Applications of Mathematics

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Applications of Mathematics, Vol. 67 (2022), No. 5, 657–674

Persistent URL: `http://dml.cz/dmlcz/151030`

ABSOLUTE VALUE EQUATIONS WITH TENSOR PRODUCT STRUCTURE: UNIQUE SOLVABILITY AND NUMERICAL SOLUTION

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Received August 20, 2021. Published online June 10, 2022.

Abstract. We consider the absolute value equations (AVEs) with a certain tensor product structure. Two aspects of this kind of AVEs are discussed in detail: the solvability and approximate solution. More precisely, first, some sufficient conditions are provided which guarantee the unique solvability of this kind of AVEs. Furthermore, a new iterative method is constructed for solving AVEs and its convergence properties are investigated. The validity of established theoretical results and performance of the proposed iterative scheme are examined numerically.

Keywords: iterative method; absolute value equation; convergence; tensor (Kronecker) product

MSC 2020: 65F10, 15A69

1. Introduction

This paper is mainly concerned with the solution of absolute value equation (AVE) in the form

$$\mathcal{A}x - |x| := (I_l \otimes I_n \otimes A + I_l \otimes B \otimes I_m + C \otimes I_n \otimes I_m)x - |x| = b, \tag{1.1}$$

where the coefficient matrices $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{l\times l}$ and the right-hand side $b \in \mathbb{R}^t$ are given and the vector $x = (x_1, x_2, \ldots, x_t)^\top \in \mathbb{R}^t$ is unknown with $t = mnl$. The notation $|x| \in \mathbb{R}^t$ stands for the vector whose ith component is the absolute value of x_i for $i = 1, 2, \ldots, t$. In this work, the matrices A, B and C are assumed to be positive definite but not necessarily symmetric.

The absolute value equations (AVEs) play an important role in study of linear and convex quadratic programming, linear complementary problems and other areas of optimization, scientific computing and engineering, see [1], [20], [21], [22], [27], [30] for

DOI: 10.21136/AM.2022.0169-21

more details. So far, several research works were devoted to discussing the (unique) solvability of AVEs in the literature, see [18], [19], [28], [29] and the references therein for more details. In general, it is well-known that the condition $\|\mathcal{A}^{-1}\| < 1$ ensures the existence of a unique solution for $\mathcal{A}x - |x| = b$. The subject of establishing necessary and/or sufficient conditions for solvability of AVEs, based on the Kronecker product structure of their coefficient matrices, has been also receiving increasing interest recently. For instance, Hashemi [10] presented sufficient condition for the unique solvability of the matrix equation

$$AXB + C|X|D = E, \tag{1.2}$$

which is mathematically equivalent to the following generalized AVE:

$$(B^\top \otimes A)x + (D^\top \otimes C)|x| = e.$$

Here $x = \mathrm{vec}(X)$ and $e = \mathrm{vec}(E)$, where the notation "$\mathrm{vec}(\cdot)$" stands for the vectorization operator which converts a given matrix into a column vector. More recently, two new sufficient conditions for the unique solvability of (1.2) were also established by Wang and Li [26].

Studying the performance of iterative techniques for solving AVEs can be regarded an active area of research, for more details see [14], [15], [18], [23], [24], [25], [30], [33] and the references therein. In particular, the Picard iteration for solving $\mathcal{A}x - |x| = b$ is given by

$$x^{(k+1)} = \mathcal{A}^{-1}(|x^{(k)}| + b), \quad k = 0, 1, 2, \ldots, \tag{1.3}$$

see [23] for more details. Using an iterative scheme as the act of $\mathcal{A}^{-1}$ is more beneficial in comparison with employing direct solver when the coefficient matrix $\mathcal{A}$ is ill-conditioned or has a large size. Bai and Yang [4] proposed the Picard-HSS iteration method to solve a class of nonlinear systems which is derived by applying the Hermitian and skew-Hermitian splitting (HSS) method [3] at each step of the Picard iteration for implementing the inverse of $\mathcal{A}$. The Picard-HSS iteration method was further exploited for solving AVEs in [24]. Alternative class of iterative methods, incorporating the Picard iterative method, was further analyzed for solving $\mathcal{A}x - |x| = b$ in [8], [9], [14], [25]. More precisely, in these works, the main problem is first rewritten into an equivalent linear system $\mathcal{A}_1\mathbf{z} = \mathbf{b}$, where

$$\mathcal{A}_1 = \begin{bmatrix} \mathcal{A} & -I \\ -D(x) & I \end{bmatrix}.$$

Then, iterative methods were extracted from the block splittings of the (preconditioned) coefficient matrix $\mathcal{A}_1$ and their convergence properties were scrutinized. In particular, Ke [14] presented the following iterative method:

$$\begin{cases} x^{(k+1)} = \mathcal{A}^{-1}(y^{(k)} + b), \\ y^{(k+1)} = (1-\tau)y^{(k)} + \tau|x^{(k+1)}|, \end{cases} \tag{1.4}$$

where the initial guess $x^{(0)}$ is given and $y^{(0)} = |x^{(0)}|$. More recently, Shams et al. [25] developed the iterative scheme

$$\begin{cases} x^{(k+1)} = \mathcal{A}^{-1}(y^{(k)} + b), \\ y^{(k+1)} = (1-\tau)|x^{(k)}| + \tau|x^{(k+1)}|, \end{cases} \tag{1.5}$$

for $k = 0, 1, 2, \ldots$, where the initial guess $x^{(0)}$ is given and $y^{(0)} = |x^{(0)}|$. Both iterative methods (1.4) and (1.5) construct the sequence $\{(x^{(k)}; y^{(k)})\}_{k=0}^{\infty}$ as approximations of $(x^*; |x^*|)$, where the parameter τ is a prescribed positive constant.

In the first part of this paper, we derive some sufficient conditions, which are computationally cheap when $\mathcal{A}$ has tensor product structure. Basically, we establish some bounds for the norm of $\mathcal{A}^{-1}$ with respect to coefficient matrices A, B and C in (1.1) under certain conditions. The second part is devoted to presenting an iterative scheme for solving AVEs. We mention here that for iterative method (1.5), a strategy to choose suitable parameter τ is provided by Shams et al. [25], whereas the optimum values of τ in iterative method (1.4) were only found experimentally. In continuation of the results presented in [25], here, our goals are both extending the convergence interval to include (possibly) negative values of τ and examining inexact solvers as the act of $\mathcal{A}^{-1}$. To do so, given a parameter τ, we consider the following block splitting for $\mathcal{A}_1$, where $\mathcal{A} = M - N$:

$$\mathcal{A}_1 = \begin{bmatrix} M & 0 \\ -\tau D(x) & I \end{bmatrix} - \begin{bmatrix} N & I \\ (1-\tau)D(x) & 0 \end{bmatrix}$$

and construct a new iterative scheme.

The remainder of this paper is organized as follows: Before ending this section, we present some notations used throughout this work. In Section 2, we derive some sufficient conditions which guarantee the unique solvability of AVEs whose coefficient matrices have the mentioned tensor product structure. A new iterative scheme for solving AVEs is proposed in Section 3 and its convergence properties are discussed. In Section 4, some numerical results are reported to numerically confirm the established theoretical results and demonstrate that the proposed method is feasible for solving (1.1). We finish the paper with some concluding remarks in Section 5.

Notations. Given an arbitrary real matrix W, the notation $\sigma_{\max}(W)$ ($\sigma_{\min}(W)$) refers to the maximum (smallest nonzero) singular value of W. If the matrix W is a square matrix with real eigenvalues, its minimum and maximum eigenvalues are denoted by $\lambda_{\min}(W)$ and $\lambda_{\max}(W)$, respectively. The notation $\varrho(W)$ stands for the spectral radius of W. The symmetric and skew-symmetric parts of a square matrix W are respectively given by

$$H(W) := \frac{1}{2}(W + W^\top) \quad \text{and} \quad S(W) := \frac{1}{2}(W - W^\top). \tag{1.6}$$

In the sequel, $W = H(W) + S(W)$ is called symmetric and skew-symmetric (SS) splitting. Throughout this paper, the norm $\|\cdot\|$ refers to vector/matrix 2-norm (i.e., $\|x\|^2 = x^\top x$ for all real vectors x and $\|W\|^2 = \varrho(WW^\top)$). Moreover, we write $(x; y)$ to denote the vector $[x^\top y^\top]^\top$.

2. Solvability of AVEs with tensor product structure

The coefficient matrix $\mathcal{A}$ in (1.1) is of size "mnl", which can be very large even for moderate values of m, n and l. Therefore, checking the condition $\|\mathcal{A}^{-1}\| < 1$ is challenging in general cases and may be too expensive. Here, we derive two types of computationally cheap sufficient conditions which guarantee $\|\mathcal{A}^{-1}\| < 1$. To present the first sufficient condition, we need to prove the following two simple propositions.

Proposition 2.1. *Let $\mathcal{A}$ be nonsingular and $\mathcal{A} = M - N$ be a splitting of A such that $\|M^{-1}\| + \|M^{-1}N\| < 1$. Then $\|\mathcal{A}^{-1}\| < 1$.*

Proof. Evidently, we have $\mathcal{A} = M - N = M(I - M^{-1}N)$. It is not difficult to verify that

$$\mathcal{A}^{-1} = (I - M^{-1}N)^{-1}M^{-1} = \left(\sum_{k=0}^{\infty}(M^{-1}N)^k\right)M^{-1},$$

which implies

$$\|\mathcal{A}^{-1}\| \leqslant \left(\sum_{k=0}^{\infty}\|M^{-1}N\|^k\right)\|M^{-1}\| = \frac{\|M^{-1}\|}{1 - \|M^{-1}N\|}.$$

Now the assertion can be deduced from the assumption immediately. $\square$

Proposition 2.2. *Let $\mathcal{A}$ be nonsingular and $\mathcal{A} = M - N$ be a splitting of $\mathcal{A}$. If M is a symmetric positive definite matrix and $\lambda_{\min}(M) > 1 + \|N\|$, then $\|\mathcal{A}^{-1}\| < 1$.*

P r o o f. According to the assumption, we have

$$\lambda_{\min}(M)=\frac{1}{\lambda_{\max}(M^{-1})}=\frac{1}{\|M^{-1}\|}>1+\|N\|.$$

Consequently, from the last inequality we can observe that

$$\|M^{-1}\|(1+\|N\|)=\|M^{-1}\|+\|M^{-1}\|\|N\|<1,$$

which concludes $\|M^{-1}\|+\|M^{-1}N\|<1$. Now the conclusion follows from the previous proposition. $\square$

In this work, we mainly assume that A, B and C are positive definite matrices, i.e., the matrices $H(A)$, $H(B)$ and $H(C)$ are symmetric positive definite. In this case, we use Proposition 2.2 to obtain a sufficient condition for solvability of AVEs in the form (1.1). To present the condition, we need to recall the subsequent proposition proved by Beik et al. [5], Proposition 3.6.

Proposition 2.3. *Let* $\mathcal{A}=I_l\otimes I_n\otimes A+I_l\otimes B\otimes I_m+C\otimes I_n\otimes I_m$. *Then the spectral norm of its skew-symmetric part is given by* $\|S(\mathcal{A})\|=\|S(A)\|+\|S(B)\|+\|S(C)\|$.

R e m a r k 2.4. It is immediate to observe that $\mathcal{A}=H(\mathcal{A})+S(\mathcal{A})$, where

$$H(\mathcal{A})=I_l\otimes I_n\otimes H(A)+I_l\otimes H(B)\otimes I_m+H(C)\otimes I_n\otimes I_m, \tag{2.1}$$

$$S(\mathcal{A})=I_l\otimes I_n\otimes S(A)+I_l\otimes S(B)\otimes I_m+S(C)\otimes I_n\otimes I_m. \tag{2.2}$$

When $H(A)$, $H(B)$ and $H(C)$ are symmetric positive definite matrices, one can immediately conclude that $H(\mathcal{A})$ is symmetric positive definite. In view of Propositions 2.2 and 2.3, we can conclude that if

$$\lambda_{\min}(H(A))+\lambda_{\min}(H(B))+\lambda_{\min}(H(C))>1+\|S(A)\|+\|S(B)\|+\|S(C)\|,$$

then $\|\mathcal{A}^{-1}\|<1$.

In the rest of this section, we derive an alternative sufficient condition under which $\|\mathcal{A}^{-1}\|<1$. Again, our emphasis is to obtain the condition with respect to coefficient matrices A, B and C having moderate sizes. To do so, we first establish the following theorem:

Theorem 2.5. *Let* $A\otimes B$ *be a non-Hermitian positive definite matrix. If the inequality*

$$\sigma_{\min}(A)\sigma_{\min}(B)>\|H(B)\|\|S(A)\|+\|H(A)\|\|S(B)\| \tag{2.3}$$

holds, then

$$\lambda_{\min}(H(A\otimes B))\geqslant\sigma_{\min}(A)\sigma_{\min}(B)-(\|H(A)\|\|S(B)\|+\|H(B)\|\|S(A)\|). \tag{2.4}$$

Proof. For national simplicity, we set $\mathcal{H}_{A\otimes B} := H(A\otimes B)$ and $\mathcal{S}_{A\otimes B} := S(A\otimes B)$. Considering the SS splitting of $A\otimes B$, i.e., $A\otimes B = \mathcal{H}_{A\otimes B} + \mathcal{S}_{A\otimes B}$, we can see that

$$A\otimes B = \mathcal{H}_{A\otimes B}(I + \mathcal{H}_{A\otimes B}^{-1}\mathcal{S}_{A\otimes B}),$$

which results in

$$\mathcal{H}_{A\otimes B}^{-1} = (I + \mathcal{H}_{A\otimes B}^{-1}\mathcal{S}_{A\otimes B})(A^{-1}\otimes B^{-1}). \tag{2.5}$$

It is known that $\mathcal{S}_{A\otimes B} = S(A)\otimes H(B) + H(A)\otimes S(B)$ and

$$\|A^{-1}\otimes B^{-1}\| = \frac{1}{\sqrt{\lambda_{\min}(AA^T\otimes BB^T)}} = \frac{1}{\sigma_{\min}(A)\sigma_{\min}(B)}.$$

Hence, one can deduce that

$$\begin{aligned} \|\mathcal{S}_{A\otimes B}\| &\leqslant \|S(A)\otimes H(B)\| + \|H(A)\otimes S(B)\| \\ &\leqslant \|S(A)\|\|H(B)\| + \|H(A)\|\|S(B)\|. \end{aligned} \tag{2.6}$$

In view of the above inequality, (2.3) implies that $1 - \|A^{-1}\otimes B^{-1}\|\|\mathcal{S}_{A\otimes B}\| \geqslant 0$. Now, by using (2.5), we obtain

$$\|\mathcal{H}_{A\otimes B}^{-1}\| \leqslant \|A^{-1}\otimes B^{-1}\|(1 + \|\mathcal{H}_{A\otimes B}^{-1}\|\|\mathcal{S}_{A\otimes B}\|),$$

which is equivalent to saying that

$$\frac{1 - \|A^{-1}\otimes B^{-1}\|\|\mathcal{S}_{A\otimes B}\|}{\|A^{-1}\otimes B^{-1}\|} \leqslant \|\mathcal{H}_{A\otimes B}^{-1}\|^{-1}. \tag{2.7}$$

From (2.7) we derive

$$\begin{aligned} \sigma_{\min}(A)\sigma_{\min}(B) - \|\mathcal{S}_{A\otimes B}\| &= \frac{1}{\|A^{-1}\otimes B^{-1}\|} - \|\mathcal{S}_{A\otimes B}\| \\ &= \frac{1 - \|A^{-1}\otimes B^{-1}\|\|\mathcal{S}_{A\otimes B}\|}{\|A^{-1}\otimes B^{-1}\|} \\ &\leqslant \frac{1}{\|\mathcal{H}_{A\otimes B}^{-1}\|} = \lambda_{\min}(\mathcal{H}_{A\otimes B}), \end{aligned} \tag{2.8}$$

where the last equality follows from the fact that $\mathcal{H}_{A\otimes B}$ is symmetric. Now from (2.6) and (2.8), we obtain

$$\begin{aligned} \sigma_{\min}(A)\sigma_{\min}(B) - (\|S(A)\|\|H(B)\| + \|H(A)\|\|S(B)\|) &\leqslant \sigma_{\min}(A)\sigma_{\min}(B) - \|\mathcal{S}_{A\otimes B}\| \\ &\leqslant \lambda_{\min}(\mathcal{H}_{A\otimes B}), \end{aligned}$$

which completes the proof. □

Evidently, a lower bound for minimum eigenvalue of $\mathcal{A}\mathcal{A}^\top$ gives an upper bound for $\|\mathcal{A}^{-1}\|$. The following proposition provides an upper bound for $\lambda_{\min}(\mathcal{A}\mathcal{A}^\top)$.

Proposition 2.6 ([5], Proposition 2.6). *Let* $\mathcal{A} = I_l \otimes I_n \otimes A + I_l \otimes B \otimes I_m + C \otimes I_n \otimes I_m$. *Then*

$$\lambda_{\max}(\mathcal{A}\mathcal{A}^\top) \leqslant (\sigma_{\max}(A) + \sigma_{\max}(B) + \sigma_{\max}(C))^2. \tag{2.9}$$

Moreover, assume that $\mathcal{A}$ *is invertible. Then if* $B^\top \otimes A$, $C^\top \otimes A$, *and* $C^\top \otimes B$ *are positive definite, we have*

$$\lambda_{\min}(\mathcal{A}\mathcal{A}^\top) \geqslant \sigma^2_{\min}(A) + \sigma^2_{\min}(B) + \sigma^2_{\min}(C). \tag{2.10}$$

Under some stronger assumptions than those used in the above proposition, we can establish a sharper upper bound for $\lambda_{\min}(\mathcal{A}\mathcal{A}^\top)$ in comparison with (2.10). To this end, we use the result established in Theorem 2.5. Basically, we assume that the following three conditions holds:

$$\sigma_{\min}(A)\sigma_{\min}(B) > \|H(B)\|\|S(A)\| + \|H(A)\|\|S(B)\|, \tag{2.11a}$$

$$\sigma_{\min}(A)\sigma_{\min}(C) > \|H(C)\|\|S(A)\| + \|H(A)\|\|S(C)\|, \tag{2.11b}$$

$$\sigma_{\min}(C)\sigma_{\min}(B) > \|H(C)\|\|S(B)\| + \|H(B)\|\|S(C)\|. \tag{2.11c}$$

Proposition 2.7. *Assume that* $\mathcal{A} = I_l \otimes I_n \otimes A + I_l \otimes B \otimes I_m + C \otimes I_n \otimes I_m$ *is invertible and the matrices* $B^\top \otimes A$, $C^\top \otimes A$, *and* $C^\top \otimes B$ *are positive definite. If inequalities* (2.11a)–(2.11c) *are satisfied, then*

$$\begin{aligned} &\lambda_{\min}(\mathcal{A}\mathcal{A}^\top) \\ &\quad\geqslant (\sigma_{\min}(A) + \sigma_{\min}(B) + \sigma_{\min}(C))^2 - 2(\|H(A)\|\|S(B)\| + \|H(B)\|\|S(A)\|) \\ &\qquad - 2(\|H(B)\|\|S(C)\| + \|H(C)\|\|S(B)\|) - 2(\|H(C)\|\|S(A)\| + \|H(A)\|\|S(C)\|). \end{aligned} \tag{2.12}$$

Proof. From (2.4) we obtain

$$\begin{aligned} &\lambda_{\min}(\mathcal{A}\mathcal{A}^\top) \\ &\quad\geqslant \sigma^2_{\min}(A) + \sigma^2_{\min}(B) + \sigma^2_{\min}(C) + 2\lambda_{\min}(H(I \otimes B^\top \otimes A)) \\ &\qquad + 2\lambda_{\min}(H(C^\top \otimes I \otimes A)) + 2\lambda_{\min}(H(C^\top \otimes B \otimes I)) \\ &\quad\geqslant \sigma^2_{\min}(A) + \sigma^2_{\min}(B) + \sigma^2_{\min}(C) \\ &\qquad + 2\sigma_{\min}(A)\sigma_{\min}(B) - 2(\|H(A)\|\|S(B)\| + \|H(B)\|\|S(A)\|) \\ &\qquad + 2\sigma_{\min}(B)\sigma_{\min}(C) - 2(\|H(B)\|\|S(C)\| + \|H(C)\|\|S(B)\|) \\ &\qquad + 2\sigma_{\min}(C)\sigma_{\min}(A) - 2(\|H(C)\|\|S(A)\| + \|H(A)\|\|S(C)\|) \\ &\quad= (\sigma_{\min}(A) + \sigma_{\min}(B) + \sigma_{\min}(C))^2 - 2(\|H(A)\|\|S(B)\| + \|H(B)\|\|S(A)\|) \\ &\qquad - 2(\|H(B)\|\|S(C)\| + \|H(C)\|\|S(B)\|) - 2(\|H(C)\|\|S(A)\| + \|H(A)\|\|S(C)\|). \end{aligned}$$

Now the result follows immediately. $\square$

R e m a r k 2.8. Under hypotheses of Proposition 2.7, if

$$\sigma_{\min}(A) + \sigma_{\min}(B) + \sigma_{\min}(C) > \sqrt{1+2T}, \tag{2.13}$$

where

$$\begin{aligned} T = &\|H(A)\|\|S(B)\| + \|H(B)\|\|S(A)\| + \|H(B)\|\|S(C)\| \\ &+ \|H(C)\|\|S(B)\| + \|H(A)\|\|S(C)\| + \|H(C)\|\|S(A)\|, \end{aligned}$$

then $\lambda_{\min}(\mathcal{A}\mathcal{A}^\top) > 1$, which ensures that $\|\mathcal{A}^{-1}\| < 1$.

We end this part by briefly discussing the case when the solution set of (1.1) is empty. To do so, similarly to [10], we first need a monotonicity property given in the following lemma; see [11].

Lemma 2.9. *Let A and B be two $m \times n$ matrices. If $|A| \leqslant B$, then $\|A\| \leqslant \|B\|$.*

The above lemma leads us to the following result which determines a simple sufficient condition for the nonexistence of solutions to (1.1) with nonnegative right-hand side.

Proposition 2.10. *Let $b \geqslant 0$ and $\sigma_{\max}(A) + \sigma_{\max}(B) + \sigma_{\max}(C) < 1$. Then linear system* (1.1) *has no solution.*

P r o o f. Assume by contradiction that x^* is a solution of (1.1). Since the right-hand side of (1.1) is nonnegative, we have $|x^*| \leqslant \mathcal{A}x^*$. Adding the norm to both sides of $|x^*| \leqslant \mathcal{A}x^*$ and using Lemma 2.9, in view of the assumption and the fact that $\|\mathcal{A}\| \leqslant \sigma_{\max}(A) + \sigma_{\max}(B) + \sigma_{\max}(C)$, we get $\|x^*\| \leqslant \|\mathcal{A}\|\|x^*\| < \|x^*\|$, which is a contradiction. □

3. An iterative method and its convergence analysis

This section mainly deals with developing a new iterative method for solving AVEs with unique solution. For deriving the method, in the first part, we do not limit the results to the special tensor (Kronecker) product structure of the coefficient matrix. We briefly discuss the implementation of the proposed method in tensor framework for solving AVEs in the form (1.1) in a separate subsection.

3.1. Proposed iterative method. Using the splitting $\mathcal{A} = M - N$, we aim to construct the sequence of approximations for $(x^*; |x^*|)$, where x^* is a unique solution of $\mathcal{A}x - |x| = b$, i.e., $\mathcal{A}x^* - |x^*| = b$. Basically, the sequence $\{(x^{(k)}; y^{(k)})\}$ is produced

($k = 0, 1, 2, \ldots$) using the following iterative scheme in which τ is a prescribed real constant:

$$\begin{cases} x^{(k+1)} = M^{-1}(Nx^{(k)} + y^{(k)} + b), \\ y^{(k+1)} = \tau|x^{(k+1)}| + (1-\tau)|x^{(k)}|, \end{cases} \tag{3.1}$$

where the initial guess $x^{(0)}$ is given and $y^{(0)} = |x^{(0)}|$.

For notational simplicity, in the sequel, we set $\nu = \|M^{-1}\|$, $\omega = \|M^{-1}N\|$ and $\mathcal{E}^{(k)} = (e_x^{(k)}; e_y^{(k)})$ with $e_x^{(k)} = x^* - x^{(k)}$ and $e_y^{(k)} = |x^*| - y^{(k)}$, where $x^{(k)}$ is the kth approximation of x^* determined by (3.1). It is immediate to see that

$$\begin{aligned} e_x^{(k+1)} &= M^{-1}Ne_x^{(k)} + M^{-1}e_y^{(k)}, \\ e_y^{(k+1)} &= \tau(|x^*| - |x^{(k+1)}|) + (1-\tau)(|x^*| - |x^{(k)}|). \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} \|e_x^{(k+1)}\| &\leqslant \omega\|e_x^{(k)}\| + \nu\|e_y^{(k)}\|, \\ \|e_y^{(k+1)}\| &\leqslant |\tau|(\omega\|e_x^{(k)}\| + \nu\|e_y^{(k)}\|) + |1-\tau|\|e_x^{(k)}\|. \end{aligned}$$

Or equivalently, we can observe that

$$\begin{bmatrix} \|e_x^{(k+1)}\| \\ \|e_y^{(k+1)}\| \end{bmatrix} \leqslant \underbrace{\begin{bmatrix} \omega & \nu \\ |1-\tau| + |\tau|\omega & |\tau|\nu \end{bmatrix}}_{\mathcal{G}} \begin{bmatrix} \|e_x^{(k)}\| \\ \|e_y^{(k)}\| \end{bmatrix}. \tag{3.2}$$

Notice that $\lim\limits_{k\to\infty} \mathcal{G}^k = 0$ if and only if $\varrho(\mathcal{G}) < 1$. Therefore, if $\varrho(\mathcal{G}) < 1$, then the proposed iterative method is convergent for any initial guess. Using this fact, we establish an interval for τ under which the proposed method converges. To this end, we need the following lemma from [31].

Lemma 3.1. *Consider the quadratic equation $x^2 - bx + c = 0$, where b and c are real numbers. Both roots of the equation are less than one in modulus if and only if $|c| < 1$, $|b| < 1 + c$.*

Theorem 3.2. *Let $\mathcal{A} = M - N$ and $\mathcal{A}x - |x| = b$ have a unique solution such that M is invertible and $\nu + \omega < 1$, where $\nu = \|M^{-1}\|$ and $\omega = \|M^{-1}N\|$. The iterative scheme* (3.1) *converges to the solution of $\mathcal{A}x - |x| = b$ for any initial guess $x^{(0)}$ if the parameter τ belongs to the following interval $\mathcal{I}$*

$$\mathcal{I} = \Big(\frac{\omega + \nu - 1}{2\nu}, \frac{1 + \nu - \omega}{2\nu}\Big). \tag{3.3}$$

Proof. Setting $\mathcal{E}^{(k)} = (\|e_x^{(k)}\|; \|e_y^{(k)}\|)$, from (3.2) we get $\mathcal{E}^{(k+1)} \leqslant \mathcal{G}\mathcal{E}^{(k)}$. To conclude the assertion, we only need to verify that the spectral radius of $\mathcal{G}$ is strictly lower than one, i.e., $\varrho(\mathcal{G}) < 1$ when $\tau \in \mathcal{I}$. Let λ be an arbitrary eigenvalue of $\mathcal{G}$ with the corresponding eigenvector $[\tilde{x}; \tilde{y}]$. As a result, we have

$$\begin{bmatrix} \omega & \nu \\ |1-\tau| + |\tau|\omega & |\tau|\nu \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \lambda \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}.$$

Or equivalently,

$$\nu\tilde{y} = (\lambda - \omega)\tilde{x}, \tag{3.4}$$

$$(|1-\tau| + |\tau|\omega)\tilde{x} + |\tau|\nu\tilde{y} = \lambda\tilde{y}. \tag{3.5}$$

Without loss of generality, since $\tilde{x}$ is necessarily nonzero, we may assume that $\|\tilde{x}\| = 1$. Multiplying both sides of (3.5) by ν and substituting $\nu\tilde{y}$ from (3.4) into (3.5), we derive

$$(|1-\tau| + |\tau|\omega)\nu\tilde{x} + |\tau|(\lambda-\omega)\nu\tilde{x} = \lambda(\lambda-\omega)\tilde{x}.$$

It is immediate to see that λ satisfies the following quadratic equation:

$$\lambda^2 - \lambda(\omega + |\tau|\nu) - |1-\tau|\nu = 0.$$

By the assumption, we have $(\omega+\nu-1)/(2\nu) < \tau < (1+\nu-\omega)/(2\nu)$. Notice that $(1+\nu-\omega)/(2\nu) < 1 + 1/\nu$ and $1 - 1/\nu < (\omega+\nu-1)/(2\nu)$, which imply that

$$|1-\tau| < \frac{1}{\nu}.$$

In view of the above relation, to conclude the result from Lemma 3.1, we only need to show that

$$\omega + |\tau|\nu < 1 - \nu|1-\tau|.$$

In fact, we need to verify that

$$\omega + \tau\nu < 1 - \nu(\tau - 1) \quad \text{for } \tau > 1, \tag{3.6a}$$

$$\omega + \tau\nu < 1 - \nu(1 - \tau) \quad \text{for } 0 < \tau < 1, \tag{3.6b}$$

$$\omega - \tau\nu < 1 - \nu(1 - \tau) \quad \text{for } \tau < 0. \tag{3.6c}$$

Inequalities (3.6a) and (3.6c) are respectively deduced from the assumption that

$$\tau < \frac{1+\nu-\omega}{2\nu} \quad \text{and} \quad \frac{\omega+\nu-1}{2\nu} < \tau.$$

Equation (3.6b) is a direct conclusion of $\omega + \nu < 1$. $\square$

Notice that the iterative method (3.1) reduces to (1.5) for $M = \mathcal{A}$. In this case the following theorem is established in [25], Theorem 2.8.

Theorem 3.3. *Let $\mathcal{A}$ be a nonsingular matrix such that $\nu = \|\mathcal{A}^{-1}\| < 1$. The iterative scheme* (1.5) *converges to the unique solution of $\mathcal{A}x - |x| = b$ for any initial guess $x^{(0)}$ if*

$$0 < \tau < \frac{1+\nu}{2\nu}. \tag{3.7}$$

We end this section by starting that Theorem 3.2 gives a wider convergence interval than (3.7). In fact, setting $\omega = 0$ in (3.3), one can reach the following convergence interval for iterative method (1.5):

$$\frac{1-\nu}{2\nu} < \tau < \frac{1+\nu}{2\nu}.$$

It is immediate to see that $[0, (1+\nu)/(2\nu)] \subset [(1-\nu)/(2\nu), (1+\nu)/(2\nu)]$.

3.2. Implementing the proposed method in tensor framework. In this part, we briefly discuss the implementation of proposed method in tensor form; for more details on the employing iterative method in tensor framework see [6], [12], [13], [17]. First, we need to recall the following product between a matrix and tensor.

Definition 3.4 (Kolda and Bader [16]). The i-mode (matrix) product of a tensor $\mathscr{X} \in \mathbb{R}^{s_1 \times s_2 \times \ldots \times s_N}$ with a matrix $U \in \mathbb{R}^{p_i \times s_i}$ is denoted by $\mathscr{X} \times_i U$ and is of size

$$s_1 \times \ldots \times s_{i-1} \times p_i \times s_{i+1} \times \ldots \times s_N,$$

and its elements are defined as

$$(\mathscr{X} \times_i U)_{w_1 \ldots w_{i-1} j w_{i+1} \ldots w_N} = \sum_{w_i=1}^{s_i} x_{w_1 w_2 \ldots w_N} u_{j w_i}$$

with $1 \leqslant w_k \leqslant s_k$ and $1 \leqslant j \leqslant p_i$ for $k = 1, 2, \ldots, N$.

In fact, the following tensor equation is mathematically equivalent to (1.1):

$$\mathscr{X} \times_1 A + \mathscr{X} \times_2 B + \mathscr{X} \times_3 C - |\mathscr{X}| = \mathscr{B}. \tag{3.8}$$

For the right-hand side vector b in (1.1), we have $b = \mathrm{vec}(\mathscr{B})$. In fact, the vector b corresponds to the vectorization of mode-1 unfolding (matricization) of tensor $\mathscr{B}$ in (3.8); for more details see [16]. The inverse act of "vec$(\cdot)$" mapping is denoted by "unvec$(\cdot)$".

For notational simplicity, we use the following linear operator:

$$\begin{aligned} \mathcal{M}\colon\ \mathbb{R}^{m\times n\times l} &\to \mathbb{R}^{m\times n\times l}, \\ \mathscr{X} \mapsto \mathcal{M}(\mathscr{X}) &:= \mathscr{X}\times_1 A + \mathscr{X}\times_2 B + \mathscr{X}\times_3 C. \end{aligned} \tag{3.9}$$

Notice that (1.1) can be reformulated as follows:

$$\mathcal{M}(\mathscr{X}) - |\mathscr{X}| = \mathscr{B}. \tag{3.10}$$

It is not difficult to observe that iterative method (3.1) can be equivalently applied for solving (3.8) instead of (1.1). In the sequel, we rewrite the proposed method based on the tensor format. To do so, one can first consider the succeeding splittings

$$A = M_1 - N_1, \quad B = M_2 - N_2 \quad \text{and} \quad C = M_3 - N_3.$$

Then, for $k = 1, 2, \ldots$ the algorithm can be implemented in the following two steps.

Step 1. Find the new approximation by solving the Sylvester tensor equation

$$\mathscr{X}_{k+1}\times_1 M_1 + \mathscr{X}_{k+1}\times_2 M_2 + \mathscr{X}_{k+1}\times_3 M_3 = \mathcal{F}(\mathscr{X}_k) + \mathscr{Y}_k + \mathscr{B}, \tag{3.11}$$

where $\mathcal{F}(\mathscr{X}_k) = \mathscr{X}_k\times_1 N_1 + \mathscr{X}_k\times_2 N_2 + \mathscr{X}_k\times_3 N_3$ and $\mathscr{X}_0$ is given and $\mathscr{Y}_0 = |\mathscr{X}_0|$.

Step 2. Set $\mathscr{Y}_{k+1} = \tau|\mathscr{X}_{k+1}| + (1-\tau)|\mathscr{X}_k|$.

In this work, we are mainly interested in the case where coefficient matrices A, B and C are positive definite matrices whose symmetric parts are dominant to their skew-symmetric part. Therefore, in Section 4, we only present our experimental reports for the case where M_1, M_2 and M_3 are respectively symmetric parts of A, B and C.

In practice, we solve the Sylvester tensor equation (3.11) inexactly by using conjugate gradient method. Invoking the idea used for applying inexact HSS method [3], for more efficient implementation, we first find the approximate solution $\mathscr{Z}_k$ of the Sylvester tensor equation

$$\mathscr{Z}\times_1 M_1 + \mathscr{Z}\times_2 M_2 + \mathscr{Z}\times_3 M_3 = \mathscr{R}_k + \mathscr{Y}_k - |\mathscr{X}_k|,$$

where $\mathscr{R}_k = \mathscr{B} - \mathcal{M}(\mathscr{X}_k) + |\mathscr{X}_k|$. Then the new approximation is computed by $\mathscr{X}_{k+1} = \mathscr{Z}_k + \mathscr{X}_k$. The proposed approach is summarized in Algorithm 2. Similarly to the observations made in [6], our numerical experiments illustrate that this kind of reformulation helps to apply the method efficiently using a loose inner tolerance (being larger than outer tolerance). In particular, the proposed algorithm provides a suitable approximation when $\eta \leqslant 0.01$ for our test problems. Therefore, we set $\eta = 0.01$ in Algorithm 1 since the smaller choice for η results in more computational costs.

Algorithm 1. Conjugate gradient based on tensor format (CG_BTF) [6].

Input: Coefficients A, B, C, the right-hand side $\mathscr{B}$, the initial guess $\mathscr{X}_0$ and tolerance η.

Compute $\mathscr{R}_0 = \mathscr{B} - \mathcal{M}(\mathscr{X}_0)$. Set $\delta = 1$ and $\mathscr{P}_0 = \mathscr{R}_0$.

begin

  while $\delta > \eta$ **do**

    $\alpha_j = \langle \mathscr{R}_j, \mathscr{R}_j \rangle / \langle \mathcal{M}(\mathscr{P}_j), \mathscr{P}_j \rangle$;

    $\mathscr{X}_{j+1} = \mathscr{X}_j + \alpha_j \mathscr{P}_j$;

    $\mathscr{R}_{j+1} = \mathscr{R}_j - \alpha_j \mathcal{M}(\mathscr{P}_j)$;

    $\beta_j = \langle \mathscr{R}_{j+1}, \mathscr{R}_{j+1} \rangle / \langle \mathscr{R}_j, \mathscr{R}_j \rangle$;

    $\mathscr{P}_{j+1} = \mathscr{R}_{j+1} + \beta_j \mathscr{P}_j$;

    $\delta = \|\mathscr{R}_{j+1}\| / \|\mathscr{R}_j\|$;

  end

end

Algorithm 2. Proposed method for solving (1.1) in tensor framework.

Input: Tolerances ε and η, the initial guess $\mathscr{X}(0) \in \mathbb{R}^{m\times n\times l}$, coefficient matrices $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{l\times l}$ and the right-hand side $\mathscr{B} = \mathrm{unvec}(b)$.

Compute $\mathscr{R}(0) = \mathscr{B} + |\mathscr{X}(0)| - \mathcal{M}(\mathscr{X}(0))$. Set $\mathscr{Y}(0) = |\mathscr{X}(0)|$.

begin

  for $k = 0, 1, 2, \ldots$ **do**

    Find approximate solution $\mathscr{Z}_k$ by solving Sylvester tensor equation

      $\mathscr{Z} \times_1 H(A) + \mathscr{Z} \times_2 H(B) + \mathscr{Z} \times_3 H(C) = \mathscr{R}_k + \mathscr{Y}_k - |\mathscr{X}_k|$

    using Algorithm 1 and tolerance η;

    Set $\mathscr{X}_{k+1} = \mathscr{Z}_k + \mathscr{X}_k$;

    Set $\mathscr{Y}_{k+1} = (1-\tau)|\mathscr{X}_k| + \tau|\mathscr{X}_{k+1}|$;

    Compute $\mathscr{R}_{k+1} = \mathscr{B} + |\mathscr{X}_{k+1}| - \mathcal{M}(\mathscr{X}_{k+1})$;

    if $\|\mathscr{R}_{k+1}\| \leqslant \varepsilon \|\mathscr{R}_0\|$ **then**

      Stop.

    end

  end

end

Following the discussions in [25], Subsection 2.2, we observed that the parameter

$$\tau^* = \frac{2}{1 + \sqrt{1 - \overline{\nu}}}, \tag{3.12}$$

for $\overline{\nu} = 1/(\lambda_{\min}(H(A)) + \lambda_{\min}(H(B)) + \lambda_{\min}(H(C)))$, gives an appropriate approximation for optimum value of parameter τ in the proposed method when $\overline{\nu} < 1$ and

$$\lambda_{\min}(H(A)) + \lambda_{\min}(H(B)) + \lambda_{\min}(H(C)) > \|S(A)\| + \|S(B)\| + \|S(C)\|.$$

4. Numerical experiments

All of the following reported numerical experiments were performed on a 64-bit 3.50 GHz core i7 processor and 24.00GB RAM using some matlab codes from MATLAB R2018a. The iterative methods for solving (1.1) are implemented in tensor framework using the tensor Toolbox, see [2], [16] for more details.

As pointed earlier, we apply the proposed iterative method in tensor form. To this end, we solve the absolute value equation (3.10) using Algorithm 2 setting $\eta = 0.01$ as tolerance for inner iterations. Two test problems were considered and the right-hand side $\mathscr{B}$ is constructed so that the tensor $\mathscr{X}^*$ is the exact solution of (3.10), where

$$\mathscr{X}^*_{ijk} = \frac{(-1)^{ijk}}{(n+1)^3} ijk \quad \text{for } i,j,k = 1,\dots,n.$$

The initial guess $\mathscr{X}_0$ is taken to be zero vector and the algorithm is terminated once

$$\frac{\|\mathcal{M}(\mathscr{X}_k) - |\mathscr{X}_k| - \mathscr{B}\|}{\|\mathscr{B}\|} \leqslant 10^{-12},$$

where $\mathscr{X}_k$ is the k-th approximate solution. For more details, we further report the relative error associated with the obtained approximate solution in tables, i.e.,

$$\text{Err} := \frac{\|\mathscr{X}^* - \mathscr{X}_k\|}{\|\mathscr{X}^*\|}.$$

It should be said that the values of τ^* in Tables 2, 3 and 5 are computed by (3.12). We further recall that the proposed method is rescued to the (inexact) Picard iteration for $\tau = 1$.

Example 4.1. The Sylvester tensor equation in the form $\mathcal{M}(\mathscr{X}) = \mathscr{B}$ appears in the discretization of a 3D convection-diffusion partial differential equation by standard finite differences on a uniform grid for the diffusion term and a second-order convergent scheme (Fromm's scheme) for the convection term with the coefficient matrices

$$A_i = \frac{1}{h^2}\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} + \frac{c_i}{4h}\begin{bmatrix} 3 & -5 & 1 & & & \\ 1 & 3 & -5 & \ddots & & \\ & \ddots & \ddots & \ddots & 1 \\ & & 1 & 3 & -5 \\ & & & 1 & 3 \end{bmatrix}, \tag{4.1}$$

see [6], Example 5.4 for more details. Here, we are interested in solving AVEs of the form (3.8), whose coefficient matrices $A = A_1$, $B = A_2$ and $C = A_3$ are $n \times n$ matrices given by (4.1). We report the obtained numerical results for two cases.

Case 1. We set $c_1 = 1, c_2 = 2$ and $c_3 = 3$. Here, the proposed method is superior to the (inexact) Picard-HSS method significantly for $n = 20, 30$ and $n = 40$ in terms of both the number of required iteration numbers and the CPU-time for convergence. More precisely, we observed that the (inexact) Picard-HSS does not work properly for dimensions greater than 50. Therefore, we do not report the performance of Picard-HSS method. The numerical results for the proposed approach are disclosed for some values of τ. Our observations show that the optimum value of τ can be negative, see Table 1 for more details.

	$n = 100$			$n = 120$			$n = 140$		
τ	Iter	CPU	ERR	Iter	CPU	ERR	Iter	CPU	ERR
1.0	32	120.9322	5.9155e−10	32	227.2430	5.1886e−10	31	408.9474	9.5242e−10
0.5	30	110.9425	5.1228e−10	30	212.4999	4.4501e−10	29	384.1434	8.6469e−10
−0.9	25	91.0535	2.5205e−10	24	166.5939	5.2736e−10	24	310.3702	4.8487e−10
−2.1	21	73.4839	1.6930e−10	20	145.5711	4.2906e−10	20	260.3470	3.7966e−10
−3.5	19	63.7312	1.1892e−10	19	132.8229	6.0510e−11	18	220.1260	2.2163e−10
−4.1	21	71.6981	6.6745e−10	21	139.7254	4.8588e−11	20	255.09134	1.7343e−10

Table 1. Numerical results for Example 4.1 (Case 1).

Case 2. In this case we set $c_1 = 0.01, c_2 = 0.02$ and $c_3 = 0.03$. In Tables 2 and 3, we respectively present the numerical results with respect to $\nu = 0.1$ and $\nu = 0.5$. We comment that the provided sufficient condition in Proposition 2.2 is satisfied when $\nu = 0.5$.

	$n = 100$			$n = 120$			$n = 140$		
τ	Iter	CPU	ERR	Iter	CPU	ERR	Iter	CPU	ERR
1.5000	20	50.8420	1.1515e−09	19	94.1938	1.4560e−13	18	171.7152	2.7201e−09
τ^*	12	33.5028	3.4878e−10	12	67.5018	3.2424e−13	12	126.6122	2.8514e−10
1.0000	16	41.2511	4.6144e−10	15	75.6591	4.5855e−13	15	143.0978	2.0314e−09
0.5000	24	56.0450	1.0016e−09	24	111.1164	8.0318e−13	23	202.4417	2.4638e−09

Table 2. Numerical results for Example 4.1 with $\nu = 0.1$ and $\tau^* = 1.1028$ (Case 2).

	$n = 100$			$n = 120$			$n = 140$		
τ	Iter	CPU	ERR	Iter	CPU	ERR	Iter	CPU	ERR
1.5000	12	31.9757	2.3834e−10	12	64.3919	1.9634e−10	12	120.6708	1.4693e−10
τ^*	7	20.2654	4.1149e−10	7	39.3771	3.9156e−10	7	73.3012	4.0106e−10
1.0000	8	21.6383	4.1074e−10	8	42.5056	4.9199e−10	8	79.2318	5.7192e−10
0.5000	12	30.8383	1.0100e−09	12	60.2294	1.0853e−09	12	113.9157	1.1535e−09

Table 3. Numerical results for Example 4.1 with $\nu = 0.5$ and $\tau^* = 1.0175$ (Case 2).

Example 4.2. We consider AVE (3.8) such that

$$A = B = C = M + 2rL + \frac{10^4}{(n+1)^2} I_n,$$

where M and L are $n \times n$ tridiagonal matrices with $M = \text{tridiag}(-1, 2, -1)$, $L = \text{tridiag}(0.5, 0, -0.5)$, and $r = 0.01$. These matrices are discussed in [32].

	$n=100$			$n=120$			$n=140$		
τ	Iter	CPU	ERR	Iter	CPU	ERR	Iter	CPU	ERR
1.5	28	3.1587	8.9150e−13	33	6.8105	1.6552e−12	40	14.2699	6.0092e−12
1.2	20	2.2902	3.2688e−13	22	4.5672	1.1576e−12	27	8.6355	3.5663e−12
1	22	2.4406	2.7648e-12	30	5.7939	4.2618e−12	44	14.0218	6.0617e−12
0.8	29	3.2128	1.2308e-12	38	7.4241	3.4098e−12	55	17.5554	4.5999e−12

Table 4. Numerical results for Example 4.2 with respect to different values of τ.

τ	$n=100$ ($\tau^*=1.1034$)			$n=120$ ($\tau^*=1.1656$)			$n=140$ ($\tau^*=1.2648$)		
	Iter	CPU	(α,β)	Iter	CPU	(α,β)	Iter	CPU	(α,β)
τ^*	17	2.0985	(8.6659,2.1945)	21	4.3223	(4.2069,2.1234)	28	10.4853	(2.2815,2.0803)

Table 5. Numerical results for Example 4.2 with respect to τ^* and reporting the validity of the condition in Remark 2.8.

We report the experimentally obtained results corresponding to the proposed method with respect to different values of τ in Table 4. For more details, in Table 5, we also disclose the performance of the proposed method for τ^* and the values of

$$\alpha := (\sigma_{\min}(A) + \sigma_{\min}(B) + \sigma_{\min}(C))^2 \quad \text{and} \quad \beta = 1 + 2T.$$

As seen, the following sufficient condition in Remark 2.8 holds, i.e., we have

$$(\sigma_{\min}(A) + \sigma_{\min}(B) + \sigma_{\min}(C)^2 > 1 + 2T.$$

5. Conclusions and future works

We mainly considered the AVEs in the form $\mathcal{A}x - |x| = b$ in which the coefficient matrix $\mathcal{A}$ has a special tensor product structure. The solvability of these AVEs was studied and two types of computationally inexpensive sufficient conditions were established for unique solvability. Moreover, a new iterative technique was proposed for finding their approximate solutions of AVEs. In general, our proposed method does not rely on the special tensor structure of $\mathcal{A}$ in theoretical point of view. Furthermore, the convergence analysis of the method in our mentioned form had been left as a project in [FILOMAT 34 (2020), 4171–4188] to be undertaken.

It was observed both theoretically and numerically that the method can possibly work well with negative values of its parameter in some cases. Nevertheless, determining (possibly negative) optimal parameters in these cases needs more research. In practice, the proposed method was applied inexactly. Although some theoretical results could be stated similarly to [24], Theorem 1 and [7], Theorem 2.3, analyzing the inexact implementation of the approach is left for a future project. In addition, we applied our proposed method under two limitations in practical implementation, i.e., tensor form of $\mathcal{A}$ and symmetric and skew-symmetric splittings of coefficient matrices A, B and C in (1.1). Omitting any of these restrictions can be considered a new project in the future.

A c k n o w l e d g m e n t s. The authors would like to sincerely thank the anonymous referee for reading the manuscript carefully and providing several helpful suggestions.

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