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STABLE TUBES IN EXTRIANGULATED CATEGORIES

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Abstract. Let $\mathcal{X}$ be a semibrick in an extriangulated category. If $\mathcal{X}$ is a τ-semibrick, then the Auslander-Reiten quiver $\Gamma(\mathcal{F}(\mathcal{X}))$ of the filtration subcategory $\mathcal{F}(\mathcal{X})$ generated by $\mathcal{X}$ is $\mathbb{Z}\mathbb{A}_\infty$. If $\mathcal{X}=\{X_i\}_{i=1}^t$ is a τ-cycle semibrick, then $\Gamma(\mathcal{F}(\mathcal{X}))$ is $\mathbb{Z}\mathbb{A}_\infty/\tau_{\mathbb{A}}^t$.

Keywords: extriangulated category; semibrick; Auslander-Reiten quiver

MSC 2020: 18E05

1. Introduction

In representation theory of algebras, the notion of simple modules is fundamental. By Schur's lemma, the endomorphism ring of a simple module is a division algebra; and there exists no nonzero homomorphism between two nonisomorphic simple modules. We say that a module is a brick if its endomorphism ring is a division algebra. Clearly, this notion is a generalization of simple modules. For each set of isoclasses of pairwise Hom-orthogonal bricks, we call it a semibrick. By Simson and Skowronski (see [5]), the filtration subcategory $\mathcal{F}(\mathcal{X})$ of a semibrick $\mathcal{X}$ in the module category is an exact abelian subcategory. Let $\mathcal{X}=\{X_i\}_{i=1}^t$ be a τ-cycle semibrick in the module category of a hereditary algebra. An interesting and significant result says that the indecomposable objects in $\mathcal{F}(\mathcal{X})$ are uniserial, and the Auslander-Reiten quiver of $\mathcal{F}(\mathcal{X})$ is a stable tube of rank t, cf. [4], [5].

Recently, Nakaoka and Palu in [3] introduced an extriangulated category by extracting properties on triangulated categories and exact categories. Iyama, Nakaoka and Palu in [2] developed the Auslander-Reiten theory for extriangulated categories.

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In this paper, we continue our study on semibricks in an extriangulated category in [6] and investigate the Auslander-Reiten quiver of the filtration subcategory generated by a semibrick.

The paper is organized as follows: We summarize the definition and some properties of an extriangulated category, its Auslander-Reiten theory and the filtration subcategory in Section 2. In Section 3, we describe the Auslander-Reiten quiver of the filtration subcategory generated by a semibrick in an extriangulated category.

Throughout this paper, we assume, unless otherwise stated, that all considered categories are skeletally small, Hom-finite, Krull-Schmidt, linear over a fixed field k, and subcategories are full and closed under isomorphisms. We denote by $\mathbb{D}$ the k-dual.

2. Preliminaries

2.1. Extriangulated categories. Let us recall some notions concerning extriangulated categories from [3].

Let $\mathscr{C}$ be an additive category and let $\mathbb{E}\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to Ab$ be a biadditive functor. For any pair of objects A, $C \in \mathscr{C}$, an element $\delta \in \mathbb{E}(C, A)$ is called an $\mathbb{E}$-*extension.* The zero element $0 \in \mathbb{E}(C, A)$ is called the *split* $\mathbb{E}$-*extension.* For any morphism $a \in \mathscr{C}(A, A')$ and $c \in \mathscr{C}(C', C)$ we have $\mathbb{E}(C, a)(\delta) \in \mathbb{E}(C, A')$ and $\mathbb{E}(c, A)(\delta) \in \mathbb{E}(C', A)$. We simply denote them by $a_*\delta$ and $c^*\delta$, respectively. Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$. A morphism $(a, c)\colon \delta \to \delta'$ of $\mathbb{E}$-extensions is a pair of morphisms $a \in \mathscr{C}(A, A')$ and $c \in \mathscr{C}(C, C')$ satisfying the equality $a_*\delta = c^*\delta'$.

By Yoneda's lemma, any $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$ induces natural transformations

$$\delta_\sharp\colon \mathscr{C}(-, C) \to \mathbb{E}(-, A) \quad \text{and} \quad \delta^\sharp\colon \mathscr{C}(A, -) \to \mathbb{E}(C, -).$$

For any $X \in \mathscr{C}$, these $(\delta_\sharp)_X$ and $(\delta^\sharp)_X$ are defined by $(\delta_\sharp)_X\colon \mathscr{C}(X, C) \to \mathbb{E}(X, A)$, $f \mapsto f^*\delta$ and $(\delta^\sharp)_X\colon \mathscr{C}(A, X) \to \mathbb{E}(C, X)$, $g \mapsto g_*\delta$.

Two sequences of morphisms $A \xrightarrow{x} B \xrightarrow{y} C$ and $A \xrightarrow{x'} B' \xrightarrow{y'} C$ in $\mathscr{C}$ are said to be *equivalent* if there exists an isomorphism $b \in \mathscr{C}(B, B')$ such that the diagram

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \| & & b \downarrow \simeq & & \| \\ A & \xrightarrow{x'} & B' & \xrightarrow{y'} & C \end{array}$$

is commutative. We denote the equivalence class of $A \xrightarrow{x} B \xrightarrow{y} C$ by $[A \xrightarrow{x} B \xrightarrow{y} C]$. In addition, for any $A, C \in \mathscr{C}$ we denote

$$0 = [A \xrightarrow{\binom{1}{0}} A \oplus C \xrightarrow{(0\,1)} C].$$

For any two classes $[A \xrightarrow{x} B \xrightarrow{y} C]$ and $[A' \xrightarrow{x'} B' \xrightarrow{y'} C']$ we denote

$$[A \xrightarrow{x} B \xrightarrow{y} C] \oplus [A' \xrightarrow{x'} B' \xrightarrow{y'} C'] = [A \oplus A' \xrightarrow{x \oplus x'} B \oplus B' \xrightarrow{y \oplus y'} C \oplus C'].$$

Definition 2.1. Let $\mathfrak{s}$ be a correspondence which associates an equivalence class $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ to any $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$. This $\mathfrak{s}$ is called a *realization* of $\mathbb{E}$ if for any morphism $(a, c)\colon\ \delta \to \delta'$ with $\mathfrak{s}(\delta) = [\Delta_1]$ and $\mathfrak{s}(\delta') = [\Delta_2]$, there is a commutative diagram as follows:

$$\begin{array}{ccc} \Delta_1 \\ \big\downarrow \\ \Delta_2 \end{array} \qquad \begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ {\scriptstyle a}\big\downarrow & & {\scriptstyle b}\big\downarrow & & {\scriptstyle c}\big\downarrow \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C'. \end{array}$$

A realization $\mathfrak{s}$ of $\mathbb{E}$ is said to be *additive* if it satisfies the following conditions:

(a) For any $A, C \in \mathscr{C}$, the split $\mathbb{E}$-extension $0 \in \mathbb{E}(C, A)$ satisfies $\mathfrak{s}(0) = 0$.

(b) $\mathfrak{s}(\delta \oplus \delta') = \mathfrak{s}(\delta) \oplus \mathfrak{s}(\delta')$ for any pair of $\mathbb{E}$-extensions δ and δ'.

Let $\mathfrak{s}$ be an additive realization of $\mathbb{E}$. If $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$, then the sequence $A \xrightarrow{x} B \xrightarrow{y} C$ is called a *conflation*, x is called an *inflation* and y is called a *deflation*. In this case, we say that $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ is an $\mathbb{E}$-triangle. We write $A = \mathrm{cocone}(y)$ and $C = \mathrm{cone}(x)$ if necessary. We say an $\mathbb{E}$-triangle is *splitting* if it realizes 0.

Definition 2.2 ([3], Definition 2.12)**.** We call the triplet $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ an *extriangulated category* if it satisfies the following conditions:

(ET1) $\mathbb{E}\colon\ \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to Ab$ is a biadditive functor.

(ET2) $\mathfrak{s}$ is an additive realization of $\mathbb{E}$.

(ET3) Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of $\mathbb{E}$-extensions, realized as $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$, $\mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$. For any commutative square

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ {\scriptstyle a}\big\downarrow & & {\scriptstyle b}\big\downarrow & & \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \end{array}$$

in $\mathscr{C}$ there exists a morphism $(a, c)\colon\ \delta \to \delta'$ which is realized by (a, b, c).

(ET3)$^{\mathrm{op}}$ Dual of (ET3).

(ET4) Let $\delta \in \mathbb{E}(D,A)$ and $\delta' \in \mathbb{E}(F,B)$ be $\mathbb{E}$-extensions realized by $A \overset{f}{\to} B \overset{f'}{\to} D$ and $B \overset{g}{\to} C \overset{g'}{\to} F$, respectively. Then there exist an object $E \in \mathscr{C}$, a commutative diagram

(2.1)
$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{f'} & D \\ \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle d} \\ A & \xrightarrow{h} & C & \xrightarrow{h'} & E \\ & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle e} \\ & & F & = & F \end{array}$$

in $\mathscr{C}$, and an $\mathbb{E}$-extension $\delta'' \in \mathbb{E}(E,A)$ realized by $A \overset{h}{\to} C \overset{h'}{\to} E$, which satisfy the following compatibilities:

(i) $D \overset{d}{\to} E \overset{e}{\to} F$ realizes $\mathbb{E}(F,f')(\delta')$,

(ii) $\mathbb{E}(d,A)(\delta'') = \delta$,

(iii) $\mathbb{E}(E,f)(\delta'') = \mathbb{E}(e,B)(\delta')$,

(ET4)$^{\mathrm{op}}$ dual of (ET4).

The higher positive and negative extensions $\mathbb{E}^n$ in an extriangulated category have been defined in [1].

Proposition 2.3 ([1], Theorem 3.5). *For any $\mathbb{E}$-triangle $A \to B \to C \overset{\delta}{\dashrightarrow}$, the following sequences of natural transformations are exact:*

$$\begin{aligned} &\mathscr{C}(C,-) \to \mathscr{C}(B,-) \to \mathscr{C}(A,-) \overset{\delta^\sharp}{\to} \mathbb{E}(C,-) \\ &\qquad \to \mathbb{E}(B,-) \to \mathbb{E}(A,-) \to \mathbb{E}^2(C,-) \to \dots, \\ &\mathscr{C}(-,A) \to \mathscr{C}(-,B) \to \mathscr{C}(-,C) \overset{\delta_\sharp}{\to} \mathbb{E}(-,A) \\ &\qquad \to \mathbb{E}(-,B) \to \mathbb{E}(-,C) \to \mathbb{E}^2(-,A) \to \dots \end{aligned}$$

In what follows, we always assume that $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is an extriangulated category.

2.2. Auslander-Reiten theory. Recently, Iyama, Nakaoka and Palu developed the Auslander-Reiten theory for extriangulated categories in [2].

Definition 2.4. A nonsplit extension $\delta \in \mathbb{E}(C,A)$ is said to be *almost split* if it satisfies the following conditions:

(1) $f_*\delta = 0$ for any nonsection $f \in \mathrm{Hom}(A,A')$.

(2) $g^*\delta = 0$ for any nonretraction $g \in \mathrm{Hom}(C',C)$.

The $\mathbb{E}$-triangle $A \to B \to C \overset{\delta}{\dashrightarrow}$ for an almost split extension δ is called an *almost split sequence* or *Auslander-Reiten $\mathbb{E}$-triangle* in the sense of [7].

Definition 2.5. We say that $\mathscr{C}$ *has almost split extensions* if it satisfies the following conditions:

(1) For any indecomposable nonprojective object $A \in \mathscr{C}$ there exists an almost split extension $\delta \in \mathbb{E}(A, B)$ for some $B \in \mathscr{C}$.

(2) For any indecomposable noninjective object $B \in \mathscr{C}$ there exists an almost split extension $\delta \in \mathbb{E}(A, B)$ for some $A \in \mathscr{C}$.

We denote by $\mathcal{P}(\mathscr{C})$ the ideal of $\mathscr{C}$ consisting of all morphisms f such that $\mathbb{E}(f, -) = 0$, and define the ideal quotient $\underline{\mathscr{C}} = \mathscr{C}/\mathcal{P}(\mathscr{C})$. Dually, we define the ideal $\mathcal{I}(\mathscr{C})$ of $\mathscr{C}$ and the ideal quotient $\overline{\mathscr{C}} = \mathscr{C}/\mathcal{I}(\mathscr{C})$. In order to study the existence of almost split extensions, Iyama, Nakaoka and Palu in [2] introduced the notion of the Auslander-Reiten Serre duality. More explicitly, the Auslander-Reiten Serre duality is a pair (τ, η) of an additive functor $\tau\colon \underline{\mathscr{C}} \to \overline{\mathscr{C}}$ and a natural isomorphism η such that

$$\eta_{B,A}\colon \mathbb{D}\mathbb{E}(B, \tau A) \cong \underline{\mathscr{C}}(A, B)$$

for any $A, B \in \mathscr{C}$. By [2], Theorem 3.6, $\mathscr{C}$ has almost split extensions if and only if $\mathscr{C}$ has the Auslander-Reiten Serre duality.

Let $\mathscr{C}$ be an extriangulated category with Auslander-Reiten Serre duality. Denote by $\operatorname{ind}(\mathscr{C})$ the set of isoclasses of indecomposable objects in $\mathscr{C}$. Given $X, Y \in \operatorname{ind}(\mathscr{C})$, set $\operatorname{Irr}(X, Y) = \operatorname{rad}(X, Y)/\operatorname{rad}^2(X, Y)$, and it is an $\operatorname{End}(Y)$-$\operatorname{End}(X)$-bimodule. We set $d_{XY} = \dim_k \operatorname{Irr}(X, Y)$. The *Auslander-Reiten quiver* $\Gamma(\mathscr{C}) = (Q_0, Q_1, \tau)$ of $\mathscr{C}$ is defined as follows:

- ▷ The set Q_0 of vertices is $\operatorname{ind}(\mathscr{C})$.
- ▷ For $X, Y \in Q_0$ there exists d_{XY} arrows $X \to Y$ in Q_1.
- ▷ The functor τ, called the *Auslander-Reiten translation*, is such that $X = \tau Y$ if and only if there exists an almost split extension $\delta \in \mathbb{E}(Y, X)$.

It is well-known that the Auslander-Reiten quiver of $\mathscr{C}$ has a close relationship with sink and source morphisms. To be precise, if $f\colon X \to Y$ is a source morphism, then $Y \cong \oplus Y_i^{d_{XY_i}}$ for all $Y_i \in \operatorname{ind}(\mathscr{C})$. If $f\colon X \to Y$ is a sink morphism, then $X \cong \oplus X_i^{d_{X_iY}}$ for all $X_i \in \operatorname{ind}(\mathscr{C})$.

2.3. Filtration subcategories. We recall some preliminary properties about filtration subcategories from [6].

Let $\mathcal{X}$ be a collection of objects in $\mathscr{C}$. The *filtration subcategory* $\mathcal{F}(\mathcal{X})$ consists of all objects M admitting a finite filtration of the from

$$0 = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \to \ldots \xrightarrow{f_{n-1}} X_n = M$$

with f_i being an inflation and $\operatorname{cone}(f_i) \in \mathcal{X}$ for any $0 \leqslant i \leqslant n-1$.

In this case, we say that M possesses an $\mathcal{X}$-*filtration* of length n and the minimal length of such a filtration is called the $\mathcal{X}$-*length* of M, which is denoted by $l_{\mathcal{X}}(M)$.

Remark 2.6. Let $\mathcal{X}$ and $\mathcal{Y}$ be two collections of objects in $\mathscr{C}$.

▷ $\mathcal{F}(\mathcal{X})$ is the smallest extension-closed subcategory containing $\mathcal{X}$ in $\mathscr{C}$.
▷ For any $\mathbb{E}$-triangle $A \to B \to C \dashrightarrow$ in $\mathcal{F}(\mathcal{X})$, we have that $l_{\mathcal{X}}(B) \leqslant l_{\mathcal{X}}(A)+l_{\mathcal{X}}(C)$.
▷ If $\mathrm{Hom}(\mathcal{X},\mathcal{Y})=0$, then $\mathrm{Hom}(\mathcal{F}(\mathcal{X}),\mathcal{F}(\mathcal{Y}))=0$.
▷ If $\mathbb{E}(\mathcal{X},\mathcal{Y})=0$, then $\mathbb{E}(\mathcal{F}(\mathcal{X}),\mathcal{F}(\mathcal{Y}))=0$.

Proposition 2.7. *Let $\mathcal{X}$ be a collection of objects in $\mathscr{C}$. If $M \in \mathcal{F}(\mathcal{X})$, then there exists two $\mathbb{E}$-triangles*

$$X_i \to M \to M' \dashrightarrow \quad \text{and} \quad M'' \to M \to X_j \dashrightarrow$$

with $X_i, X_j \in \mathcal{X}$ and $l_{\mathcal{X}}(M') = l_{\mathcal{X}}(M'') = l_{\mathcal{X}}(M) - 1$.

Proof. It is easily proved by [6], Lemma 2.9. □

Let M be an object in $\mathscr{C}$, we say that M is a *brick* if $\mathrm{End}(M) \cong k$. A set $\mathcal{X}$ of mutually nonisomorphic bricks in $\mathscr{C}$ is called a *semibrick* if $\mathrm{Hom}(X_1,X_2)=0$ for any two nonisomorphic objects X_1, X_2 in $\mathcal{X}$.

The following result will be frequently used in what follows, see [6], Lemmas 3.5, 5.4 and Corollary 3.6.

Proposition 2.8. *Let $\mathcal{X}$ be a semibrick in $\mathscr{C}$.*

(1) *If $f\colon X \to M$ is a nonzero morphism in $\mathcal{F}(\mathcal{X})$ with $X \in \mathcal{X}$, then f is an inflation such that $l_{\mathcal{X}}(\mathrm{cone}(f)) = l_{\mathcal{X}}(M) - 1$.*
(2) *If $f\colon M \to X$ is a nonzero morphism in $\mathcal{F}(\mathcal{X})$ with $X \in \mathcal{X}$, then f is a deflation such that $l_{\mathcal{X}}(\mathrm{cocone}(f)) = l_{\mathcal{X}}(M) - 1$.*
(3) *$\mathcal{F}(\mathcal{X})$ is closed under direct summands in $\mathscr{C}$.*
(4) *For any object $X \in \mathcal{F}(\mathcal{X})$, if $X = A \oplus B$, then $l_{\mathcal{X}}(X) = l_{\mathcal{X}}(A) + l_{\mathcal{X}}(B)$.*

3. The Auslander-Reiten quivers of filtration subcategories

In what follows, we assume that $\mathscr{C}$ is an extriangulated category with Auslander-Reiten Serre duality (τ,η).

Definition 3.1. A semibrick $\mathcal{X} = \{X_i\}_{i\in\mathbb{Z}}$ is said to be τ-*semibrick* if it satisfies the following conditions:

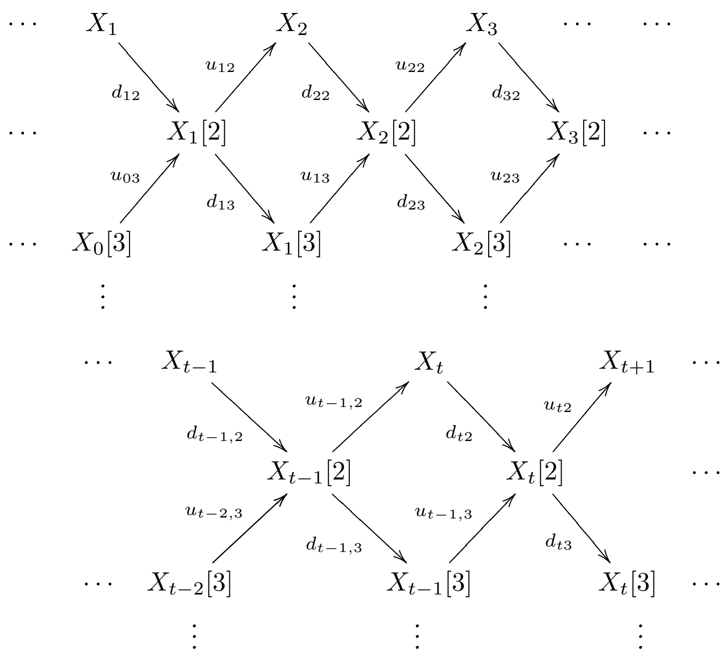
(1) $\tau X_i = X_{i-1}$ for $i \in \mathbb{Z}$.
(2) $\mathbb{E}^2(X_i,X_j)=0$ for $i,j \in \mathbb{Z}$.

If $\mathscr{C}$ has positive global dimension 1 in the sense of [1], Definition 3.28, then (2) is satisfied. Denote by $\mathbb{Z}\mathbb{A}_\infty$ the infinite translation quiver of the infinite quiver $\mathbb{A}_\infty$, $\tau_{\mathbb{A}}$ is an automorphism of $\mathbb{Z}\mathbb{A}_\infty$. For explicit definitions, we refer to [5], Section 1 of Chapter X. Now we are able to present our main results of this paper.

Theorem 3.2. *Let $\mathcal{X}$ be a τ-semibrick in $\mathscr{C}$. Then $\Gamma(\mathcal{F}(\mathcal{X})) \cong \mathbb{Z}\mathbb{A}_\infty$.*

Before proving Theorem 3.2, we need some preparations.

Lemma 3.3. *Let $\mathcal{X} = \{X_i\}_{i\in\mathbb{Z}}$ be a τ-semibrick in $\mathscr{C}$. We set $X_i[0] = 0$, $X_i[1] = X_i$ for $i \in \mathbb{Z}$. Then there exists an infinite diagram*



satisfying the following conditions.

(1) *For each $X_i[j]$, with $i \in \mathbb{Z}$ and $j \geqslant 2$, there exist two $\mathbb{E}$-triangles*

$$X_i \xrightarrow{d'_{ij}} X_i[j] \xrightarrow{u_{ij}} X_{i+1}[j-1] \overset{\mu_{ij}}{\dashrightarrow} \quad \text{and} \quad X_i[j-1] \xrightarrow{d_{ij}} X_i[j] \xrightarrow{u'_{ij}} X_{i+j-1} \overset{\nu_{ij}}{\dashrightarrow},$$

where $d'_{ij} = d_{i,j}\dots d_{i2}$ and $u'_{ij} = u_{i+j-2,2}\dots u_{ij}$.

(2) *For each $X_i[j]$ with $i \in \mathbb{Z}$ and $j \geqslant 1$, there exists an $\mathbb{E}$-triangle*

$$X_i[j] \xrightarrow{\binom{u_{ij}}{d_{i,j+1}}} X_{i+1}[j-1] \oplus X_i[j+1] \xrightarrow{(d_{i+1,j}\ u_{i,j+1})} X_{i+1}[j] \overset{\varrho_{ij}}{\dashrightarrow}$$

(3) *For any $f \in \mathrm{Hom}(X_i[j], X_l)$ with $i, l \in \mathbb{Z}$ and $j \geqslant 2$, we have that $fd_{ij} = 0$.*

(4) *For any $f \in \mathrm{Hom}(X_l, X_i[j])$ with $i, l \in \mathbb{Z}$ and $j \geqslant 2$, we have that $u_{ij}f = 0$.*

P r o o f. We proceed the proofs of (1) and (2) by induction on j. For $i \in \mathbb{Z}$, we have that

$$\begin{aligned}1 &\leqslant \dim_k \mathbb{E}(X_{i+1}, X_i)\\ &= \dim_k \mathbb{E}(X_{i+1}, \tau X_{i+1})\\ &= \dim_k \mathbb{D}\underline{\mathscr{C}}(X_{i+1}, X_{i+1})\\ &\leqslant \dim_k \operatorname{End}(X_{i+1}) = 1.\end{aligned}$$

Thus, $\dim_k \mathbb{E}(X_{i+1}, X_i) = 1$ and there exists a unique nonsplit extension $\varrho_{i1} \in \mathbb{E}(X_{i+1}, X_i)$, which is also an almost split extension. Hence, there exists an Auslander-Reiten $\mathbb{E}$-triangle

$$X_i \xrightarrow{d_{i2}} X_i[2] \xrightarrow{u_{i2}} X_{i+1} \overset{\varrho_{i1}}{\dashrightarrow} .$$

Since $d_{i2} \neq 0$, by Proposition 2.8, we have that $l_{\mathcal{X}}(X_i[2]) = 1 + l_{\mathcal{X}}(X_{i+1}) = 2$. In this case, we take $\mu_{i2} = \nu_{i2} = \varrho_{i1}$.

For $j \geqslant 2$, by induction, there exist two $\mathbb{E}$-triangles

$$X_i \xrightarrow{d'_{ij}} X_i[j] \xrightarrow{u_{ij}} X_{i+1}[j-1] \overset{\mu_{ij}}{\dashrightarrow}$$

and

$$X_{i+1}[j-1] \xrightarrow{d_{i+1,j}} X_{i+1}[j] \xrightarrow{u'_{i+1,j}} X_{i+j} \overset{\nu_{i+1,j}}{\dashrightarrow} \tag{3.1}$$

with $d'_{ij} = d_{ij} \dots d_{i2}$ and $u'_{i+1,j} = u_{i+j-1,2} \dots u_{i+1,j}$. Applying the functor $\operatorname{Hom}(-, X_i)$ to (3.1), we obtain an exact sequence

$$\mathbb{E}(X_{i+j}, X_i) \to \mathbb{E}(X_{i+1}[j], X_i) \to \mathbb{E}(X_{i+1}[j-1], X_i) \to 0.$$

Hence, there exists an extension $\gamma \in \mathbb{E}(X_{i+1}[j], X_i)$ such that the diagram

$$\begin{array}{ccccccc} X_i & \xrightarrow{d'_{ij}} & X_i[j] & \xrightarrow{u_{ij}} & X_{i+1}[j-1] & \overset{\mu_{ij}}{\dashrightarrow} \\ \Vert & & \downarrow & & \downarrow{\scriptstyle d_{i+1,j}} & \\ X_i & \longrightarrow & X_i[j+1] & \xrightarrow{-u_{i,j+1}} & X_{i+1}[j] & \overset{\gamma}{\dashrightarrow} \end{array}$$

is commutative. Using (3.1) together with $(\text{ET4})^{\text{op}}$, we obtain a commutative diagram

$$
(3.2)\qquad
\begin{array}{ccccccc}
X_i & \xrightarrow{d'_{ij}} & X_i[j] & \xrightarrow{u_{ij}} & X_{i+1}[j-1] & \overset{\mu_{ij}}{\dashrightarrow} & \\
\| & & {\scriptstyle h}\downarrow{\scriptstyle \cong} & & \| & & \\
X_i & \longrightarrow & H & \longrightarrow & X_{i+1}[j-1] & \overset{\mu_{ij}}{\dashrightarrow} & \\
\| & & {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle d_{i+1,j}} & & \\
X_i & \longrightarrow & X_i[j+1] & \xrightarrow{-u_{i,j+1}} & X_{i+1}[j] & \overset{\gamma}{\dashrightarrow} & \\
 & & \downarrow & & \downarrow{\scriptstyle u'_{i+1,j}} & & \\
 & & X_{i+j} & = & X_{i+j}. & &
\end{array}
$$

Set $lh = d_{i,j+1}$, by (3.2), we obtain two $\mathbb{E}$-triangles

$$X_i \xrightarrow{d'_{i,j+1}} X_i[j+1] \xrightarrow{u_{i,j+1}} X_{i+1}[j] \overset{\mu_{i,j+1}}{\dashrightarrow}$$

and

$$X_i[j] \xrightarrow{d_{i,j+1}} X_i[j+1] \xrightarrow{u'_{i,j+1}} X_{i+j} \overset{\nu_{i,j+1}}{\dashrightarrow}\ ,$$

where $d'_{i,j+1} = lhd'_{ij} = d_{i,j+1}d_{ij}\dots d_{i2}$ and

$$u'_{i,j+1} = u'_{i+1,j}u_{i,j+1} = u_{i+j-1,2}\dots u_{i+1,j}u_{i,j+1}.$$

Moreover, by [3], Corollary 3.16, there is an $\mathbb{E}$-triangle

$$X_i[j] \xrightarrow{\binom{u_{ij}}{d_{i,j+1}}} X_{i+1}[j-1] \oplus X_i[j+1] \xrightarrow{(d_{i+1,j},\, u_{i,j+1})} X_{i+1}[j] \overset{\varrho_{ij}}{\dashrightarrow}$$

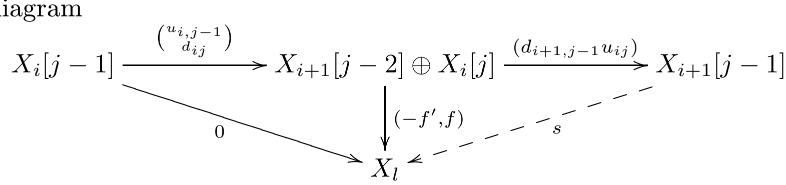
with $\varrho_{ij} = {d'_{ij}}_*\gamma$.

(3) For $j = 2$, $fd_{i2} \in \operatorname{Hom}(X_i, X_l)$. If $l \neq i$, then $fd_{i2} = 0$. If $l = i$, either $fd_{i2} = 0$ or fd_{i2} is an isomorphism. For the latter, we obtain that $d_{i,2}$ is a section and $\varrho_{i1} = 0$, which is a contradiction.

For $j > 2$, by induction, $fd_{ij}d_{i,j-1} \in \operatorname{Hom}(X_i[j-2], X_l) = 0$. Thus, we obtain the commutative diagram

$$
\begin{array}{ccccc}
X_i[j-2] & \xrightarrow{d_{i,j-1}} & X_i[j-1] & \xrightarrow{u'_{i,j-1}} & X_{i+j-2} \\
 & {\scriptstyle 0}\searrow & \downarrow{\scriptstyle fd_{ij}} & \swarrow{\scriptstyle s} & \\
 & & X_l & &
\end{array}
$$

with $fd_{ij} = su'_{i,j-1} = su_{i+j-3,2}\dots u_{i+1,j-2}u_{i,j-1}$. Let $f' = su_{i+j-3,2}\dots u_{i+1,j-2}\colon X_{i+1}[j-2]\to X_l$ and $fd_{ij} = f'u_{i,j-1}$. Since $(-f', f)\binom{u_{i,j-1}}{d_{ij}} = 0$, there is a commutative diagram



with $-f' = sd_{i+1,j-1}$. By induction, we get that $-f' = 0$ and $fd_{ij} = 0$. The proof of (4) is similar. □

In what follows, we keep the notation used in Lemma 3.3.

Remark 3.4. Note that $X_i[j]\neq 0$ for $i\in\mathbb{Z}$ and $j\geqslant 1$. Clearly, $X_i[j]\neq 0$ for $i\in\mathbb{Z}$ and $1\leqslant j\leqslant 2$. For $j>2$, there exists an $\mathbb{E}$-triangle

$$X_i[j]\xrightarrow{d_{i+1,j}} X_i[j+1]\xrightarrow{u'_{i,j+1}} X_{i+j}\overset{\nu_{i,j+1}}{\dashrightarrow}\ .$$

If $X_i[j]=0$, then $u'_{i,j+1}$ is an isomorphism and $u_{i+j-1,2}$ is a retraction, which is a contradiction.

Lemma 3.5.

(1) *d_{ij} and u_{ij} are nonzero for $i\in\mathbb{Z}$ and $j\geqslant 2$.*

(2) *If $\mathrm{Hom}(X_i, X_j[k])\neq 0$ for $i,j\in\mathbb{Z}$ and $k\geqslant 1$, then $j=i$.*

(2′) *If $\mathrm{Hom}(X_j[k], X_i)\neq 0$ for $i,j\in\mathbb{Z}$ and $k\geqslant 1$, then $j=i-k+1$.*

(3) *$d'_{ij} = d_{ij}\dots d_{i2}\neq 0$ for $i\in\mathbb{Z}$ and $j\geqslant 2$.*

(3′) *$u'_{ij} = u_{i+j-2,2}\dots u_{ij}\neq 0$ for $i\in\mathbb{Z}$ and $j\geqslant 2$.*

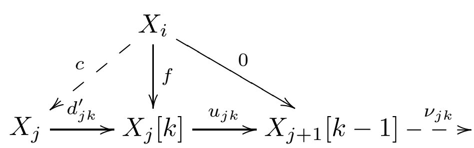
(4) *$l_{\mathcal{X}}(X_i[j]) = j$ for $i\in\mathbb{Z}$ and $j\geqslant 1$.*

Proof. (1) For $j\geqslant 2$ and $i\in\mathbb{Z}$, by Lemma 3.3, we have an $\mathbb{E}$-triangle

$$X_i\xrightarrow{d'_{ij}} X_i[j]\xrightarrow{u_{ij}} X_{i+1}[j-1]\overset{\mu_{ij}}{\dashrightarrow},$$

where $d'_{ij} = d_{ij}\dots d_{i2}$. Assume that $u_{ij}=0$, then d'_{ij} is a retraction. Since X_i is indecomposable and $X_i[j]\neq 0$, we conclude that d'_{ij} is an isomorphism and $X_{i+1}[j-1]\cong 0$, which is a contradiction. Similarly, one gets that $d_{ij}\neq 0$.

(2) Assume that $0\neq f\in\mathrm{Hom}(X_i, X_j[k])$. By Lemma 3.3 (4), there is a commutative diagram



since $u_{jk}f = 0$. As $f \neq 0$, we know that $c \neq 0$. It follows that $j = i$. The proof of (2′) is similar.

(3) The case of $j = 2$ follows from (1). If $j > 2$ and $d'_{ij} = 0$, there is a diagram

(3.3)
$$\begin{array}{ccccc} X_i & \xrightarrow{d_{i2}} & X_i[2] & \xrightarrow{u_{i2}} & X_{i+1} \overset{\varrho_{i1}}{\dashrightarrow} \\ & \searrow^{0} & \downarrow{\scriptstyle d'} & \swarrow_{s} & \\ & & X_i[j] & & \end{array}$$

such that $d' = d_{ij} \ldots d_{i3} = su_{i2}$. By (2), we know that $s \in \mathrm{Hom}(X_{i+1}, X_i[j]) = 0$ and $d' = 0$. Take $d'' = d_{ij} \ldots d_{i4}$, then $d''d_{i3} = d' = 0$. Replacing ϱ_{i1} by υ_{i3} in (3.3), there exists a morphism $s'\colon X_{i+2} \to X_i[j]$ such that $s'u'_{i3} = d''$. It follows that $d'' = 0$. Repeating the process, one has that $d_{ij} = s''u'_{i,j-1}$, where s'' is a morphism from X_{i+j-2} to $X_i[j]$. Thus, $s'' = 0$ and $d_{ij} = 0$, which contradicts to (1). The proof of (3′) is similar.

(4) By Lemma 3.3 and (3), there is an $\mathbb{E}$-triangle

$$X_i \xrightarrow{d'_{ij}} X_i[j] \to X_{i+1}[j-1] \dashrightarrow$$

with $d'_{ij} \neq 0$. By Proposition 2.8, we obtain that $l_{\mathcal{X}}(X_i[j]) = 1 + l_{\mathcal{X}}(X_{i+1}[j-1]) = 1 + j - 1 = j$. □

Lemma 3.6.

(1) *If $f\colon X_s \to X_i[j]$ is a nonzero morphism for $i, s \in \mathbb{Z}$ and $j \geqslant 1$, then $s = i$ and f is an inflation such that* $\mathrm{cone}(f) = X_{i+1}[j-1]$.
(2) *$X_i[j]$ is indecomposable for $i \in \mathbb{Z}$ and $j \geqslant 1$.*
(3) *$\mu_{ij} \neq 0$ and $\nu_{ij} \neq 0$ for $i \in \mathbb{Z}$ and $j \geqslant 2$.*
(4) *$\varrho_{ij} \neq 0$ for $i \in \mathbb{Z}$ and $j \geqslant 2$.*
(5) *$\mathrm{Hom}(X_{i+1}[j], X_i[j+1]) = 0$ for $i \in \mathbb{Z}$ and $j \geqslant 1$.*
(6) *If $X_{i+1}[j] = \tau X_{i+2}[j]$, then $\mathbb{E}(X_{i+1}[j+1], X_{i+1}[j]) = 0$ for $i \in \mathbb{Z}$ and $j \geqslant 1$.*

Proof. (1) By Lemma 3.5 (2) and Proposition 2.8, $s = i$ and there is an $\mathbb{E}$-triangle

$$X_i \xrightarrow{f} X_i[j] \to M \dashrightarrow$$

with $l_{\mathcal{X}}(M) = l_{\mathcal{X}}(X_i[j]) - 1 = j - 1$. By Lemma 3.3 (4), there is a commutative diagram

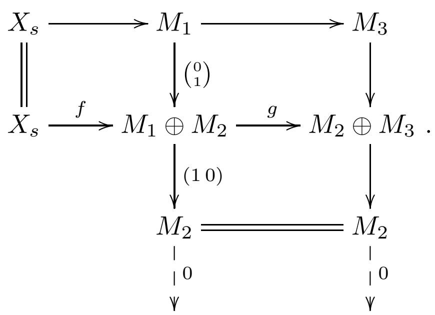
$$\begin{array}{ccccc} X_i & \xrightarrow{f} & X_i[j] & \longrightarrow & M \dashrightarrow \\ \downarrow{\scriptstyle h} & & \| & & \downarrow{\scriptstyle h'} \\ X_i & \xrightarrow{d'_{ij}} & X_i[j] & \xrightarrow{u_{ij}} & X_{i+1}[j-1] \dashrightarrow \end{array}$$

since $u_{ij}f = 0$. Note that $f \neq 0$, then $h \neq 0$. Thus, h is an isomorphism and so is h'.

(2) If $j=1$, then X_i is indecomposable since X_i is a brick. Assume that $X_i[l]$ is indecomposable for $i \in \mathbb{Z}$ and $1 \leqslant l \leqslant j-1$. If $X_i[j] = M_1 \oplus M_2$ with $M_1, M_2 \neq 0$, by Propositions 2.7 and 2.8, $M_1 \in \mathcal{F}(\mathcal{X})$ and there exists an $\mathbb{E}$-triangle

$$X_s \xrightarrow{f} M_1 \to M_3 \dashrightarrow$$

with $l_{\mathcal{X}}(M_3) = l_{\mathcal{X}}(M_1) - 1$ for some $s \in \mathbb{Z}$. Since $l_{\mathcal{X}}(M_1) > l_{\mathcal{X}}(M_3)$, we have that $f \neq 0$. We have the following commutative diagram by (ET4):

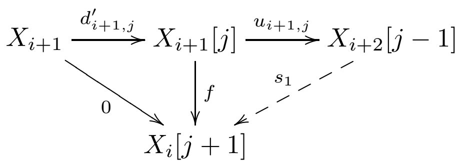


Since $f \neq 0$, by (1), we obtain that $M_2 \oplus M_3 \cong X_{i+1}[j-1]$, which is a contradiction.

(3) By (2).

(4) If $j=1$, then ϱ_{i1} is an almost split extension for $i \in \mathbb{Z}$. If $j \geqslant 2$, we claim that $\varrho_{ij} \neq 0$. Indeed, if $\varrho_{ij} = 0$, then $X_{i+1}[j-1] \oplus X_i[j+1] \cong X_i[j] \oplus X_{i+1}[j]$. It follows that $X_i[j+1]$ is a direct summand of $X_{i+1}[j]$ or $X_i[j]$. Thus, Lemma 3.5 (4) implies that $j+1 \leqslant j$, which is a contradiction.

(5) Let $f \in \mathrm{Hom}(X_{i+1}[j], X_i[j+1])$. By Lemma 3.5 (2), we obtain that $f d'_{i+1,j} = 0$ and then we have the commutative diagram



such that $f = s_1 u_{i+1,j}$. By Lemma 3.5 (2) again, we know that

$$s_1 d'_{i+2,j-1} \in \mathrm{Hom}(X_{i+2}, X_i[j+1]) = 0$$

and there exists a morphism $s_2\colon X_{i+3}[j-2] \to X_i[j+1]$ such that $s_2 u_{i+2,j-1} = s_1$ and $f = s_2 u_{i+2,j-1} u_{i+1,j}$. Repeating the process, we obtain that

$$f = s_{j-2} u_{i+j-2,3} \ldots u_{i+2,j-1} u_{i+1,j},$$

where $s_{j-2} \in \operatorname{Hom}(X_{i+j-1}[2], X_i[j+1])$. Since

$$s_{j-2}d_{i+j-1,2} \in \operatorname{Hom}(X_{i+j-1}, X_i[j+1]) = 0,$$

there exists a morphism $s_{j-1} \in \operatorname{Hom}(X_{i+j}, X_i[j+1]) = 0$ such that

$$f = s_{j-1}u_{i+j-1,2}\dots u_{i+2,j-1}u_{i+1,j} = 0.$$

(6) By (5), we have that

$$\begin{aligned} \dim_k \mathbb{E}(X_{i+1}[j+1], X_{i+1}[j]) &= \dim_k \mathbb{D}\underline{\mathscr{C}}(\tau^{-1}X_{i+1}[j], X_{i+1}[j+1]) \\ &= \dim_k \mathbb{D}\underline{\mathscr{C}}(X_{i+2}[j], X_{i+1}[j+1]) \\ &\leqslant \dim_k \operatorname{Hom}(X_{i+2}[j], X_{i+1}[j+1]) = 0. \end{aligned}$$

Therefore, we complete the proof. □

Lemma 3.7. *For each $X_i[j]$ with $i \in \mathbb{Z}$ and $j \geqslant 1$, the sequence*

$$X_i[j] \xrightarrow{\binom{u_{ij}}{d_{i,j+1}}} X_{i+1}[j-1] \oplus X_i[j+1] \xrightarrow{(d_{i+1,j}\, u_{i,j+1})} X_{i+1}[j] \overset{\varrho_{ij}}{\dashrightarrow}$$

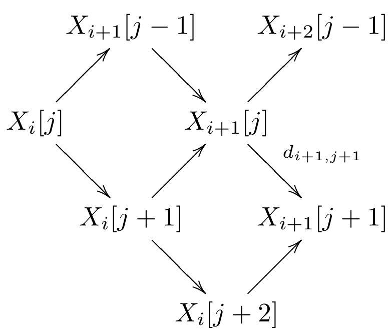
is an Auslander-Reiten $\mathbb{E}$-triangle.

P r o o f. We proceed the proof by induction on j. The proof of $j = 1$ follows from Lemma 3.3. Assume that ϱ_{il} is an almost split extension for $i \in \mathbb{Z}$ and $1 \leqslant l \leqslant j$.

By Lemma 3.6, we know that $X_{i+1}[j+1]$ is an indecomposable nonprojective object in $\mathscr{C}$. Then there exists an Auslander-Reiten $\mathbb{E}$-triangle

$$\tau X_{i+1}[j+1] \to E \to X_{i+1}[j+1] \dashrightarrow$$

in $\mathscr{C}$. By Lemma 3.3, there is the following diagram:



Since $d_{i+1,j+1}$ is an irreducible morphism, there exists an irreducible morphism $s\colon \tau X_{i+1}[j+1] \to X_{i+1}[j]$. It means that $\tau X_{i+1}[j+1]$ is a direct summand of $X_{i+1}[j-1] \oplus X_i[j+1]$. Then either $\tau X_{i+1}[j+1] \cong X_i[j+1]$ or $\tau X_{i+1}[j+1] \cong X_{i+1}[j-1]$. If $\tau X_{i+1}[j+1] \cong X_{i+1}[j-1]$, then $X_{i+1}[j+1] \cong \tau^{-1} X_{i+1}[j-1] = X_{i+2}[j-1]$. By Lemma 3.5 (4), we have that

$$j+1 = l_{\mathcal{X}}(X_{i+1}[j+1]) = l_{\mathcal{X}}(X_{i+2}[j-1]) = j-1,$$

which is a contradiction. Hence, $\tau X_{i+1}[j+1] \cong X_i[j+1]$. There is a commutative diagram

$$\begin{array}{ccccccc}
X_i[j+1] & \xrightarrow{\binom{u_{i,j+1}}{d_{i,j+2}}} & X_{i+1}[j] \oplus X_i[j+2] & \xrightarrow{(d_{i+1,j+1}u_{i,j+2})} & X_{i+1}[j+1] & \overset{\varrho_{i,j+1}}{\dashrightarrow} & \\
{\scriptstyle s}\downarrow & & \downarrow & & \| & & \\
X_i[j+1] & \longrightarrow & E & \longrightarrow & X_{i+1}[j+1] & \overset{\sigma'}{\dashrightarrow} & ,
\end{array}$$

since $(d_{i+1,j+1}u_{i,j+2})$ is not a retraction.

Assume that s is not an isomorphism. For $d'_{i,j+1} = d_{i,j+1} \dots d_{i2}$, we claim that $sd'_{i,j+1} = 0$. If $sd'_{i,j+1} \neq 0$, by Lemma 3.6 (1), there is a commutative diagram

$$\begin{array}{ccccccc}
X_i & \xrightarrow{d'_{i,j+1}} & X_i[j+1] & \xrightarrow{u_{i,j+1}} & X_{i+1}[j] & \overset{\mu_{i,j+1}}{\dashrightarrow} & \\
\| & & \downarrow{\scriptstyle s} & & {\scriptstyle h}\downarrow & & \\
X_i & \longrightarrow & X_i[j+1] & \longrightarrow & X_{i+1}[j] & \overset{\theta}{\dashrightarrow} & .
\end{array}$$

Since $\operatorname{End}(X_{i+1}[j])$ is local and s is not an isomorphism, then $h^n = 0$ for some $n \in \mathbb{N}$. Observe that ${h^{n-1}}^* u_{i,j+1} = {h^n}^* \theta = 0$, we have that there exists a morphism $s'\colon X_{i+1}[j] \to X_i[j+1]$ such that $h^{n-1} = u_{i,j+1}s'$. By Lemma 3.6 (5), we have that $s' = 0$ and $h^{n-1} = 0$. Repeating the process, we have that $h^*\theta = \mu_{i,j+1} = 0$, which is a contradiction. Therefore, we conclude that $sd'_{i,j+1} = 0$ and there is a diagram

$$\begin{array}{ccccc}
X_i & \xrightarrow{d'_{i,j+1}} & X_i[j+1] & \xrightarrow{u_{i,j+1}} & X_{i+1}[j] \\
 & & \downarrow{\scriptstyle s} & \swarrow{\scriptstyle q} & \\
 & & X_i[j+1] & &
\end{array}$$

with $s = qu_{i,j+1}$. By Lemma 3.6 (6), we get that

$$\sigma' = s_*\varrho_{i,j+1} = (qu_{i,j+1})_*\varrho_{i,j+1} = q_*{u_{i,j+1}}_*\varrho_{i,j+1} = 0,$$

since ${u_{i,j+1}}_*\varrho_{i,j+1} \in \mathbb{E}(X_{i+1}[j+1], X_{i+1}[j]) = 0$. This contradicts the fact that σ' is an almost split extension. Hence, s is an isomorphism and $\varrho_{i,j+1} = \sigma'$. □

Lemma 3.8. *Let $M \in \mathcal{F}(\mathcal{X})$ with $l_{\mathcal{X}}(M) \geqslant j \geqslant 1$, and $h \in \mathrm{Hom}(X_i[j], M)$ such that $hd'_{ij} \neq 0$ for $i \in \mathbb{Z}$. Then there exists an $\mathbb{E}$-triangle*

$$X_i[j] \to M \to M'' \dashrightarrow$$

with $l_{\mathcal{X}}(M'') = l_{\mathcal{X}}(M) - j$.

P r o o f. If $j = 2$, by Proposition 2.8, there is a commutative diagram

$$\begin{array}{ccccccc}
X_i & \xrightarrow{d_{i2}} & X_i[2] & \xrightarrow{u_{i2}} & X_{i+1} & \overset{\varrho_{i1}}{\dashrightarrow} & \\
\| & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h'} & & \\
X_i & \xrightarrow{hd_{i2}} & M & \longrightarrow & M' & \overset{\theta}{\dashrightarrow} &
\end{array}$$

with $l_{\mathcal{X}}(M') = l_{\mathcal{X}}(M) - 1$. If $h' = 0$, then $\varrho_{i1} = h'^{*}\theta = 0$, which is a contradiction. Hence, by Proposition 2.8 again, h' is an inflation and we have the following commutative diagram by $(\mathrm{ET4})^{\mathrm{op}}$:

(3.4)
$$\begin{array}{ccccc}
X_i & \xrightarrow{d_{i2}} & X_i[2] & \xrightarrow{u_{i2}} & X_{i+1} \\
\| & & \downarrow{\scriptstyle h''} & & \downarrow{\scriptstyle h'} \\
X_i & \longrightarrow & M & \longrightarrow & M' \\
 & & \downarrow & & \downarrow \\
 & & M'' & = & M''
\end{array}$$

with $l_{\mathcal{X}}(M'') = l_{\mathcal{X}}(M) - 2$. So the second column in (3.4) gives a desired $\mathbb{E}$-triangle.

For $j > 2$, by diagram (3.2) in Lemma 3.3, there is a commutative diagram

$$\begin{array}{ccccccc}
X_i & \xrightarrow{d_{i2}} & X_i[2] & \xrightarrow{u_{i2}} & X_{i+1} & \overset{\varrho_{i1}}{\dashrightarrow} & \\
\| & & \downarrow{\scriptstyle d_{ij}\ldots d_{i3}} & & \downarrow{\scriptstyle d_{i+1,j-1}\ldots d_{i+1,2}} & & \\
X_i & \xrightarrow{d'_{ij}} & X_i[j] & \xrightarrow{u_{ij}} & X_{i+1}[j-1] & \overset{\mu_{ij}}{\dashrightarrow} & \\
\| & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h'} & & \\
X_i & \longrightarrow & M & \longrightarrow & M' & \overset{\theta}{\dashrightarrow} &
\end{array}$$

with $l_{\mathcal{X}}(M') = l_{\mathcal{X}}(M) - 1$. Since $\varrho_{i1} \neq 0$, then $h'd_{i+1,j-1} \ldots d_{i+1,2} \neq 0$. By induction, we know that h' is an inflation such that

$$l_{\mathcal{X}}(\mathrm{cone}(h')) = l_{\mathcal{X}}(M') - j + 1 = l_{\mathcal{X}}(M) - j.$$

Applying (ET4)$^{\rm op}$ yields an exact commutative diagram

$$\begin{array}{ccccc} X_i & \xrightarrow{d'_{ij}} & X_i[j] & \xrightarrow{u_{ij}} & X_{i+1}[j-1] \\ \| & & \downarrow{\scriptstyle h''} & & \downarrow{\scriptstyle h'} \\ X_i & \longrightarrow & M & \longrightarrow & M' \\ & & \downarrow & & \downarrow \\ & & \operatorname{cone}(h') & = & \operatorname{cone}(h'). \end{array} \tag{3.5}$$

So the second column in (3.5) gives a desired $\mathbb{E}$-triangle. □

Now we are in the position to prove Theorem 3.2.

Proof of Theorem 3.2. By Lemmas 3.3 and 3.7, it remains to show that each indecomposable object M in $\mathcal{F}(\mathcal{X})$ has the form $X_i[j]$ for some $i \in \mathbb{Z}$ and $j \geqslant 1$.

Assume that M is an indecomposable object with $l_{\mathcal{X}}(M) = j$. By Proposition 2.7, there is a nonsplit $\mathbb{E}$-triangle

$$X_i \xrightarrow{a} M \to M_1 \dashrightarrow$$

with $l_{\mathcal{X}}(M_1) = j - 1$ for some $i \in \mathbb{Z}$. Since $l_{\mathcal{X}}(M) > l_{\mathcal{X}}(M_1)$, we get that $a \neq 0$. Since a is not a section and

$$X_i \xrightarrow{d_{i2}} X_i[2] \xrightarrow{u_{i2}} X_{i+1} \overset{\varrho_{i1}}{\dashrightarrow}$$

is an Auslander-Reiten $\mathbb{E}$-triangle, there exists a morphism $a'_2\colon X_i[2] \to M$ such that $a = a'_2 d_{i2} \neq 0$. By Lemma 3.8, there exists an $\mathbb{E}$-triangle

$$X_i[2] \xrightarrow{a_2} M \to M_2 \dashrightarrow$$

with $l_{\mathcal{X}}(M_2) = j - 2$. It is clear that $a_2 \neq 0$ and a_2 is not a section. Since

$$X_i[2] \xrightarrow{\binom{u_{i2}}{d_{i3}}} X_{i+1} \oplus X_i[3] \xrightarrow{(d_{i+1,2}\ u_{i3})} X_{i+1}[2] \overset{\varrho_{i2}}{\dashrightarrow}$$

is an Auslander-Reiten $\mathbb{E}$-triangle, there exists a morphism $(s_1, s_2)\colon X_{i+1} \oplus X_i[3] \to M$ such that $s_1 u_{i2} + s_2 d_{i3} = a_2$. Hence, $s_2 d_{i3} d_{i2} = s_2 d_{i3} d_{i2} + s_1 u_{i2} d_{i2} = a_2 d_{i2}$.

We claim that $a_2 d_{i2} \neq 0$. Indeed, applying (ET4) yields an exact commutative diagram

$$\begin{array}{ccccc} X_i & \xrightarrow{d_{i2}} & X_i[2] & \xrightarrow{u_{i2}} & X_{i+1} \\ \| & & \downarrow{\scriptstyle a_2} & & \downarrow \\ X_i & \longrightarrow & M & \longrightarrow & M' \\ & & \downarrow & & \downarrow \\ & & M_2 & = & M_2. \end{array}$$

By Remark 2.6, we have that $l_{\mathcal{X}}(M') \leqslant 1 + l_{\mathcal{X}}(M_2) = j - 1$. If $a_2 d_{i2} = 0$, then M is a direct summand of M' and $l_{\mathcal{X}}(M) \leqslant l_{\mathcal{X}}(M') \leqslant j - 1$, which is a contradiction. Therefore, we conclude that $s_2 d'_{i3} = s_2 d_{i3} d_{i2} = a_2 d_{i2} \neq 0$. By Lemma 3.8, there exists an $\mathbb{E}$-triangle

$$X_i[3] \xrightarrow{a'_3} M \to M_3 \dashrightarrow$$

with $l_{\mathcal{X}}(M_3) = j - 3$. Note that a'_3 is not a section and ϱ_{i3} is an almost split extension. Repeating the process, we obtain an $\mathbb{E}$-triangle

$$X_i[j] \xrightarrow{a_4} M \to M_4 \dashrightarrow$$

with $l_{\mathcal{X}}(M_4) = j - j = 0$. So $X_i[j] \cong M$.

Let $\Gamma(\mathcal{F}(\mathcal{X})) = (Q_0, Q_1, \tau)$ be the Auslander-Reiten quiver of $\mathcal{F}(\mathcal{X})$. Then

$$Q_0 = \{X_i[j] \colon i \in \mathbb{Z} \text{ and } j \geqslant 1\}.$$

For any $a = X_i[j]$, $b \in Q_0$, by Lemma 3.7, we know that $d_{ab} \neq 0$ if and only if $b = X_{i+1}[j-1]$ or $b = X_i[j+1]$. Then the arrows in Q_1 starting at a are

$$X_i[j] \xrightarrow{u_{ij}} X_{i+1}[j-1] \quad \text{and} \quad X_i[j] \xrightarrow{d_{i,j+1}} X_i[j+1]\ .$$

Similarly, the arrows in Q_1 ending at a are

$$X_{i-1}[j] \xrightarrow{d_{ij}} X_i[j] \quad \text{and} \quad X_{i-1}[j+1] \xrightarrow{u_{i-1,j+1}} X_i[j]\ .$$

Therefore, we obtain that $\Gamma(\mathcal{F}(\mathcal{X}))$ is the diagram in Lemma 3.3, and it is isomorphic to $\mathbb{Z}\mathbb{A}_\infty$. □

Definition 3.9. A finite semibrick $\mathcal{X} = \{X_i\}_{i=1}^t$ is said to be τ-*cycle* if it satisfies the following conditions.

(1) $\tau X_2 = X_1$, $\tau X_3 = X_2, \ldots, \tau X_t = X_{t-1}$ and $\tau X_1 = X_t$.

(2) $\mathbb{E}^2(X_i, X_j) = 0$ for $i, j \in [1, t]$.

Theorem 3.10. *Let $\mathcal{X} = \{X_i\}_{i=1}^t$ be a τ-cycle semibrick. Then $\Gamma(\mathcal{F}(\mathcal{X})) \cong \mathbb{Z}\mathbb{A}_\infty/\tau_{\mathbb{A}}^t$.*

P r o o f. It is proved by the analogous arguments as those for proving Theorem 3.2. □

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