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WEIGHTED ERDŐS-KAC TYPE THEOREM OVER QUADRATIC FIELD IN SHORT INTERVALS

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Abstract. Let $\mathbb{K}$ be a quadratic field over the rational field and $a_{\mathbb{K}}(n)$ be the number of nonzero integral ideals with norm n. We establish Erdős-Kac type theorems weighted by $a_{\mathbb{K}}(n)^l$ and $a_{\mathbb{K}}(n^2)^l$ of quadratic field in short intervals with $l \in \mathbb{Z}^+$. We also get asymptotic formulae for the average behavior of $a_{\mathbb{K}}(n)^l$ and $a_{\mathbb{K}}(n^2)^l$ in short intervals.

Keywords: ideal counting function; Erdős-Kac theorem; quadratic field; short intervals; mean value

MSC 2020: 11N60, 11N37, 11N45

1. Introduction

Let $\mathbb{K}$ be a number field of degree $q \geqslant 2$ and let $\mathcal{O}_{\mathbb{K}}$ be its ring of algebraic integers. Then the Dedekind zeta function of $\mathbb{K}$ is defined as

$$\zeta_{\mathbb{K}}(s) = \sum_{\mathfrak{a}} \frac{1}{(\mathfrak{N}\mathfrak{a})^s} = \sum_{n\geqslant 1} \frac{a_{\mathbb{K}}(n)}{n^s}, \quad \Re s > 1,$$

where $\mathfrak{a}$ varies over nonzero integral ideals in $\mathcal{O}_{\mathbb{K}}$, $a_{\mathbb{K}}(n)$ is called the *ideal counting function* which is defined as the number of nonzero integral ideals in $\mathcal{O}_{\mathbb{K}}$ with norm n and $\mathfrak{N}\mathfrak{a}$ is the norm of $\mathfrak{a}$.

The function $a_{\mathbb{K}}(n)$ is very important in algebraic number theory. Since its behavior is irregular, one often tries to study the asymptotic behavior of $a_{\mathbb{K}}(n)$. In 1927, Landau in [6] gave an asymptotic formula for the average behavior of $a_{\mathbb{K}}(n)$. It is hard to improve Landau's result. In 1993, Nowak in [10] gave a more precise error term for any algebraic number field of degree $q \geqslant 3$.

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In 2010, Lü in [8] gave the average behavior of powers of $a_{\mathbb{K}}(n)$. They showed that if $\mathbb{K}$ is a Galois extension over $\mathbb{Q}$ of degree $q \geqslant 2$, then for any $\varepsilon > 0$ and any integer $l \geqslant 2$,

$$\sum_{n \leqslant x} a_{\mathbb{K}}(n)^l = xP_l(\log x) + O(x^{1-3/(q^l+6)+\varepsilon}),$$

where $P_l(t)$ denotes a suitable polynomial in t of degree $q^{l-1} - 1$. This improved the error term $O(x^{1-2/(q^l)+\varepsilon})$ of Chandraseknaran and Good (see [2]) for $q \geqslant 3$ and $l \geqslant 2$.

In 2011, Lü and Yang in [9] gave the asymptotic behavior of $a_{\mathbb{K}}(n)$ in quadratic field over square numbers. They showed that if $\mathbb{K}$ is a quadratic field, then for any integer $l \geqslant 1$,

$$\sum_{n \leqslant x} a_{\mathbb{K}}(n^2)^l = xP_\beta(\log x) + O(x^{1-3/(2\beta+2)+\varepsilon}),$$

where

$$\beta = \frac{3^l + 1}{2},$$

$P_\beta(t)$ is a polynomial in t of degree $\beta - 1$ and $\varepsilon > 0$ is an arbitrarily small constant. Moreover, when $l \geqslant 3$, they got an asymptotic formula with a more precise error term.

In 2015, for a quadratic number field $\mathbb{K}$ with discriminant $d(\mathbb{K})$, Zhai in [14] gave the short interval estimate

$$\sum_{x < n \leqslant x+y} a_{\mathbb{K}}(n)^2 \sim B_0 y \log x, \tag{1.1}$$

which holds if $y = o(x)$ and $y/(x^{1/2} \log x) \to \infty$ as $x \to \infty$, where

$$B_0 = \frac{6}{\pi^2} L^2(1, \chi') \prod_{p \mid d(\mathbb{K})} \frac{p}{p+1},$$

where p is a rational prime number and $L(s, \chi')$ is a Dirichlet L-function with respect to a certain nonprincipal real character modulo $|d(\mathbb{K})|$.

The discussion of the distribution of arithmetic functions can also be viewed from the perspective of probability. In 1939, Erdős and Kac in [4] proved that the distribution of $\omega(n)$ on the set $\{n \in \mathbb{N}\colon\ n \leqslant x\}$ is approximately Gaussian, with mean $\log \log x$ and standard deviation $(\log \log x)^{1/2}$, i.e., for any $\lambda \in \mathbb{R}$ we have

$$\frac{1}{x} \sum_{\substack{n \leqslant x \\ \omega(n) - \log\log x \leqslant \lambda (\log \log x)^{1/2}}} 1 \to \Phi(\lambda), \quad x \to \infty,$$

where $\omega(n)$ is the number of distinct prime divisors of n and $\Phi(\lambda)$ is the normal distribution function.

We now assume that $\mathbb{K} = \mathbb{Q}(\sqrt{d})$, where d is a square-free integer. Assume the set $\Omega_{x,y} = \{n \in \mathbb{N}\colon\ x \leqslant n \leqslant y\}$, where x is a sufficiently large number and $y = o(x)$. We will define two new uniform probability measures associated with $a_{\mathbb{K}}(n)$ in $\Omega_{x,y}$ and study the Erdős-Kac type theorem as follows. In fact, the authors have proved a special case for Gaussian field in paper, see [7]. In this paper, the method is similar to the previous article, but has a difference in dealing with $a_{\mathbb{K}}(n)$ since it is complicated in general quadratic field.

1.1. Erdős-Kac type theorem with weight $a_{\mathbb{K}}(n)^l$ over quadratic field in short intervals. Define the summatory function in short intervals as

$$V_l(x,y) := \sum_{x<n\leqslant x+y} a_{\mathbb{K}}(n)^l,$$

where $l \in \mathbb{Z}^+$, $x^{7/12+\varepsilon} \leqslant y \leqslant x$. We prove the following theorem

Theorem 1.1. *Let $\varepsilon \in (0, \frac{5}{12})$. Then for any real number λ and any integer $l \in \mathbb{Z}^+$ we have*

$$\frac{1}{V_l(x,y)} \sum_{\substack{x<n\leqslant x+y \\ \omega(n)-2^{l-1}\log\log x \leqslant \lambda(2^{l-1}\log\log x)^{1/2}}} a_{\mathbb{K}}(n)^l = \Phi(\lambda) + O_{l,\varepsilon}\Big(\frac{1}{\sqrt{\log\log x}}\Big) \tag{1.2}$$

uniformly for $x \geqslant 3$ and $x^{7/12+\varepsilon} \leqslant y \leqslant x$, where the implied constant depends on l and ε only. The error term in (1.2) *is optimal.*

Remark 1.1. The exponent $\frac{7}{12}$ in Theorem 1.1 comes from Huxley's zero-density bound for the Riemann ζ-function. This constant can be reduced to $\frac{1}{2}$ if we assume the zero-density hypothesis.

In order to prove that the error term in (1.2) is optimal, we need to consider the local distribution of $a_{\mathbb{K}}(n)^l$ in short intervals. For $l \in \mathbb{Z}^+$ and $k \in \mathbb{N}$, define

$$V_{k,l}(x,y) := \sum_{\substack{x<n\leqslant x+y \\ \omega(n)=k}} a_{\mathbb{K}}(n)^l. \tag{1.3}$$

We have the following result.

Theorem 1.2. *Let $l \in \mathbb{Z}^+$, $k \in \mathbb{N}$, $B > 0$ and $\varepsilon \in (0, \frac{5}{12})$. We have*

$$\begin{aligned} V_{k,l}(x,y) = {} & \frac{y}{\log x}\frac{(2^{l-1}\log\log x)^{k-1}}{(k-1)!} \\ & \times \Big\{\pi_j\Big(\frac{k-1}{2^{l-1}\log\log x}\Big) + O_{l,B,\varepsilon}\Big(\frac{k-1}{(\log\log x)^2} + \frac{\log\log x}{k\log x}\Big)\Big\} \end{aligned} \tag{1.4}$$

uniformly for $x \geqslant 3$, $x^{7/12+\varepsilon} \leqslant y \leqslant x$ and $1 \leqslant k \leqslant 2^{l-1}B\log\log x$, where

$$\text{(1.5)}\quad \pi_1(z) := \frac{2^{l-1}}{\Gamma(2^{l-1}z+1)}\Big(1+\sum_{v\geqslant 1}\frac{(v+1)^l z}{2^v}\Big)\Big(\frac{1}{2}\Big)^{2^{l-1}z} v(z) \quad \textit{if } d \equiv 1 \pmod 8,$$

$$\pi_2(z) := \frac{2^{l-1}}{\Gamma(2^{l-1}z+1)}\Big(1+\frac{z}{3}\Big)\Big(\frac{1}{2}\Big)^{2^{l-1}z} v(z) \quad \textit{if } d \equiv 5 \pmod 8,$$

$$\pi_3(z) := \frac{2^{l-1}}{\Gamma(2^{l-1}z+1)} v(z) \quad \textit{if } d \equiv 2,\ 3 \pmod 4$$

with

$$v(z) := \prod_{\substack{p\nmid d(\mathbb{K})\\ p\geqslant 3\\ (d/p)=1}}\Big(1+\sum_{v\geqslant 1}\frac{(v+1)^l z}{p^v}\Big)\Big(1-\frac{1}{p}\Big)^{2^{l-1}z} \prod_{\substack{p\nmid d(\mathbb{K})\\ p\geqslant 3\\ (d/p)=-1}}\Big(1+\frac{z}{p^2-1}\Big)\Big(1-\frac{1}{p}\Big)^{2^{l-1}z}$$
$$\times \prod_{p\mid d(\mathbb{K})}\Big(1+\frac{z}{p-1}\Big)\Big(1-\frac{1}{p}\Big)^{2^{l-1}z}$$

and the implied constant depends on l, B and ε only.

In Section 4.1, we will establish the connection between $V_{k,l}(x,y)$ and the error term in formula (1.2). By using the estimation of $V_{k,l}(x,y)$ in Theorem 1.2, we proved that the error term in formula (1.2) is optimal.

1.2. Erdős-Kac type theorem with weight $a_{\mathbb{K}}(n^2)^l$ over quadratic field in short intervals. Similarly to the functions $V_l(x,y)$ and $V_{k,l}(x,y)$ for $a_{\mathbb{K}}(n)^l$, we define the following functions for $a_{\mathbb{K}}(n^2)^l$:

$$\text{(1.6)}\qquad U_l(x,y) := \sum_{x<n\leqslant x+y} a_{\mathbb{K}}(n^2)^l, \quad U_{k,l}(x,y) := \sum_{\substack{x<n\leqslant x+y\\ \omega(n)=k}} a_{\mathbb{K}}(n^2)^l,$$

where $l \in \mathbb{Z}^+$, $k \in \mathbb{N}$ and $x^{7/12+\varepsilon} \leqslant y \leqslant x$.

Our results are as follows.

Theorem 1.3. *Let $\varepsilon \in (0, \frac{5}{12})$. Then for any real number λ and any integer $l \in \mathbb{Z}^+$ we have*

$$\text{(1.7)}\qquad \frac{1}{U_l(x,y)}\sum_{\substack{x<n\leqslant x+y\\ \omega(n)-\beta\log\log x\leqslant \lambda(\beta\log\log x)^{1/2}}} a_{\mathbb{K}}(n^2)^l = \Phi(\lambda) + O_{l,\varepsilon}\Big(\frac{1}{\sqrt{\log\log x}}\Big)$$

uniformly for $x \geqslant 3$ and $x^{7/12+\varepsilon} \leqslant y \leqslant x$, where the implied constant depends on l and ε only. The error term in (1.7) is optimal.

Theorem 1.4. *Let $l \in \mathbb{Z}^+$, $k \in \mathbb{N}$, $B > 0$ and $\varepsilon \in (0, \frac{5}{12})$. We have*

$$U_{k,l}(x,y) = \frac{y}{\log x} \frac{(\beta \log\log x)^{k-1}}{(k-1)!} \times \Big\{ \lambda_j\Big(\frac{k-1}{\beta \log\log x}\Big) + O_{l,B,\varepsilon}\Big(\frac{k-1}{(\log\log x)^2} + \frac{\log\log x}{k \log x}\Big)\Big\} \tag{1.8}$$

uniformly for $x \geqslant 3$, $x^{7/12+\varepsilon} \leqslant y \leqslant x$ and $1 \leqslant k \leqslant \beta B \log\log x$, where

$$\begin{aligned} \lambda_1(z) &:= \frac{\beta}{\Gamma(\beta z+1)}\Big(1 + \sum_{v \geqslant 1} \frac{(2v+1)^l z}{2^v}\Big)\Big(\frac{1}{2}\Big)^{\beta z} u(z) \quad \text{if } d \equiv 1 \pmod 8, \\ \lambda_2(z) &:= \frac{\beta}{\Gamma(\beta z+1)}(1+z)\Big(\frac{1}{2}\Big)^{\beta z} u(z) \quad \text{if } d \equiv 5 \pmod 8, \\ \lambda_3(z) &:= \frac{\beta}{\Gamma(\beta z+1)} u(z) \quad \text{if } d \equiv 2,\ 3 \pmod 4 \end{aligned} \tag{1.9}$$

with

$$\begin{aligned} u(z) := &\prod_{\substack{p \nmid d(\mathbb{K}) \\ p \geqslant 3 \\ (d/p)=1}} \Big(1 + \sum_{v \geqslant 1} \frac{(2v+1)^l z}{p^v}\Big)\Big(1 - \frac{1}{p}\Big)^{\beta z} \prod_{\substack{p \nmid d(\mathbb{K}) \\ p \geqslant 3 \\ (d/p)=-1}} \Big(1 + \frac{z}{p-1}\Big)\Big(1 - \frac{1}{p}\Big)^{\beta z} \\ &\times \prod_{p \mid d(\mathbb{K})} \Big(1 + \frac{z}{p-1}\Big)\Big(1 - \frac{1}{p}\Big)^{\beta z} \end{aligned}$$

and the implied constant depends on l, B and ε only.

To prove the above conclusions, we will study the partial holomorphic continuation of functions $\sum_{n \geqslant 1} a_{\mathbb{K}}(n)^l z^{\omega(n)} n^{-s}$ and $\sum_{n \geqslant 1} a_{\mathbb{K}}(n^2)^l z^{\omega(n)} n^{-s}$, $z \in \mathbb{C}$, $l \in \mathbb{Z}^+$, into the critical strip $0 < \Re s < 1$. We prove the asymptotic estimations in the short intervals of the arithmetic functions $a_{\mathbb{K}}(n)^l z^{\omega(n)}$ and $a_{\mathbb{K}}(n^2)^l z^{\omega(n)}$ in Lemmas 2.3 and 2.4, respectively. These two estimations will play a key role in the proof of Theorems 1.1–1.4.

In particular, let $z = 1$ in Lemma 2.3. We have

$$V_l(x,y) = \sum_{x < n \leqslant x+y} a_{\mathbb{K}}(n)^l = y(\log x)^{2^{l-1}-1}\Big\{\pi_j(1) + O_\varepsilon\Big(\frac{1}{\log x}\Big)\Big\}.$$

This generalizes the result of Lü and Wang (see [8]) in short intervals over quadratic field. And when $l = 2$, we give a more precise asymptotic formula which improves the result of Zhai, see [14].

We can also deduce that

$$U_l(x,y) = \sum_{x<n\leqslant x+y} a_{\mathbb{K}}(n^2)^l = y(\log x)^{\beta}\Big\{\lambda_j(1) + O_\varepsilon\Big(\frac{1}{\log x}\Big)\Big\}$$

by taking $z = 1$ in Lemma 2.4. That is, we generalize Lü and Yang's result (see [9]) to the case of short intervals over a quadratic field.

2. Some preliminary lemmas

In this section, let us fix some notations:

- ▷ ε is an arbitrarily small positive constant,
- ▷ $r \in \mathbb{N}$, $\alpha > 0$, $\delta \geqslant 0$, $A \geqslant 0$, $M > 0$ (constants),
- ▷ $\boldsymbol{z} := (z_1, \ldots, z_r) \in \mathbb{C}^r$ and $\boldsymbol{\omega} := (\omega_1, \ldots, \omega_r) \in \mathbb{C}^r$,
- ▷ $\boldsymbol{\kappa} := (\kappa_1, \ldots, \kappa_r) \in (\mathbb{R}^{+*})^r$ with $1 \leqslant \kappa_1 < \ldots < \kappa_r \leqslant 2\kappa_1$,
- ▷ $\boldsymbol{\chi} := (\chi_1, \ldots, \chi_r)$ and the χ_i are non principal Dirichlet characters,
- ▷ $\mathbf{B} := (\mathbf{B_1}, \ldots, \mathbf{B_r}) \in (\mathbb{R}^{+*})^{\mathbf{r}}$ and $\mathbf{C} := (\mathbf{C_1}, \ldots, \mathbf{C_r}) \in (\mathbb{R}^{+*})^{\mathbf{r}}$,
- ▷ the notation $|\boldsymbol{z}| \leqslant \mathbf{B}$ means that $|z_i| \leqslant B_i$ for $1 \leqslant i \leqslant r$.

Assume $f(n)$ is an arithmetic function and its associated Dirichlet series is defined by

$$\mathcal{F}(s) := \sum_{n\geqslant 1} f(n) n^{-s}.$$

The Dirichlet series $\mathcal{F}(s)$ is called of type $\mathscr{P}(\boldsymbol{\kappa}, \boldsymbol{z}, \boldsymbol{\omega}, \mathbf{B}, \mathbf{C}, \alpha, \delta, \mathbf{A}, \mathbf{M})$ if it satisfies the following conditions:

(a) For any $\varepsilon > 0$ we have

$$|f(n)| \ll_\varepsilon M n^{\varepsilon}, \quad n \geqslant 1,$$

where the implied constant depends only on ε.

(b) We have

$$\sum_{n\geqslant 1} |f(n)| n^{-\sigma} \leqslant M\Big(\sigma - \frac{1}{\kappa_1}\Big)^{-\alpha}, \quad \sigma > \frac{1}{\kappa_1}.$$

(c) The Dirichlet series $\mathcal{F}(s)$ has the expression

$$\mathcal{F}(s) = \boldsymbol{\zeta}(\boldsymbol{\kappa} s)^{\boldsymbol{z}} \boldsymbol{L}(\boldsymbol{\kappa} s; \boldsymbol{\chi})^{\boldsymbol{\omega}} G(s),$$

where

$$\boldsymbol{\zeta}(\boldsymbol{\kappa} s)^{\boldsymbol{z}} := \prod_{1\leqslant i\leqslant r} \zeta(\kappa_i s)^{z_i}, \quad \boldsymbol{L}(\boldsymbol{\kappa} s; \boldsymbol{\chi})^{\boldsymbol{\omega}} := \prod_{1\leqslant i\leqslant r} L(\kappa_i s, \chi_i)^{\omega_i}$$

and the Dirichlet series $G(s)$ is a holomorphic function in (some open set containing) $\sigma \geqslant (2\kappa_1)^{-1}$. Moreover, in this region, $G(s)$ satisfies the bound

$$|G(s)| \leqslant M(|\tau|+1)^{\max\{\delta(1-\kappa_1\sigma),0\}} \log^A(|\tau|+1)$$

uniformly for $|\boldsymbol{z}| \leqslant \boldsymbol{B}$ and $|\boldsymbol{\omega}| \leqslant \boldsymbol{C}$. In the sequel, we implicitly define the real numbers σ and τ through the relation $s = \sigma + \mathrm{i}\tau$ and choose the principal value of the complex logarithm.

Usually, we remember that $N(\sigma, T)$ is the number of zeros of $\zeta(s)$ in the region $\Re s \geqslant \sigma$ and $|\Im m s| \leqslant T$. It is well known that there are two constants ψ and η such that

$$N(\sigma, T) \ll T^{\psi(1-\sigma)}(\log T)^\eta$$

for $\frac{1}{2} \leqslant \sigma \leqslant 1$ and $T \geqslant 2$. Huxley in [5] showed that $\psi = \frac{12}{5}$ and $\eta = 9$ are admissible.

The following result is Corollary 1.2 of [11], which constitutes one of the key tools.

Lemma 2.1. *If the Dirichlet series $\mathcal{F}(s)$ is of type $\mathscr{P}(\boldsymbol{\kappa}, \boldsymbol{z}, \boldsymbol{\omega}, \mathbf{B}, \mathbf{C}, \alpha, \delta, \mathbf{A}, \mathbf{M})$, then for any $\varepsilon > 0$ we have*

$$\sum_{x < n \leqslant x + x^{1-1/\kappa_1} y} f(n) = y'(\log x)^{z-1}\Big\{\lambda_0(\boldsymbol{\kappa}, \boldsymbol{z}, \boldsymbol{\omega}, \boldsymbol{\chi}) + O\Big(\frac{M}{\log x}\Big)\Big\}$$

uniformly for $x \geqslant 3$, $x^{(1-(\psi+\delta)^{-1})/\kappa_1+\varepsilon} \leqslant y \leqslant x^{1/\kappa_1}$, $|\boldsymbol{z}| \leqslant \boldsymbol{B}$, $|\boldsymbol{\omega}| \leqslant \boldsymbol{C}$, where

$$y' := \kappa_1((x + x^{1-1/\kappa_1} y)^{1/\kappa_1} - x^{1/\kappa_1}),$$
$$\lambda_0(\boldsymbol{\kappa}, \boldsymbol{z}, \boldsymbol{\omega}, \boldsymbol{\chi}) := \frac{G(1/\kappa_1)}{\kappa_1^{z_1}\Gamma(z_1)} \prod_{2\leqslant i\leqslant r} \zeta\Big(\frac{\kappa_i}{\kappa_1}\Big)^{z_i} \prod_{1\leqslant i\leqslant r} L\Big(\frac{\kappa_i}{\kappa_1}, \chi_i\Big)^{\omega_i}$$

and the implied constant in the O-term depends only on $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, α, δ and ε. Note that $\psi = \frac{12}{5}$ is admissible.

We will give a formula for $a_{\mathbb{K}}(n)$ which will be used in Lemma 2.3.

Lemma 2.2. *Let $\mathbb{K}/\mathbb{Q}$ be a Galois extension of degree q, m be an integer satisfying $(m, q) = r$. Then*

$$a_{\mathbb{K}}(p^m) = \begin{cases} \dbinom{(q+m)/f - 1}{m/f} & \text{if } f \mid r, \\ 0 & \text{if } f \nmid r \end{cases}$$

holds true for all unramified primes p.

P r o o f. See Lemma 2.2 in [9]. □

Lemma 2.3. *Let $B > 0$ and $\varepsilon \in (0, \frac{5}{12})$. Then we have*

$$\sum_{x<n\leqslant x+y} a_{\mathbb{K}}(n)^l z^{\omega(n)} = y(\log x)^{2^{l-1}z-1}\Big\{z\pi_j(z) + O_{B,\varepsilon}\Big(\frac{1}{\log x}\Big)\Big\} \tag{2.1}$$

uniformly for $x \geqslant 3$, $x^{7/12+\varepsilon} \leqslant y \leqslant x$ and $|z| \leqslant B$, where $\pi_j(z)$ $(j = 1, 2, 3)$ are defined as in (1.5). *In particular, taking $z = 1$ we get*

$$V_l(x,y) = \sum_{x<n\leqslant x+y} a_{\mathbb{K}}(n)^l = y(\log x)^{2^{l-1}-1}\Big\{\pi_j(1) + O_\varepsilon\Big(\frac{1}{\log x}\Big)\Big\},\ j = 1, 2, 3 \tag{2.2}$$

uniformly for $x \geqslant 3$ and $x^{7/12+\varepsilon} \leqslant y \leqslant x$, where $\pi_j(z)$ are defined in Theorem 1.2.

Proof. *Case 1*: $d \equiv 1 \pmod 4$.

Subcase 1.1: $d \equiv 1 \pmod 8$.

Since the function $a_{\mathbb{K}}(n)$ is multiplicative for $\Re s > 1$ we can write

$$\begin{aligned}
\mathcal{F}_1(s;z) &:= \sum_{n\geqslant 1} \frac{a_{\mathbb{K}}(n)^l z^{\omega(n)}}{n^s} = \prod_p \Big(1 + \frac{za_{\mathbb{K}}(p)^l}{p^s} + \frac{za_{\mathbb{K}}(p^2)^l}{p^{2s}} + \frac{za_{\mathbb{K}}(p^3)^l}{p^{3s}} + \ldots\Big)\\
&= \Big(1 + \frac{2^l z}{2^s} + \frac{3^l z}{2^{2s}} + \frac{4^l z}{2^{3s}} + \ldots\Big) \times \prod_{p\mid d(\mathbb{K})} \Big(1 + \frac{z}{p^s} + \frac{z}{p^{2s}} + \frac{z}{p^{3s}} + \ldots\Big)\\
&\quad \times \prod_{\substack{p\nmid d(\mathbb{K})\\ p\geqslant 3\\ (d/p)=1}} \Big(1 + \frac{2^l z}{p^s} + \frac{3^l z}{p^{2s}} + \frac{4^l z}{p^{3s}} + \ldots\Big)\\
&\quad \times \prod_{\substack{p\nmid d(\mathbb{K})\\ p\geqslant 3\\ (d/p)=-1}} \Big(1 + \frac{z}{p^{2s}} + \frac{z}{p^{4s}} + \frac{z}{p^{6s}} + \ldots\Big)\\
&= \Big(1 + \sum_{v\geqslant 1} \frac{(v+1)^l z}{2^{vs}}\Big) \times \prod_{\substack{p\nmid d(\mathbb{K})\\ p\geqslant 3\\ (d/p)=1}} \Big(1 + \sum_{v\geqslant 1} \frac{(v+1)^l z}{p^{vs}}\Big)\\
&\quad \times \prod_{\substack{p\nmid d(\mathbb{K})\\ p\geqslant 3\\ (d/p)=-1}} \Big(1 + \frac{z}{p^{2s}-1}\Big) \times \prod_{p\mid d(\mathbb{K})} \Big(1 + \frac{z}{p^s - 1}\Big).
\end{aligned}$$

Let χ' be the real primitive Dirichlet character of modulo $|d(\mathbb{K})|$ and $L(s, \chi')$ be the Dirichlet L-function corresponding to χ'. According to the discusssion in Bump's book (see [1], Chapter 1, Section 7), we have

$$a_{\mathbb{K}}(p) = \sum_{m\mid p} \chi'(m) = 1 + \chi'(p).$$

Then for $\Re s > 1$ we have

$$L(s,\chi') = \prod_{p}\Big(1-\frac{\chi'(p)}{p^s}\Big)^{-1} = \Big(1-\frac{1}{2^s}\Big)^{-1}\prod_{\substack{p\nmid d(\mathbb{K})\\ p\geqslant 3\\ (d/p)=1}}\Big(1-\frac{1}{p^s}\Big)^{-1}\prod_{\substack{p\nmid d(\mathbb{K})\\ p\geqslant 3\\ (d/p)=-1}}\Big(1+\frac{1}{p^s}\Big)^{-1}.$$

For simplicity, we put

$$\gamma = \frac{3^l-1}{2}$$

in what follows. Thus, we can write

$$F_1(s;z) = \zeta(s)^{2^{l-1}z}\zeta(2s)^{(\beta-2^{l-1})z-2^{2l-2}z^2}L(s,\chi')^{2^{l-1}z}L(2s,\chi')^{\gamma z-2^{2l-2}z^2}G(s;z),$$

where $G(s;z) = G_1(s;z)G_2(s;z)G_3(s;z)G_4(s;z)$ and

$$G_1(s;z) := \Big(1+\sum_{v\geqslant 1}\frac{(v+1)^l z}{2^{vs}}\Big)\Big(1-\frac{1}{2^s}\Big)^{2^l z}\Big(1-\frac{1}{2^{2s}}\Big)^{(3^l-2^{l-1})z-2^{2l-1}z^2},$$

$$G_2(s;z) := \prod_{\substack{p\nmid d(\mathbb{K})\\ p\geqslant 3\\ (d/p)=1}}\Big(1+\sum_{v\geqslant 1}\frac{(v+1)^l z}{p^{vs}}\Big)\Big(1-\frac{1}{p^s}\Big)^{2^l z}\Big(1-\frac{1}{p^{2s}}\Big)^{(3^l-2^{l-1})z-2^{2l-1}z^2},$$

$$G_3(s;z) := \prod_{\substack{p\nmid d(\mathbb{K})\\ p\geqslant 3\\ (d/p)=-1}}\Big(1+\frac{z}{p^{2s}-1}\Big)\Big(1+\frac{1}{p^{2s}}\Big)^{-z}\Big(1-\frac{1}{p^{4s}}\Big)^{\beta z-2^{2l-2}z^2},$$

$$G_4(s;z) := \prod_{p\mid d(\mathbb{K})}\Big(1+\frac{z}{p^s-1}\Big)\Big(1-\frac{1}{p^s}\Big)^{2^{l-1}z}\Big(1-\frac{1}{p^{2s}}\Big)^{(\beta-2^{l-1})z-2^{2l-2}z^2}.$$

It is clear that there is a positive constant $M_1 = M_1(B)$ depending on B such that

$$|G_1(s;z)| \leqslant M_1 \tag{2.3}$$

uniformly for $\Re s \geqslant \frac{1}{2}$ and $|z| \leqslant B$.

The Euler product $G_2(s;z)$ is expandable as a Dirichlet series

$$G_2(s;z) = \sum_{n\geqslant 1}\frac{g_2(n)}{n^s} = \prod_{\substack{p\nmid d(\mathbb{K})\\ p\geqslant 3\\ (d/p)=1}}\Big(1+\sum_{v\geqslant 1}\frac{g_2(p^v)}{p^{vs}}\Big),$$

where $g_2(n)$ is the multiplicative function whose values on prime powers are given by the identity

$$1+\sum_{v\geqslant 1}g_2(p^v)\xi^v = \Big(1+\sum_{v\geqslant 1}(v+1)^l z\xi^v\Big)(1-\xi)^{2^l z}(1-\xi^2)^{(3^l-2^{l-1})z-2^{2l-1}z^2} \quad (|\xi|<1).$$

In particular, for all the prime numbers that satisfy the conditions $p \nmid d(\mathbb{K})$, $p \geqslant 3$ and $(d/p) = 1$, we have

$$g_2(p) = g_2(p^2) = 0 \tag{2.4}$$

and

(2.5)

$$\begin{aligned}|g_2(p^v)| &= \left|\frac{1}{2\pi \mathrm{i}} \oint_{|\xi|=2^{-1/6}} \Big(1 + \sum_{v\geqslant 1} (v+1)^l z\xi^v\Big) \frac{(1-\xi)^{2^l z}(1-\xi^2)^{(3^l-2^{l-1})z-2^{2l-1}z^2}}{\xi^{v+1}}\,\mathrm{d}\xi\right| \\ &\leqslant M_0(B) 2^{v/6} \quad (v \geqslant 3,\ |z| \leqslant B)\end{aligned}$$

with

$$M_0(B) := \max_{|z|\leqslant B} \max_{|\xi|=2^{-1/6}} \left|\Big(1 + \sum_{v\geqslant 1}(v+1)^l z\xi^v\Big)(1-\xi)^{2^l z}(1-\xi^2)^{(3^l-2^{l-1})z-2^{2l-1}z^2}\right|.$$

In view of (2.4), we easily deduce that

$$\log|G_2(s;z)| = \log\left|\prod_{\substack{p\nmid d(\mathbb{K})\\ p\geqslant 3\\ (d/p)=1}} \Big(1 + \sum_{v\geqslant 3} \frac{g_2(p^v)}{p^{vs}}\Big)\right| \leqslant \sum_{\substack{p\nmid d(\mathbb{K})\\ p\geqslant 3\\ (d/p)=1}} \sum_{v\geqslant 3} \frac{|g_2(p^v)|}{p^{v\sigma}}.$$

Therefore, when $\sigma > \frac{1}{3}$ and $|z| \leqslant B$, by (2.5), we have

$$\log|G_2(s;z)| \leqslant M_0(B) \sum_{\substack{p\nmid d(\mathbb{K})\\ p\geqslant 3\\ (d/p)=1}} \sum_{v\geqslant 3} (2^{(1/6)} p^{-\sigma})^v \leqslant \frac{M_0(B)2^{1/2}}{1-2^{-1/6}} \sum_{\substack{p\nmid d(\mathbb{K})\\ p\geqslant 3\\ (d/p)=1}} \frac{1}{p^{3\sigma}}.$$

This shows that the Euler product $G_2(s;z)$ converges absolutely for $\sigma > \frac{1}{3}$ and

$$|G_2(s;z)| \leqslant M_2 \quad (\sigma \geqslant \tfrac{1}{2},\ |z| \leqslant B) \tag{2.6}$$

with

$$M_2 = M_2(B) := \exp\Bigg(\frac{M_0(B)2^{1/2}}{1-2^{-1/6}} \sum_{\substack{p\nmid d(\mathbb{K})\\ p\geqslant 3\\ (d/p)=1}} \frac{1}{p^{3/2}}\Bigg).$$

Similarly, we can prove that there is a positive constant $M_3 = M_3(B)$ such that

$$|G_3(s;z)| \leqslant M_3 \quad (\sigma \geqslant \tfrac{1}{2},\ |z| \leqslant B), \tag{2.7}$$

and since the ramified primes are finite, we can deduce that

$$|G_4(s;z)| \leqslant M_4 \quad (\sigma \geqslant \tfrac{1}{2},\ |z| \leqslant B), \tag{2.8}$$

where $M_4 = M_4(B)$ is a positive constant.

Combining (2.3), (2.6), (2.7) and (2.8), we have

$$|G(s;z)| \leqslant M_1M_2M_3M_4 =: M$$

for $\Re s \geqslant \frac{1}{2}$ and $|z| \leqslant B$.

From the above proof we get:

(a) For any $\varepsilon > 0$,

$$|z^{\omega(n)}| \leqslant B^{(1+o(1))\log n/\log \log n} = n^{(1+o(1))\log B/\log\log n} \ll n^{\varepsilon/2}.$$

Chandrasekharan and Narasimhan in [3] proved that $a_{\mathbb{K}}(n)$ is a multiplicative function and $a_{\mathbb{K}}(n) \ll (d(n))^{q-1}$. When $\mathbb{K} = \mathbb{Q}(\sqrt{d})$, we have $a_{\mathbb{K}}(n)^l \ll d(n)^l \ll n^{\varepsilon/2}$. Then

$$|a_{\mathbb{K}}(n)^l z^{\omega(n)}| \ll_\varepsilon Mn^\varepsilon \quad (n \geqslant 1).$$

(b)

$$\sum_{n\geqslant 1} \frac{|a_{\mathbb{K}}(n)^l z^{\omega(n)}|}{n^\sigma} \leqslant M(\sigma-1)^{-2^{l-1}|z|} \quad (\sigma > 1).$$

(c) The Dirichlet series $G(s;z)$ satisfies the bound $|G(s;z)| \leqslant M$ for $\Re s \geqslant \frac{1}{2}$ and $|z| \leqslant B$.

This shows that the Dirichlet series $\mathcal{F}_1(s;z)$ is of type $\mathscr{P}(\boldsymbol{\kappa}, \boldsymbol{z}, \boldsymbol{\omega}, \boldsymbol{B}, \boldsymbol{C}, 2^{l-1}|z|, 0, 0, M)$ with $\boldsymbol{\kappa} = (1,2)$, $\boldsymbol{z} = (2^{l-1}z, (\beta-2^{l-1})z - 2^{2l-2}z^2)$, $\boldsymbol{\omega} = (2^{l-1}z, \gamma z - 2^{2l-2}z^2)$, $\boldsymbol{B} = (2^{l-1}B, (\beta-2^{l-1})B + 2^{2l-2}B^2)$, $\boldsymbol{C} = (2^{l-1}B, \gamma B + 2^{2l-2}B^2)$, $\boldsymbol{\chi} = (\chi', \chi')$.

Thus, substituting $\psi = \frac{12}{5}$ into Lemma 2.1, we get the case where $j = 1$ in the required asymptotic formula (2.1).

Subcase 1.2: $d \equiv 5 \pmod 8$

$$\begin{aligned}\mathcal{F}_2(s;z) :=& \left(1+\frac{z}{2^{2s}-1}\right) \prod_{\substack{p\nmid d(\mathbb{K})\\ p\geqslant 3\\ (d/p)=1}} \left(1+\sum_{v\geqslant 1}\frac{(v+1)^l z}{p^{vs}}\right) \prod_{\substack{p\nmid d(\mathbb{K})\\ p\geqslant 3\\ (d/p)=-1}} \left(1+\frac{z}{p^{2s}-1}\right)\\ &\times \prod_{p\mid d(\mathbb{K})}\left(1+\frac{z}{p^s-1}\right)\\ =& \zeta(s)^{2^{l-1}z}\zeta(2s)^{(\beta-2^{l-1})z-2^{2l-2}z^2} L(s,\chi')^{2^{l-1}z} L(2s,\chi')^{\gamma z - 2^{2l-2}z^2} G'(s;z),\end{aligned}$$

where

$$\begin{aligned}G'(s;z) &= \left(1+\frac{z}{2^{2s}-1}\right)\left(1+\frac{1}{2^{2s}}\right)^{-z}\left(1-\frac{1}{2^{4s}}\right)^{\beta z - 2^{2l-2}z^2} G_2(s;z)G_3(s;z)G_4(s;z)\\ &= G_1'(s;z)G_2(s;z)G_3(s;z)G_4(s;z).\end{aligned}$$

Similarly to the estimation of $G_1(s;z)$, we can prove that there is a positive constant $M_1' = M_1'(B)$ such that

$$|G_1'(s;z)| \leqslant M_1' \quad (\sigma \geqslant \tfrac{1}{2},\, |z| \leqslant B). \tag{2.9}$$

In Subcase 1.1, we have shown that

$$|G_2(s;z)| \leqslant M_2, \quad |G_3(s;z)| \leqslant M_3, \quad |G_4(s;z)| \leqslant M_4 \quad (\sigma \geqslant \tfrac{1}{2},\, |z| \leqslant B).$$

Thus, we can deduce that

$$|G'(s;z)| \leqslant M_1' M_2 M_3 M_4 =: M'$$

for $\Re s \geqslant \frac{1}{2}$ and $|z| \leqslant B$.

This shows that the Dirichlet series $\mathcal{F}_2(s;z)$ is of type $\mathscr{P}(\boldsymbol{\kappa}, \boldsymbol{z}, \boldsymbol{\omega}, \boldsymbol{B}, \boldsymbol{C}, 2^{l-1}|z|, 0, 0, M')$ and the values of $\boldsymbol{\kappa}$, $\boldsymbol{z}$, $\boldsymbol{\omega}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{\chi}$ are the same as in Subcase 1.1.

Thus, substituting $\psi = \frac{12}{5}$ into Lemma 2.1, we get the case, where $j = 2$ in the required asymptotic formula (2.1).

Case 2: $d \equiv 2, 3 \pmod 4$

$$\begin{aligned}\mathcal{F}_3(s;z) &:= \prod_{\substack{p\nmid d(\mathbb{K})\\ p\geqslant 3\\ (d/p)=1}} \Big(1 + \sum_{v\geqslant 1} \frac{(v+1)^l z}{p^{vs}}\Big) \prod_{\substack{p\nmid d(\mathbb{K})\\ p\geqslant 3\\ (d/p)=-1}} \Big(1 + \frac{z}{p^{2s}-1}\Big) \prod_{p\mid d(\mathbb{K})} \Big(1 + \frac{z}{p^s - 1}\Big)\\ &= \zeta(s)^{2^{l-1}z} \zeta(2s)^{(\beta - 2^{l-1})z - 2^{2l-2}z^2} L(s,\chi')^{2^{l-1}z} L(2s,\chi')^{\gamma z - 2^{2l-2}z^2} G''(s;z),\end{aligned}$$

where

$$G''(s;z) = G_2(s;z) G_3(s;z) G_4(s;z).$$

Thus, we can deduce that

$$|G''(s;z)| \leqslant M_2 M_3 M_4 =: M''$$

for $\Re s \geqslant \frac{1}{2}$ and $|z| \leqslant B$.

This shows that the Dirichlet series $\mathcal{F}_3(s;z)$ is of type $\mathscr{P}(\boldsymbol{\kappa}, \boldsymbol{z}, \boldsymbol{\omega}, \boldsymbol{B}, \boldsymbol{C}, 2^{l-1}|z|, 0, 0, M'')$ and the values of $\boldsymbol{\kappa}$, $\boldsymbol{z}$, $\boldsymbol{\omega}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{\chi}$ are the same as in Subcase 1.1.

Thus, substituting $\psi = \frac{12}{5}$ into Lemma 2.1, we get the case, where $j = 3$ in the required asymptotic formula (2.1). □

Lemma 2.4. *Let $B > 0$ and $\varepsilon \in (0, \frac{5}{12})$. Then we have*

$$\sum_{x<n\leqslant x+y} a_{\mathbb{K}}(n^2)^l z^{\omega(n)} = y(\log x)^{\beta z-1}\Big\{z\lambda_j(z) + O_{B,\varepsilon}\Big(\frac{1}{\log x}\Big)\Big\} \tag{2.10}$$

uniformly for $x \geqslant 3$, $x^{7/12+\varepsilon} \leqslant y \leqslant x$ and $|z| \leqslant B$, where $\lambda_j(z)$ $(j = 1, 2, 3)$ are defined as in (1.9). *In particular, taking $z = 1$ we get the following result:*

$$U_l(x,y) = \sum_{x<n\leqslant x+y} a_{\mathbb{K}}(n^2)^l = y(\log x)^{\gamma}\Big\{\lambda_j(1) + O_{\varepsilon}\Big(\frac{1}{\log x}\Big)\Big\}, \quad j = 1, 2, 3 \tag{2.11}$$

uniformly for $x \geqslant 3$ and $x^{7/12+\varepsilon} \leqslant y \leqslant x$, where $\lambda_j(z)$ are defined in Theorem 1.4.

Since the proofs of Lemmas 2.3 and 2.4 are very similar, the proof of Lemma 2.4 can be derived by analogy with Lemma 2.3.

The next lemma is the Berry-Esseen inequality, see [10], Theorem II.7.14.

Lemma 2.5. *Let F, G be two distribution functions with respective characteristic functions f and g. Suppose that G is differentiable and that G' is bounded on $\mathbb{R}$. Then we have*

$$\|F - G\|_{\infty} \leqslant 16\frac{\|G'\|_{\infty}}{T} + 6\int_{-T}^{T}\Big|\frac{f(\tau) - g(\tau)}{\tau}\Big|\,\mathrm{d}\tau$$

for all $T > 0$, where $\|F\|_{\infty} := \sup_{\lambda\in\mathbb{R}} |F(\lambda)|$.

3. Proof of Theorem 1.2

By the definition of $V_{k,l}(x,y)$ in (1.3), we have

$$\sum_{x<n\leqslant x+y} a_{\mathbb{K}}(n)^l z^{\omega(n)} = \sum_k V_{k,l}(x,y)z^k.$$

Applying Cauchy formula, it can be deduced that

$$V_{k,l}(x,y) = \frac{1}{2\pi\mathrm{i}}\oint_{|z|=r}\Big(\sum_{x<n\leqslant x+y} a_{\mathbb{K}}(n)^l z^{\omega(n)}\Big)\frac{\mathrm{d}z}{z^{k+1}},$$

where $r := k/2^{l-1}\log\log x$.

By Lemma 2.3, it follows that

$$V_{k,l}(x,y) = \frac{y}{\log x} J_{k,l}(x;r) + O_{l,B,\varepsilon}\Big(\frac{y}{(\log x)^2}\frac{(2^{l-1}\log\log x)^k}{k!}\Big) \tag{3.1}$$

uniformly for $x \geqslant 3$, $x^{7/12+\varepsilon} \leqslant y \leqslant x$ and $1 \leqslant k \leqslant 2^{l-1}B\log\log x$, where

$$J_{k,l}(x;r) := \frac{1}{2\pi \mathrm{i}}\oint_{|z|=r} \frac{(\log x)^{2^{l-1}z}\pi_j(z)}{z^k}\,\mathrm{d}z, \quad j=1,2,3.$$

Using the Stirling formula, the error term of $V_{k,l}(x,y)$ in (3.1) can be estimated as

$$\begin{aligned}
&\ll \oint_{|z|=r} \frac{y}{(\log x)^2}(\log x)^{\Re(2^{l-1}z)}\frac{|\mathrm{d}z|}{|z|^{k+1}} \ll \frac{y}{(\log x)^2}\frac{1}{r^k}\int_0^{2\pi} \mathrm{e}^{k\cos\theta}\,\mathrm{d}\theta \\
&\ll \frac{y}{(\log x)^2}\frac{1}{r^k}\bigg(\int_0^{\pi/2} \mathrm{e}^{k\cos\theta}\,\mathrm{d}\theta + 1\bigg) \quad (t = k(1-\cos\theta)) \\
&\ll \frac{y}{(\log x)^2}\frac{1}{r^k}\bigg(\frac{\mathrm{e}^k}{\sqrt{k}}\int_0^k \mathrm{e}^{-t}t^{-1/2}\,\mathrm{d}t + 1\bigg) \ll \frac{(2^{l-1}\log\log x)^k}{k!}\frac{y}{(\log x)^2}.
\end{aligned}$$

Then we will estimate the $J_{k,l}(x;r)$ of the main term in the case of $k=1$ and $k \geqslant 2$, separately.

When $k=1$, since $z \mapsto \pi_j(z)$ $(j=1,2,3)$ is analytic for $|z| \leqslant B$, we have

$$J_{1,l}(x;r) = \frac{1}{2\pi \mathrm{i}}\oint_{|z|=r} \frac{(\log x)^{2^{l-1}z}\pi_j(z)}{z}\,\mathrm{d}z = \pi_j(0) = 2^{l-1}. \tag{3.2}$$

When $k \geqslant 2$, since $z \mapsto \pi_j(z)$ $(j=1,2,3)$ is analytic for $|z| \leqslant B$, we have $J_{k,l}(x;r) = J_{k,l}(x;r_0)$ with $r_0 := (k-1)/(2^{l-1}\log\log x)$ and the Taylor expansion of $\pi_j(z)$ at $z = r_0$:

$$\pi_j(z) = \pi_j(r_0) + \pi_j'(r_0)(z-r_0) + (z-r_0)^2\int_0^1 (1-t)\pi_j''(r_0+t(z-r_0))\,\mathrm{d}t \tag{3.3}$$

we shall estimate the contributions of three terms on the right-hand side of (3.3) as $J_{k,l}(x;r_0)$. Firstly, we will estimate the first two terms separately:

$$\frac{\pi_j(r_0)}{2\pi i}\oint_{|z|=r} \frac{(\log x)^{2^{l-1}z}}{z^k}\,\mathrm{d}z = \frac{(2^{l-1}\log\log x)^{k-1}}{(k-1)!}\pi_j\Big(\frac{k-1}{2^{l-1}\log\log x}\Big), \tag{3.4}$$

and

$$\begin{aligned}
&\frac{\pi_j'(r_0)}{2\pi i}\oint_{|z|=r} \frac{(\log x)^{2^{l-1}z}(z-r_0)}{z^k}\,\mathrm{d}z \\
&\qquad = \pi_j'(r_0)\Big(\frac{(2^{l-1}\log\log x)^{k-2}}{(k-2)!} - r_0\frac{(2^{l-1}\log\log x)^{k-1}}{(k-1)!}\Big) = 0.
\end{aligned} \tag{3.5}$$

For $0 \leqslant t \leqslant 1$ and $|z| = r_0$ we have

$$|r_0 + t(z - r_0)| = |r_0(1-t) + tz| \leqslant r_0(1-t) + t|z| = r_0 \leqslant B.$$

Since $z \mapsto \pi_j(z)$ is analytic for $|z| \leqslant B$, there is a positive constant C_l such that $|\pi_j''(z)| \leqslant C_l$ for $|z| \leqslant B$. Thus, the contribution of the third term on the right-hand side of (3.3) to $J_{k,l}(x; r_0)$ is

$$\begin{aligned}
&\ll \oint_{|z|=r_0} \frac{(\log x)^{\Re(2^{l-1}z)}|\mathrm{e}^{\mathrm{i}\theta} - 1|^2}{|z|^{k-2}} |\,\mathrm{d}z| \ll \frac{1}{r_0^{k-3}} \int_0^{2\pi} \mathrm{e}^{(k-1)\cos\theta}(1-\cos\theta)\,\mathrm{d}\theta \\
&\ll \frac{1}{r_0^{k-3}} \biggl(\int_0^{\pi/2} \mathrm{e}^{(k-1)\cos\theta}(1-\cos\theta)\,\mathrm{d}\theta + 1 \biggr) \quad (t = (k-1)(1-\cos\theta)) \\
&\ll \frac{1}{{r_0}^{k-3}} \biggl(\frac{\mathrm{e}^{k-1}}{(k-1)^{3/2}} \int_0^{k-1} \mathrm{e}^{-t} t^{1/2}\,\mathrm{d}t + 1 \biggr) \ll \frac{(2^{l-1}\log\log x)^{k-1}}{(k-1)!} \frac{k-1}{(2^{l-1}\log\log x)^2}.
\end{aligned}$$

By combining (3.2), (3.4), (3.5) and (3.6), it follows that

$$\begin{aligned}
\text{(3.6)} \qquad J_{k,l}(x; r) = {} & \frac{(2^{l-1}\log\log x)^{k-1}}{(k-1)!} \pi_j \Bigl(\frac{k-1}{2^{l-1}\log\log x} \Bigr) \\
& + O_{l,B,\varepsilon} \Bigl(\frac{k-1}{(2^{l-1}\log\log x)^2} \frac{(2^{l-1}\log\log x)^{k-1}}{(k-1)!} \Bigr)
\end{aligned}$$

for $x \geqslant 3$ and $1 \leqslant k \leqslant 2^{l-1} B \log\log x$.

Therefore, the asymptotic formula (1.4) can be given by (3.1) and (3.7).

4. Proof of Theorem 1.1

Put

$$F_{x,y}(\lambda) := \frac{1}{V_l(x,y)} \sum_{\substack{x < n \leqslant x+y \\ \omega(n) - 2^{l-1}\log\log x \leqslant \lambda(2^{l-1}\log\log x)^{1/2}}} a_{\mathbb{K}}(n)^l.$$

And $\varphi_{x,y}(\tau)$ be the characteristic function of $F_{x,y}(\lambda)$, i.e.,

$$\begin{aligned}
\text{(4.1)} \qquad \varphi_{x,y}(\tau) &:= \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\tau\lambda}\,\mathrm{d}F_{x,y}(\lambda) \\
&= \frac{1}{V_l(x,y)} \sum_{x < n \leqslant x+y} a_{\mathbb{K}}(n)^l \exp\Bigl\{ \mathrm{i}\tau \frac{\omega(n) - 2^{l-1}\log\log x}{(2^{l-1}\log\log x)^{1/2}} \Bigr\} \\
&= \frac{\mathrm{e}^{-\mathrm{i}\tau T}}{V_l(x,y)} \sum_{x < n \leqslant x+y} a_{\mathbb{K}}(n)^l \mathrm{e}^{\mathrm{i}(\tau/T)\omega(n)},
\end{aligned}$$

where $T := (2^{l-1}\log\log x)^{1/2}$.

Let $(F,G)=(F_{x,y},\Phi)$, by using Lemma 2.5, it follows that

$$\|F_{x,y}-\Phi\|_\infty \leqslant \frac{16}{\sqrt{2\pi}T}+6\int_{-T}^{T}\Big|\frac{\varphi_{x,y}(\tau)-\mathrm{e}^{-\tau^2/2}}{\tau}\Big|\,\mathrm{d}\tau.$$

Thus, it suffices to show that

$$\int_{-T}^{T}\Big|\frac{\varphi_{x,y}(\tau)-\mathrm{e}^{-\tau^2/2}}{\tau}\Big|\,\mathrm{d}\tau \ll \frac{1}{T} \tag{4.2}$$

uniformly for $x\geqslant 3$ and $x^{7/12+\varepsilon}\leqslant y\leqslant x$.

Applying Lemma 2.3 with $z=\mathrm{e}^{\mathrm{i}t}$ we have

$$\frac{1}{V_l(x,y)}\sum_{x<n\leqslant x+y} a_{\mathbb{K}}(n)^l\mathrm{e}^{\mathrm{i}t\omega(n)}=(\log x)^{2^{l-1}(\mathrm{e}^{\mathrm{i}t}-1)}A(\mathrm{e}^{\mathrm{i}t})+O_\varepsilon\Big(\frac{1}{\log x}\Big)$$

uniformly for $t\in\mathbb{R}$, $x\geqslant 3$, $x^{7/12+\varepsilon}\leqslant y\leqslant x$, and

$$A(z):=\frac{z\pi_j(z)}{\pi_j(1)}$$

is an entire function of z such that $A(1)=1$, where $\pi_j(z)$ $(j=1,2,3)$ are defined as in (1.5) . Taking $t=\tau/T$, we can deduce that

$$\varphi_{x,y}(\tau)=(\log x)^{2^{l-1}(\mathrm{e}^{\mathrm{i}(\tau/T)}-1)}A(\mathrm{e}^{\mathrm{i}(\tau/T)})\mathrm{e}^{-\mathrm{i}\tau T}+O_\varepsilon\Big(\frac{1}{\log x}\Big) \tag{4.3}$$

uniformly for $x\geqslant 3$, $x^{7/12+\varepsilon}\leqslant y\leqslant x$ and $|\tau|\leqslant T$.

In view of the inequality $\cos t-1\leqslant -2(t/\pi)^2$ $(|t|\leqslant 1)$, we have

$$|(\log x)^{2^{l-1}(\mathrm{e}^{\mathrm{i}(\tau/T)}-1)}|=\mathrm{e}^{(\cos(\tau/T)-1)T^2}\leqslant \mathrm{e}^{-2(\tau/\pi)^2},$$

from which we can conclude that

$$\varphi_{x,y}(\tau)\ll_\varepsilon \mathrm{e}^{-2(\tau/\pi)^2}$$

uniformly for $x\geqslant 3$, $x^{7/12+\varepsilon}\leqslant y\leqslant x$ and $|\tau|\leqslant T$. Thus,

$$\begin{aligned}\int_{\pm T^{1/3}}^{\pm T}\Big|\frac{\varphi_{x,y}(\tau)-\mathrm{e}^{-\tau^2/2}}{\tau}\Big|\,\mathrm{d}\tau &\ll \int_{T^{1/3}}^{T}\mathrm{e}^{-2(\tau/\pi)^2}+\mathrm{e}^{-\tau^2/2}\,\mathrm{d}\tau\\ &\ll \int_{T^{1/3}}^{T}\mathrm{e}^{-2(\tau/\pi)^2}\,\mathrm{d}\tau \ll \frac{1}{T}.\end{aligned} \tag{4.4}$$

By the Taylor developments

$$A(\mathrm{e}^{\mathrm{i}(\tau/T)}) = 1 + O(|\tau|/T), \quad \mathrm{e}^{\mathrm{i}(\tau/T)} - 1 = \mathrm{i}(\tau/T) - \tfrac{1}{2}(\tau/T)^2 + O((|\tau|/T)^3),$$

when $|\tau| \leqslant T^{1/3}$, we can deduce that

$$\begin{aligned}(\log x)^{2^{l-1}(\mathrm{e}^{\mathrm{i}(\tau/T)}-1)} A(\mathrm{e}^{\mathrm{i}(\tau/T)})\mathrm{e}^{-\mathrm{i}\tau T} &= \mathrm{e}^{-\tau^2/2+O(|\tau|^3/T)}\Big\{1 + O\Big(\frac{|\tau|}{T}\Big)\Big\} \\ &= \mathrm{e}^{-\tau^2/2}\Big\{1 + O\Big(\frac{|\tau|^3+|\tau|}{T}\Big)\Big\}.\end{aligned}$$

Inserting this into (4.3), it follows that

$$\varphi_{x,y}(\tau) = \mathrm{e}^{-\tau^2/2}\Big\{1 + O\Big(\frac{|\tau|^3+|\tau|}{T}\Big)\Big\} + O_\varepsilon\Big(\frac{1}{\log x}\Big) \tag{4.5}$$

uniformly for $x \geqslant 3$, $x^{7/12+\varepsilon} \leqslant y \leqslant x$ and $|\tau| \leqslant T^{1/3}$. With the help of this evaluation, we can conclude that

$$\begin{aligned}\int_{\pm 1/\log x}^{\pm T^{1/3}} \Big|\frac{\varphi_{x,y}(\tau) - \mathrm{e}^{-\tau^2/2}}{\tau}\Big| \,\mathrm{d}\tau &\ll \int_{1/\log x}^{T^{1/3}} \Big(\mathrm{e}^{-\tau^2/2}\frac{\tau^2+1}{T} + \frac{1}{\tau \log x}\Big)\,\mathrm{d}\tau \\ &\ll \frac{1}{T} + \frac{\log\log x}{\log x} \ll \frac{1}{T}.\end{aligned} \tag{4.6}$$

For $|\tau| \leqslant (\log x)^{-1}$ we have

$$\Big|\frac{\tau(\omega(n) - 2^{l-1}\log\log x)}{(2^{l-1}\log\log x)^{1/2}}\Big| \ll \frac{|\tau|\log x}{T},$$

thus, we can write

$$\exp\Big\{\mathrm{i}\tau\frac{\omega(n) - 2^{l-1}\log\log x}{(2^{l-1}\log\log x)^{1/2}}\Big\} = 1 + O\Big(\frac{|\tau|\log x}{T}\Big).$$

Inserting this into (4.1), it follows that

$$\varphi_{x,y}(\tau) = 1 + O\Big(\frac{|\tau|\log x}{T}\Big). \tag{4.7}$$

From this and the relation $\mathrm{e}^{-\tau^2/2} = 1 + O(\tau^2)$, we deduce that

$$\int_{-1/\log x}^{1/\log x} \Big|\frac{\varphi_{x,y}(\tau) - \mathrm{e}^{-\tau^2/2}}{\tau}\Big| \,\mathrm{d}\tau \ll \int_{-1/\log x}^{1/\log x} \Big(\frac{\log x}{T} + |\tau|\Big)\,\mathrm{d}\tau \ll \frac{1}{T}. \tag{4.8}$$

Now (4.2) follows from (4.4), (4.6) and (4.8) immediately.

4.1. Optimality of the error term. In this subsection, we will prove that the error term in (1.2) is optimal. For $\lambda \in \mathbb{R}$, define

$$R_\lambda(x,y) := \frac{1}{V_l(x,y)} \sum_{\substack{x<n\leqslant x+y \\ \omega(n)-2^{l-1}\log\log x\leqslant \lambda(2^{l-1}\log\log x)^{1/2}}} a_{\mathbb{K}}(n)^l - \Phi(\lambda),$$

and

$$R(x,y) := \sup_{\lambda\in\mathbb{R}} |R_\lambda(x,y)|.$$

Let $k := [2^{l-1}\log\log x]$ and $\theta := k - 2^{l-1}\log\log x$, then we have

$$\begin{aligned}
\frac{V_{k,l}(x,y)}{V_l(x,y)} &= F_{x,y}\bigg(\frac{\theta}{\sqrt{2^{l-1}\log\log x}}\bigg) - F_{x,y}\bigg(\frac{\theta-1/2\sqrt{2\pi}}{\sqrt{2^{l-1}\log\log x}}\bigg) \\
&\leqslant \Phi\bigg(\frac{\theta}{\sqrt{2^{l-1}\log\log x}}\bigg) - \Phi\bigg(\frac{\theta-1/2\sqrt{2\pi}}{\sqrt{2^{l-1}\log\log x}}\bigg) + 2R(x,y) \\
&\leqslant \int_{(\theta-1/2\sqrt{2\pi})/\sqrt{2^{l-1}\log\log x}}^{\theta/\sqrt{2^{l-1}\log\log x}} \mathrm{e}^{-\tau^2/2}\,\mathrm{d}\tau + 2R(x,y) \\
&\leqslant \frac{1}{2\sqrt{\pi 2^l \log\log x}} + 2R(x,y).
\end{aligned} \tag{4.9}$$

On the other hand, using the Stirling formula, it can be deduced from Theorem 1.2 and formula (2.2) that

$$\frac{V_{k,l}(x,y)}{V_l(x,y)} \sim \frac{(2^{l-1}\log\log x)^{k-1}}{(\log x)^{2^{l-1}}(k-1)!} \sim \frac{1}{\sqrt{\pi 2^l\log\log x}}. \tag{4.10}$$

From (4.9) and (4.10), we derive that

$$R(x,y) \geqslant \frac{1+o(1)}{2\sqrt{\pi 2^l\log\log x}} - \frac{1}{4\sqrt{\pi 2^l\log\log x}} = \frac{1+o(1)}{4\sqrt{\pi 2^l\log\log x}}$$

uniformly for $x \geqslant 3$ and $x^{7/12+\varepsilon} \leqslant y \leqslant x$.

This completes the proof of Theorem 1.1. □

The proofs of Theorems 1.4 and 1.3 are very similar to those of Theorems 1.2 and 1.1, we only point out the differences.

5. Proof of Theorem 1.4

Similarly to (3.1), by Lemma 2.4 we can deduce that

$$U_{k,l}(x,y) = \frac{y}{\log x} I_{k,l}(x;r) + O_{l,B,\varepsilon}\Big(\frac{y}{(\log x)^2}\frac{(\beta\log\log x)^k}{k!}\Big) \tag{5.1}$$

uniformly for $x \geqslant 3$, $x^{7/12+\varepsilon} \leqslant y \leqslant x$ and $1 \leqslant k \leqslant \beta B\log\log x$ (i.e., $r := k/\beta\log\log x$), where

$$I_{k,l}(x;r) := \frac{1}{2\pi \mathrm{i}}\oint_{|z|=r}\frac{(\log x)^{\beta z}\lambda_j(z)}{z^k}\,\mathrm{d}z, \quad j=1,2,3.$$

On the other hand, by the same argument for evaluating $J_{k,l}(x;r)$, we can prove that

$$\begin{aligned} I_{k,l}(x;r) = {} & \frac{(\beta\log\log x)^{k-1}}{(k-1)!}\lambda_j\Big(\frac{k-1}{\beta\log\log x}\Big) \\ & + O_{l,B,\varepsilon}\Big(\frac{k-1}{(\beta\log\log x)^2}\frac{(\beta\log\log x)^{k-1}}{(k-1)!}\Big) \end{aligned} \tag{5.2}$$

for $x \geqslant 3$ and $1 \leqslant k \leqslant \beta B\log\log x$.

Therefore, the asymptotic formula (1.8) can be given by (5.1) and (5.2).

6. Proof of Theorem 1.3

Put

$$G_{x,y}(\lambda) := \frac{1}{U_l(x,y)}\sum_{\substack{x<n\leqslant x+y\\ \omega(n)-\beta\log\log x\leqslant\lambda(\beta\log\log x)^{1/2}}} a_{\mathbb{K}}(n^2)^l.$$

Then its characteristic function is given by

$$\varphi_{x,y}(\tau) := \frac{\mathrm{e}^{-\mathrm{i}\tau T}}{U_l(x,y)}\sum_{x<n\leqslant x+y} a_{\mathbb{K}}(n^2)^l \mathrm{e}^{\mathrm{i}(\tau/T)\omega(n)},$$

where $T := (\beta\log\log x)^{1/2}$.

Similarly to (4.4), (4.6) and (4.8), we can prove that

$$\int_{-T}^{T}\Big|\frac{\varphi_{x,y}(\tau)-\mathrm{e}^{-\tau^2/2}}{\tau}\Big|\,\mathrm{d}\tau \ll \frac{1}{T}. \tag{6.1}$$

Now the required result (1.7) follows from (6.1) thanks to the Berry-Esseen inequality.

Finally, with Theorem 1.4 and (2.11) we can prove, as before, that the error term in (1.7) is optimal.

References

[1] *D. Bump*: Automorphic Forms and Representations. Cambridge Studies in Advanced Mathematics 55. Cambridge University Press, Cambridge, 1997. zbl MR doi

[2] *K. Chandrasekharan, A. Good*: On the number of integral ideals in Galois extensions. Monatsh. Math. *95* (1983), 99–109. zbl MR doi

[3] *K. Chandrasekharan, R. Narasimhan*: The approximate functional equation for a class of zeta-functions. Math. Ann. *152* (1963), 30–64. zbl MR doi

[4] *P. Erdős, M. Kac*: On the Gaussian law of errors in the theory of additive functions. Proc. Natl. Acad. Sci. USA *25* (1939), 206–207. zbl doi

[5] *M. N. Huxley*: On the difference between consecutive primes. Invent. Math. *15* (1972), 164–170. zbl MR doi

[6] *E. Landau*: Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale. B. G. Teubner, Leipzig, 1927. (In German.) zbl MR

[7] *X.-L. Liu, Z.-S. Yang*: Weighted Erdős-Kac type theorems over Gaussian field in short intervals. Acta Math. Hung. *162* (2020), 465–482. zbl MR doi

[8] *G. Lü, Y. Wang*: Note on the number of integral ideals in Galois extensions. Sci. China, Math. *53* (2010), 2417–2424. zbl MR doi

[9] *G. Lü, Z. Yang*: The average behavior of the coefficients of Dedekind zeta function over square numbers. J. Number Theory *131* (2011), 1924–1938. zbl MR doi

[10] *W. G. Nowak*: On the distribution of integer ideals in algebraic number fields. Math. Nachr. *161* (1993), 59–74. zbl MR doi

[11] *J. Wu, Q. Wu*: Mean values for a class of arithmetic functions in short intervals. Math. Nachr. *293* (2020), 178–202. zbl MR doi

[12] *W. Zhai*: Asymptotics for a class of arithmetic functions. Acta Arith. *170* (2015), 135–160. zbl MR doi

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