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ON THE INVARIANCE OF CERTAIN TYPES
OF GENERALIZED COHEN-MACAULAY MODULES UNDER
FOXBY EQUIVALENCE

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Abstract. Let R be a local ring and C a semidualizing module of R . We investigate the behavior of certain classes of generalized Cohen-Macaulay R -modules under the Foxby equivalence between the Auslander and Bass classes with respect to C . In particular, we show that generalized Cohen-Macaulay R -modules are invariant under this equivalence and if M is a finitely generated R -module in the Auslander class with respect to C such that $C \otimes_R M$ is surjective Buchsbaum, then M is also surjective Buchsbaum.

Keywords: Auslander class; Bass class; Buchsbaum module; dualizing module; generalized Cohen-Macaulay module; local cohomology; semidualizing module; surjective Buchsbaum module

MSC 2020: 13C14, 13D05, 13D45

1. INTRODUCTION

Throughout this paper, (R, \mathfrak{m}, k) is a commutative Noetherian local ring with nonzero identity. Let $\mathcal{P}(R)$ and $\mathcal{I}(R)$ denote the full subcategory of finitely generated R -modules of finite projective and injective dimension, respectively.

Let R be Cohen-Macaulay with a dualizing module ω_R . By virtue of [12], Theorem 2.9, there is an equivalence of categories

$$\mathcal{P}(R) \begin{array}{c} \xrightarrow{\omega_R \otimes_R -} \\ \xleftarrow{\text{Hom}_R(\omega_R, -)} \end{array} \mathcal{I}(R).$$

Let $\mathcal{CM}(R)$ denote the full subcategory of Cohen-Macaulay R -modules. By [7], Theorem 3.3 (i), the above equivalence leads to the following one

$$\mathcal{P}(R) \cap \mathcal{CM}(R) \begin{array}{c} \xrightarrow{\omega_R \otimes_R -} \\ \xleftarrow{\text{Hom}_R(\omega_R, -)} \end{array} \mathcal{I}(R) \cap \mathcal{CM}(R).$$

The notion of generalized Cohen-Macaulay modules is one of the most natural extensions of the notion of Cohen-Macaulay modules. Surjective Buchsbaum modules and Buchsbaum modules are special classes of generalized Cohen-Macaulay modules. Kawasaki in [7], Theorem 3.3 (iii) showed that if an R -module $M \in \mathcal{P}(R)$ is such that the R -module $\omega_R \otimes_R M$ is surjective Buchsbaum, then M itself is so. Also, he presented an example to show that the converse does not hold, see [7], Proposition 4.1.

Let (R, \mathfrak{m}, k) be an arbitrary local ring and C a semidualizing module of R . Let $\mathcal{A}_C(R)$ and $\mathcal{B}_C(R)$ denote the Auslander and Bass classes with respect to C , respectively. It is known that $\mathcal{P}(R) \subseteq \mathcal{A}_C(R)$ and $\mathcal{I}(R) \subseteq \mathcal{B}_C(R)$. Also, these classes are closely related to Gorenstein homological dimensions. Namely, if R is Cohen-Macaulay with a dualizing module ω_R , then $\mathcal{A}_{\omega_R}(R)$ or $\mathcal{B}_{\omega_R}(R)$ is precisely the class of all R -modules with finite Gorenstein projective or Gorenstein injective dimension, respectively, see [4], Theorems 4.1 and 4.4. By Foxby equivalence, there is an equivalence of categories:

$$\mathcal{A}_C(R) \begin{array}{c} \xrightarrow{C \otimes_R -} \\ \xleftarrow{\text{Hom}_R(C, -)} \end{array} \mathcal{B}_C(R).$$

In this article, we investigate the behavior of the classes of Gorenstein, maximal Cohen-Macaulay, Cohen-Macaulay, surjective Buchsbaum, Buchsbaum and generalized Cohen-Macaulay R -modules under the later equivalence. More precisely, if $\mathcal{X}(R)$ is any of these classes of modules, we examine the existence of an equivalence

$$\mathcal{A}_C(R) \cap \mathcal{X}(R) \begin{array}{c} \xrightarrow{C \otimes_R -} \\ \xleftarrow{\text{Hom}_R(C, -)} \end{array} \mathcal{B}_C(R) \cap \mathcal{X}(R).$$

To ease the reading of the paper, in Section 2, we will recall the needed definitions and bring some elementary results. In Section 3, we will consider the classes of Gorenstein, maximal Cohen-Macaulay and Cohen-Macaulay R -modules, see Theorem 3.3. In Section 4, which is the core of this paper, we will consider the classes of surjective Buchsbaum, Buchsbaum and generalized Cohen-Macaulay R -modules, see Theorems 4.3, 4.7 and Corollary 4.9. Also, we provide an example of a Cohen-Macaulay local ring R with a dualizing module ω_R and a Buchsbaum R -module $M \in \mathcal{A}_{\omega_R}(R)$ such that $\omega_R \otimes_R M$ is not Buchsbaum, see Example 4.8.

2. PREREQUISITES

We begin with recalling the definitions of weak sequences and filter regular sequences.

Definition 2.1. Let M be a finitely generated R -module and $\underline{x} = x_1, \dots, x_n$ be a sequence of elements of \mathfrak{m} .

(i) Following [15], page 39, we say \underline{x} is a *weak M -sequence* if

$$(\langle x_1, x_2, \dots, x_{i-1} \rangle_M : x_i) \subseteq (\langle x_1, x_2, \dots, x_{i-1} \rangle_M : \mathfrak{m})$$

for every $i = 1, \dots, n$.

(ii) Following [11], we say \underline{x} is a *filter M -regular sequence* if

$$(\langle x_1, x_2, \dots, x_{i-1} \rangle_M : x_i) \subseteq \bigcup_{l \in \mathbb{N}} (\langle x_1, x_2, \dots, x_{i-1} \rangle_M : \mathfrak{m}^l)$$

for every $i = 1, \dots, n$.

By the definition, it is clear that any weak M -sequence is a filter M -regular sequence.

The following lemma is well-known and easily verified.

Lemma 2.2. Let M be a finitely generated R -module and $\underline{x} = x_1, \dots, x_n \in \mathfrak{m}$. The following statements are equivalent:

- (i) \underline{x} is a filter M -regular sequence.
- (ii) For each $1 \leq i \leq r$, the element x_i does not belong to the union of the elements of $\text{Ass}_R(M/\langle x_1, x_2, \dots, x_{i-1} \rangle_M) \setminus \{\mathfrak{m}\}$.
- (iii) $x_1/1, \dots, x_n/1$ is a weak $M_{\mathfrak{p}}$ -regular sequence for every $\mathfrak{p} \in \text{Supp}_R(M) \setminus \{\mathfrak{m}\}$.

In the sequel, we denote the length of an R -module M by $l_R(M)$.

Definition 2.3. Let M be a finitely generated R -module.

(i) We say that M is *Gorenstein* if either $M = 0$ or M is nonzero and $\text{id}_R M = \text{depth}_R M$.

(ii) We say that M is *maximal Cohen-Macaulay* if $\text{depth}_R M = \dim R$.

(iii) We say that M is *surjective Buchsbaum* if the natural map

$$\varphi_M^i: \text{Ext}_R^i(k, M) \rightarrow H_{\mathfrak{m}}^i(M)$$

is surjective for every $i < \dim_R M$.

(iv) We say that M is *Buchsbaum* if every system of parameters x_1, \dots, x_d of M forms a weak M -sequence.

(v) We say that M is *generalized Cohen-Macaulay* if $l_R(H_{\mathfrak{m}}^i(M)) < \infty$ for every $i < \dim_R M$.

Remark 2.4.

- (i) Every Cohen-Macaulay R -module is surjective Buchsbaum.
- (ii) The classes of modules introduced in the above definition are subsets of each other from top to bottom, respectively.
- (iii) If M is a generalized Cohen-Macaulay R -module, then every system of parameters of M is a filter M -regular sequence, see [13], Appendix, Remark 11 and Theorem 14.
- (iv) If R is a quotient of a Cohen-Macaulay local ring, then an R -module M is generalized Cohen-Macaulay if and only if every system of parameters of M is a filter M -regular sequence, see [13], Appendix, Proposition 16.

Definition 2.5. A finitely generated R -module C is called *semidualizing* if it satisfies the following conditions:

- (i) the homothety map $\chi_C^R: R \rightarrow \text{Hom}_R(C, C)$ is an isomorphism and
- (ii) $\text{Ext}_R^i(C, C) = 0$ for all $i > 0$.

Definition 2.6. Let C be a semidualizing module of R .

- (i) The Auslander class $\mathcal{A}_C(R)$ is the class of all R -modules M for which the natural map $\gamma_M^C: M \rightarrow \text{Hom}_R(C, C \otimes_R M)$ is an isomorphism and

$$\text{Tor}_i^R(C, M) = 0 = \text{Ext}_R^i(C, C \otimes_R M)$$

for all $i \geq 1$.

- (ii) The Bass class $\mathcal{B}_C(R)$ is the class of all R -modules M for which the evaluation map $\xi_M^C: C \otimes_R \text{Hom}_R(C, M) \rightarrow M$ is an isomorphism and

$$\text{Ext}_R^i(C, M) = 0 = \text{Tor}_i^R(C, \text{Hom}_R(C, M))$$

for all $i \geq 1$.

3. COHEN-MACAULAY MODULES

In Theorem 3.3, we show that Cohen-Macaulay R -modules are invariant under Foxby equivalence. To prove it, we need the following two lemmas.

Lemma 3.1. *Let C be a semidualizing module of R and M a finitely generated R -module. Then:*

- (i) $\text{Supp}_R C = \text{Spec } R$.
- (ii) $\text{Supp}_R(C \otimes_R M) = \text{Supp}_R M = \text{Supp}_R(\text{Hom}_R(C, M))$. In particular,

$$\dim_R(C \otimes_R M) = \dim_R M = \dim_R(\text{Hom}_R(C, M)).$$

- (iii) $C \otimes_R M \neq 0$ if and only if $M \neq 0$ if and only if $\text{Hom}_R(C, M) \neq 0$.

P r o o f. (i) It is well-known and obvious, because

$$\text{Spec } R = \text{Supp}_R(\text{Hom}_R(C, C)) \subseteq \text{Supp}_R C \subseteq \text{Spec } R.$$

(ii) By (i), one has

$$\text{Supp}_R(C \otimes_R M) = \text{Supp}_R M \cap \text{Supp}_R C = \text{Supp}_R M.$$

On the other hand, [1], page 267, Proposition 10 implies that

$$\text{Ass}_R(\text{Hom}_R(C, M)) = \text{Supp}_R C \cap \text{Ass}_R M = \text{Ass}_R M,$$

and so $\text{Supp}_R M = \text{Supp}_R(\text{Hom}_R(C, M))$.

(iii) is clear by (ii). □

Lemma 3.2. *Let C be a semidualizing module of R and $\underline{x} = x_1, \dots, x_n \in \mathfrak{m}$. Assume that $M \in \mathcal{A}_C(R)$ and $N \in \mathcal{B}_C(R)$ are two finitely generated R -modules. Then:*

- (i) $\text{Ass}_R M = \text{Ass}_R(C \otimes_R M)$ and $\text{Ann}_R M = \text{Ann}_R(C \otimes_R M)$.
- (ii) \underline{x} is a (weak) M -regular sequence if and only if it is a (weak) $C \otimes_R M$ -regular sequence. In particular, $\text{depth}_R M = \text{depth}_R(C \otimes_R M)$.
- (iii) $\text{Ass}_R N = \text{Ass}_R(\text{Hom}_R(C, N))$ and $\text{Ann}_R N = \text{Ann}_R(\text{Hom}_R(C, N))$.
- (iv) \underline{x} is a (weak) N -regular sequence if and only if it is a (weak) $\text{Hom}_R(C, N)$ -regular sequence. In particular, $\text{depth}_R N = \text{depth}_R(\text{Hom}_R(C, N))$.

P r o o f. (i) By [1], page 267, Proposition 10, one has

$$\text{Ass}_R M = \text{Ass}_R(\text{Hom}_R(C, C \otimes_R M)) = \text{Supp}_R C \cap \text{Ass}_R(C \otimes_R M) = \text{Ass}_R(C \otimes_R M).$$

By considering the R -isomorphism $M \cong \text{Hom}_R(C, C \otimes_R M)$, one can easily deduce that $\text{Ann}_R M = \text{Ann}_R(C \otimes_R M)$.

(ii) By Lemma 3.1 (iii), it follows that $M/\underline{x}M \neq 0$ if and only if

$$(C \otimes_R M)/\underline{x}(C \otimes_R M) \neq 0.$$

So, it is enough to show that \underline{x} is a weak M -regular sequence if and only if it is a weak $C \otimes_R M$ -regular sequence. We prove this by induction on n . The case $n = 1$ follows by (i). Note that for a finitely generated R -module L , the union of members of $\text{Ass}_R L$ is the set of all elements of R which are zero-divisor on L .

Next, assume that $n > 1$ and the claim holds for $n - 1$. By the case $n = 1$, without loss of generality, we can and do assume that x_1 is regular on both M and $C \otimes_R M$. Set $\overline{M} := M/x_1M$. Applying [10], Proposition 3.1.7 (a) on the exact sequence

$$0 \rightarrow M \xrightarrow{x_1} M \rightarrow \overline{M} \rightarrow 0$$

yields that $\overline{M} \in \mathcal{A}_C(R)$. By induction hypothesis, x_2, \dots, x_n is a weak \overline{M} -regular sequence if and only if x_2, \dots, x_n is a weak $C \otimes_R \overline{M}$ -regular sequence. Since

$$C \otimes_R \overline{M} \cong (C \otimes_R M)/x_1(C \otimes_R M),$$

the proof is complete.

As $N \in \mathcal{B}_C(R)$, one has $N \cong C \otimes_R \text{Hom}_R(C, N)$ and $\text{Hom}_R(C, N) \in \mathcal{A}_C(R)$. Thus, (iii) and (iv) are immediate by (i) and (ii), respectively. \square

Part (i) of the next result extends [7], Theorem 3.3 (i).

Theorem 3.3. *Let C be a semidualizing module of R and M a finitely generated R -module in $\mathcal{A}_C(R)$. Then:*

- (i) *M is Cohen-Macaulay if and only if $C \otimes_R M$ is Cohen-Macaulay.*
- (ii) *M is maximal Cohen-Macaulay if and only if $C \otimes_R M$ is maximal Cohen-Macaulay.*
- (iii) *If M is Gorenstein, then $C \otimes_R M$ is also Gorenstein.*

Proof. (i) By Lemma 3.1 (ii) and Lemma 3.2 (ii), one has $\dim_R M = \dim_R(C \otimes_R M)$ and $\text{depth}_R M = \text{depth}_R(C \otimes_R M)$. So (i) is immediate.

(ii) It is immediate, because by Lemma 3.2 (ii) we have

$$\text{depth}_R M = \text{depth}_R(C \otimes_R M).$$

(iii) Assume that M is Gorenstein and set $N := C \otimes_R M$. If $M = 0$, then the claim is obvious. So we assume that $M \neq 0$. Then $N \neq 0$ and $N \in \mathcal{B}_C(R)$. As $N \neq 0$, $\text{depth}_R N < \infty$. Now, as $\text{Ext}_R^i(C, N) = 0$ for all $i \geq 1$ and $\text{Hom}_R(C, N) \cong M$ has finite injective dimension, [3], Theorem 8.1 yields that $C \cong R$. Thus, $C \otimes_R M \cong M$, and so $C \otimes_R M$ is Gorenstein. \square

The following example indicates that the converse of Theorem 3.3 (iii) does not hold.

Example 3.4. Let (R, \mathfrak{m}, k) be a Cohen-Macaulay local ring which is not Gorenstein with a dualizing module ω_R . Then $\omega_R \otimes_R R$ is Gorenstein, but R is not Gorenstein.

4. GENERALIZED COHEN-MACAULAY MODULES

In this section, we first prove that generalized Cohen-Macaulay R -modules are invariant under Foxby equivalence. To this end, we need the following two lemmas.

Lemma 4.1. *Let C be a semidualizing module of R , M an n -dimensional finitely generated R -module and $\underline{x} = x_1, \dots, x_n \in \mathfrak{m}$. Then \underline{x} is a system of parameters of M if and only if it is a system of parameters of $C \otimes_R M$.*

Proof. First note that by Lemma 3.1 (ii), we have $\dim_R(C \otimes_R M) = \dim_R M$. Applying Lemma 3.1 (ii) again yields that

$$\begin{aligned} \dim_R((C \otimes_R M)/\langle x_1, x_2, \dots, x_n \rangle(C \otimes_R M)) &= \dim_R(C \otimes_R (M/\langle x_1, x_2, \dots, x_n \rangle M)) \\ &= \dim_R(M/\langle x_1, x_2, \dots, x_n \rangle M). \end{aligned}$$

This completes the argument, because for an n -dimensional finitely generated R -module N , it is known that \underline{x} is a system of parameters of N if and only if $\dim_R(N/\langle \underline{x} \rangle N) = 0$. □

Lemma 4.2. *Let C be a semidualizing module of R . Let M be a finitely generated R -module in $\mathcal{A}_C(R)$ and $\underline{x} = x_1, \dots, x_n \in \mathfrak{m}$. Then \underline{x} is a filter M -regular sequence if and only if \underline{x} is a filter $C \otimes_R M$ -regular sequence.*

Proof. Let $\mathfrak{p} \in \text{Spec } R$. Then by [10], Propositions 2.2.3 and 3.5.3, respectively, $C_{\mathfrak{p}}$ is a semidualizing module of the local ring $R_{\mathfrak{p}}$ and $M_{\mathfrak{p}} \in \mathcal{A}_{C_{\mathfrak{p}}}(R_{\mathfrak{p}})$. Thus, Lemma 3.2 (ii) yields that $x_1/1, \dots, x_n/1$ is a weak $M_{\mathfrak{p}}$ -regular sequence if and only if $x_1/1, \dots, x_n/1$ is a weak $C_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$ -regular sequence. Now, Lemma 2.2 completes the proof. Note that $C_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \cong (C \otimes_R M)_{\mathfrak{p}}$ and by Lemma 3.1 (ii), $\text{Supp}_R M = \text{Supp}_R(C \otimes_R M)$. □

The next result generalizes [7], Theorem 3.3 (ii).

Theorem 4.3. *Let C be a semidualizing module of R and M a finitely generated R -module in $\mathcal{A}_C(R)$. Then M is generalized Cohen-Macaulay if and only if $C \otimes_R M$ is generalized Cohen-Macaulay.*

Proof. By [10], Propositions 2.2.1 and 3.4.7, \widehat{C} is a semidualizing module of \widehat{R} and $\widehat{M} \in \mathcal{A}_{\widehat{C}}(\widehat{R})$. On the other hand, it is routine to check that a finitely generated R -module N is generalized Cohen-Macaulay if and only if the \widehat{R} -module \widehat{N} is so. Also, there is a natural \widehat{R} -isomorphism $\widehat{C} \otimes_{\widehat{R}} \widehat{M} \cong \widehat{C \otimes_R M}$. Thus, we may and do assume that R is complete. Now, the assertion follows by Lemmas 4.1 and 4.2 and Remark 2.4 (iv). Note that by the Cohen Structure Theorem, every complete local ring is a homomorphic image of a regular local ring. □

Let \mathfrak{a} be an ideal of R and M and N two R -modules. The i th generalized local cohomology module of M and N with respect to \mathfrak{a} is defined by $H_{\mathfrak{a}}^i(M, N) := \varinjlim_n \text{Ext}_R^i(M/\mathfrak{a}^n M, N)$, see [6].

Lemma 4.4. *Let \mathfrak{a} be an ideal of R and C a semidualizing module of R . Then for every R -module N in $\mathcal{B}_C(R)$, there is a natural R -isomorphism $H_{\mathfrak{a}}^i(C, N) \cong H_{\mathfrak{a}}^i(\text{Hom}_R(C, N))$ for all $i \geq 0$.*

Proof. For every finitely generated R -module L and every injective R -module E , we claim that $H_{\mathfrak{a}}^i(\text{Hom}_R(L, E)) = 0$ for all $i \geq 1$. To this end, let F_{\circ} be a free resolution of L consisting of finitely generated free R -modules. As the R -module E is injective, it follows that $\text{Hom}_R(F_{\circ}, E)$ is an injective resolution of the R -module $\text{Hom}_R(L, E)$. Since $\Gamma_{\mathfrak{a}}(E)$ is an injective R -module, for each natural integer i , one has

$$H_{\mathfrak{a}}^i(\text{Hom}_R(L, E)) = H^i(\Gamma_{\mathfrak{a}}(\text{Hom}_R(F_{\circ}, E))) = H^i(\text{Hom}_R(F_{\circ}, \Gamma_{\mathfrak{a}}(E))) = 0.$$

Let I° be an injective resolution of N . Since N and all modules in the complex I° belong to $\mathcal{B}_C(R)$, it follows that the functor $\text{Hom}_R(C, -)$ leaves the exact sequence

$$0 \rightarrow N \rightarrow I^0 \rightarrow \dots \rightarrow I^j \rightarrow \dots$$

exact. Hence, $\text{Hom}_R(C, I^{\circ})$ is a $\Gamma_{\mathfrak{a}}$ -acyclic resolution of the R -module $\text{Hom}_R(C, N)$. This implies the first isomorphism in the following display:

$$\begin{aligned} H_{\mathfrak{a}}^i(\text{Hom}_R(C, N)) &\cong H^i(\Gamma_{\mathfrak{a}}(\text{Hom}_R(C, I^{\circ}))) \\ &\cong H^i(\varinjlim_n \text{Hom}_R(R/\mathfrak{a}^n, \text{Hom}_R(C, I^{\circ}))) \\ &\cong \varinjlim_n H^i(\text{Hom}_R(R/\mathfrak{a}^n, \text{Hom}_R(C, I^{\circ}))) \\ &\cong \varinjlim_n H^i(\text{Hom}_R(C/\mathfrak{a}^n C, I^{\circ})) \\ &\cong \varinjlim_n \text{Ext}_R^i(C/\mathfrak{a}^n C, N) = H_{\mathfrak{a}}^i(C, N). \end{aligned}$$

□

Corollary 4.5. *Let C be a semidualizing module of R and M a finitely generated R -module in $\mathcal{A}_C(R)$. Then there is a spectral sequence*

$$\text{Ext}_{\mathfrak{m}}^p(C, H_{\mathfrak{m}}^q(C \otimes_R M)) \Rightarrow H_{\mathfrak{m}}^{p+q}(M).$$

Proof. Let L and N be two finitely generated R -modules. Consider the covariant functors $G(-) := \Gamma_{\mathfrak{m}}(-)$ and $F(-) := \text{Hom}_R(L, -)$ on the category of R -modules and R -homomorphisms. Then for every injective R -module I , the R -module $G(I)$ is injective and $H_{\mathfrak{m}}^i(L, N) \cong R^i(FG)(N)$ for all $i \in \mathbb{N}_0$. Hence, applying [9], Theorem 10.47 on the functors $G(-)$ and $F(-)$ yields a spectral sequence

$$\text{Ext}_R^p(L, H_{\mathfrak{m}}^q(N)) \Rightarrow_p H_{\mathfrak{m}}^{p+q}(L, N).$$

Letting $L := C$ and $N := C \otimes_R M$, by Lemma 4.4, one deduces that

$$H_{\mathfrak{m}}^{p+q}(C, C \otimes_R M) \cong H_{\mathfrak{m}}^{p+q}(\text{Hom}_R(C, C \otimes_R M)) \cong H_{\mathfrak{m}}^{p+q}(M).$$

□

Recall that for each nonnegative integer n , the n th Betti (or Bass) number of an R -module M is defined as $\beta_n^R(M) := \text{Vdim}_k(\text{Tor}_R^n(k, M))$ (or $\mu_R^n(M) := \text{Vdim}_k(\text{Ext}_R^n(k, M))$). If there is no ambiguity about the underlying ring, we then show these invariants with $\beta_n(M)$ and $\mu^n(M)$, respectively.

Lemma 4.6. *Let M be a generalized Cohen-Macaulay R -module with $d := \dim_R M > 0$. Then we have*

$$\mu^n(M) \leq \sum_{j=0}^n \beta_j(k) l_R(H_{\mathfrak{m}}^{n-j}(M))$$

for all $n \leq d$. Furthermore, the following statements are equivalent:

- (i) M is a surjective Buchsbaum R -module.
- (ii) The equality in the above holds for all $n < d$.

Proof. See [7], Lemma 2.2, [8], Corollary 1.14 and [16], Theorem 1.2. □

The next result is a far reach generalization of [7], Theorem 3.3 (iii).

Theorem 4.7. *Let C be a semidualizing module of R and M a finitely generated R -module in $\mathcal{A}_C(R)$. Assume that $C \otimes_R M$ is surjective Buchsbaum. Then M is also surjective Buchsbaum.*

Proof. Remark 2.4 (ii) and Theorem 4.3 imply that M is generalized Cohen-Macaulay. Clearly, we may assume that $d := \dim_R M > 0$. So by Lemma 4.6, we have

$$(*) \quad \mu^n(M) \leq \sum_{j=0}^n \beta_j(k) l_R(H_{\mathfrak{m}}^{n-j}(M))$$

and

$$(**) \quad \mu^n(C \otimes_R M) = \sum_{j=0}^n \beta_j(k) l_R(H_{\mathfrak{m}}^{n-j}(C \otimes_R M))$$

for all $n < d$. Note that by Lemma 3.1 (ii), $\dim_R M = \dim_R(C \otimes_R M)$. Let $0 \leq n < d$ be an integer. Consider the spectral sequence

$$\mathrm{Ext}_R^p(C, \mathrm{H}_m^q(C \otimes_R M)) \Rightarrow \mathrm{H}_m^{p+q}(M),$$

see Corollary 4.5. There is a filtration

$$0 = \mathrm{H}^{-1} \subseteq \mathrm{H}^0 \subseteq \dots \subseteq \mathrm{H}^{n-1} \subseteq \mathrm{H}^n = \mathrm{H}_m^n(M)$$

such that $\mathrm{H}^p / \mathrm{H}^{p-1} \cong \mathrm{E}_\infty^{p, n-p}$ for all $p = 0, \dots, n$. Hence,

$$l_R(\mathrm{H}_m^n(M)) = \sum_{p=0}^n l_R(\mathrm{E}_\infty^{p, n-p}) \leq \sum_{p=0}^n l_R(\mathrm{E}_2^{p, n-p}).$$

Note that $\mathrm{E}_\infty^{p, n-p}$ is a subquotient of $\mathrm{E}_2^{p, n-p}$ for all $p = 0, \dots, n$. Let F_\bullet be a minimal free resolution of C . Then $\mathrm{E}_2^{p, n-p} = \mathrm{Ext}_R^p(C, \mathrm{H}_m^{n-p}(C \otimes_R M))$ is a subquotient of $\mathrm{Hom}_R(F_p, \mathrm{H}_m^{n-p}(C \otimes_R M))$, and so

$$(\dagger) \quad l_R(\mathrm{H}_m^n(M)) \leq \sum_{p=0}^n \beta_p(C) l_R(\mathrm{H}_m^{n-p}(C \otimes_R M)).$$

As $M \cong \mathrm{Hom}_R(C, C \otimes_R M)$ and $\mathrm{Ext}_R^i(C, C \otimes_R M) = 0$ for all $i \geq 1$, we get $M \simeq R \mathrm{Hom}_R(C, C \otimes_R M)$, and so [5], Chapter 13, Result 19 yields that

$$\mu^n(M) = \sum_{p=0}^n \beta_p(C) \mu^{n-p}(C \otimes_R M).$$

Thus,

$$\begin{aligned} \mu^n(M) &\stackrel{(**)}{=} \sum_{p=0}^n \beta_p(C) \left(\sum_{j=0}^{n-p} \beta_j(k) l_R(\mathrm{H}_m^{n-p-j}(C \otimes_R M)) \right) \\ &= \sum_{j=0}^n \beta_j(k) \left(\sum_{p=0}^{n-j} \beta_p(C) l_R(\mathrm{H}_m^{n-j-p}(C \otimes_R M)) \right) \\ &\stackrel{(\dagger)}{\geq} \sum_{j=0}^n \beta_j(k) l_R(\mathrm{H}_m^{n-j}(M)) \stackrel{(*)}{\geq} \mu^n(M). \end{aligned}$$

So,

$$\mu^n(M) = \sum_{j=0}^n \beta_j(k) l_R(\mathrm{H}_m^{n-j}(M))$$

for all $n < d$. Therefore, M is surjective Buchsbaum by Lemma 4.6. \square

Next, we provide an example of a Cohen-Macaulay local ring R with a dualizing module ω_R and a Buchsbaum R -module M of finite projective dimension such that the R -module $\omega_R \otimes_R M$ is not Buchsbaum.

Example 4.8. Let (T, \mathfrak{n}, k) be a 4-dimensional regular local ring with k infinite. Let $\mathfrak{a} \subset \mathfrak{n}^2$ be an ideal of T with 3 generators such that $R := T/\mathfrak{a}$ is a 2-dimensional Cohen-Macaulay local ring which is not Gorenstein. (For an explicit realization, let k be an infinite field, $T := k[[X, Y, Z, W]]$ and $R := k[[X, Y, Z, W]]/\langle X^4 - Y^3, X^5 - Z^3, Y^5 - Z^4 \rangle$.)

The Auslander-Buchsbaum formula yields that $\text{pd}_T R = 2$. Let $\mu(\mathfrak{a})$ denote the minimum number of generators of \mathfrak{a} . Then

$$2 = \dim T - \dim R = \text{hta} \leq \mu(\mathfrak{a}) \leq 3.$$

We claim that $\mu(\mathfrak{a}) = 3$. On the contrary, suppose that \mathfrak{a} can be generated by two elements z_1 and z_2 . Then by [2], Corollary 1.6.19, z_1, z_2 is a T -regular sequence, and so R is Gorenstein. From this contradiction we conclude that $\mu(\mathfrak{a}) = 3$.

As $\mu(\mathfrak{a}) = 3$, the minimal free resolution of the T -module R has the form

$$0 \rightarrow T^n \rightarrow T^3 \rightarrow T \rightarrow 0.$$

We show that $n = 2$. To this end, let K denote the quotient field of the domain T . As $\mathfrak{a} \neq 0$, we immediately see that $K \otimes_T \mathfrak{a} \cong K$. Hence, by applying the exact functor $K \otimes_T -$ to the exact sequence

$$0 \rightarrow T^n \rightarrow T^3 \rightarrow \mathfrak{a} \rightarrow 0,$$

it turns out that $n = 2$.

Let ω_R denote the dualizing module of R . Then by [2], Corollary 3.3.9, the minimal free resolution of the T -module ω_R has the form

$$0 \rightarrow T \rightarrow T^3 \rightarrow T^2 \rightarrow 0.$$

Applying the right exact functor $R \otimes_T -$ to the exact sequence $T^3 \rightarrow T^2 \rightarrow \omega_R \rightarrow 0$ implies the exact sequence $R^3 \rightarrow R^2 \rightarrow \omega_R \rightarrow 0$. Thus, $R^3 \rightarrow R^2 \rightarrow 0$ is the beginning of the minimal free resolution of the R -module ω_R , and so $\beta_0^R(\omega_R) = \beta_0^T(\omega_R) = 2$ and $\beta_1^R(\omega_R) = \beta_1^T(\omega_R) = 3$.

By [7], Proposition 4.1 and its proof, there exists a 2-dimensional surjective Buchsbaum R -module M of finite projective dimension such that the R -module $\omega_R \otimes_R M$ is not surjective Buchsbaum and

$$(4.1) \quad l_T(\mathbf{H}_n^i(\omega_R \otimes_R M)) = \begin{cases} \beta_1^T(R), & i = 1, \\ \beta_0^T(R), & i = 0, \end{cases}$$

and

$$(4.2) \quad l_T(\mathbf{H}_n^i(M)) = \begin{cases} \beta_1^T(\omega_R), & i = 1, \\ \beta_0^T(\omega_R), & i = 0. \end{cases}$$

We claim that the R -module $\omega_R \otimes_R M$ is not even Buchsbaum. Suppose the contrary holds. Then by [13], Chapter I, Lemma 1.6, M and $\omega_R \otimes_R M$ are also Buchsbaum as T -modules. As T is regular, [13], Chapter I, Corollary 2.16 yields that M and $\omega_R \otimes_R M$ are surjective Buchsbaum T -modules. In particular, by Lemma 4.6, one has the following equalities:

$$(4.3) \quad \begin{aligned} \mu_T^1(\omega_R \otimes_R M) &= \beta_0^T(k)l_T(\mathbf{H}_n^1(\omega_R \otimes_R M)) + \beta_1^T(k)l_T(\mathbf{H}_n^0(\omega_R \otimes_R M)) \\ &= \beta_0^T(k)\beta_1^T(R) + \beta_1^T(k)\beta_0^T(R) = 3 + \beta_1^T(k) \end{aligned}$$

and

$$(4.4) \quad \begin{aligned} \mu_T^1(M) &= \beta_0^T(k)l_T(\mathbf{H}_n^1(M)) + \beta_1^T(k)l_T(\mathbf{H}_n^0(M)) \\ &= \beta_0^T(k)\beta_1^T(\omega_R) + \beta_1^T(k)\beta_0^T(\omega_R) = 3 + 2\beta_1^T(k). \end{aligned}$$

Since $\text{pd}_R M$ is finite, it follows that $M \in \mathcal{A}_{\omega_R}(R)$. In particular, $\text{Tor}_i^R(\omega_R, M) = 0$ for all $i \geq 0$, and so one has the quism $\omega_R \otimes_R M \simeq \omega_R \otimes_R^L M$. By [2], Exercise 3.3.26, we have $\mu_T^1(\omega_R \otimes_R M) = \beta_3^T(\omega_R \otimes_R M)$ and $\mu_T^1(M) = \beta_3^T(M)$. So,

$$(4.5) \quad \begin{aligned} \mu_T^1(\omega_R \otimes_R M) &= \text{Vdim}_k(\mathbf{H}_3((\omega_R \otimes_R M) \otimes_T^L k)) \\ &= \text{Vdim}_k(\mathbf{H}_3((\omega_R \otimes_R^L M) \otimes_T^L k)) \\ &= \text{Vdim}_k(\mathbf{H}_3(\omega_R \otimes_R^L (M \otimes_T^L k))) \\ &= \sum_{i+j=3} \text{Vdim}_k(\mathbf{H}_i(\omega_R \otimes_R^L k)) \times \text{Vdim}_k(\mathbf{H}_j(M \otimes_T^L k)) \\ &\geq \text{Vdim}_k(\mathbf{H}_0(\omega_R \otimes_R^L k)) \times \text{Vdim}_k(\mathbf{H}_3(M \otimes_T^L k)) \\ &= \beta_0^R(\omega_R)\beta_3^T(M) = 2\mu_T^1(M). \end{aligned}$$

The fourth equality holds by [5], Chapter 10, Result 7. Now, (4.3), (4.4) and (4.5) yield the desired contradiction. Hence, the R -module $\omega_R \otimes_R M$ is not Buchsbaum.

Corollary 4.9. *Let C be a semidualizing module of R and M a finitely generated R -module in $\mathcal{B}_C(R)$.*

- (i) *M is generalized Cohen-Macaulay if and only if $\text{Hom}_R(C, M)$ is generalized Cohen-Macaulay.*
- (ii) *If M is surjective Buchsbaum, then $\text{Hom}_R(C, M)$ is also surjective Buchsbaum.*
- (iii) *M is Cohen-Macaulay if and only if $\text{Hom}_R(C, M)$ is Cohen-Macaulay.*
- (iv) *M is maximal Cohen-Macaulay if and only if $\text{Hom}_R(C, M)$ is maximal Cohen-Macaulay.*
- (v) *If the R -module $\text{Hom}_R(C, M)$ is Gorenstein, then M is also Gorenstein.*

Proof. In view of the natural isomorphism $M \cong C \otimes_R \text{Hom}_R(C, M)$ and the fact that $\text{Hom}_R(C, M) \in \mathcal{A}_C(R)$, these claims follow immediately by Theorems 3.3, 4.3 and 4.7. \square

Let C be a semidualizing module of R and n be a nonnegative integer. Let $\mathcal{P}_C(R)$ and $\mathcal{I}_C(R)$ denote, respectively, the classes of C -projective and C -injective finitely generated R -modules. Set $\widehat{\mathcal{P}}_C(R)_{\leq n}$, $\widehat{\mathcal{P}}(R)_{\leq n}$, $\widehat{\mathcal{I}}_C(R)_{\leq n}$ and $\widehat{\mathcal{I}}(R)_{\leq n}$ to be the classes of finitely generated R -modules of C -projective, projective, C -injective and injective dimension of at most n , respectively. Then by [14], Theorem 2.12, there are the following Foxby equivalences of categories:

$$\begin{array}{ccc}
 \mathcal{P}(R) & \begin{array}{c} \xrightarrow{C \otimes_R -} \\ \xleftarrow{\text{Hom}_R(C, -)} \end{array} & \mathcal{P}_C(R) \\
 \downarrow & & \downarrow \\
 \widehat{\mathcal{P}}(R)_{\leq n} & \begin{array}{c} \xrightarrow{C \otimes_R -} \\ \xleftarrow{\text{Hom}_R(C, -)} \end{array} & \widehat{\mathcal{P}}_C(R)_{\leq n} \\
 \downarrow & & \downarrow \\
 \mathcal{A}_C(R) & \begin{array}{c} \xrightarrow{C \otimes_R -} \\ \xleftarrow{\text{Hom}_R(C, -)} \end{array} & \mathcal{B}_C(R) \\
 \uparrow & & \uparrow \\
 \widehat{\mathcal{I}}_C(R)_{\leq n} & \begin{array}{c} \xrightarrow{C \otimes_R -} \\ \xleftarrow{\text{Hom}_R(C, -)} \end{array} & \widehat{\mathcal{I}}(R)_{\leq n} \\
 \uparrow & & \uparrow \\
 \mathcal{I}_C(R) & \begin{array}{c} \xrightarrow{C \otimes_R -} \\ \xleftarrow{\text{Hom}_R(C, -)} \end{array} & \mathcal{I}(R)
 \end{array}$$

In view of Theorems 3.3, 4.3 and 4.7 and Corollary 4.9, we deduce the following corollary.

Corollary 4.10. *Let C be a semidualizing module of R and n a nonnegative integer. Let $(\mathcal{Y}(R), \mathcal{Z}(R))$ be any of the pairs $(\mathcal{P}(R), \mathcal{P}_C(R))$, $(\widehat{\mathcal{P}}(R)_{\leq n}, \widehat{\mathcal{P}}_C(R)_{\leq n})$, $(\mathcal{A}_C(R), \mathcal{B}_C(R))$, $(\widehat{\mathcal{I}}_C(R)_{\leq n}, \widehat{\mathcal{I}}(R)_{\leq n})$; or $(\mathcal{I}_C(R), \mathcal{I}(R))$. Then there is the following display of equivalences and functors:*

$$\begin{array}{ccc}
 \mathcal{Y}(R) \cap \mathcal{G}(R) & \xrightarrow{C \otimes_R -} & \mathcal{Z}(R) \cap \mathcal{G}(R) \\
 \downarrow & & \downarrow \\
 \mathcal{Y}(R) \cap \mathcal{MCM}(R) & \begin{array}{c} \xrightarrow{C \otimes_R -} \\ \xleftarrow{\text{Hom}_R(C, -)} \end{array} & \mathcal{Z}(R) \cap \mathcal{MCM}(R) \\
 \downarrow & & \downarrow \\
 \mathcal{Y}(R) \cap \mathcal{CM}(R) & \begin{array}{c} \xrightarrow{C \otimes_R -} \\ \xleftarrow{\text{Hom}_R(C, -)} \end{array} & \mathcal{Z}(R) \cap \mathcal{CM}(R) \\
 \downarrow & & \downarrow \\
 \mathcal{Y}(R) \cap \mathcal{SB}(R) & \xleftarrow{\text{Hom}_R(C, -)} & \mathcal{Z}(R) \cap \mathcal{SB}(R) \\
 \downarrow & & \downarrow \\
 \mathcal{Y}(R) \cap \mathcal{GCM}(R) & \begin{array}{c} \xrightarrow{C \otimes_R -} \\ \xleftarrow{\text{Hom}_R(C, -)} \end{array} & \mathcal{Z}(R) \cap \mathcal{GCM}(R)
 \end{array}$$

Here, $\mathcal{G}(R)$ and $\mathcal{MCM}(R)$ denote the classes of Gorenstein and maximal Cohen-Macaulay R -modules, respectively, and \mathcal{SB} and \mathcal{GCM} stand for the full subcategory of surjective Buchsbaum and generalized Cohen-Macaulay R -modules, respectively.

We end the paper by proposing the following natural question.

Question 4.11. Let C be a semidualizing module of R and M a finitely generated R -module in $\mathcal{A}_C(R)$ such that $C \otimes_R M$ is Buchsbaum. Then, is M Buchsbaum?

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