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ON THE AVERAGE NUMBER OF SYLOW SUBGROUPS IN FINITE GROUPS

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Abstract. We prove that if the average number of Sylow subgroups of a finite group is less than $\frac{41}{5}$ and not equal to $\frac{29}{4}$, then G is solvable or $G/F(G) \cong A_5$. In particular, if the average number of Sylow subgroups of a finite group is $\frac{29}{4}$, then $G/N \cong A_5$, where N is the largest normal solvable subgroup of G. This generalizes an earlier result by Moretó et al.

Keywords: Sylow number; non-solvable group

MSC 2020: 20D20, 20D15

1. Introduction

In this paper, groups under consideration are finite. We denote by $\pi(G)$ the set of prime divisors of the order of the group G, and by $n_p(G)$, the number of Sylow $p-$subgroups of G. The Fitting subgroup of G denoted by $F(G)$, is defined as the largest nilpotent normal subgroup of G. All further unexplained notations are standard and can be found in [2].

Let $S = S(G) = \{p \in \pi(G)\colon n_p(G) > 1\}$. In [5], the average number of Sylow subgroups of a finite group G is defined as $\operatorname{asn}(G) = \sum\limits_{p\in S(G)} n_p(G)/|S|$. In [1], $\sum\limits_{p\in S(G)} n_p(G)$ is denoted by $\delta_0(G)$. So, we have $\operatorname{asn}(G) = \delta_0(G)/|S|$. When $\operatorname{asn}(G) < 7$, it is proved that G is a solvable group, see [5], Theorem A. Also, in [4], it is proved that if G is a finite non-solvable group such that $\operatorname{asn}(G) < \frac{29}{4}$, then $G/F(G) \cong A_5$. In fact, they proved that $G/N \cong A_5$, where N is the largest normal solvable subgroup of G, and then they proved that N is nilpotent. So $N = F(G)$. Authors in [4] pose the following open question.

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Question. What is the best possible value of $C > \frac{29}{4}$ such that if G is non-solvable and $\mathrm{asn}(G) < C$, then we still get that $G/F(G) \cong A_5$?

In our main theorem, we show that if $C < \frac{41}{5}$ and $\mathrm{asn}(G) \neq \frac{29}{4}$, then $G/F(G) \cong A_5$, and if $C = \frac{29}{4}$, then $G/N \cong A_5$, where N is the largest normal solvable subgroup of G, but N is not nilpotent and then $N \neq F(G)$. For example, let N be $2^3 : 7$, the extension of an elementary abelian of order 8 by a group of order 7 (Frobenius group) and $G = A_5 \times N$. Then N is the largest normal solvable subgroup of G, $n_2(G) = 5$, $n_3(G) = 10$, $n_5(G) = 6$ and $n_7(G) = 8$. Thus, $\mathrm{asn}(G) = \frac{29}{4}$, $G/N \cong A_5$, but $N \neq F(G)$.

Our main theorem follows.

Theorem 1.1. *If G is a finite non-solvable group with $\mathrm{asn}(G) < \frac{41}{5}$ and $\mathrm{asn}(G) \neq \frac{29}{4}$, then $G/F(G) \cong A_5$. If $\mathrm{asn}(G) = \frac{29}{4}$, then $G/N \cong A_5$, where N is the largest normal solvable subgroup of G.*

We need the following lemmas to prove the main theorem.

Lemma 1.1 ([6], Lemma 1)**.** *Let G be a group and N be a normal subgroup of G. Then $n_p(N)n_p(G/N) \mid n_p(G)$ for every prime p.*

Lemma 1.2 ([3], Theorem 9.3.1)**.** *Let G be a finite solvable group and $|G| = m \cdot n$, where $m = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$, $(m, n) = 1$. Let $\pi = \{p_1, \ldots, p_r\}$ and h_m be the number of π-Hall subgroups of G. Then $h_m = q_1^{\beta_1} \ldots q_s^{\beta_s}$ satisfies the following condition for all $i \in \{1, 2, \ldots, s\}$: $q_i^{\beta_i} \equiv 1 \pmod{p_j}$ for some p_j.*

2. Proof of the main theorem

Since G is a finite non-solvable group, it has the normal series $1 \trianglelefteq N \trianglelefteq H \trianglelefteq G$ such that H/N is a direct product of isomorphic simple groups, and N is the maximal solvable normal subgroup of G. We show that $H/N \cong A_5$. If $|S| \geqslant 8$, then $\mathrm{asn}(G) \geqslant \frac{3+4+6+8+12+14+18+20}{8} > \frac{41}{5}$, a contradiction. Hence, $|S| \leqslant 7$. Since $\mathrm{asn}(G) = S(G)/|S| < \frac{41}{5}$, we have $\delta_0(G) < 7 \times \frac{41}{5}$. It follows that $\delta_0(G) < 57.4$. By [1], Corollary 1.8, there exists a prime $p \in \pi(G)$ such that $n_p(H/N)^2 > |H/N|$. Since $\delta_0(G) < 57.4$ and $n_p(H/N) \leqslant n_p(G)$, we have $n_p(H/N) < 58$. Hence, $|H/N| < 58^2 = 3364 < 60^2 = |A_5|^2$. But by [2], the simple non-abelian groups of order less than 3364 are: A_5, A_6, A_7, $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,8)$, $\mathrm{PSL}(2,11)$, $\mathrm{PSL}(2,13)$ and $\mathrm{PSL}(2,17)$. It is easy to check that $\delta_0(T) > 58$ when $T = A_6$, A_7, $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,8)$, $\mathrm{PSL}(2,11)$, $\mathrm{PSL}(2,13)$ and $\mathrm{PSL}(2,17)$. Therefore, $H/N \cong A_5$, $\mathrm{PSL}(2,7)$.

If $H/N \cong \mathrm{PSL}(2,7)$, then $\{2,3,7\} \subseteq \pi(G)$ and by Lemma 1.1, $n_2(\mathrm{PSL}(2,7)) = 21 \mid n_2(G)$, $n_3(\mathrm{PSL}(2,7)) = 28 \mid n_3(G)$, $n_7(\mathrm{PSL}(2,7)) = 8 \mid n_7(G)$. On the other hand, we have $|S| \leqslant 7$.

If $|S| = 3$, then $\mathrm{asn}(G) \geqslant \frac{21+28+8}{3} = \frac{57}{3} > \frac{41}{5}$, a contradiction.

If $|S| = 4$, then $\mathrm{asn}(G) \geqslant \frac{21+28+6+8}{4} = \frac{63}{4} > \frac{41}{5}$, a contradiction.

If $|S| = 5$, then $\mathrm{asn}(G) \geqslant \frac{21+28+6+8+12}{5} = \frac{75}{5} > \frac{41}{5}$, a contradiction.

If $|S| = 6$, then $\mathrm{asn}(G) \geqslant \frac{21+28+6+8+12+14}{6} = \frac{89}{6} > \frac{41}{5}$, a contradiction.

If $|S| = 7$, then $\mathrm{asn}(G) \geqslant \frac{21+28+6+8+12+14+18}{7} = \frac{107}{7} > \frac{41}{5}$, a contradiction.

Therefore, $H/N \cong A_5$. Now, if we set $\overline{H} := H/N \cong A_5$ and $\overline{G} := G/N$, then

$$A_5 \cong \overline{H} \cong \overline{H}C_{\overline{G}}(\overline{H})/C_{\overline{G}}(\overline{H}) \leqslant \overline{G}/C_{\overline{G}}(\overline{H}) = N_{\overline{G}}(\overline{H})/C_{\overline{G}}(\overline{H}) \leqslant \mathrm{Aut}(\overline{H}).$$

Put $K = \{x \in G\colon\ xN \in C_{\overline{G}}(\overline{H})\}$, so $G/K \cong \overline{G}/C_{\overline{G}}(\overline{H})$. Hence,

$$A_5 \leqslant G/K \leqslant \mathrm{Aut}(A_5) \cong S_5.$$

Let $G/K \cong S_5$. By Lemma 1.1, $n_2(S_5) = 15 \mid n_2(G)$, $n_3(S_5) = 10 \mid n_3(G)$, $n_5(S_5) = 6 \mid n_7(G)$. Since $|S| \leqslant 7$, we consider the following cases.

$|S| = 3$: Then $\mathrm{asn}(G) \geqslant \frac{15+10+6}{3} = \frac{31}{3} > \frac{41}{5}$, a contradiction.

$|S| = 4$: Then $\mathrm{asn}(G) \geqslant \frac{15+10+6+8}{4} = \frac{39}{4} > \frac{41}{5}$, a contradiction.

$|S| = 5$: Then $\mathrm{asn}(G) \geqslant \frac{15+10+6+8+12}{5} = \frac{51}{5} > \frac{41}{5}$, a contradiction.

$|S| = 6$: Then $\mathrm{asn}(G) \geqslant \frac{15+10+6+8+12+14}{5} = \frac{65}{6} > \frac{41}{5}$, a contradiction.

$|S| = 7$: Then $\mathrm{asn}(G) \geqslant \frac{15+10+6+8+12+14+18}{7} = \frac{83}{7} > \frac{41}{5}$, a contradiction.

Therefore, $G/K \cong A_5$.

We show that $K = N$. Suppose that $K \neq N$. Since $C_{\overline{G}}(\overline{H}) \cong K/N$ and N is the maximal solvable normal subgroup of G, we have that K is a non-solvable normal subgroup of G. On the other hand, by Lemma 1.1, $n_2(A_5) = 5 \mid n_2(G)$, $n_3(A_5) = 10 \mid n_3(G)$ and $n_5(A_5) = 6 \mid n_5(G)$. By Lemma 1.1, we have $n_p(K)n_p(A_5) \mid n_p(G)$ for every prime $p \in \pi(G)$. Since K is a non-solvable group, we have $n_2(K) \geqslant 3$. Thus, $n_2(G) \geqslant 15$. Similarly to the case $G/K \cong S_5$, since $\mathrm{asn}(G) < \frac{41}{5}$ and $|S| \leqslant 7$, we get a contradiction. Therefore, $K = N$.

Now, we show that if $\mathrm{asn}(G) \neq \frac{29}{4}$, then $N = F(G)$. It is sufficient to prove that N is nilpotent. By the above discussion, $n_p(N) = 1$ for every prime $p \in \{2,3,5\}$. Assume that there exists a prime $r \geqslant 17$ that divides $|G|$. If $n_r(G) > 1$, then $n_r(G) \geqslant 18$. Since $\mathrm{asn}(G) \geqslant \frac{5+10+6+18}{4} = \frac{41}{5}$, we get a contradiction. It follows that $n_r(G) = 1$ for every $r \geqslant 17$. Therefore, $n_r(N) = 1$ for every $r \geqslant 17$. Assume that there exists a prime $r \leqslant 13$ such that $n_r(N) \neq 1$. So, $r = 7$, 11 or 13.

If $r = 13$, then $n_{13}(N) = 14$. Because if $n_{13}(N) \geqslant 27$, then $n_{13}(G) \geqslant 27$, we get a contradiction by $\mathrm{asn}(G) < \frac{41}{5}$. Hence, $n_{13}(N) = 14$. By Lemma 1.2, $2 \equiv 1 \bmod 13$, a contradiction.

If $r = 11$, then $n_{11}(N) = 12$. Because if $n_{11}(N) \geqslant 23$, then $n_{11}(G) \geqslant 23$, we get a contradiction by $\operatorname{asn}(G) < \frac{41}{5}$. Hence, $n_{11}(N) = 12$. By Lemma 1.2, $4 \equiv 1 \bmod 11$, a contradiction.

If $r = 7$, then $n_7(N) = 8$ or 15. Because if $n_7(N) \geqslant 22$, then $n_7(G) \geqslant 22$, we get a contradiction by $\operatorname{asn}(G) < \frac{41}{5}$. Hence, $n_7(N) = 8$ or 15. In the case of $n_7(N) = 15$, by Lemma 1.2, $3 \equiv 1 \bmod 7$, a contradiction. Thus, $n_7(N) = 8$. Since $n_7(N) \mid n_7(G) = 1 + 7k$, we have $n_7(G) = 8$. On the other hand, since $5 \mid n_2(G)$, $10 \mid n_3(G)$, $6 \mid n_5(G)$ and $\operatorname{asn}(G) < \frac{41}{5}$, we have $n_2(G) = 5$, $n_3(G) = 10$, $n_5(G) = 6$, and there is no prime $p \in \pi(G)$ with $n_p(G) > 1$. Hence, $\operatorname{asn}(G) = \frac{5+10+6+8}{4} = \frac{29}{4}$, a contradiction. Therefore, N is a nilpotent subgroup of G, and so $N = F(G)$.

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