqslant 0$:

$$F(x,\eta) = \int_{x-\eta}^{x+\eta} q(t)\,\mathrm{d}t, \quad \eta \geqslant 0,\ x \in \mathbb{R}.$$

Lemma 4.12 ([2]). *Suppose that* (2.4) *holds. Then for every* $x \in \mathbb{R}$ *the inequality* $\eta \geqslant d(x)$ (*or* $0 \leqslant \eta \leqslant d(x)$) *holds if and only if* $F(x,\eta) \geqslant 2$ (*or* $F(x,\eta) \leqslant 2$), *respectively.*

Below, together with (4.34), we use the following notation (see (3.17) and (4.22)):

$$l(x) = \varepsilon(x) + \varkappa(x), \quad x \in \mathbb{R}. \tag{4.49}$$

Clearly, from the hypotheses of Theorem 3.11 it follows (see (3.16), (4.23)) that

$$\lim_{|x|\to\infty} l(x) = 0. \tag{4.50}$$

Lemma 4.13. *For* $|x| \gg 1$ *the function* $d(x)$, $x \in \mathbb{R}$, *satisfies the inequalities*

$$\frac{1-l(x)}{q_1(x)} \leqslant d(x) \leqslant \frac{1+l(x)}{q_1(x)}. \tag{4.51}$$

P r o o f. Let $t \in \omega(x) = [x - 2/q_1(x), x + 2/q_1(x)]$, $|x| \gg 1$, see (4.34). Then we have the relations, see the proof of Lemma 4.4

(4.52)

$$\Big|\frac{1}{q_1(t)} - \frac{1}{q_1(x)}\Big| \leqslant \Big|\int_x^t \frac{|q_1'(\xi)|}{q_1(\xi)^2}\,\mathrm{d}\xi\Big| \leqslant \Big(\sup_{\xi\in\omega(x)} \frac{|q_1'(\xi)}{q_1(\xi)^2}\Big)|t-x| \leqslant \frac{\varepsilon(x)}{2}|t-x| \leqslant \frac{\varepsilon(x)}{q_1(x)}$$

$$\Rightarrow \frac{1-\varepsilon(x)}{q_1(x)} \leqslant \frac{1}{q_1(t)} \leqslant \frac{1+\varepsilon(x)}{q_1(x)} \quad \text{if } t \in \omega(x)$$

$$\Rightarrow 1-\varepsilon(x) \leqslant \frac{1}{1+\varepsilon(x)} \leqslant \frac{q_1(t)}{q_1(x)} \leqslant \frac{1}{1-\varepsilon(x)} \quad \text{for } t \in \omega(x),\ |x| \gg 1.$$

Set now

$$\eta(x) = \frac{1+l(x)}{q_1(x)}, \quad |x| \gg 1.$$

Then $[x-\eta(x), x+\eta(x)] \subseteq \omega(x)$ (see (4.50)) and we get

$$\begin{aligned} F(x,\eta(x)) &= \int_{x-\eta(x)}^{x+\eta(x)} q(t)\,\mathrm{d}t = \int_{x-\eta(x)}^{x+\eta(x)} q_1(t)\,\mathrm{d}t + \int_{x-\eta(x)}^{x+\eta(x)} q_2(t)\,\mathrm{d}t \\ &\geqslant \int_{x-\eta(x)}^{x+\eta(x)} \frac{q_1(t)}{q_1(x)} q_1(x)\,\mathrm{d}t - \varkappa(x) \geqslant \frac{2\eta(x)}{1+\varepsilon(x)} q_1(x) - \varkappa(x) \\ &= 2 + \frac{1-\varepsilon(x)}{1+\varepsilon(x)}\varkappa(x) \geqslant 2, \end{aligned}$$

see (3.17), (4.52). Hence, $\eta(x) \geqslant d(x)$ for $|x| \gg 1$ by Lemma 4.12. Set now

$$\eta(x) = \frac{1 - l(x)}{q_1(x)}, \quad |x| \gg 1.$$

Then, taking into account (4.49), (4.50) and (4.51), we get

$$\begin{aligned} F(x, \eta(x)) &= \int_{x-\eta(x)}^{x+\eta(x)} q(t)\,\mathrm{d}t = \int_{x-\eta(x)}^{x+\eta(x)} q_1(t)\,\mathrm{d}t + \int_{x-\eta(x)}^{x+\eta(x)} q_2(t)\,\mathrm{d}t \\ &\leqslant \int_{x-\eta(x)}^{x+\eta(x)} \frac{q_1(t)}{q_1(x)} q_1(x)\,\mathrm{d}t + \varkappa(x) \leqslant \frac{2\eta(x)}{1-\varepsilon(x)} q_1(x) + \varkappa(x) \\ &= 2 - \frac{1+\varepsilon(x)}{1-\varepsilon(x)} \varkappa(x) \leqslant 2. \end{aligned}$$

Hence, $0 \leqslant \eta(x) \leqslant d(x)$ for $|x| \gg 1$ by Lemma 4.12. □

Corollary 4.14. *We have*

$$\frac{1}{2} \leqslant \frac{d(t)}{d(x)} \leqslant 2 \quad \textit{for } t \in \Delta(x) = [x - d(x), x + d(x)],\ |x| \gg 1. \tag{4.53}$$

Proof. The upper estimate in (4.53) holds for all $x \in \mathbb{R}$, see (2.7). Furthermore, from (4.51) and (4.50) it follows that $\Delta(x) \subseteq \omega(x)$ if $|x| \gg 1$. Then, if $t \in \Delta(x)$ and $|x| \gg 1$, by (4.35) and Lemma 4.13, we have

$$\frac{1-l(t)}{q_1(t)} \leqslant d(t) \leqslant \frac{1+l(t)}{q_1(t)} \quad \text{if } t \in \Delta(x),\ |x| \gg 1. \tag{4.54}$$

Hence, for $t \in \Delta(x)$ and $|x| \gg 1$, by (4.50), (4.51), (4.52) and (4.54), we get

$$\frac{d(t)}{d(x)} \geqslant \frac{1-l(t)}{q_1(t)} \frac{q_1(x)}{1+l(x)} \geqslant \frac{1-l(t)}{1+l(x)}(1-\varepsilon(x)) \geqslant \frac{1}{2}.$$

□

Corollary 4.15. *There exists $\delta \in (0, 2]$ such that for every $t \in \mathbb{R}$ inequalities* (4.40) *hold.*

Proof. The upper estimate in (4.40) follows from (2.7). Furthermore, there exists $x_0 \gg 1$ such that inequalities (4.40) hold for $\delta = \frac{1}{2}$ and $|x| \geqslant x_0$, see (4.53). Let $|x| \leqslant x_0$. Since the function $d(\cdot)$ is continuous and positive (see (2.5), (2.6), (2.7)), the segments $\Delta(t)$, $\Delta(x)$ are finite for $t \in \Delta(x)$, $|x| \leqslant x_0$, and therefore there exist $m \in (0, \infty)$, $M \in (0, \infty)$ such that $0 < m \leqslant d(t)$, $d(x) \leqslant M$, for $t \in \Delta(x)$, $|x| \leqslant x_0$.

Hence, $d(t)/d(x) > \delta_1 > 0$ for $t \in \Delta(x)$, $|x| \leqslant x_0$, $\delta_1 = m/M$. Clearly, for $\delta = \min\{\frac{1}{2}; \delta_1\}$ we get (4.40). □

Corollary 4.16. *There exists a constant* $c \in [1, \infty)$ *such that*

$$c^{-1} \leqslant q_1(x)d(x) \leqslant c \quad \text{for } x \in \mathbb{R}. \tag{4.55}$$

Proof. Let $f(x) = d(x)q_1(x)$, $x \in \mathbb{R}$. Then there exists $x_0 \gg 1$ such that $f(x) \in [2^{-1}, 2]$ if $|x| \geqslant x_0$ (see (4.51)). On the other hand, the function $f(x)$ is positive and continuous for $x \in [-x_0, x_0]$ (see (2.7) and Definition 3.10). This implies (4.55). □

Thus, under the hypotheses of Theorem 3.11, Corollaries 4.15 and 4.16 hold. Let $\delta > 0$ and $c \in [1, \infty)$ be the constants defined there, put

$$c_1 = \max\Big\{2, c; \frac{4}{\delta}; \frac{c}{\delta}\Big\}. \tag{4.56}$$

Corollary 4.17. *For* $x \in \mathbb{R}$ *and* $p \in (1, \infty)$, $p' = p(p-1)^{-1}$, *we have the inequalities (see* (3.6), (3.7)*):*

$$\exp(-c_1 p) \int_{-\infty}^{x} \mu(t)^p \exp\Big(-c_1 p \int_t^x q_1(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}t \leqslant J_p^{(-)}(x) \leqslant \exp(c_1 p) \int_{-\infty}^{x} \mu(t)^p \exp\Big(-\frac{p}{c_1} \int_t^x q_1(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}t, \tag{4.57}$$

$$\exp(-c_1 p') \int_x^{\infty} \frac{1}{\theta(t)^{p'}} \exp\Big(-c_1 p' \int_x^t q_1(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}t \leqslant J_{p'}^{(+)}(x) \leqslant \exp(c_1 p') \int_x^{\infty} \frac{1}{\theta(t)^{p'}} \exp\Big(-\frac{p'}{c_1} \int_x^t q_1(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}t. \tag{4.58}$$

Proof. Both inequalities are proved in the same way, so below we only consider (4.58). Since equation (1.1) is quasi-standard, by Lemma 4.10 we have a well-defined function $d(x)$, $x \in \mathbb{R}$, and there exists an $\mathbb{R}(x, d(\cdot))$-covering of the semi-axis $[x, \infty)$. By the estimates (4.47), (4.48), (4.55) and (4.56) this implies that

$$\begin{aligned} J_{p'}^{(+)}(x) &\geqslant \exp\Big(-\frac{4p'}{\delta}\Big) \int_x^{\infty} \frac{1}{\theta(t)^{p'}} \exp\Big(-2p' \int_x^t \frac{\mathrm{d}\xi}{d(\xi)}\Big)\,\mathrm{d}t \\ &\geqslant \exp(-c_1 p') \int_x^{\infty} \frac{1}{\theta(t)^{p'}} \exp\Big(-c_1 p' \int_x^t q_1(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}t, \\ J_{p'}^{(+)}(x) &\leqslant \exp(2p') \int_x^{\infty} \frac{1}{\theta(t)^{p'}} \exp\Big(-p'\delta \int_x^t \frac{\mathrm{d}\xi}{d(\xi)}\Big)\,\mathrm{d}t \\ &\leqslant \exp(c_1 p') \int_x^{\infty} \frac{1}{\theta(t)^{p'}} \exp\Big(-\frac{p'}{c_1} \int_x^t q_1(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}t. \end{aligned}$$

□

Remark 4.18. By the assumption of Theorem 3.11, the pair $\{L_{p,\mu}; L_{p,\theta}\}$ agrees with equation (3.11), i.e., for $\alpha > 0$ there exists a constant $c(\alpha) \in [1, \infty)$ such that estimates (3.14) hold, see Definition 3.8. Thus, in the proof of Lemma 4.19 below we need different constants $\alpha > 0$ and constants $c(\alpha)$ corresponding to each other by (3.14). All these pairs are collected in the following table.

α	p'	p	$\frac{p'}{2c_1}$	$\frac{p}{2c_1}$
$c(\alpha)$	$c_2(p')$	$c_3(p)$	$c_4\Big(\frac{p'}{2c_1}\Big)$	$c_5\Big(\frac{p}{2c_1}\Big)$

Table 2.

Here $p \in (1, \infty)$, $p' = p(p-1)^{-1}$ and the constant c_1 is defined in (4.56). Throughout the sequel we use the notation from Table 2 without any additional comments.

Lemma 4.19. *For $p \in (1, \infty)$ and $x \in \mathbb{R}$, we have the inequalities* (*see* (3.6), (3.7), (4.57) *and* (4.58))*:*

$$\frac{\exp(-c_1 p)}{(1+c_1)pc_3(p)}\frac{\mu(x)^p}{q_1(x)} \leqslant J_p^{(-)}(x) \leqslant \frac{2c_1}{p}c_5(p)\exp(c_1 p)\frac{\mu(x)^p}{q_1(x)}, \tag{4.59}$$

$$\frac{\exp(-c_1 p')}{(1+c_1)p'c_2(p')}\frac{1}{\theta(x)^{p'}q_1(x)} \leqslant J_p^{(+)}(x) \leqslant \frac{2c_1}{p'}c_4\Big(\frac{p'}{2c_1}\Big)\exp(c_1 p')\frac{1}{\theta(x)^{p'}q_1(x))}. \tag{4.60}$$

P r o o f. Inequalities (4.59) and (4.60) are continuations of estimates (4.57) and (4.58) and are proved in the same way. Therefore, below we only consider (4.60). Since the pair $\{L_{p,\mu}; L_{p,\theta}\}$ agrees with equation (3.11), according to Table 2, (4.58) and Lemma 4.1, we have for all $x \in \mathbb{R}$ that

$$\begin{aligned}
J_{p'}^{(+)}(x) &\geqslant \frac{\exp(-c_1 p')}{\theta(x)^{p'}q_1(x)}\int_x^\infty \Big[\Big(\frac{\theta(x)}{\theta(t)}\Big)^{p'}\frac{q_1(x)}{q_1(t)}\Big]q_1(t)\exp\Big(-c_1 p'\int_x^t q_1(\xi)\,d\xi\Big)\,dt\\
&\geqslant \frac{\exp(-c_1 p')}{c_2(p')\theta(x)^{p'}q_1(x)}\int_x^\infty q_1(t)\exp\Big(-(1+c_2)p'\int_x^t q_1(\xi)\,d\xi\Big)\,dt\\
&= \frac{\exp(-c_1 p')}{(1+c_1)p'c_2(p')}\frac{1}{\theta(x)^{p'}q_1(x)}\Big[-\exp\Big(-(1+c_1)p'\int_x^t q_1(\xi)\Big)\,d\xi\Big|_{t=x}^{t=\infty}\Big]\\
&= \frac{\exp(-c_1 p')}{(1+c_1)p'c_2(p')}\frac{1}{\theta(x)^{p'}q_1(x)}.
\end{aligned}$$

Similarly, using the upper estimate in (4.58), we obtain the second inequality in (4.60):

$$\begin{aligned}
J_{p'}^{(+)}(x) &\leqslant \frac{\exp(c_1p')}{\theta(x)^{p'}q_1(x)}\int_x^\infty\Big[\Big(\frac{\theta(x)}{\theta(t)}\Big)^{p'}\frac{q_1(x)}{q_1(t)}\Big]q_1(t)\exp\Big(-\frac{p'}{c_1}\int_x^t q_1(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}t\\
&\leqslant \frac{\exp(c_1p')\exp(p'/2c_1)}{\theta(x)^{p'}q_1(x)}\int_x^\infty q_1(t)\exp\Big(-\frac{p'}{2c_1}\int_x^t q_1(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}t\\
&= \frac{2c}{p'}\exp(c_1p')c_4\Big(\frac{p'}{2c_1}\Big)\frac{1}{\theta(x)^{p'}q_1(x)}\Big[-\exp\Big(-\frac{p'}{2c_1}\int_x^t q_1(\xi)\,\mathrm{d}\xi\Big)\Big|_{t=x}^{t=\infty}\Big]\\
&= \frac{2c_1}{p'}c_4\Big(\frac{p'}{2c_1}\Big)\exp(c_1p')\frac{1}{\theta(x)^{p'}q_1(x)}.
\end{aligned}$$

□

Finally, let us turn to Theorem 3.11. We have to prove that if the conditions

(1) $p\in(1,\infty)$ and equality (3.1) holds,

(2) equation (1.1) is quasi-standard,

(3) the pair $\{L_{p,\mu};L_{p,\theta}\}$ agrees with equation (3.11)

are satisfied then the pair $\{L_{p,\mu};L_{p,\theta}\}$ is admissible for equation (1.1) if and only if $m(q(\cdot),\mu(\cdot),\theta(\cdot))<\infty$, see (3.18).

P r o o f of Theorem 3.11. *Necessity.* Suppose that under conditions (1), (2), (3) the pair $\{L_{p,\mu};L_{p,\theta}\}$ is admissible for equation (1.1). Then $\|S\|_{p\to p}<\infty$ by Theorem 3.3 and therefore $S_p<\infty$ by Lemma 3.2. Therefore, from conditions (2), (3) and Lemma 4.19 we obtain the relations

$$\begin{aligned}
\infty > S_p &= \sup_{x\in\mathbb{R}}(J_p^{(-)}(x))^{1/p}(J_{p'}^{(+)}(x))^{1/p'}\\
&\geqslant \frac{\exp(-2c_1)(p)^{-1/p}(p')^{-1/p'}}{(1+c_1)c_2(p')^{1/p'}c_3(p)^{1/p}}\sup_{x\in\mathbb{R}}\Big(\frac{\mu(x)}{\theta(x)}\frac{1}{q_1(x)}\Big),
\end{aligned}$$

i.e., $m(q_1(\cdot),\mu(\cdot),\theta(\cdot))<\infty$ as required.

Sufficiency. Since $p\in(1,\infty)$, conditions (2), (3) hold and $m(q_1(\cdot),\mu(\cdot),\theta(\cdot))<\infty$, we have $S_p<\infty$ (see (3.5)) by Lemma 4.19:

$$\begin{aligned}
S_p &= \sup_{x\in\mathbb{R}}(J_p^{(-)}(x))^{1/p}(J_{p'}^{(+)}(x))^{1/p'}\\
&\leqslant 2c_1(p')^{-1/p'}(p)^{-1/p}(\exp(2c_1))c_4\Big(\frac{p'}{2c_1}\Big)^{1/p'}c_5\Big(\frac{p}{2c_1}\Big)^{1/p}\sup_{x\in\mathbb{R}}\Big(\frac{\mu(x)}{\theta(x)}\frac{1}{q_1(x)}\Big)<\infty.
\end{aligned}$$

Then $\|S\|_{p\to p}<\infty$ by Lemma 3.2. Since condition (3.1) also holds by assumption, it remains to refer to Theorem 3.3. □

5. Examples

Below we consider problem (I)–(II) in the case of equation (1.1) with coefficients $q(\cdot)$ given in the form (3.9), where

$$q_1(x) = (1+x^2)^\alpha, \quad q_2(x) = (1+x^2)^\alpha \cos x^2, \quad x \in \mathbb{R},\ \alpha \in \mathbb{R}. \tag{5.1}$$

In the sequel, whenever we mention (1.1), we only mean the case (5.1).

Our goal is as follows: for every value of the parameter $\alpha \in \mathbb{R}$, find at least one pair $\{L_{p,\mu}; L_{p,\theta}\}$ admissible for (1.1). Here, according to the strength of our statements, we subdivide the study of problem (I)–(II) into the following cases:

Case $\alpha \geqslant 0$: We apply Theorem 1.2 and show that in this situation the pair $\{L_p; L_p\}$, $p \in (1,\infty)$, is admissible for (1.1). To this end, we introduce the function

$$\varphi(x) = \int_{x-1}^{x+1} (1+t^2)^\alpha \cos t^2 \, dt, \quad x \geqslant 2. \tag{5.2}$$

The integral (5.2) can be written in a different way:

$$\varphi(x) = \int_{x-1}^{x+1} \frac{(1+t^2)^\alpha}{2t} (\sin t^2)' \, dt, \quad x \geqslant 2. \tag{5.3}$$

It is easy to see that for a given $\alpha \geqslant 0$, there exists $x_0(\alpha) \gg 1$ such that for $x \geqslant x(\alpha)$ and $t \in [x-1, x+1]$, the function

$$\psi(t) = \frac{(1+t^2)^\alpha}{2t}, \quad t \in [x-1, x+1],\ x \geqslant x_0(\alpha),$$

is monotone. Indeed, since

$$\psi'(t) = [(2\alpha - 1)t^2 - 1] \frac{(1+t^2)^{\alpha-1}}{2t}, \quad t \gg 1,$$

we have $\psi'(t) > 0$ for $\alpha > \frac{1}{2}$ and $t \gg 1$, and $\psi'(t) < 0$ for $\alpha \in [0, \frac{1}{2}]$ and $t \gg 1$, as required. Furthermore, for $t \in [x-1, x+1]$ and $x \geqslant x_0(\alpha) \geqslant 2$, we have

$$\frac{1}{2} \leqslant 1 - \frac{1}{x} \leqslant \frac{t}{x} \leqslant 1 + \frac{1}{x} \leqslant 2 \tag{5.4}$$

$$\Rightarrow \frac{1}{4} \leqslant \min\Big\{1; \Big(\frac{t}{x}\Big)^2\Big\} \leqslant \frac{1+t^2}{1+x^2} \leqslant \max\Big\{1; \Big(\frac{t}{x}\Big)^2\Big\} \leqslant 4. \tag{5.5}$$

By the second mean value theorem (see [9], Chapter X, Section 5), (5.4) and (5.5), this implies that for $x \geqslant x(\alpha) \gg 1$ we have

$$
\begin{aligned}
(5.6) \qquad |\varphi(x)| &= \left|\int_{x-1}^{x+1} \psi(t)(\sin t^2)'\,\mathrm{d}t\right| \\
&\leqslant 2 \max_{t\in[x-1,x+1]}(|\psi(t)|) \sup_{\beta_1,\beta_2\in[x-1,x+1]}\left|\int_{\beta_1}^{\beta_2}(\sin t^2)'\,\mathrm{d}t\right| \\
&\leqslant 4 \max_{t\in[x-1,x+1]}|\psi(t)| = 4\max_{t\in[x-1,x+1]}\frac{1}{2}|t|^{2\alpha-1}\Big(1+\frac{1}{t^2}\Big)^{\alpha} \leqslant 2^{3\alpha}x^{2\alpha-1}.
\end{aligned}
$$

Hence, for $x \geqslant x_0(\alpha) \gg 2$, we have (see (5.4), (5.5) and (5.6))

$$
\begin{aligned}
(5.7) \qquad \int_{x-1}^{x+1} q(t)\,\mathrm{d}t &= \int_{x-1}^{x+1} q_1(t)\,\mathrm{d}t + \int_{x-1}^{x+1} q_2(t)\,\mathrm{d}t \\
&\geqslant \int_{x-1}^{x+1}\Big(\frac{1+t^2}{1+x^2}\Big)^{\alpha}(1+x^2)^{\alpha}\,\mathrm{d}t - \left|\int_{x-1}^{x+1}(1+t^2)^{\alpha}\cos t^2\,\mathrm{d}t\right| \\
&\geqslant \frac{1}{c}x^{2\alpha} - cx^{2\alpha-1} \geqslant \frac{1}{2}, \qquad x \geqslant x_0(\alpha) \gg 2,\ c \gg 1.
\end{aligned}
$$

Furthermore, the function

$$
Q(x) = \int_{x-1}^{x+1} q(t)\,\mathrm{d}t, \qquad x \in [0, x_0]
$$

is continuous and positive on $[0, x_0]$, and therefore $\inf\limits_{x\in[0,x_0]} Q(x) > 0$. Together with (5.7) (taking into account that the function $q(t)$, $t \in \mathbb{R}$, is even), this implies that $q_0(1) > 0$ (see (1.6)) and it remains to refer to Theorem 1.2.

Case $\alpha \in (-\frac{1}{2}, 0)$: It is easy to see that $q_1(\cdot) \in K$ for $\alpha \in (-\frac{1}{2}, 0)$, see (3.13). To check (3.16) as $x \to \infty$ (the case where $x \to -\infty$ is similar because the functions $q_1(\cdot)$ and $q_2(\cdot)$ are even), we put

$$
\Theta(x) = [x - 2^{1+|\alpha|}|x|^{2|\alpha|}, x + 2^{1+|\alpha|}|x|^{2|\alpha|}], \qquad x \gg 1.
$$

Then, obviously, $[x - s, x + s] \subseteq \Theta(x)$ for $|s| \leqslant 2/q_1(x)$ and $x \gg 1$. In addition, for $t \in \Theta(x)$ and $x \gg 1$, inequalities (5.4) and (5.5) remain true. Below, for $x \gg 1$, once again we apply the second mean value theorem (see [9], Chapter X, Section 5) and inequalities (5.4) and (5.5):

$$
\begin{aligned}
\varkappa(x) &= \sup_{|s|\leqslant 2/q_1(x)}\left|\int_{x-s}^{x+s} q_2(t)\,\mathrm{d}t\right| \\
&\leqslant \sup_{[x-s,x-s]\subseteq\Theta(x)}\left|\int_{x-s}^{x+s}\frac{1}{2t(1+t^2)^{|\alpha|}}(\sin t^2)'\,\mathrm{d}t\right| \\
&\leqslant \max_{t\in\Theta(x)}\frac{1}{t(1+t^2)^{|\alpha|}} \leqslant \frac{c}{x(1+x^2)^{|\alpha|}},
\end{aligned}
$$

i.e., equality (3.16) holds, and therefore one can apply Theorem 3.11. Set

$$\begin{cases} \mu(x) = (1+x^2)^{\beta}, & x \in \mathbb{R},\ \beta \geqslant -\dfrac{1}{2p},\ p \in (1,\infty), \\ \theta(x) = (1+x^2)^{\beta+|\alpha|}, & x \in \mathbb{R},\ \alpha \in \Big(-\dfrac{1}{2}, 0\Big). \end{cases} \tag{5.8}$$

Then (3.1) holds, and from Lemma 3.9 it follows that the pair $\{L_{p,\mu}; L_{p,\theta}\}$ with weights (5.8) agrees with (3.11) and $m(q_1(\cdot),\mu(\cdot),\theta(\cdot)) \equiv 1$. Hence, by Theorem 3.11, the pair $\{L_{p,\mu}; L_{p,\theta}\}$ is admissible for (1.1).

Note, in addition, that this result can be obtained in a different way. Indeed, it is easy to see that for $\alpha \in (-\frac{1}{2}, 0)$ the integral

$$\int_{-\infty}^{\infty} q_2(t)\,\mathrm{d}t = 2\int_0^{\infty} (1+t^2)^{\alpha}\cos t^2\,\mathrm{d}t = 2\int_1^{\infty} \frac{(\sin t^2)'\,\mathrm{d}t}{2t(1+t^2)^{|\alpha|}} + 2\int_0^1 (1+t^2)^{\alpha}\cos t^2\,\mathrm{d}t$$

converges by Dirichlet's test, and therefore $\mathbb{P} < \infty$, see (3.10). It remains to repeat the aforesaid regarding weights (5.8) and equation (3.11) and refer to Corollary 3.5.

Case $\alpha = -\frac{1}{2}$: Consider the union of Cases 5.2 and 5.3, i.e., below we have $\alpha \in [-\frac{1}{2}, 0)$. Set

$$\mu(x) = \frac{1}{(1+x^2)^{|\alpha|/p}}, \quad \theta(x) = (1+x^2)^{|\alpha|/p'}, \quad x \in \mathbb{R}.$$

Then, obviously, (3.1) holds and $\mathbb{P} < \infty$, see (3.10). This inequality is checked, as above, with the help of Dirichlet's test. Therefore, with such a choice of weights, the pair $\{L_{p,\mu}; L_{p,\theta}\}$ is admissible for (1.1) by Corollary 3.6.

Case $\alpha < -\frac{1}{2}$: Clearly, the functions $q_1(\cdot)$, $q_2(\cdot)$ (and hence $q(\cdot)$) belong to $L_1(\mathbb{R})$ for $\alpha < -\frac{1}{2}$. Therefore, by Corollary 3.4, if condition (3.1) holds and $A_p < \infty$ (see (3.8)), then any pair $\{L_{p,\mu}; L_{p,\theta}\}$ is admissible for (1.1).

To conclude, we give another example of applying Theorem 3.3: we show that it implies Theorem 1.2.

Proof of Theorem 1.2. *Necessity.* Let the pair $\{L_p; L_p\}$ be admissible for (1.1). Since (3.1) automatically holds and $\|S\|_{p\to p} < \infty$ by Theorem 3.3, we have $S_p < \infty$ by Lemma 3.2. This implies equalities (2.4). Indeed, assume, say, that $\|q\|_{L_1(0,\infty)} < \infty$. Then, since $J_p^{(-)}(0) > 0$ (recall that here $\mu(\cdot) \equiv 1$), we get

$$\begin{aligned} \infty > S_p &= \sup_{x\in\mathbb{R}} (J_p^{(-)}(x))^{1/p} (J_{p'}^{(+)}(x))^{1/p'} \\ &\geqslant (J_p^{(-)}(0))^{1/p} (J_{p'}^{(+)}(0))^{1/p'} \\ &\geqslant (J_p^{(-)}(0))^{1/p} \Big(\int_0^{\infty} \exp(-p'\|q\|_{L_1(0,\infty)})\,\mathrm{d}t\Big)^{1/p'} \\ &= \infty. \end{aligned}$$

We get a contradiction. Hence, equalities (2.4) hold, and therefore the function $d(x)$, $x \in \mathbb{R}$, is well-defined. It is easy to see that $a < \infty$, where $a = \sup_{x\in\mathbb{R}} d(x)$. Indeed, we use the inequality $S_p < \infty$ once again and get (see (2.6)):

$$\begin{aligned}
\infty > S_p &= \sup_{x\in\mathbb{R}} \left[\int_{-\infty}^{x} \exp\left(-p\int_t^x q(\xi)\,\mathrm{d}\xi\right)\mathrm{d}t\right]^{1/p} \\
&\quad\times \left[\int_x^{\infty} \exp\left(-p'\int_x^t q(\xi)\,\mathrm{d}\xi\right)\mathrm{d}t\right]^{1/p'} \\
&\geqslant \sup_{x\in\mathbb{R}} \left[\int_{x-d(x)}^{x} \exp\left(-p\int_{x-d(x)}^{x+d(x)} q(\xi)\,\mathrm{d}\xi\right)\mathrm{d}t\right]^{1/p} \\
&\quad\times \left[\int_x^{x+d(x)} \exp\left(-p'\int_{x-d(x)}^{x+d(x)} q(\xi)\,\mathrm{d}\xi\right)\mathrm{d}t\right]^{1/p'} \\
&\geqslant \exp(-4)\sup_{x\in\mathbb{R}} d(x) \\
\Rightarrow a &= \sup_{x\in\mathbb{R}} d(x) \leqslant \exp(4) S_p < \infty.
\end{aligned}$$

This implies that $q_0(a) \geqslant 2 > 0$ as needed:

$$q_0(a) = \inf_{x\in\mathbb{R}} \int_{x-a}^{x+a} q(t)\,\mathrm{d}t \geqslant \inf_{x\in\mathbb{R}} \int_{x-d(x)}^{x+d(x)} q(t)\,\mathrm{d}t = 2 > 0.$$

Sufficiency. Suppose that $q_0(a) > 0$ for some $a \in (0,\infty)$ (see (1.6)). Then for $p \in (1,\infty)$ we have the inequalities (see (3.6), (3.7)):

(5.9) $$\sup_{x\in\mathbb{R}} J_p^{(-)}(x) < \infty, \quad \sup_{x\in\mathbb{R}} J_{p'}^{(+)}(x) < \infty.$$

Since both inequalities in (5.9) are checked in the same way, we only prove the second case. Fix $x \in \mathbb{R}$ and define segments

$$\{X_n\}_{n=1}^{\infty}, \quad X_n = [X_n^{(-)}, X_n^{(+)}], \quad X_n^{(\pm)} = x_n \pm a, \quad n \geqslant 1$$

with the properties

$$X_1^{(-)} = x, \quad X_n^{(+)} = X_{n+1}^{(-)}, \quad n \geqslant 1.$$

Then we have

(5.10) $$\int_{X_1^{(-)}}^{X_n^{(-)}} q(\xi)\,\mathrm{d}\xi = \sum_{n=1}^{n} \int_{X_k} q(\xi)\,\mathrm{d}\xi \geqslant (n-1)q_0(a), \quad n \geqslant 1.$$

From (5.10) it now follows that

$$
\begin{aligned}
J_{p'}^{(+)}(x) &= \int_x^\infty \exp\Big(-p'\int_x^t q(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}t\\
&= \sum_{n=1}^\infty \int_{X_n} \exp\Big(-p'\int_{X_1^{(-)}}^t q(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}t\\
&\leqslant 2a + \sum_{n=2}^\infty \int_{X_n} \exp\Big(-p'\int_{X_1^{(-)}}^t q(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}t\\
&\leqslant 2a + \sum_{n=2}^\infty 2a\exp\Big(-p'\int_{X_1^{(-)}}^{X_n^{(-)}} q(\xi)\,\mathrm{d}\xi\Big)\\
&\leqslant 2a\Big(1+\sum_{n=2}^\infty \exp(-p'(n-1)q_0(a))\Big)\\
&= 2a[1+(1-\exp(-p'q_0(a)))^{-1}]\\
&< \infty.
\end{aligned}
$$

Thus, by (5.9) and Lemma 3.2, we have

$$
\|S\|_{p\to p} \leqslant (p)^{1/p}(p')^{1/p'}S_p = (p)^{1/p}(p')^{1/p'}\sup_{x\in\mathbb{R}}(J_p^{(-)}(x))^{1/p}(J_{p'}^{(+)}(x))^{1/p'} < \infty.
$$

Hence, the pair $\{L_p; L_p\}$ is admissible for (1.1) by Theorem 3.3. $\square$

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