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APPLICATION OF ROTHE'S METHOD TO A PARABOLIC INVERSE PROBLEM WITH NONLOCAL BOUNDARY CONDITION

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Abstract. We consider an inverse problem for the determination of a purely time-dependent source in a semilinear parabolic equation with a nonlocal boundary condition. An approximation scheme for the solution together with the well-posedness of the problem with the initial value $u_0 \in H^1(\Omega)$ is presented by means of the Rothe time-discretization method. Further approximation scheme via Rothe's method is constructed for the problem when $u_0 \in L^2(\Omega)$ and the integral kernel in the nonlocal boundary condition is symmetric.

Keywords: Rothe's method; nonlocal boundary condition; semilinear parabolic equation; inverse source problem

MSC 2020: 65M20, 35K58, 35R30

1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be a bounded domain with the boundary Γ of class $C^{0,1}$ and $T > 0$. We consider a problem of finding functions $u\colon \overline{\Omega} \times [0,T] \to \mathbb{R}$ and $p\colon [0,T] \to \mathbb{R}$ obeying the semilinear parabolic equation

$$\partial_t u - \Delta u = p(t)h(x,t) + f(x,t,u,\nabla u) \quad \text{in } \Omega \times (0,T) \tag{1.1}$$

with the initial condition

$$u(x,0) = u_0(x) \quad \text{in } \Omega \tag{1.2}$$

and the nonlocal boundary condition

$$u(x,t) = \int_\Omega k(x,y,t)u(y,t)\,\mathrm{d}y \quad \text{on } \Gamma \times (0,T), \tag{1.3}$$

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subject to the additional measurement

$$\int_\Omega u(x,t)\omega(x)\,\mathrm{d}x = q(t), \quad t \in (0,T), \tag{1.4}$$

where h, u_0, k, ω, q, f are given.

The parabolic equations with nonlocal integral conditions arise in thermoelasticity, ion-diffusion in channels, the technology of integral circuits, etc. (see [8], [22], [3] and the references therein). The problem (1.1)–(1.4) describes, for example, the quasi-static flexure of a thermoelastic rod, where u is entropy and the integral over-determination condition (1.4) means the average entropy over the domain Ω, see [8], p. 469–471, and [16], p. 378.

A number of methods for solving such nonlocal direct and inverse problems (IPs) are known, see, e.g., [4], [6], [13] and [7], [1].

The Rothe time-discretization method (or method of lines) as an approximate approach gives a simple numerical scheme together with the existence of solution for a wide range of evolution problems, see, e.g., the monographs by Kačur [12] and Rektorys [17]. Recently, this method was applied to parabolic IPs with classical boundary conditions, e.g., in [9], [10], [11], [19], [21] and to parabolic direct problems with nonlocal integral conditions, e.g., in [20], [5], [14].

Slodička [20] considered the unique solvability of the direct problem (1.1)–(1.3) with $f = f(\nabla u)$ using Rothe's method; more precisely, a solution u in the function space

$$V = \{v\colon\ v \in C([0,T];\, L^2(\Omega)) \cap L^\infty(0,T; H^1(\Omega)),\ \partial_t v \in L^2(0,T; L^2(\Omega))\}$$

was obtained under the assumption $u_0 \in H^2(\Omega)$; the assumption was regarded important for solvability of the considered problem. One can take notice that the regularity assumption $u_0 \in H^2(\Omega)$ for obtaining such a solution in [20] is stronger than required for the second order parabolic problems with classical boundary conditions (this appears also in [9], [21]).

On the other hand, Kozhanov [13], employing the parameter continuation method, showed the existence of a solution $u \in W_2^{2,1}(\Omega \times (0,T)) \cap L^\infty(0,T; H^1(\Omega))$ to the direct problem (1.1)–(1.3) under the condition $u_0 \in H^1(\Omega)$ but with an additional strong assumption $k(x,y,t) = 0$, $y \in \Gamma$.

The aim of this paper is to establish the Rothe time-discretization method for the parabolic IP (1.1)–(1.4) under weaker regularity than H^2 for the initial value u_0.

First, we find the solution $u \in V$ to the IP (1.1)–(1.4) under the assumption $u_0 \in H^1(\Omega)$ without further assumptions on the other data than [20]. The H^1-regularity of u_0 requires test functions different from [20]. We construct a time-discretization scheme to find an approximate solution. We choose suitable test functions taking account of the compatibility condition on the initial value, which together with the obvious and efficient inequality (2.20) yields estimates for solutions of the discrete scheme. For the proof of uniqueness of a solution, we use Rothe's method as well, which also distinguishes the paper from the above-mentioned references, where the uniqueness of the solution was proved just using the energy method, irrespective of the Rothe method. We use *a priori* the Rothe method to obtain the estimate of $\|\partial_t(u^{(1)} - u^{(2)})\|_{L^2(0,T;L^2(\Omega))}$ for two solutions $u^{(1)}, u^{(2)}$ to (1.1)–(1.4). Thus we can use a suitable test function to prove the uniqueness by the energy method, which is also crucial for weakening the regularity of u_0. See Remark 2.2 for more details.

Next, in this paper, we further address the Rothe method for (1.1)–(1.4) under the assumption $u_0 \in L^2(\Omega)$. To this end, we modify the above discrete scheme and apply the symmetry condition $k(x,y,t) = k(y,x,t)$ for the integral kernel (see [8], p. 471) to get the required estimates for its solutions.

Finally, we refer to [2] and [10], where the Rothe methods were proposed weakening the Lipschitz continuity of nonlinear term and the regularity of integral overdetermination value, respectively, as compared with the previous papers which also had applied the method.

Notations: We use the standard notation $L^2(\Omega), L^2(0,T), H^1(\Omega), H_0^1(\Omega)$ for Lebesgue and Sobolev spaces. By $H^{-1}(\Omega)$ we denote the dual space of $H_0^1(\Omega)$. Moreover, $L^p(0,T;X)$ for $1 \leqslant p \leqslant \infty$ and a Banach space X denotes the standard Bochner spaces. The symbol $\|\cdot\|_X$ denotes the norm of the normed space X. Moreover, $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_1 = \|\cdot\|_{H^1(\Omega)}$, $\|\cdot\|_\Gamma = \|\cdot\|_{L^2(\Gamma)}$. Throughout the paper, (μ,ν) denotes the usual scalar product in $L^2(\Omega)$, that is, $(\mu,\nu) = \int_\Omega \mu(x)\nu(x)\,\mathrm{d}x$, and $\|\mu\| = \sqrt{(\mu,\mu)}$. Letter C with subscript denotes different positive constants which are dependent upon the domain Ω or the length T of the time interval, or given functions. In particular, the positive constants C_ε, C_{ε_i}, $i \in \mathbb{N}$, depend also on the positive constants $\varepsilon, \varepsilon_i$, respectively.

The paper is organized as follows. In Section 2, we prove the convergence of Rothe's method and the uniqueness of the solution for $u_0 \in H^1(\Omega)$. Section 3 is devoted to the case of $u_0 \in L^2(\Omega)$.

2. Rothe's method I (case of $u_0 \in H^1(\Omega)$)

We make the following assumptions on the known functions:

(A1) $u_0 \in H^1(\Omega); \quad u_0(x) = \int_\Omega k(x,y,0)u_0(y)\,\mathrm{d}y, \quad x \in \Gamma.$

(A2) $\omega \in H_0^1(\Omega), \quad q \in C^1[0,T]; \quad h(t) \in L^2(\Omega),$

$$\left|\int_\Omega h(x,t)\omega(x)\,\mathrm{d}x\right| \geqslant C_{h1} > 0, \quad t \in [0,T];$$

$$\|h(t) - h(t')\| \leqslant C_{h2}|t-t'|, \quad t,t' \in [0,T].$$

(A3) $f(t,v,\nabla v) \in L^2(\Omega)$ for $t \in [0,T]$, $v \in H^1(\Omega)$;

$$\|f(t,v,\nabla v) - f(t',w,\nabla w)\| \leqslant C_f[\|v-w\|_1 + |t-t'|(1+\|v\|_1+\|w\|_1)] \quad \text{for } t,t' \in [0,T],\ v,w \in H^1(\Omega).$$

(A4)
$$\sqrt{\int_\Omega\int_\Omega k^2(x,y,t)\,\mathrm{d}y\,\mathrm{d}x} \leqslant C_{k1} < 1,$$

$$\sqrt{\int_\Omega\int_\Omega (\partial_t k)^2(x,y,t)\,\mathrm{d}y\,\mathrm{d}x} \leqslant C_{k2},$$

$$\sqrt{\int_\Omega\int_\Omega |\nabla_x k(x,y,t)|^2\,\mathrm{d}y\,\mathrm{d}x} \leqslant C_{k2},$$

$$\sqrt{\int_\Omega\int_\Omega |\nabla_x \partial_t k(x,y,t)|^2\,\mathrm{d}y\,\mathrm{d}x} \leqslant C_{k2}, \quad t \in [0,T].$$

Here we remark that the restriction on smallness of the integral kernel k like (A4) is common in the treatment of parabolic equations with the nonlocal boundary condition (1.3), see [6], [7], [13], [20].

We use the formal notation

$$Kv(x,t) := \int_\Omega k(x,y,t)v(y,t)\,\mathrm{d}y, \quad (x,t) \in \overline{\Omega} \times [0,T],$$
$$Fv(x,t) := f(x,t,v(x,t),\nabla v(x,t)), \quad (x,t) \in \Omega \times [0,T],$$

for a function $v(x,t)$. Multiplying formally the equation (1.1) by a test function $\varphi \in L^2(0,T;H_0^1(\Omega))$ and integrating the result over $\Omega \times (0,T)$, we have

$$\int_0^T (\partial_t u(t),\varphi(t))\,\mathrm{d}t + \int_0^T (\nabla u(t),\nabla\varphi(t))\,\mathrm{d}t = \int_0^T p(t)(h(t),\varphi(t))\,\mathrm{d}t + \int_0^T (Fu(t),\varphi(t))\,\mathrm{d}t \quad \forall\,\varphi \in L^2(0,T;H_0^1(\Omega)). \tag{2.1}$$

This yields the following definition of the solution.

Definition 2.1. A pair of functions (u, p) is called a *solution to the IP* (1.1)–(1.4) if $(u, p) \in V \times L^2(0, T)$ satisfies the following assumptions:

(i) for a.e. $t \in (0, T)$ and all $\phi \in H_0^1(\Omega)$,

$$(2.2) \qquad (\partial_t u(t), \phi) + (\nabla u(t), \nabla\phi) = p(t)(h(t), \phi) + (Fu(t), \phi),$$

(ii) for a.e. $t \in (0, T)$,

$$(2.3) \qquad p(t)(h(t), \omega) = q'(t) + (\nabla u(t), \nabla\omega) - (Fu(t), \omega),$$

(iii) u satisfies (1.2) and (1.3) in the trace sense.

Remark 2.1. (i) Choosing the test function $\phi = \omega$ in (2.2) and using the additional condition (1.4) lead to (2.3).

(ii) Obviously, the solution $(u, p) \in V \times L^2(0, T)$ by Definition 2.1 satisfies (2.1) pointwise. In particular, one has $\Delta u \in L^2(0, T; L^2(\Omega))$. However, $u \in L^2(0, T; H^2(\Omega))$ is not guaranteed in general since we do not know whether the right-hand side of (1.3) could belong to $L^2(0, T; H^{3/2}(\Omega))$ under the assumption (A4).

2.1. Time-discretization scheme and existence of a solution. Rothe's method is based on a semi-discretization with respect to the time variable. We divide the time interval $[0, T]$ into $n \in \mathbb{N}$ subintervals $[t_{i-1}, t_i]$, $i = 1, \ldots, n$, where $t_i = i\tau$ and $\tau = T/n$.

Put

$$\begin{aligned} Kv_i(x) &:= \int_\Omega k(x, y, t_i) v_i(y) \, \mathrm{d}y, \quad x \in \overline{\Omega}, \\ Fv_i(x) &:= f(x, t_i, v_i(x), \nabla v_i(x)), \quad x \in \Omega, \\ \delta v_i(x) &:= \frac{v_i(x) - v_{i-1}(x)}{\tau}, \quad x \in \overline{\Omega}, \end{aligned}$$

for a function $v_i(x)$.

On the basis of (1.2), (1.3), (2.2) and (2.3) we construct the following recurrent system of time-discretized problems to find $u_i(x)\colon \overline{\Omega} \to \mathbb{R}$ and $p_i \in \mathbb{R}$ from $u_{i-1}(x)\colon \overline{\Omega} \to \mathbb{R}$ for $i = 1, \ldots, n$:

$$(2.4) \qquad (\delta u_i, \phi) + (\nabla u_i, \nabla\phi) = p_i(h_i, \phi) + (Fu_{i-1}, \phi) \quad \forall\, \phi \in H_0^1(\Omega),$$

$$(2.5) \qquad p_i(h_i, \omega) = q_i' + (\nabla u_{i-1}, \nabla\omega) - (Fu_{i-1}, \omega),$$

$$(2.6) \qquad u_i(x) = Ku_{i-1}(x), \quad x \in \Gamma,$$

where $u_0(x)$ is given by (1.2), and $h_i(x) = h(x, t_i)$ and $q_i' = q'(t_i)$.

The following lemma shows the well-posedness of the scheme (2.4)–(2.6).

Lemma 2.1. *Let* (A1)–(A4) *be satisfied. Then, for all* $n \in \mathbb{N}$ *and* $i = 1, \ldots, n$, *there exists the unique pair* $(u_i, p_i) \in H^1(\Omega) \times \mathbb{R}$, *satisfying* (2.4)–(2.6).

Proof. Let $u_{i-1} \in H^1(\Omega)$ be given. Then (2.5) determines the unique p_i. We can rewrite the equation (2.4) as

$$(u_i, \phi) + \tau(\nabla u_i, \nabla \phi) = (u_{i-1}, \phi) + \tau p_i (h_i, \phi) + \tau(F u_{i-1}, \phi) \quad \forall\, \phi \in H_0^1(\Omega). \tag{2.7}$$

Substituting $u_i(x) = v_i(x) + K u_{i-1}(x)$ into the left-hand side (LHS) of (2.7) yields

$$\begin{aligned}(v_i, \phi) + \tau(\nabla v_i, \nabla \phi) &= (u_{i-1}, \phi) + \tau p_i (h_i, \phi) + \tau(F u_{i-1}, \phi) \\ &\quad - (K u_{i-1}, \phi) - \tau(\nabla K u_{i-1}, \nabla \phi) \quad \forall\, \phi \in H_0^1(\Omega).\end{aligned} \tag{2.8}$$

From the Lax-Milgram lemma, we immediately obtain the existence and uniqueness of a solution $v_i \in H_0^1(\Omega)$ of (2.8). Thus there exists the unique solution $u_i \in H^1(\Omega)$ of (2.4), (2.6). □

We derive estimates for $u_i(x)$, p_i, $i = 1, \ldots, n$, satisfying (2.4)–(2.6).

Lemma 2.2. *Let* (A1)–(A4) *be satisfied. Then there exist* $\tau_0 > 0$ *and* $C > 0$ *such that for all* $n > T/\tau_0$ *the solutions* (u_j, p_j), $j = 1, \ldots, n$, *to* (2.4)–(2.6) *satisfy*

$$\|u_j\|_1^2 \leqslant C, \tag{2.9}$$

$$\tau \sum_{i=1}^{j} \|\delta u_i\|^2 \leqslant C, \tag{2.10}$$

$$\sum_{i=1}^{j} \|u_i - u_{i-1}\|_1^2 \leqslant C, \tag{2.11}$$

$$p_j^2 \leqslant C. \tag{2.12}$$

Proof. Let $\delta K u_0(x) \equiv 0$. Then it follows from assumption (A1) and (2.6) that $(\delta u_i - \delta K u_{i-1})\tau \in H_0^1(\Omega)$, $i = 1, \ldots, n$. If we set $\phi = (\delta u_i - \delta K u_{i-1})\tau$ in (2.4) and sum it up for $i = 1, \ldots, j$ keeping $1 \leqslant j \leqslant n$, we obtain

$$\begin{aligned}&\tau \sum_{i=1}^{j} (\delta u_i, \delta u_i) + \tau \sum_{i=1}^{j} (\nabla u_i, \nabla \delta u_i) \\ &\qquad = \tau \sum_{i=1}^{j} p_i (h_i, \delta u_i - \delta K u_{i-1}) + \tau \sum_{i=1}^{j} (F u_{i-1}, \delta u_i - \delta K u_{i-1}) \\ &\qquad\quad + \tau \sum_{i=1}^{j} (\delta u_i, \delta K u_{i-1}) + \tau \sum_{i=1}^{j} (\nabla u_i, \nabla \delta K u_{i-1}).\end{aligned} \tag{2.13}$$

On the other hand, from (A3) we get

$$\|Fu_{i-1}\| \leqslant C_f(\|u_{i-1}\| + \|\nabla u_{i-1}\|) + M_f, \tag{2.14}$$

where $M_f := \max_{t\in[0,T]} \|f(t,0,0)\|$.

Applying the Cauchy inequality and (2.14) to (2.5) yields

$$|p_i| \leqslant \frac{|q_i'| + |(\nabla u_{i-1}, \nabla\omega)| + |(Fu_{i-1}, \omega)|}{|(h_i, \omega)|} \leqslant C_1 + C_2\|u_{i-1}\| + C_3\|\nabla u_{i-1}\|, \tag{2.15}$$

where

$$C_1 = \frac{1}{C_{h1}}(M_f\|\omega\| + \max_{t\in[0,T]} |q'(t)|), \quad C_2 = \frac{C_f\|\omega\|}{C_{h1}} \quad \text{and} \quad C_3 = \frac{C_f\|\omega\| + \|\nabla\omega\|}{C_{h1}}.$$

Moreover, it follows from (A4) that

$$\tau\|\delta Ku_{i-1}\| = \|Ku_{i-1} - Ku_{i-2}\| \leqslant \tau(C_{k1}\|\delta u_{i-1}\| + C_{k2}\|u_{i-2}\|), \tag{2.16}$$

$$\tau\|\nabla\delta Ku_{i-1}\| = \|\nabla Ku_{i-1} - \nabla Ku_{i-2}\| \leqslant \tau C_{k2}(\|\delta u_{i-1}\| + \|u_{i-2}\|). \tag{2.17}$$

The application of the identity

$$2\sum_{i=1}^{j} a_i(a_i - a_{i-1}) = a_j^2 - a_0^2 + \sum_{i=1}^{j}(a_i - a_{i-1})^2 \quad \forall\, a_i \in \mathbb{R},\ i = 0, \ldots, j, \tag{2.18}$$

to the second term in the LHS of (2.13) says that

$$\tau\sum_{i=1}^{j}(\nabla u_i, \nabla\delta u_i) = \frac{1}{2}\|\nabla u_j\|^2 - \frac{1}{2}\|\nabla u_0\|^2 + \frac{1}{2}\sum_{i=1}^{j}\|\nabla u_i - \nabla u_{i-1}\|^2.$$

Using (2.16) and Young's inequality, the third term in the right-hand side (RHS) of (2.13) can be estimated as

$$\begin{aligned}\left|\tau\sum_{i=1}^{j}(\delta u_i, \delta Ku_{i-1})\right| &\leqslant C_{k1}\tau\sum_{i=2}^{j}\|\delta u_i\|\|\delta u_{i-1}\| + C_{k2}\tau\sum_{i=2}^{j}\|\delta u_i\|\|u_{i-2}\| \\ &\leqslant C_{\varepsilon_1} + C_{k1}\tau\sum_{i=1}^{j}\|\delta u_i\|^2 + C_{\varepsilon_1}\tau\sum_{i=1}^{j}\|u_i\|^2 + \varepsilon_1\tau\sum_{i=1}^{j}\|\delta u_i\|^2.\end{aligned}$$

Estimating, similarly to above, the other terms in the RHS of (2.13) by (2.14)–(2.17) and applying the obtained relations to (2.13), we get

$$\begin{aligned}(1 - C_{k1} - \varepsilon_2)\tau\sum_{i=1}^{j}\|\delta u_i\|^2 + \frac{1}{2}\|\nabla u_j\|^2 + \frac{1}{2}\sum_{i=1}^{j}\|\nabla u_i - \nabla u_{i-1}\|^2 \\ \leqslant C_{\varepsilon_2} + C_{\varepsilon_2}\tau\sum_{i=1}^{j}\|u_i\|^2 + C_{\varepsilon_2}\tau\sum_{i=1}^{j}\|\nabla u_i\|^2.\end{aligned} \tag{2.19}$$

On the other hand, applying (2.18) to the LHS and Young's inequality to the RHS of the inequality

$$\tau\sum_{i=1}^{j}(\delta u_i,u_i)\leqslant\Bigl|\tau\sum_{i=1}^{j}(\delta u_i,u_i)\Bigr|,$$

we have

$$\frac{1}{2}\|u_j\|^2-\frac{1}{2}\|u_0\|^2+\frac{1}{2}\sum_{i=1}^{j}\|u_i-u_{i-1}\|^2\leqslant\frac{1}{\varepsilon_3}\tau\sum_{i=1}^{j}\|u_i\|^2+\varepsilon_3\tau\sum_{i=1}^{j}\|\delta u_i\|^2. \tag{2.20}$$

Putting (2.19) and (2.20) together, we arrive at

$$(1-C_{k1}-\varepsilon)\tau\sum_{i=1}^{j}\|\delta u_i\|^2+\frac{1}{2}\|u_j\|_1^2+\frac{1}{2}\sum_{i=1}^{j}\|u_i-u_{i-1}\|_1^2\leqslant C_\varepsilon+C_\varepsilon\tau\sum_{i=1}^{j}\|u_i\|_1^2. \tag{2.21}$$

If we select ε such that $0<\varepsilon<1-C_{k1}$ and choose τ_0 so as to satisfy $0<\tau_0<1/(2C_\varepsilon)$ in (2.21), we obtain (2.9)–(2.11) by Grönwall's lemma (cf. [14]).

Squaring both sides of (2.15) and taking into account (2.9) yield (2.12).

It is obvious that the constant C in (2.9)–(2.12) depends also on $\|u_0\|_1$, that is, $C=C(T,C_f,M_f,C_{h1},M_h,M_q,C_{k1},C_{k2},\|\omega\|_1,\|u_0\|_1)$, where $M_h:=\max_{t\in[0,T]}\|h(t)\|$, $M_q:=\max_{t\in[0,T]}|q'(t)|$. □

Now we introduce the following piecewise linear in time function $\hat{u}_n\colon[0,T]\to H^1(\Omega)$ and piecewise constant in time functions $\bar{u}_n\colon[0,T]\to H^1(\Omega)$ and $\bar{p}_n\colon[0,T]\to\mathbb{R}$:

$$\hat{u}_n(t)=\begin{cases}u_0, & t=0,\\ u_{i-1}+(t-t_{i-1})\delta u_i, & t\in(t_{i-1},t_i],\ 1\leqslant i\leqslant n,\end{cases} \tag{2.22}$$

$$\bar{u}_n(t)=\begin{cases}u_0, & t=0,\\ u_i, & t\in(t_{i-1},t_i],\ 1\leqslant i\leqslant n,\end{cases} \tag{2.23}$$

$$\bar{p}_n(t)=\begin{cases}p_1, & t=0,\\ p_i, & t\in(t_{i-1},t_i],\ 1\leqslant i\leqslant n.\end{cases} \tag{2.24}$$

In the same way we can define the functions $\bar{h}_n,\bar{q_n'}$ which are piecewise constant in time. Then we can rewrite (2.4)–(2.6) at $t\in(0,T]$ as

$$(\partial_t\hat{u}_n(t),\phi)+(\nabla\bar{u}_n(t),\nabla\phi)=\bar{p}_n(t)(\bar{h}_n(t),\phi)+(F\bar{u}_n(t^{(n)}),\phi), \tag{2.25}$$

$$\bar{p}_n(t)(\bar{h}_n(t),\omega)=\bar{q_n'}(t)+(\nabla\bar{u}_n(t^{(n)}),\nabla\omega)-(F\bar{u}_n(t^{(n)}),\omega), \tag{2.26}$$

$$\bar{u}_n(x,t)=K\bar{u}_n(x,t^{(n)}),\quad x\in\Gamma, \tag{2.27}$$

where $\partial_t\hat{u}_n(t)=\delta u_i$ and $t^{(n)}=t_{i-1}$ for $t\in(t_{i-1},t_i]$, $1\leqslant i\leqslant n$.

Now we show that the convergent subsequences of $\{\hat{u}_n\}$, $\{\bar{u}_n\}$, $\{\bar{p}_n\}$, $n \in \mathbb{N}$, tend to the solution of the IP (1.1)–(1.4).

Theorem 2.1. *Let* (A1)–(A4) *be satisfied. Then there exists a solution* $(u, p) \in V \times L^2(0, T)$ *to the problem* (1.1)–(1.4).

P r o o f. By Lemma 2.2, see (2.9)–(2.12), there exist constants $\tau_0 > 0$ and $C = C(T, C_f, M_f, C_{h1}, M_h, M_q, C_{k1}, C_{k2}, \|\omega\|_1, \|u_0\|_1) > 0$ such that, for all $n > T/\tau_0$,

$$\|\bar{u}_n(t)\|_1 \leqslant C \quad \forall\, t \in [0, T], \quad \|\partial_t \hat{u}_n\|_{L^2(0,T;L^2(\Omega))} \leqslant C, \quad \|\hat{u}_n\|_{L^2(0,T;H^1(\Omega))} \leqslant C,$$
$$\|\hat{u}_n - \bar{u}_n\|^2_{L^2(0,T;H^1(\Omega))} \leqslant C\tau,$$
$$\int_0^T \|\bar{u}_n(t^{(n)}) - \bar{u}_n(t)\|_1^2 \,\mathrm{d}t \leqslant C\tau, \quad \|\bar{p}_n\|_{L^2(0,T)} \leqslant C.$$

Thus, by [12], Lemma 1.3.13 there exists a sequence $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that

$$\begin{aligned} \hat{u}_{n_k} &\to u && \text{in } C([0,T]; L^2(\Omega)),\\ \partial_t \hat{u}_{n_k} &\rightharpoonup \partial_t u && \text{in } L^2(0,T; L^2(\Omega)),\\ \bar{u}_{n_k} &\overset{*}{\rightharpoonup} u && \text{in } L^\infty(0,T; H^1(\Omega)),\\ \bar{p}_{n_k} &\rightharpoonup p && \text{in } L^2(0,T). \end{aligned}$$

Obviously, $\bar{u}_{n_k} \to u$ in $L^2(0, T; L^2(\Omega))$ and $u \in V$.

On the other hand, similarly to [20], Lemma 4.1, from (2.25) we can get

$$\int_0^T \|\nabla \bar{u}_n(t) - \nabla \bar{u}_m(t)\|^2 \,\mathrm{d}t \leqslant C_1\Big(\frac{1}{n} + \frac{1}{m} + \int_0^T \|\bar{u}_n(t) - \bar{u}_m(t)\|^2 \,\mathrm{d}t\Big).$$

Due to the fact that $\{\bar{u}_{n_k}\}$ is a Cauchy sequence in $L^2(0, T; L^2(\Omega))$, it follows from the above inequality that $\{\nabla \bar{u}_{n_k}\}$ is a Cauchy sequence in $L^2(0, T; L^2(\Omega))$. Hence,

$$\bar{u}_{n_k} \to u, \quad \hat{u}_{n_k} \to u \quad \text{in } L^2(0,T; H^1(\Omega)).$$

From now on, n_k is written as n for simplicity.

First let us see that u satisfies (1.3) in $L^\infty(0, T; L^2(\Gamma))$. Using (2.27), the trace theorem and the inequality ([15])

$$\|v\|^2_\Gamma \leqslant \varepsilon \|\nabla v\|^2 + C_\varepsilon \|v\|^2 \quad \forall\, v \in H^1(\Omega), \quad \forall\, \varepsilon > 0,\ C_\varepsilon > 0,$$

we obtain

$$\begin{aligned}
(2.28)\quad \|u(t)-Ku(t)\|_\Gamma^2 &\leqslant 2\|\bar u_n(t)-u(t)\|_\Gamma^2+2\|K\bar u_n(t^{(n)})-Ku(t)\|_\Gamma^2\\
&\leqslant 2\|\bar u_n(t)-u(t)\|_\Gamma^2+C_2\|\bar u_n(t^{(n)})-u(t)\|^2+C_2\tau^2\|u(t)\|^2\\
&\leqslant \varepsilon_1\|\nabla\bar u_n(t)-\nabla u(t)\|^2+C_{\varepsilon_1}\|\bar u_n(t)-u(t)\|^2\\
&\quad +C_{\varepsilon_1}\|\bar u_n(t^{(n)})-\bar u_n(t)\|^2+C_{\varepsilon_1}\tau^2\|u(t)\|^2\\
&\leqslant \varepsilon_1\|\nabla\bar u_n(t)-\nabla u(t)\|^2\\
&\quad +2C_{\varepsilon_1}\|\hat u_n(t)-\bar u_n(t)\|^2+2C_{\varepsilon_1}\|\hat u_n(t)-u(t)\|^2\\
&\quad +C_{\varepsilon_1}\|\bar u_n(t^{(n)})-\bar u_n(t)\|^2+C_{\varepsilon_1}\tau^2\|u(t)\|^2.
\end{aligned}$$

On the other hand, by (2.10) we have

$$\begin{gathered}
\|\hat u_n(t)-\bar u_n(t)\|=\|(t-t_i)\delta u_i\|\leqslant\|u_i-u_{i-1}\|=\sqrt{\tau}\sqrt{\tau\|\delta u_i\|^2}\leqslant C_3\sqrt{\tau},\\
\|\bar u_n(t^{(n)})-\bar u_n(t)\|=\|u_i-u_{i-1}\|\leqslant C_3\sqrt{\tau}
\end{gathered}$$

for all $t\in(t_{i-1},t_i]$, $1\leqslant i\leqslant n$, thus, from (2.28) we derive

$$\operatorname*{ess\,sup}_{t\in[0,T]}\|u(t)-Ku(t)\|_\Gamma^2\leqslant\varepsilon_2+C_{\varepsilon_2}(\tau^2+\tau)+C_{\varepsilon_2}\operatorname*{ess\,sup}_{t\in[0,T]}\|\hat u_n(t)-u(t)\|^2.$$

Passing to the limit $n\to\infty$ in the inequality above and taking into account that $\varepsilon_2>0$ is an arbitrary constant, we get

$$\operatorname*{ess\,sup}_{t\in[0,T]}\|u(t)-Ku(t)\|_\Gamma^2=0.$$

Next we see that u,p satisfy (2.2) and (2.3). Multiplying (2.25) by an arbitrary $\psi\in L^2(0,T)$ and integrating it over $(0,T)$, we obtain

$$\begin{aligned}
(2.29)\qquad \int_0^T(\partial_t\hat u_n(t),\phi)\psi(t)\,\mathrm{d}t&+\int_0^T(\nabla\bar u_n(t),\nabla\phi)\psi(t)\,\mathrm{d}t\\
&=\int_0^T\bar p_n(t)(\bar h_n(t),\phi)\psi(t)\,\mathrm{d}t+\int_0^T(F\bar u_n(t^{(n)}),\phi)\psi(t)\,\mathrm{d}t.
\end{aligned}$$

Passing to the limit $n\to\infty$ in (2.29), it follows that

$$\begin{aligned}
\int_0^T(\partial_t u(t),\phi)\psi(t)\,\mathrm{d}t&+\int_0^T(\nabla u(t),\nabla\phi)\psi(t)\,\mathrm{d}t\\
&=\int_0^T p(t)(h(t),\phi)\psi(t)\,\mathrm{d}t+\int_0^T(Fu(t),\phi)\psi(t)\,\mathrm{d}t
\end{aligned}$$

(here the Lipschitz continuity of $h(t)$ is used).

Since $\psi\in L^2(0,T)$ is arbitrary, we can see that u and p satisfy (2.2). In the same way, starting from (2.26), we can show that u and p satisfy (2.3). □

2.2. Uniqueness of the solution. Let $(u^{(1)}, p^{(1)})$ and $(u^{(2)}, p^{(2)})$ be two solutions to the IP (1.1)–(1.4). Let $\tilde{u} := u^{(1)} - u^{(2)}$ and $\tilde{p} := p^{(1)} - p^{(2)}$.

Before we verify the uniqueness of the solution, we consider the following direct problem with regard to an unknown function w:

$$
(2.30)\qquad \begin{cases} (\partial_t w(t), \phi) + (\nabla w(t), \nabla \phi) = P(t)(h(t), \phi) + (F(t), \phi), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. } t \in (0, \xi) \quad \forall\, \phi \in H_0^1(\Omega), \\ w(x, 0) = 0, \quad x \in \Omega, \\ w(x,t) = K\tilde{u}(x,t) \quad \text{in } L^2(0, \xi; H^{\frac{1}{2}}(\Gamma)) \cap L^\infty(0, \xi; L^2(\Gamma)), \end{cases}
$$

where $\xi \in (0, T]$, $F(x,t) = Fu^{(1)}(x,t) - Fu^{(2)}(x,t)$ and

$$
P(t) = \frac{(\nabla \tilde{u}(t), \nabla \omega) - (F(t), \omega)}{(h(t), \omega)}.
$$

We show the existence and estimate for the solution to the problem (2.30) by using Rothe's method as in Section 3.

Divide the time interval $[0, \xi]$ into $n \in \mathbb{N}$ subintervals $[t_{i-1}, t_i]$, $i = 1, \ldots, n$, where $t_i = i\tau$ and $\tau = \xi/n$. We construct the following recurrent system of the time-discretized problem to get $w_i(x)$ from $w_{i-1}(x)$ ($w_0(x) = 0$) for all $i = 1, \ldots, n$:

$$
(2.31)\qquad \begin{cases} (\delta w_i, \phi) + (\nabla w_i, \nabla \phi) = P_i(h_i, \phi) + (F_i, \phi) \quad \forall\, \phi \in H_0^1(\Omega), \\ w_i(x) = K\tilde{u}_i(x), \quad x \in \Gamma, \end{cases}
$$

where $z_i = z(t_i)$ for the function $z \neq w$.

We can prove the existence and uniqueness of the solution $w_i \in H^1(\Omega)$, $i = 1, \ldots, n$, to the problem (2.31) as in Lemma 2.1 when (A2)–(A4) are satisfied.

Now, let us derive the estimates for $w_i(x)$.

Lemma 2.3. *Let* (A2)–(A4) *be satisfied. Then there exist constants* ε, C_ε, τ_0, $C > 0$ *such that, for all* $n > \xi/\tau_0$ *and all* $j = 1, \ldots, n$,

$$
(2.32)\qquad \|w_j\|_1^2 \leqslant C, \quad \tau \sum_{i=1}^{j} \|\delta w_i\|^2 \leqslant C, \quad \sum_{i=1}^{j} \|w_i - w_{i-1}\|_1^2 \leqslant C,
$$

$$
(2.33)\qquad \Big(1 - \frac{C_{k1}}{2} - \varepsilon\Big)\tau \sum_{i=1}^{n} \|\delta w_i\|^2 \leqslant \Big(\frac{C_{k1}}{2} + \varepsilon\Big)\tau \sum_{i=1}^{n} \|\delta \tilde{u}_i\|^2 + C_\varepsilon \tau \sum_{i=1}^{n} \|\tilde{u}_i\|_1^2 + C_\varepsilon \tau \sum_{i=1}^{n} \|w_i\|_1^2,
$$

where $\varepsilon, C_\varepsilon$ *are independent of the choice of* ξ *and* $0 < \varepsilon < (1 - C_{k1})/2$.

P r o o f. Setting $\phi = (\delta w_i - \delta K\tilde{u}_i)\tau$ in the equation of (2.31) and summing it up for $i = 1, \ldots, j$ yield

$$\begin{aligned}(2.34)\qquad \tau\sum_{i=1}^{j}(\delta w_i, \delta w_i) + \tau\sum_{i=1}^{j}(\nabla w_i, \nabla\delta w_i) &= \tau\sum_{i=1}^{j}P_i(h_i, \delta w_i - \delta K\tilde{u}_i) + \tau\sum_{i=1}^{j}(F_i, \delta w_i - \delta K\tilde{u}_i)\\ &\quad + \tau\sum_{i=1}^{j}(\delta w_i, \delta K\tilde{u}_i) + \tau\sum_{i=1}^{j}(\nabla w_i, \nabla\delta K\tilde{u}_i).\end{aligned}$$

Estimating (2.34) in a similar way as in Lemma 2.2 and adding it to the inequality (2.20) for w_i, we are led to

$$\begin{aligned}(2.35)\qquad \Big(1 - \frac{C_{k1}}{2} - \varepsilon\Big)\tau\sum_{i=1}^{j}\|\delta w_i\|^2 + \frac{1}{2}\|w_j\|_1^2 + \frac{1}{2}\sum_{i=1}^{j}\|w_i - w_{i-1}\|_1^2\\ \leqslant \Big(\frac{C_{k1}}{2} + \varepsilon\Big)\tau\sum_{i=1}^{j}\|\delta\tilde{u}_i\|^2 + C_\varepsilon\tau\sum_{i=1}^{j}\|\tilde{u}_i\|_1^2 + C_\varepsilon\tau\sum_{i=1}^{j}\|w_i\|_1^2.\end{aligned}$$

If we select ε such that $0 < \varepsilon < (1 - C_{k1})/2$ and choose τ_0 satisfying $0 < \tau_0 < 1/(2C_\varepsilon)$ in (2.35), we get (2.32) by Grönwall's lemma.

The estimate (2.33) is obtained from (2.35) directly. $\square$

Lemma 2.4. *Let* (A2)–(A4) *be satisfied. Then the problem* (2.30) *has the unique solution* $w \in V$*. Furthermore,*

$$\begin{aligned}(2.36)\qquad \Big(1 - \frac{C_{k1}}{2} - \varepsilon\Big)\int_0^\xi\|\partial_t w(t)\|^2\,\mathrm{d}t \leqslant \Big(\frac{C_{k1}}{2} + \varepsilon\Big)\int_0^\xi\|\partial_t\tilde{u}(t)\|^2\,\mathrm{d}t\\ + C_\varepsilon\int_0^\xi\|\tilde{u}(t)\|_1^2\,\mathrm{d}t + C_\varepsilon\int_0^\xi\|w(t)\|_1^2\,\mathrm{d}t\end{aligned}$$

holds, where the constants $\varepsilon, C_\varepsilon$ *are the same as in Lemma* 2.3.

P r o o f. In the same way as in (2.22) and (2.23), we introduce a piecewise linear in time function $\widehat{w}_n\colon [0,\xi] \to H^1(\Omega)$ and piecewise constant in time function $\overline{w}_n\colon [0,\xi] \to H^1(\Omega)$. Using (2.32) and [12], Lemma 1.3.13, we see the existence of a function $w \in V$ and subsequences of $\{\widehat{w}_n\}$ and $\{\overline{w}_n\}$ such that

$$\begin{aligned}\widehat{w}_{n_k} &\to w &&\text{in } C([0,\xi]; L^2(\Omega)),\\ \partial_t\widehat{w}_{n_k} &\rightharpoonup \partial_t w &&\text{in } L^2(0,\xi; L^2(\Omega)),\\ \overline{w}_{n_k} &\overset{*}{\rightharpoonup} w &&\text{in } L^\infty(0,\xi; H^1(\Omega)).\end{aligned}$$

We can easily find that $w \in V$ satisfies (2.30). The uniqueness of the solution to (2.30) is obvious.

Finally, passing to the limit $k \to \infty$ in (2.33) for $n = n_k$, we get (2.36). □

Now, we are in a position to prove the uniqueness of solution to the IP (1.1)–(1.4).

Theorem 2.2. *Let* (A2)–(A4) *be satisfied. Then the solution* $(u, p) \in V \times L^2(0, T)$ *to the IP* (1.1)–(1.4) *is unique.*

Proof. Subtracting the corresponding (2.2), (2.3) for $(u^{(1)}, p^{(1)})$ and $(u^{(2)}, p^{(2)})$ from each other, for a.e. $t \in (0, T)$ we have

$$(\partial_t \tilde{u}(t), \phi) + (\nabla \tilde{u}(t), \nabla \phi) = \tilde{p}(t)(h(t), \phi) + (F(t), \phi) \quad \forall\, \phi \in H_0^1(\Omega), \tag{2.37}$$

$$\tilde{p}(t)(h(t), \omega) = (\nabla \tilde{u}(t), \nabla \omega) - (F(t), \omega). \tag{2.38}$$

Due to (A3) and (2.38) the following inequalities hold:

$$\|F(t)\| \leqslant C_f(\|\tilde{u}(t)\| + \|\nabla \tilde{u}(t)\|), \tag{2.39}$$

$$|\tilde{p}(t)| C_{h1} \leqslant |\tilde{p}(t)(h(t), \omega)| \leqslant C_1(\|\tilde{u}(t)\| + \|\nabla \tilde{u}(t)\|). \tag{2.40}$$

Now, setting $\phi = \tilde{u}(t) - K\tilde{u}(t)$ in (2.37) and integrating it over $(0, \xi), \xi \in (0, T]$, we obtain

$$\begin{aligned} \frac{1}{2}\|\tilde{u}(\xi)\|^2 &+ \int_0^\xi \|\nabla \tilde{u}(t)\|^2 \,\mathrm{d}t \\ &= \int_0^\xi \tilde{p}(t)(h(t), \tilde{u}(t) - K\tilde{u}(t)) \,\mathrm{d}t + \int_0^\xi (F(t), \tilde{u}(t) - K\tilde{u}(t)) \,\mathrm{d}t \\ &\quad + \int_0^\xi (\nabla \tilde{u}(t), \nabla K\tilde{u}(t)) \,\mathrm{d}t + \int_0^\xi (\partial_t \tilde{u}(t), K\tilde{u}(t)) \,\mathrm{d}t. \end{aligned} \tag{2.41}$$

Estimating the RHS of (2.41) by (2.39), (2.40) and (A4) in a standard way, we have

$$\frac{1}{2}\|\tilde{u}(\xi)\|^2 + (1 - \varepsilon)\int_0^\xi \|\nabla \tilde{u}(t)\|^2 \,\mathrm{d}t \leqslant \varepsilon \int_0^\xi \|\partial_t \tilde{u}(t)\|^2 \,\mathrm{d}t + C_\varepsilon \int_0^\xi \|\tilde{u}(t)\|^2 \,\mathrm{d}t. \tag{2.42}$$

On the other hand, it follows from (2.37) and (2.38) that $\tilde{u}$ is a solution to (2.30) for all $\xi \in (0, T]$. By (2.36) and the uniqueness of the solution to the problem (2.30), there exists a constant $C > 0$ such that

$$\int_0^\xi \|\partial_t \tilde{u}(t)\|^2 \,\mathrm{d}t \leqslant C \int_0^\xi \|\tilde{u}(t)\|^2 \,\mathrm{d}t + C \int_0^\xi \|\nabla \tilde{u}(t)\|^2 \,\mathrm{d}t, \tag{2.43}$$

where C is independent of ξ. Substituting (2.43) into (2.42) and using Grönwall's lemma (cf. [14]) leads to $u^{(1)} = u^{(2)}$ and, finally, it follows from (2.40) that $p^{(1)} = p^{(2)}$. □

R e m a r k 2.2. In [20], the direct problem (1.1)–(1.3) has been studied under the assumption $u_0 \in H^2(\Omega)$ and (A3), (A4). For the convergence of Rothe's method, $u_0 \in H^2(\Omega)$ was required due to the argument setting $\delta u_0 = f(\nabla u_0) + \Delta u_0$. On the other hand, uniqueness of the solution was proved by the energy method assuming $\partial_t \tilde{u}(t) - \partial_t K\tilde{u}(t) \in H_0^1(\Omega)$, which tacitly requires $u_0 \in H^2(\Omega)$ according to the well-known theory of parabolic equations. Our result for the IP can be an extension of [20] from $u_0 \in H^2(\Omega)$ to $u_0 \in H^1(\Omega)$.

3. Rothe's method II (case of $u_0 \in L^2(\Omega)$)

Now, our study continues to establishing Rothe's method for the IP (1.1)–(1.4) under the weaker assumption $u_0 \in L^2(\Omega)$. We show that the method will succeed for a symmetric integral kernel k, i.e., $k(x, y, t) = k(y, x, t)$, and for the nonlinear term $f(x, t, u(x, t))$ in (1.1)–(1.4).

Let us replace (A1) and (A4) by (A1)′ and (A4)′, respectively. That is:

(A1)′ $u_0 \in L^2(\Omega)$.

(A4)′ In addition to (A4), $k(x, y, t) = k(y, x, t)$ holds for $(x, y, t) \in \overline{\Omega} \times \overline{\Omega} \times [0, T]$.

Define the function space W by

$$W = \{w\colon\ w \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)),\ \partial_t w \in L^2(0, T; H^{-1}(\Omega))\}.$$

Under (A1)′, (A2), (A3) and (A4)′, we find the *solution* $(u, p) \in W \times L^2(0, T)$ to the problem

$$\begin{cases} (\partial_t u(t), \phi) + (\nabla u(t), \nabla\phi) = p(t)(h(t), \phi) + (f(t, u(t)), \phi), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. } t \in (0, T) \quad \forall\, \phi \in H_0^1(\Omega), \\ p(t)(h(t), \omega) = q'(t) + (\nabla u(t), \nabla\omega) - (f(t, u(t)), \omega), \quad \text{a.e. } t \in (0, T), \\ u(x, 0) = u_0(x), \quad x \in \Omega, \\ u(x, t) = Ku(x, t) \quad \text{in } L^2(0, T; L^2(\Gamma)). \end{cases} \tag{3.1}$$

We construct the discrete scheme (2.4)–(2.6), where (2.5) is changed into the relation

$$p_i = \frac{q_i' + (\nabla u_i, \nabla\omega) - (Fu_{i-1}, \omega)}{(h_i, \omega)} \tag{2.5′}$$

in view of (A1)′ and where $Fu_{i-1}(x) = f(x, t_{i-1}, u_{i-1}(x))$.

Lemma 3.1. *Let* (A1)′, (A2), (A3), *and* (A4)′ *be satisfied. Then there exists a constant* $\tau_0 > 0$ *such that for all* $n > T/\tau_0$ *and all* $i = 1, \ldots, n$, *the problem* (2.4), (2.5′), (2.6) *has the unique solution* $(u_i, p_i) \in H^1(\Omega) \times \mathbb{R}$.

P r o o f. Let $u_{i-1} \in H^1(\Omega)(u_0 \in L^2(\Omega))$ be given. Substituting (2.5′) into (2.4) and setting $u_i(x) = v_i(x) + Ku_{i-1}(x)$ in a similar way as in the proof of Lemma 2.1, then

$$\begin{aligned} A_i^{(n)}(v_i,\phi) &:= (v_i,\phi) + \tau(\nabla v_i,\nabla\phi) - \frac{\tau}{(h_i,\omega)}(\nabla v_i,\nabla\omega)(h_i,\phi) \\ &= (u_{i-1},\phi) + \tau\frac{q_i' - (Fu_{i-1},\omega)}{(h_i,\omega)}(h_i,\phi) + \tau(Fu_{i-1},\phi) - (Ku_{i-1},\phi) \\ &\quad - \tau(\nabla Ku_{i-1},\nabla\phi) + \frac{\tau}{(h_i,\omega)}(\nabla Ku_{i-1},\nabla\omega)(h_i,\phi) \quad \forall\,\phi\in H_0^1(\Omega) \end{aligned}$$

holds. The following estimates for $A_i^{(n)}(v_i,\phi)$ are obtained:

$$\begin{aligned} |A_i^{(n)}(v_i,\phi)| &\leqslant \|v_i\|\|\phi\| + \tau\|\nabla v_i\|\|\nabla\phi\| + \tau C_h\|\nabla v_i\|\|\phi\| \leqslant C_1\|v_i\|_{H_0^1(\Omega)}\|\phi\|_{H_0^1(\Omega)}, \\ A_i^{(n)}(v_i,v_i) &\geqslant \|v_i\|^2 + \tau\|\nabla v_i\|^2 - \tau C_h\|\nabla v_i\|\|v_i\| \\ &\geqslant \Big(1 - \frac{\tau C_h}{\varepsilon}\Big)\|v_i\|^2 + \tau(1 - C_h\varepsilon)\|\nabla v_i\|^2, \end{aligned}$$

where $C_h = \|\nabla\omega\|M_h/C_{h1}$. Selecting ε such that $0 < \varepsilon < 1/C_h$ and choosing $\tau_0 = \varepsilon/C_h$, our proof is closed by the Lax-Milgram lemma. □

Next, we derive estimates for $u_i(x)$, p_i, $i = 1,\dots,n$. We remark that the assumption (A1)′ does not allow the relation (2.13) to hold since the second term in its LHS makes no sense. Thus, choosing $\phi = (u_i - Ku_{i-1})\tau$ in (2.4), we must obtain estimates for $u_i(x)$, p_i, $i = 1,\dots,n$. The following lemma shows that a symmetric integral kernel makes the possibility.

Lemma 3.2. *Let* (A1)′, (A2), (A3) *and* (A4)′ *be satisfied. Then there exist constants* $\tau_0 > 0$ *and* $C = C(T, C_f, M_f, C_{h1}, M_h, M_q, C_{k1}, C_{k2}, \|\omega\|_1, \|u_0\|) > 0$ *such that, for all* $n > T/\tau_0$ *and all* $j = 1,\dots,n$,

$$\|u_j\|^2 \leqslant C, \quad \tau\sum_{i=1}^{j}\|\nabla u_i\|^2 \leqslant C, \quad \sum_{i=1}^{j}\|u_i - u_{i-1}\|^2 \leqslant C, \tag{3.2}$$

$$\tau\sum_{i=1}^{j}\|\delta u_i\|_{H^{-1}(\Omega)}^2 \leqslant C, \tag{3.3}$$

$$\tau\sum_{i=1}^{j}p_i^2 \leqslant C. \tag{3.4}$$

P r o o f. We set $\phi = (u_i - Ku_{i-1})\tau$ in (2.4) and sum it up for $i = 1, \ldots, j$ to get

$$\tau\sum_{i=1}^{j}(\delta u_i, u_i) + \tau\sum_{i=1}^{j}(\nabla u_i, \nabla u_i) - \tau\sum_{i=1}^{j}(\delta u_i, Ku_{i-1}) - \tau\sum_{i=1}^{j}(\nabla u_i, \nabla Ku_{i-1}) = \tau\sum_{i=1}^{j}p_i(h_i, u_i - Ku_{i-1}) + \tau\sum_{i=1}^{j}(Fu_{i-1}, u_i - Ku_{i-1}). \tag{3.5}$$

Taking into account the symmetry of k and the identity

$$\sum_{i=1}^{j}a_{i-1}(b_i - b_{i-1}) = a_jb_j - a_0b_0 - \sum_{i=1}^{j}b_i(a_i - a_{i-1}) \quad \forall\, a_i, b_i \in \mathbb{R},\ i = 1, \ldots, j,$$

we can rewrite the third term in the LHS of (3.5) as

$$\begin{aligned}
&\tau\sum_{i=1}^{j}(\delta u_i, Ku_{i-1})\\
&\quad= \frac{1}{2}\int_\Omega\int_\Omega\sum_{i=1}^{j}k(x,y,t_{i-1})u_{i-1}(y)[u_i(x) - u_{i-1}(x)]\,\mathrm{d}y\,\mathrm{d}x\\
&\qquad+ \frac{1}{2}\int_\Omega\int_\Omega\sum_{i=1}^{j}k(x,y,t_{i-1})u_{i-1}(x)[u_i(y) - u_{i-1}(y)]\,\mathrm{d}y\,\mathrm{d}x\\
&\quad= \frac{1}{2}\int_\Omega\int_\Omega[k(x,y,t_j)u_j(y)u_j(x) - k(x,y,t_0)u_0(y)u_0(x)]\,\mathrm{d}y\,\mathrm{d}x\\
&\qquad- \frac{1}{2}\int_\Omega\int_\Omega\sum_{i=1}^{j}[k(x,y,t_i) - k(x,y,t_{i-1})]u_i(y)u_i(x)\,\mathrm{d}y\,\mathrm{d}x\\
&\qquad- \frac{1}{2}\int_\Omega\int_\Omega\sum_{i=1}^{j}k(x,y,t_{i-1})[u_i(y) - u_{i-1}(y)][u_i(x) - u_{i-1}(x)]\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}$$

From this relation, we obtain

$$\begin{aligned}
\Bigl|\tau\sum_{i=1}^{j}(\delta u_i, Ku_{i-1})\Bigr| &\leqslant \frac{1}{2}C_{k1}\|u_j\|^2 + \frac{1}{2}C_{k1}\|u_0\|^2 + \frac{1}{2}C_{k2}\tau\sum_{i=1}^{j}\|u_i\|^2\\
&\quad+ \frac{1}{2}C_{k1}\sum_{i=1}^{j}\|u_i - u_{i-1}\|^2.
\end{aligned}$$

We estimate the other terms of (3.5) in a similar way as in the proof of Lemma 2.2 to obtain

$$\frac{1}{2}(1-C_{k1})\|u_j\|^2+(1-\varepsilon)\tau\sum_{i=1}^{j}\|\nabla u_i\|^2+\frac{1}{2}(1-C_{k1})\sum_{i=1}^{j}\|u_i-u_{i-1}\|^2 \leqslant C_\varepsilon+C_\varepsilon\tau\sum_{i=1}^{j}\|u_i\|^2,$$

yielding (3.2) by Grönwall's lemma. The estimate (3.4) is obtained from (2.5′) and (3.2). On the other hand, it follows from (2.4) that

$$\begin{aligned}|(\delta u_i,\phi)| &\leqslant |p_i(h_i,\phi)|+|(Fu_{i-1},\phi)|+|(\nabla u_i,\nabla\phi)|\\ &\leqslant C_1(1+|p_i|+\|u_{i-1}\|+\|\nabla u_i\|)\|\phi\|_{H_0^1(\Omega)}.\end{aligned}$$

Hence,

$$\|\delta u_i\|_{H^{-1}(\Omega)}=\sup_{\|\phi\|_{H_0^1(\Omega)}\leqslant 1}|(\delta u_i,\phi)|\leqslant C_1(1+|p_i|+\|u_{i-1}\|+\|\nabla u_i\|)$$

holds and we get (3.3) by using (3.2) and (3.4). □

Theorem 3.1. *Let* (A1)′, (A2), (A3), *and* (A4)′ *be satisfied. Then there exists a solution* $(u,p)\in W\times L^2(0,T)$ *to the problem* (3.1).

P r o o f. Defining functions $\hat{u}_n$, $\bar{u}_n$, $\bar{p}_n$ as in (2.22)–(2.24), we get the following estimates from (3.2)–(3.4):

$$\begin{gathered}\|\hat{u}_n\|_{L^2(0,T;H^1(\Omega))}\leqslant C,\quad \|\partial_t\hat{u}_n\|_{L^2(0,T;H^{-1}(\Omega))}\leqslant C,\\ \|\hat{u}_n-\bar{u}_n\|^2_{L^2(0,T;L^2(\Omega))}\leqslant C\tau,\\ \int_0^T\|\bar{u}_n(t^{(n)})-\bar{u}_n(t)\|^2\,\mathrm{d}t\leqslant C\tau,\quad \|\bar{p}_n\|_{L^2(0,T)}\leqslant C,\end{gathered}$$

where $C=C(T,C_f,M_f,C_{h1},M_h,M_q,C_{k1},C_{k2},\|\omega\|_1,\|u_0\|)$. Due to Aubin's lemma [18], there exists a sequence $\{n_k\}_{k\in\mathbb{N}}\subset\mathbb{N}$ such that

$$\begin{aligned}\hat{u}_{n_k}&\rightharpoonup u &&\text{in } L^2(0,T;H^1(\Omega)),\\ \partial_t\hat{u}_{n_k}&\rightharpoonup\partial_t u &&\text{in } L^2(0,T;H^{-1}(\Omega)),\\ \hat{u}_{n_k}\to u,\ \bar{u}_{n_k}&\to u &&\text{in } L^2(0,T;L^2(\Omega)),\\ \bar{p}_{n_k}&\rightharpoonup p &&\text{in } L^2(0,T).\end{aligned}$$

Hence, $u\in W$, $p\in L^2(0,T)$. Similarly to Theorem 2.1, we can establish that (u,p) satisfies (3.1). □

Under the assumptions (A2), (A3) and (A4), the uniqueness of a solution to (3.1) can be proved as in the proof of Theorem 2.2. However, using the symmetry of the integral kernel, we present a shorter proof of the uniqueness.

Theorem 3.2. *Let* (A2), (A3), *and* (A4)′ *be satisfied. Then, the solution* $(u,p) \in W \times L^2(0,T)$ *to the problem* (3.1) *is unique.*

P r o o f. Suppose that $(u^{(1)},p^{(1)})$ and $(u^{(2)},p^{(2)})$ are two solutions. Then they satisfy (2.37), (2.38), where $F(x,t) = f(x,t,u^{(1)}(x,t)) - f(x,t,u^{(2)}(x,t))$.

Setting $\phi = \tilde u(t) - K\tilde u(t)$ in (2.37) and integrating it over $(0,\xi), \xi \in (0,T]$, we have

$$\begin{aligned}(3.6)\quad \frac{1}{2}\|\tilde u(t)\|^2 + \int_0^\xi \|\nabla \tilde u(t)\|^2 \,\mathrm{d}t - \int_0^\xi (\nabla \tilde u(t), \nabla K\tilde u(t))\,\mathrm{d}t - \int_0^\xi (\partial_t \tilde u(t), K\tilde u(t))\,\mathrm{d}t \\ = \int_0^\xi \tilde p(t)(h(t), \tilde u(t) - K\tilde u(t))\,\mathrm{d}t + \int_0^\xi (F(t), \tilde u(t) - K\tilde u(t))\,\mathrm{d}t.\end{aligned}$$

Due to the symmetry of k, the fourth term in the LHS of (3.6) leads to

$$\begin{aligned}\int_0^\xi (\partial_t \tilde u(t), K\tilde u(t))\,\mathrm{d}t &= \frac{1}{2}\int_0^\xi \mathrm{d}t \int_\Omega \int_\Omega k(x,y,t)\partial_t(\tilde u(x,t)\tilde u(y,t))\,\mathrm{d}y\,\mathrm{d}x \\ &= \frac{1}{2}\int_\Omega\int_\Omega k(x,y,\xi)\tilde u(x,\xi)\tilde u(y,\xi)\,\mathrm{d}y\,\mathrm{d}x \\ &\quad - \frac{1}{2}\int_0^\xi \mathrm{d}t \int_\Omega\int_\Omega k_t(x,y,t)\tilde u(x,t)\tilde u(y,t)\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

From the above, we obtain

$$\left|\int_0^\xi (\partial_t \tilde u(t), K\tilde u(t))\,\mathrm{d}t\right| \leqslant \frac{1}{2}C_{k1}\|\tilde u(\xi)\|^2 + \frac{1}{2}C_{k2}\int_0^\xi \|\tilde u(t)\|^2\,\mathrm{d}t.$$

Estimating the other terms of (3.6) directly by the Cauchy and Young inequalities and using the Grönwall lemma, we have $u^{(1)} = u^{(2)}$ and finally $p^{(1)} = p^{(2)}$ from (2.40). □

R e m a r k 3.1. Obviously, our result for the IP (1.1)–(1.4) suggests Rothe's method for the direct problem (1.1)–(1.3) as well.

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