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A NOTE ON MEASURE-VALUED SOLUTIONS
TO THE FULL EULER SYSTEM

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Abstract. We construct two particular solutions of the full Euler system which emanate from the same initial data. Our aim is to show that the convex combination of these two solutions form a measure-valued solution which may not be approximated by a sequence of weak solutions. As a result, the weak* closure of the set of all weak solutions, considered as parametrized measures, is not equal to the space of all measure-valued solutions. This is in stark contrast with the incompressible Euler equations.

Keywords: measure-valued solution; compressible Euler system

MSC 2020: 35B99, 35Q31

1. INTRODUCTION

In the context of fluid dynamics, measure-valued solutions were first studied by DiPerna and Majda [8], who developed an appropriate mathematical framework and showed existence for such solutions of the incompressible Euler equations. Measure-valued solutions describe the one-point statistics of a fluid, i.e., they give the probability distribution of the fluid velocity (and other state variables like density or temperature) at a given point in time and space. If we are willing to accept such a probabilistic description rather than a deterministic one (which would, of course, contain more information), then we can easily obtain a solution for any initial data, bypassing the notorious problem of non-interchangeability of weak limits and nonlinearities.

Measure-valued solutions are sometimes thought of as a “cheap way out” of the fundamental lack of compactness for inviscid fluid models, and are criticized as not

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containing enough interesting information. Yet, in recent years, the concept has been intensely studied again, as it turned out to have several merits after all: First, measure-valued solutions, despite representing a very weak notion of solution, enjoy a weak-strong stability property that entails important consequences for singular limits, numerical approximation, and long-time behaviour; this property is known for many systems of fluid mechanics, including the incompressible [2], isentropic compressible [13], and full compressible [4] Euler systems, the isentropic compressible Navier-Stokes equations [9], and the Navier-Stokes-Fourier equations [5], see also the survey [19]. One should remark that weak-strong uniqueness holds only for *admissible* measure-valued solutions, which comply with appropriate energy or entropy inequalities.

Motivated by numerical simulations, Fjordholm et al. [10] argued that measure-valued solutions provide for a more suitable notion of solution than weak (distributional) solutions; indeed, the numerical computation of unstable shear flows with randomly perturbed initial data yields highly unpredictable results on the level of weak solutions, but apparently stable and regular behavior on the measure-valued level. This phenomenon is of course very plausible in the light of phenomenological turbulence theory [11].

Measure-valued solutions seem like a vast generalization of weak solutions, but are they really? Surprisingly, the answer is ‘no’ for the incompressible Euler equations [18]. Indeed, any measure-valued solution is weakly approximated by a sequence of weak solutions (and if the measure is admissible, then the approximating sequence can also be chosen to consist of admissible weak solutions), or in other words: The set of Dirac parametrized measures is weakly* dense in the set of all measure-valued solutions. One might thus say that the notion of measure-valued solution is just a topological closure, but not a substantial extension, of the more classical concept of weak solution.

Looking at [18] from a different angle, one could view the result as an instance of a characterization of Young measures generated by sequences with specific properties (viz. being a solution of the incompressible Euler equations). The most classical result of this kind is the characterization of gradient Young measures [14], [15], where *not* every Young measure whose barycenter is a gradient is itself generated by a sequence of gradients. This already indicates that the situation for incompressible Euler is rather unusual.

The fact that every measure-valued solution of the incompressible Euler equations is generable, is related to the *wave cone* for the corresponding linear constraint; indeed, the wave cone in this case is the whole space. It turns out that this is no longer the case for compressible models. In [7], the wave cone for the isentropic compressible Euler system is determined (and it is not the whole space), and a preliminary

application to the generability of measure-valued solutions is given. This result was recently extended in [12] to yield an admissible measure-valued solution with atomic initial data that is not recovered from a sequence of weak solutions. The result relies on a refined rigidity lemma (see Theorem 3.1 below) and the construction of ‘wild’ solutions for certain Riemann data in [6]. It is shown, in addition, that the constructed measure-valued solution does not arise as a vanishing viscosity limit, and it is argued that such measure-valued solutions, therefore, should be discarded as unphysical.

The aim of the present contribution is to extend the result from [12] to the full Euler system. Our main result (Theorem 3.2 below) states that there exists a measure-valued solution to the full Euler system with non-constant entropy which is not generated by any sequence of weak solutions. Importantly, our measure-valued solution has atomic (deterministic) initial data. As in the isentropic case, our result means that there are measure-valued solutions that should be regarded as unphysical. In applications, however, measure-valued solutions are usually obtained from suitable approximate solutions such as numerical schemes or viscosity limits, and measure-valued solutions arising in such a way are not expected to behave as strangely as the one constructed in this note.

This contribution follows a similar strategy as [12], but it relies on the convex integration construction from [16] for the full Euler system rather than the isentropic construction from [6]. We have tried to keep the presentation as concise as possible; this means that we only briefly recall the relevant facts from [12], [16] with no extensive discussion. Moreover, our result could be extended in various straightforward ways to include the admissibility condition (which is actually satisfied in our construction, since the scheme in [16] produces entropy solutions), concentration measures (which we could simply take to be zero, whence any hypothetical generating sequence would automatically have equi-integrable non-linearities, cf. [12], proof of Theorem 4.13), or viscosity limits. Although the equation is considered in two space dimensions, the validity of the result in higher dimensions can be obtained simply by a trivial expansion. We have chosen, however, to keep this note as short and simple as possible.

2. PRELIMINARIES

Let $T > 0$. We consider the following system on time-space $[0, T] \times \mathbb{R}^2$:

$$(2.1) \quad \begin{aligned} \partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{v}) &= 0, \\ \partial_t(\varrho \mathbf{v}) + \operatorname{div}_x(\varrho \mathbf{v} \otimes \mathbf{v}) + \nabla p(\varrho, \theta) &= 0, \\ \partial_t \left(\frac{1}{2} \varrho |\mathbf{v}|^2 + \varrho e(\varrho, \theta) \right) + \operatorname{div}_x \left(\left(\frac{1}{2} \varrho |\mathbf{v}|^2 + \varrho e(\varrho, \theta) \right) \mathbf{v} \right) + \operatorname{div}_x(p(\varrho, \theta) \mathbf{v}) &= 0 \end{aligned}$$

with unknowns $\varrho: [0, T] \times \Omega \rightarrow \mathbb{R}_0^+$, $\mathbf{v} = (v, u): [0, T] \times \Omega \rightarrow \mathbb{R}^2$ and $\theta: [0, T] \times \Omega \rightarrow \mathbb{R}_0^+$. The functions e and p are interrelated through the Gibbs law, which gives rise also to an entropy—a function $s(\varrho, \theta)$ such that

$$\theta Ds(\varrho, \theta) = De(\varrho, \theta) + p(\varrho, \theta)D\left(\frac{1}{\varrho}\right),$$

where D stands for the gradient with respect to ϱ and θ . Throughout this paper we consider an ideal gas, i.e.

$$p(\varrho, \theta) = \varrho\theta, \quad e(\varrho, \theta) = c_v\theta, \quad s(\varrho, \theta) = \log\left(\frac{\theta^{c_v}}{\varrho}\right) \text{ for some } c_v > 0$$

and for simplicity, we assume $c_v = 1$. We rewrite (2.1) into conservative variables ϱ , $\mathbf{m} = \varrho\mathbf{v}$, $E = \frac{1}{2}\varrho|\mathbf{v}|^2 + \varrho e(\varrho, \theta)$. Our choice of state variables gives $p = E - \frac{1}{2}\mathbf{m}^2/\varrho$ and thus we get

$$(2.2) \quad \begin{aligned} \partial_t \varrho + \operatorname{div}_x \mathbf{m} &= 0, \\ \partial_t \mathbf{m} + \operatorname{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} + \mathbb{1} \left(E - \frac{1}{2} \frac{\mathbf{m}^2}{\varrho} \right) \right) &= 0, \\ \partial_t E + \operatorname{div} \left(\left(2E - \frac{1}{2} \frac{\mathbf{m}^2}{\varrho} \right) \frac{\mathbf{m}}{\varrho} \right) &= 0, \end{aligned}$$

where $\mathbb{1}$ denotes the (2×2) identity matrix. Furthermore, we may rewrite it as a linear differential system

$$(2.3) \quad \begin{aligned} \partial_t \varrho + \operatorname{div}_x \mathbf{m} &= 0, \\ \partial_t \mathbf{m} + \operatorname{div}_x U + \nabla E &= 0, \\ \partial_t E + \operatorname{div}_x r &= 0 \end{aligned}$$

with the following constraints:

$$(2.4) \quad \begin{aligned} U &= \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} - \frac{1}{2} \frac{\mathbf{m}^2}{\varrho} \mathbb{1}, \\ r &= \left(\left(2E - \frac{1}{2} \frac{\mathbf{m}^2}{\varrho} \right) \frac{\mathbf{m}}{\varrho} \right). \end{aligned}$$

We recall that according to the definition, U is a traceless symmetric matrix and thus system (2.3) may be rewritten as

$$\operatorname{div} \begin{pmatrix} \varrho & m_1 & m_2 \\ m_1 & U_{11} + E & U_{12} \\ m_2 & U_{12} & -U_{11} + E \\ E & r_1 & r_2 \end{pmatrix} = 0.$$

Here div stands for the divergence over the time-space variables (t, x) .

Definition 2.1. We say that a family of probability measures $\nu := \nu_{t,x} \in L_{w*}^\infty([0, T] \times \mathbb{R}^2, \mathcal{P}(\mathbb{R}_0^+ \times \mathbb{R}^2 \times \mathbb{R}_0^+))$ is a measure-valued solution to (2.1) with initial data $(\varrho^0, \mathbf{m}^0, E^0)$ if

$$\int_0^T \int_{\mathbb{R}^2} \varrho(t, x) \partial_t \varphi(t, x) + \mathbf{m}(t, x) \cdot \nabla \varphi(t, x) \, dx \, dt + \int_{\mathbb{R}^2} \varrho^0(x) \varphi(0, x) \, dx = 0$$

for all $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^2)$,

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^2} \mathbf{m}(t, x) \cdot \partial_t \varphi(t, x) + \left\langle \nu_{t,x}, \frac{\xi_m \otimes \xi_m}{\xi_\varrho} - \mathbb{1} \left(\xi_E - \frac{1}{2} \frac{\xi_m^2}{\xi_\varrho} \right) \right\rangle : \nabla \varphi(t, x) \, dx \, dt \\ + \int_{\mathbb{R}^2} \mathbf{m}^0(x) \cdot \varphi(0, x) \, dx = 0 \end{aligned}$$

for all $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^2)^2$,

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^2} E(t, x) \partial_t \varphi(t, x) + \left\langle \nu_{t,x}, \left(2\xi_E - \frac{1}{2} \frac{\xi_m^2}{\xi_\varrho} \right) \frac{\xi_m}{\xi_\varrho} \right\rangle \cdot \nabla \varphi(t, x) \, dx \, dt \\ + \int_{\mathbb{R}^2} E^0(x) \varphi(0, x) \, dx = 0 \end{aligned}$$

for all $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^2)$.

Here $\xi_\varrho \in \mathbb{R}_0^+$, $\xi_m \in \mathbb{R}^2$ and $\xi_E \in \mathbb{R}_0^+$ are dummy variables for ϱ , m and E , meaning that

$$\varrho(t, x) = \int_0^\infty \xi_\varrho \, d\nu_{t,x}, \quad \mathbf{m}(t, x) = \int_{\mathbb{R}^2} \xi_m \, d\nu_{t,x}, \quad E = \int_0^\infty \xi_E \, d\nu_{t,x}$$

and we use the notation

$$\langle \nu_{t,x}, f(\xi_\varrho, \xi_m, \xi_E) \rangle = \int_{\mathbb{R}_0^+ \times \mathbb{R}^2 \times \mathbb{R}_0^+} f(\xi_\varrho, \xi_m, \xi_E) \, d\nu_{t,x}.$$

For the existence of a measure-valued solution, one needs an extended notion including concentration effects; if these are taken into account, then the existence of measure-valued solutions is known owing to Březina [3]. His definition of measure-valued solution is also slightly different from ours in that he used the renormalized entropy balance and the total energy balance instead of the energy balance. In any case, every weak solution (ϱ, \mathbf{m}, E) is also a measure valued solution (in the sense of Definition 2.1) with $\nu_{t,x} = \delta_{\varrho(t,x)} \otimes \delta_{\mathbf{m}(t,x)} \otimes \delta_{E(t,x)}$. Thus, the existence of infinitely many weak solutions for certain initial data exhibited in [1] and in [16] shows a fortiori the existence of non-unique measure-valued solutions for these data.

As an immediate consequence of the definition, we obtain that every convex combination of two measure valued solutions is also a measure-valued solution, that is, if ν, μ are two measure-valued solutions, then so is $\lambda\nu + (1 - \lambda)\mu$ for any $\lambda \in [0, 1]$.

Our aim is to prove that there exists a measure valued solution to (2.1) which cannot be obtained as a limit of weak solutions. To make this precise, we say that a sequence $(z_n): \Omega \rightarrow \mathbb{R}^d$ of measurable functions generates the parametrized measure $(\nu_x)_{x \in \Omega}$ if

$$f(z_n) \rightharpoonup \langle \nu, f \rangle \quad \text{weakly in } L^1(\Omega)$$

for all continuous functions $f: \mathbb{R}^d \rightarrow \mathbb{R}$ for which $(f(z_n))$ is equi-integrable.

We will take advantage of the following theorem proved in [7]:

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz and bounded domain, $1 \leq p < \infty$, and \mathcal{A} a linear operator of the form*

$$\mathcal{A}z := \sum_{i=1}^N A^{(i)} \frac{\partial z}{\partial x_i},$$

where $A^{(i)}$ are $l \times d$ matrices and $z: \mathbb{R}^N \mapsto \mathbb{R}^d$ a vector valued function. Let $p \in (1, \infty)$ and $z_1, z_2 \in \mathbb{R}^d$, $z_1 \neq z_2$ be two constant states such that

$$z_2 - z_1 \notin \Lambda,$$

where Λ denotes the wave cone defined by

$$\Lambda = \{\bar{z} \in \mathbb{R}^d: \text{there exists } \xi \in \mathbb{R}^N \setminus \{0\} \text{ such that } \mathcal{A}(\bar{z}h(\cdot \cdot \xi)) = 0 \forall h: \mathbb{R} \rightarrow \mathbb{R}\}.$$

Let further $z_n: \Omega \rightarrow \mathbb{R}^d$ be an equi-integrable family of functions with

$$\begin{aligned} \|z_n\|_{L^p} &\leq c, \\ \mathcal{A}z_n &\rightarrow 0 \quad \text{in } W^{-1,r}(\Omega) \end{aligned}$$

for some $r \in (1, N/(N - 1))$, and assume that (z_n) generates a compactly supported Young measure such that

$$\text{supp}[\nu_x] \subset \{\lambda z_1 + (1 - \lambda)z_2, \lambda \in [0, 1]\} \quad \text{for a.a. } x \in \Omega.$$

Then there exists $z_\infty \in \mathbb{R}^d$ such that

$$z_n \rightarrow z_\infty \quad \text{in } L^p(\Omega).$$

In our setting (i.e. the state vector is $(\varrho, m_1, m_2, U_{11}, U_{12}, E, r_1, r_2)$), we have

$$A^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$A^{(3)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For a given point $z = (\varrho, m_1, m_2, U_{11}, U_{12}, E, r_1, r_2)$ define a matrix $(Z_{\mathcal{A}})_{ji}$ as follows:

$$(Z_{\mathcal{A}})_{ji} = \sum_{k=1}^d A_{jk}^{(i)} z_k, \quad j = 1, \dots, 4, \quad i = 1, \dots, 3.$$

As observed in [7], Section 3.2, $z \in \Lambda$ if and only if the corresponding $Z_{\mathcal{A}}$ satisfies $\text{rank } Z_{\mathcal{A}} < 3$. Thus, it is enough to take

$$z_1 = \left(1, 1, 0, \frac{1}{2}, 0, \frac{3}{2}, \frac{5}{2}, 0\right), \quad z_2 = \left(\gamma, 1, 0, \frac{1}{2\gamma}, 0, \frac{3}{2\gamma}, \frac{5}{2\gamma^2}, 0\right).$$

Trivially, both z_1 and z_2 are solutions to (2.2), since they are constant. Moreover, there exists γ such that $z_1 - z_2 \notin \Lambda$. Indeed, the corresponding $Z_{\mathcal{A}}$ is of the form

$$\begin{pmatrix} 1 - \gamma & 0 & 0 \\ 0 & 2\left(1 - \frac{1}{\gamma}\right) & 0 \\ 0 & 0 & 1 - \frac{1}{\gamma} \\ \frac{3}{2}\left(1 - \frac{1}{\gamma}\right) & \frac{5}{2}\left(1 - \frac{1}{\gamma^2}\right) & 0 \end{pmatrix}$$

and the determinant of the 3×3 submatrix $(Z_{\mathcal{A}})_{i,j=1}^3$ is $2(1 - \gamma)(1 - 1/\gamma)^2$ and thus is non-zero for all $\gamma \neq 1$. According to Theorem 2.2, the measure valued solution $\nu_{t,x} = \frac{1}{2}\delta_{z_1} + \frac{1}{2}\delta_{z_2}$ may not be approximated by a sequence of weak solutions. This is the cheapest way how to produce a solution of the demanded quality. However, the initial datum for this solution is already a measure and one may ask, as we did in the introduction, whether there is a measure-valued solution emanating from deterministic initial data which cannot be approximated by a sequence of weak solutions.

3. SOLUTION EMANATING FROM ‘ATOMIC’ INITIAL DATA

We present a non-constant variation of Theorem 2.2 proven in [12]:

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^3$ be an open bounded domain, \mathcal{A} a linear homogeneous constant rank differential operator of order one satisfying $l \geq 3$, and $1 \leq p < \infty$. Further, let $\bar{z}_1, \bar{z}_2 \in L^\infty(\Omega, \mathbb{R}^m)$ be such that $\bar{z}_2 - \bar{z}_1 \notin \Lambda$ a.e. in Ω , where Λ is the associated wave cone. Assume $z_n: \Omega \rightarrow \mathbb{R}^m$ is an equi-integrable family of functions such that*

$$\|z_n\|_p \leq c < \infty, \quad \mathcal{A}z_n \rightarrow 0 \quad \text{in } W^{-1,r}(\Omega)$$

for some $r \in (1, N/(N-1))$, and $\{z_n\}$ generates a compactly supported Young measure $\nu \in L^\infty_w(\Omega, \mathcal{M}^1(\mathbb{R}^m))$ such that

$$\text{supp}(\nu_x) \subset \{\lambda \bar{z}_1(x) + (1-\lambda)\bar{z}_2(x), \lambda \in [0, 1]\}$$

for a.e. $x \in \Omega$. Then for a.e. $x \in \Omega$ it holds that

$$\nu_x = \delta_{w(x)}$$

with $w \in L^1(\Omega)$ and $z_n \rightarrow w$ in $L^1(\Omega)$.

The question of the existence of the demanded measure-valued solution reduces to the question whether there are two weak solutions to (2.2) \bar{z}_1 and \bar{z}_2 emanating from the same ‘atomic’ initial conditions such that $\bar{z}_1 - \bar{z}_2 \notin \Lambda$ on an open non-empty set $\Omega \subset \mathbb{R}^3$. As noted before, we need to compute the rank of a certain matrix.

Let $\bar{z}_1 = (\varrho^\alpha, m_1^\alpha, m_2^\alpha, U_{11}^\alpha, U_{12}^\alpha, E^\alpha, r_1^\alpha, r_2^\alpha)$ and $\bar{z}_2 = (\varrho^\beta, m_1^\beta, m_2^\beta, U_{11}^\beta, U_{12}^\beta, E^\beta, r_1^\beta, r_2^\beta)$ be two solutions for which the constraint (2.4) is effective almost everywhere. The appropriate $Z_{\mathcal{A}}$ is of the form

$$(3.1) \quad \begin{pmatrix} \varrho^\alpha - \varrho^\beta & m_1^\alpha - m_1^\beta & m_2^\alpha - m_2^\beta \\ m_1^\alpha - m_1^\beta & U_{11}^\alpha - U_{11}^\beta + E^\alpha - E^\beta & U_{12}^\alpha - U_{12}^\beta \\ m_2^\alpha - m_2^\beta & U_{12}^\alpha - U_{12}^\beta & U_{11}^\beta - U_{11}^\alpha + E^\alpha - E^\beta \\ E^\alpha - E^\beta & r_1^\alpha - r_1^\beta & r_2^\alpha - r_2^\beta \end{pmatrix}.$$

Below, we show the existence of two solutions $(\varrho^\alpha, \mathbf{v}^\alpha, p^\alpha)$ and $(\varrho^\beta, \mathbf{v}^\beta, p^\beta)$ for which the appropriate matrix $Z_{\mathcal{A}}$ is of rank 3 on an open non-empty set.

3.1. The self-similar solution. Take Riemann initial data of the following form:

$$(\varrho, \mathbf{v}, p)_0 = \begin{cases} (\varrho_-, (v_K, 0), p_-) & \text{for } x_1 < 0, \\ (\varrho_K, (0, 0), p_+) & \text{for } x_1 > 0, \end{cases}$$

where

$$\varrho_K = \varrho_- \frac{p_- + 3p_+}{3p_- + p_+}, \quad v_K = \frac{\sqrt{2}}{\sqrt{\varrho_-}} \frac{p_+ - p_-}{\sqrt{p_- + 3p_+}}$$

and $\varrho_-, p_-, p_+ > 0, p_+ > p_-$ are given constants. According to [17], there is a self-similar solution consisting of a 1-shock. In particular, let

$$s = -\frac{p_+ + 3p_-}{\sqrt{2\varrho_-(p_- + 3p_+)}}.$$

Then a triple

$$(\varrho^\alpha, \mathbf{v}^\alpha, p^\alpha) = \begin{cases} (\varrho_-, (v_K, 0), p_-) & \text{for } x_1 < st, \\ (\varrho_K, (0, 0), p_+) & \text{for } x_1 > st \end{cases}$$

is a weak solution to (2.1), see Figure 1.

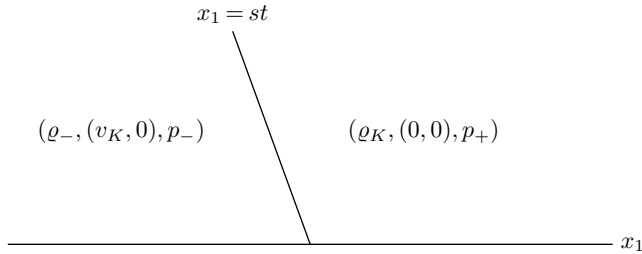


Figure 1. Self-similar solution.

3.2. The wild solution. Here we present the necessary details of the construction from [16]. First, according to [16], Section 4.3, we define a pressure $p_\delta = p_+ + \delta_p$ and a velocity $v_\delta = \delta_v$, where

$$\delta_v = \delta_p \sqrt{\frac{2}{\varrho_K(4p_+ + 3\delta_p)}}.$$

Also, set

$$\varrho_\delta = \varrho_K \frac{3p_\delta + p_+}{3p_+ + p_\delta}.$$

Note that $\delta_v = \delta_v(\delta_p)$ is a smooth function on a neighborhood of 0, $\delta_v(0) = 0$, and δ_p is a positive arbitrarily small number. The time-space is then divided into regions $\Omega_-, \Omega_1, \Omega_2, \Omega_\delta$ and Ω_+ as shown in Figure 2.

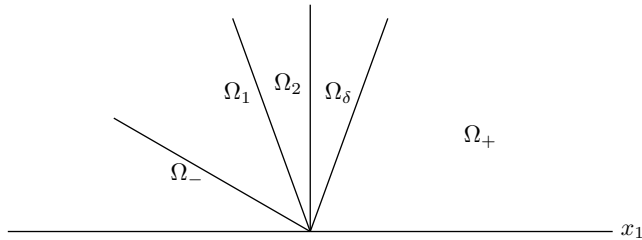


Figure 2. Fan partition.

Between Ω_δ and Ω_+ there is a 3-shock. In order to handle the region $\Omega_- \cup \Omega_1 \cup \Omega_2 \cup \Omega_\delta$, we use the Galilean transformation

$$(\varrho, (v, u), p)(t, x) \mapsto (\varrho, (v - \delta_v, u), p)(t, x + \delta_v t \mathbf{e}_1)$$

to get the right state

$$(\varrho_-, (v_K - \delta_v, 0), p_-) \quad \text{in } \Omega_-$$

and the left state

$$(\varrho_{\delta_p}, (0, 0), p_+ + \delta_p) \quad \text{in } \Omega_\delta.$$

The Galilean transformation also changes the sets Ω_- , Ω_1 , Ω_2 , and Ω_δ . However, we keep the same notation for the sake of simplicity.

According to [16], Theorem 3.1, there exist infinitely many admissible weak solutions to (2.1). We denote one of these solutions by $(\varrho^\beta, (\mathbf{v}')^\beta, p^\beta)$ —this solution has to be transformed back by Galilean transformation into the solution $(\varrho^\beta, \mathbf{v}^\beta, p^\beta)$. Such a solution fulfills

$$(\varrho^\beta, (\mathbf{v}')^\beta, p^\beta) \upharpoonright_{\Omega_-} = (\varrho_-, (v_K - \delta_v, 0), p_-), \quad (\varrho^\beta, (\mathbf{v}')^\beta, p^\beta) \upharpoonright_{\Omega_\delta} = (\varrho_\delta, (0, 0), p_+ + \delta_p).$$

Moreover, we have the following:

$$\begin{aligned} |(\mathbf{v}')^\beta \upharpoonright_{\Omega_2}|^2 &= \varepsilon_2 + \tilde{\varepsilon}_2, & \varrho^\beta \upharpoonright_{\Omega_1} &= \varrho_1 \in \mathbb{R}^+, & \varrho^\beta \upharpoonright_{\Omega_2} &= \varrho_2 \in \mathbb{R}^+, \\ p^\beta \upharpoonright_{\Omega_1} &= p_1, & p^\beta \upharpoonright_{\Omega_2} &= p_2, \end{aligned}$$

where

$$\begin{aligned} \varepsilon_2 &= \varepsilon_2(v_- - \delta_p, \varrho_1, \varrho_2, p_-, p_+), & \tilde{\varepsilon}_2 &= \tilde{\varepsilon}_2(v_- - \delta_p, \varrho_1, \varrho_2, p_-, p_+), \\ p_2 &:= p_2(\varrho_2, p_-) \end{aligned}$$

are continuous functions. Moreover, the construction is such that $\varrho_K - \varrho_1$ and $\varrho_2 - \varrho_K$ may be arbitrarily small. As δ_v and δ_p are also arbitrarily small, we verify the property that matrix (3.1) has full rank for $\varrho_1 = \varrho_2 = \varrho_K$, $v_- = v_K$ and $p_\delta = p_+$ on a set of positive measure. The aforementioned continuity then allows to use the intended approximation.

The interface between Ω_2 and Ω_δ is $\{x_1 = \delta_v t\}$. Consequently, the domain $\Omega_2 \cap \{x_1 > st\}$ is non-empty and has positive measure, since $s < 0$. On this set we take

$$\begin{aligned} z^\alpha &= \left(\varrho^\alpha, \varrho^\alpha v^\alpha, \varrho^\alpha u^\alpha, \frac{1}{2} \varrho^\alpha ((v^\alpha)^2 - (u^\alpha)^2), \varrho^\alpha u^\alpha v^\alpha, \frac{1}{2} \varrho^\alpha |\mathbf{v}^\alpha|^2 + p^\alpha, \right. \\ &\quad \left. \left(\frac{1}{2} \varrho^\alpha |\mathbf{v}^\alpha|^2 + 2p^\alpha \right) v^\alpha, \left(\frac{1}{2} \varrho^\alpha |\mathbf{v}^\alpha|^2 + 2p^\alpha \right) u^\alpha \right), \\ z^\beta &= \left(\varrho^\beta, \varrho^\beta v^\beta, \varrho^\beta u^\beta, \frac{1}{2} \varrho^\beta ((v^\beta)^2 - (u^\beta)^2), \varrho^\beta u^\beta v^\beta, \frac{1}{2} \varrho^\beta |\mathbf{v}^\beta|^2 + p^\beta, \right. \\ &\quad \left. \left(\frac{1}{2} \varrho^\beta |\mathbf{v}^\beta|^2 + 2p^\beta \right) v^\beta, \left(\frac{1}{2} \varrho^\beta |\mathbf{v}^\beta|^2 + 2p^\beta \right) u^\beta \right). \end{aligned}$$

Note that $|\mathbf{v}^\beta|^2 = |(\mathbf{v}')^\beta + \delta_p \mathbf{e}_1|^2 = \varepsilon_2 + \tilde{\varepsilon}_2 + o(\delta)$ on Ω_2 . Matrix (3.1) on the considered set is of the form

$$\begin{pmatrix} \varrho_2 - \varrho_K & \varrho_2 v^\beta & \varrho_2 u^\beta \\ \varrho_2 v^\beta & \varrho_2 |v^\beta|^2 + p_2 - p_+ & \varrho_2 u^\beta v^\beta \\ \varrho_2 u^\beta & \varrho_2 u^\beta v^\beta & \varrho_2 |u^\beta|^2 + p_2 - p_+ \\ \frac{1}{2} \varrho_2 |\mathbf{v}^\beta|^2 + p_2 - p_+ & (\frac{1}{2} \varrho_2 |\mathbf{v}^\beta|^2 + 2p_2) v^\beta & (\frac{1}{2} \varrho_2 |\mathbf{v}^\beta|^2 + 2p_2) u^\beta \end{pmatrix}.$$

The determinant of the submatrix consisting of the first, second, and third rows is

$$(p_2 - p_+) ((\varrho_2 - \varrho_K)(p_2 - p_+) - \varrho_2 \varrho_K |\mathbf{v}^\beta|^2).$$

So the corresponding matrix Z_A is of rank 3 once we know that $p_2 \neq p_+$ and

$$(3.2) \quad |\mathbf{v}^\beta|^2 \neq \frac{(\varrho_2 - \varrho_K)(p_2 - p_+)}{\varrho_2 \varrho_K}.$$

We have $p_2 \approx p_-(\varrho_K/\varrho_-)^2 \neq p_+$ once we know that $p_+ > p_-$, and so it remains to verify (3.2).

Since all above mentioned quantities are bounded and $\varrho_2 - \varrho_K$ is negligible, the right-hand side of (3.2) can be made arbitrarily close to zero. We need to show that $|\mathbf{v}^\beta|^2$ is far away from zero. According to [16], Sections 3.2 & 3.8 we have

$$|\mathbf{v}^\beta|^2 \approx \varepsilon_2(v_K, \varrho_K, \varrho_K, p_-, p_+) + \tilde{\varepsilon}_2(v_K, \varrho_K, \varrho_K, p_-, p_+) = 4 \frac{(p_+ - p_-)(p_+ + p_-)^2}{\varrho_-(3p_+ + p_-)(3p_- + p_+)}.$$

Consequently, (3.2) is fulfilled and it is allowed to consider also the intended small perturbations. The Young measure

$$\nu_{t,x} = \frac{1}{2} \delta_{(\varrho^\alpha, \varrho^\alpha \mathbf{v}^\alpha, (1/2)\varrho^\alpha |\mathbf{v}^\alpha|^2 + p^\alpha)} + \frac{1}{2} \delta_{(\varrho^\beta, \varrho^\beta \mathbf{v}^\beta, (1/2)\varrho^\beta |\mathbf{v}^\beta|^2 + p^\beta)}$$

is a measure-valued solution which, due to Theorem 3.1, cannot be generated by weak solutions.

We get the following claim as a result of the previous considerations.

Theorem 3.2. *There exists a measure-valued solution to (2.1) with non-constant entropy, emanating from certain Riemann initial data, which cannot be generated by a sequence of weak solutions.*

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