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RESTRICTED WEAK TYPE INEQUALITIES FOR THE ONE-SIDED HARDY-LITTLEWOOD MAXIMAL OPERATOR IN HIGHER DIMENSIONS

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Abstract. We give a quantitative characterization of the pairs of weights (w, v) for which the dyadic version of the one-sided Hardy-Littlewood maximal operator satisfies a restricted weak (p, p) type inequality for $1 \leqslant p < \infty$. More precisely, given any measurable set E_0, the estimate

$$w(\{x \in \mathbb{R}^n \colon M^{+,d}(\mathcal{X}_{E_0})(x) > t\}) \leqslant \frac{C[(w,v)]^p_{A_p^{+,d}(\mathcal{R})}}{t^p} v(E_0)$$

holds if and only if the pair (w, v) belongs to $A_p^{+,d}(\mathcal{R})$, that is,

$$\frac{|E|}{|Q|} \leqslant [(w,v)]_{A_p^{+,d}(\mathcal{R})} \Big(\frac{v(E)}{w(Q)}\Big)^{1/p}$$

for every dyadic cube Q and every measurable set $E \subset Q^+$. The proof follows some ideas appearing in S. Ombrosi (2005). We also obtain a similar quantitative characterization for the non-dyadic case in $\mathbb{R}^2$ by following the main ideas in L. Forzani, F. J. Martín-Reyes, S. Ombrosi (2011).

Keywords: restricted weak type; one-sided maximal operator

MSC 2020: 42B25, 28B99

1. Introduction

In 1986 Sawyer in [10] started the theory of one-sided weights. Namely, he introduced the class of weights A_p^+ and showed that this class is necessary and sufficient for the weighted boundedness of the one-sided Hardy-Littlewood maximal function.

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Some extensions and generalizations were given consequently in the articles [5], [6] and [7], among others. In [9], the author characterizes the functions w for which the one-sided Hardy-Littlewood maximal operator

$$M_v^+ f(x) = \sup_{h>0} \frac{\int_x^{x+h} |f| v}{\int_x^{x+h} v}$$

verifies a restricted weak (p,p) type on the real line, that is, a weak type inequality applied to the function $f = \mathcal{X}_E$, where E is an arbitrary measurable set. More precisely, the inequality

$$w(\{x \in \mathbb{R} \colon\ M_v^+(\mathcal{X}_E)(x) > t\}) \leqslant \frac{C}{t^p} v(E)$$

holds if and only if $w \in A_p^+(\mathcal{R})(v\,\mathrm{d}x)$. This set corresponds to the class of weights that satisfy a restricted A_p^+ condition with respect to the measure $\mathrm{d}\mu = v(x)\,\mathrm{d}x$, see the section below for details.

Although the theory in this setting was deeply developed and the main results were improved and generalized, most of the results were set on $\mathbb{R}$.

In [8], Ombrosi characterized the pair of weights (w, v) for which the inequality

$$w(\{x \in \mathbb{R}^n \colon\ M^{+,d} f(x) > t\}) \leqslant \frac{C}{t^p} \int_{\mathbb{R}^n} |f|^p v \tag{1.1}$$

holds for every positive t, and where $1 \leqslant p < \infty$. The operator $M^{+,d}$ is a dyadic version of M^+ defined on $\mathbb{R}^n$. A similar result was also obtained for $M^{-,d}$.

It is well-known that the operators $M^+ f$ and $M^{+,d} f$ are pointwise equivalent on $\mathbb{R}$, see [6]. However, this result is false in general in higher dimensions. This means that a non-dyadic version of (1.1) cannot be obtained directly from the dyadic case, and the problem of finding such an estimate remained open.

In [1], Forzani, Martín-Reyes and Ombrosi proposed a way to generalize the operators M^+ and M^- to higher dimensions and solved the problem discussed above on $\mathbb{R}^2$. The technique used, although newfangled and quite delicate, relied heavily upon the dimension. This means that the corresponding problem for $n \geqslant 3$ still remains open.

Related to strong estimates in dimension greater than one, some partial results were obtained in [4]. At this point we would also like to mention interesting applications of this theory to parabolic differential equations obtained by Kinnunen and Saari in [2] and [3].

In this article we use some ideas of [1] and [8] to give a characterization of the pairs of weights for which the one-sided Hardy-Littlewood maximal operator satisfies a restricted weak type inequality in higher dimensions.

Concretely, for the dyadic case we have the following result.

Theorem 1.1. *Let (w, v) be a pair of weights and $1 \leqslant p < \infty$. Then the following statements are equivalent:*

(a) *The operator $M^{+,d}$ is of restricted weak (p, p) type with respect to (w, v), that is, there exists a positive constant C such that the inequality*

$$w(\{x \in \mathbb{R}^n \colon M^{+,d}(\mathcal{X}_E)(x) > t\}) \leqslant \frac{C[(w,v)]^p_{A_p^{+,d}(\mathcal{R})}}{t^p} v(E)$$

holds for every positive t and every measurable set E.

(b) *(w, v) belongs to $A_p^{+,d}(\mathcal{R})$.*

For the non-dyadic case we prove the next theorem.

Theorem 1.2. *Let (w, v) be a pair of nonnegative measurable functions defined in $\mathbb{R}^2$ and $1 \leqslant p < \infty$. The following conditions are equivalent:*

(a) *The operator M^+ is of restricted weak (p, p) type with respect to (w, v), that is, there exists a positive constant C such that the inequality*

$$w(\{x \in \mathbb{R}^2 \colon M^{+}(\mathcal{X}_E)(x) > t\}) \leqslant \frac{C[(w,v)]^p_{A_p^{+,d}(\mathcal{R})}}{t^p} v(E)$$

holds for every positive t and every measurable set E.

(b) *(w, v) belongs to $A_p^{+}(\mathcal{R})$.*

The article is organized as follows. In Section 2 we give the preliminaries and definitions required for these main results. In Sections 3 and 4 we prove Theorems 1.1 and 1.2, respectively.

2. Preliminaries and basic definitions

We shall deal with dyadic cubes with sides parallel to the coordinate axes. Given a dyadic cube $Q = \prod_{i=1}^{n} [a_i, b_i)$, we denote with $Q^+ = \prod_{i=1}^{n} [b_i, 2b_i - a_i)$ and $Q^- = \prod_{i=1}^{n} [2a_i - b_i, a_i)$.

Given a positive number s, we denote $(Q)^{s,+} = \prod_{i=1}^{n} [a_i, a_i + sh)$, where $h = b_i - a_i$. Similarly, we denote $(Q)^{s,-} = \prod_{i=1}^{n} [b_i - sh, b_i)$.

For $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ and $h > 0$, we denote $Q_{x,h} = \prod_{i=1}^{n} [x_i, x_i + h)$ and $Q_{x,h^-} = \prod_{i=1}^{n} [x_i - h, x_i)$. The one-sided Hardy-Littlewood maximal operators are given by

$$M^+f(x) = \sup_{h>0} \frac{1}{|Q_{x,h}|} \int_{Q_{x,h}} |f(y)|\,\mathrm{d}y, \quad \text{and} \quad M^-f(x) = \sup_{h>0} \frac{1}{|Q_{x,h^-}|} \int_{Q_{x,h^-}} |f(y)|\,\mathrm{d}y.$$

We shall consider the dyadic version of these operators, that is,

$$M^{+,d}f(x) = \sup_{Q\ni x} \frac{1}{|Q|} \int_{Q^+} |f(y)|\,\mathrm{d}y \quad \text{and} \quad M^{-,d}f(x) = \sup_{Q\ni x} \frac{1}{|Q|} \int_{Q^-} |f(y)|\,\mathrm{d}y,$$

where the supremum is taken over dyadic cubes.

Given $1 < p < \infty$, we say that a pair of weights (w, v) belongs to A_p^+ if there exists a positive constant C such that the inequallity

$$\Big(\int_Q w\Big)\Big(\int_{Q^+} v^{1-p'}\Big)^{p-1} \leqslant C|Q|^p$$

holds for every cube Q in $\mathbb{R}^n$.

When $p = 1$, we say that (w, v) belongs to A_1^+ if there exist a positive constant C that verifies

$$M^-w(x) \leqslant Cv(x)$$

for almost every x. The smallest constant for which these inequalities hold is denoted by $[(w, v)]_{A_p^+}$.

Similarly, we say that (w, v) belongs to $A_p^{+,d}$ if the inequalities above hold for every dyadic cube Q and when $p = 1$, the involved operator is $M^{-,d}$. In this case, the corresponding smallest constant is denoted by $[(w, v)]_{A_p^{+,d}}$.

For $1 \leqslant p < \infty$ we say that $(w, v) \in A_p^{+,d}(\mathcal{R})$ if there exists a positive constant C such that the inequality

$$\frac{|E|}{|Q|} \leqslant C\Big(\frac{v(E)}{w(Q)}\Big)^{1/p} \tag{2.1}$$

holds for every dyadic cube Q and every measurable set $E \subset Q^+$. The smallest constant C for which the inequality above holds will be denoted by $[(w, v)]_{A_p^{+,d}(\mathcal{R})}$.

We say that a pair of weights (w, v) belongs to $A_p^+(\mathcal{R})$, $1 \leqslant p < \infty$, if inequality (2.1) holds for every cube Q and every measurable subset E of Q^+.

Remark 2.1. By replacing Q^+ by Q^- and M^- by M^+ we can define the A_p^- classes for $1 \leqslant p < \infty$. The dyadic version of these classes, $A_p^{-,d}$, are defined by considering dyadic cubes on their definitions. The same occurs for $A_p^-(\mathcal{R})$ and $A_p^{-,d}(\mathcal{R})$.

Throughout the paper we shall present the results for M^+, but the same arguments can be adapted to get the corresponding versions for M^-.

The novelty of considering restricted weak type inequalities relies on that although we take a particular function f, we consider a wider class of weights. This property is contained in the following proposition.

Proposition 2.1. *$A_p^+ \subset A_p^+(\mathcal{R})$ for every $1 < p < \infty$, and $A_1^+ = A_1^+(\mathcal{R})$.*

Proof. Let $1 < p < \infty$ and assume that $(w, v) \in A_p^+$. Fix a cube Q and a measurable subset E of Q^+ with $|E| > 0$. Then

$$|E| \leqslant \Big(\int_E v\Big)^{1/p} \Big(\int_{Q^+} v^{1-p'}\Big)^{1/p'} \leqslant [(w,v)]_{A_p^+}^{1/p} \Big(\frac{v(E)}{w(Q)}\Big)^{1/p} |Q|,$$

which implies that $(w, v) \in A_p^+(\mathcal{R})$ and $[(w,v)]_{A_p^+(\mathcal{R})} \leqslant [(w,v)]_{A_p^+}^{1/p}$.

On the other hand, set $p = 1$ and assume that $(w, v) \in A_1^+$. Fix a cube Q and a measurable set $E \subset Q^+$ with positive measure. Then for every $x \in E$ we have that

$$\frac{1}{|Q|} \int_{(Q^+)^-} w \leqslant [(w,v)]_{A_1^+} v(x),$$

which implies that

$$\frac{w(Q)}{|Q|} \leqslant [(w,v)]_{A_1^+} \frac{v(E)}{|E|},$$

and then $(w, v) \in A_1^+(\mathcal{R})$. Conversely, fix x and $h > 0$. Let $Q = Q_{x,h^-}$, $\lambda > \operatorname*{ess\,inf}_{Q_{x,h}} v$ and $E = \{y \in Q_{x,h} \colon v(y) < \lambda\}$. Then we have that

$$\frac{w(Q_{x,h^-})}{|Q_{x,h^-}|} \leqslant [(w,v)]_{A_1^+(\mathcal{R})} \lambda.$$

By letting $\lambda \to \operatorname*{ess\,inf}_{Q_{x,h}} v$ and then taking supremum over h we get that

$$M^- w(x) \leqslant [(w,v)]_{A_1^+(\mathcal{R})} v(x).$$

□

The following lemma states a useful property for weights on the $A_p^+(\mathcal{R})$ class.

Lemma 2.1. *Let $1 \leqslant p < \infty$, (w, v) be a pair of weights in $A_p^+(\mathcal{R})$ and a, b two positive constants. Then*

(a) *if $a \leqslant b$, then $(w_0, v_0) = (\max\{w, a\}, \max\{v, b\})$ belongs to $A_p^+(\mathcal{R})$ and $[(w_0, v_0)]_{A_p^+(\mathcal{R})} \leqslant 2\max\{1, [(w,v)]_{A_p^+(\mathcal{R})}\}$;*

(b) *$(w_1, v_1) = (\min\{w, a\}, \max\{v, b\})$ belongs to $A_p^+(\mathcal{R})$ and $[(w_0, v_0)]_{A_p^+(\mathcal{R})} \leqslant [(w,v)]_{A_p^+(\mathcal{R})}$.*

P r o o f. Let us first prove statement (a). Fix a cube Q and a measurable subset E of Q^+. We have to show that there exists a positive constant C, independent of Q and E, such that

$$\frac{w_0(Q)}{v_0(E)} \leqslant C\Big(\frac{|Q|}{|E|}\Big)^p.$$

We write

$$w_0(Q) = \int_{Q\cap\{w\geqslant a\}} w_0 + \int_{Q\cap\{w<a\}} w_0 = w(Q\cap\{w\geqslant a\}) + a|Q\cap\{w<a\}|,$$

and therefore,

$$\frac{w_0(Q)}{v_0(E)} = \frac{w(Q\cap\{w\geqslant a\})}{v_0(E)} + \frac{a|Q\cap\{w<a\}|}{v_0(E)} = I + II.$$

Now observe that

$$I \leqslant \frac{w(Q)}{v(E)} \leqslant [(w,v)]_{A_p^+(\mathcal{R})}\Big(\frac{|Q|}{|E|}\Big)^p.$$

On the other hand,

$$II \leqslant \frac{a|Q|}{b|E|} \leqslant \Big(\frac{|Q|}{|E|}\Big)^p,$$

since $a \leqslant b$ and $|Q| \geqslant |E|$. Therefore, $(w_0,v_0) \in A_p^+(\mathcal{R})$ and $[(w_0,v_0)]_{A_p^+(\mathcal{R})} \leqslant 2\max\{1,[(w,v)]_{A_p^+(\mathcal{R})}\}$.

For the proof of statement (b), observe that $w_1 \leqslant w$ and $v_1 \geqslant v$, so

$$\frac{w_1(Q)}{v_1(E)} \leqslant \frac{w(Q)}{v(E)} \leqslant [(w,v)]_{A_p^+(\mathcal{R})}\Big(\frac{|Q|}{|E|}\Big)^p,$$

which shows that $(w_1,v_1) \in A_p^+(\mathcal{R})$ with $[(w_1,v_1)]_{A_p^+(\mathcal{R})} \leqslant [(w,v)]_{A_p^+(\mathcal{R})}$. □

3. Restricted weak (p,p) type of $M^{+,d}$ in $\mathbb{R}^n$

We devote this section to proving Theorem 1.1. We start with the following lemma which will be useful for this purpose. This result is an adaptation of Lemma 2.1 in [8].

Lemma 3.1. *Let $1 \leqslant p < \infty$, $(w,v) \in A_p^{+,d}(\mathcal{R})$ and $\mu > 0$. Let E be a measurable set such that $0 < |E| < \infty$ and $\{Q_j\}_{j\in\Gamma_\mu}$ a disjoint family of dyadic cubes such that for every $j \in \Gamma_\mu$ we have*

$$\mu < \frac{|E\cap Q_j^+|}{|Q_j|} \leqslant 2\mu. \tag{3.1}$$

Then we have that

$$\sum_{j\in\Gamma_\mu} w(Q_j) \leqslant \frac{C[(w,v)]^p_{A_p^{+,d}(\mathcal{R})}}{\mu^p} v\Big(E\cap\Big(\bigcup_{j\in\Gamma_\mu} Q_j^+\Big)\Big).$$

P r o o f. For $m \geqslant 0$ we define the sets

$$i_m = \{j \in \Gamma_\mu\colon \text{ there exist exactly } m \text{ cubes } Q_s^+\colon\ Q_j^+ \subsetneq Q_s^+ \text{ with } s \in \Gamma_\mu\}$$

and also

$$\sigma_m = \bigcup_{j \in i_m} Q_j^+.$$

Also, we define $E_j^+ = E \cap Q_j^+$ and $F_m = \bigcup\limits_{j \in i_m} E_j^+$.

Notice that $\Gamma_\mu = \bigcup\limits_{m \geqslant 0} i_m$ and if j_1 and j_2 belong to i_m for some m, then $Q_{j_1}^+ \cap Q_{j_2}^+ = \emptyset$. This yields

$$|F_m| = \sum_{j \in i_m} |E_j^+|.$$

On the other hand, $\sigma_{m+1} \subset \sigma_m$ for every $m \geqslant 0$, so

$$F_{m+1} \subset F_m \quad \text{and} \quad |F_{m+1}| \leqslant |F_m|. \tag{3.2}$$

For fixed m_0 and $j_0 \in i_{m_0}$, if $Q_j^+ \subsetneq Q_{j_0}^+$, then $j \in i_m$ with $m > m_0$ and $Q_j \subset Q_{j_0}^{2,+}$. Therefore,

$$\bigcup_{m > m_0} \ \bigcup_{j \in i_m\colon\ Q_j^+ \subsetneq Q_{j_0}^+} Q_j \subset (Q_{j_0})^{2,+}$$

and this implies that

$$\sum_{m > m_0} \ \sum_{j \in i_m\colon\ Q_j^+ \subsetneq Q_{j_0}^+} |Q_j| \leqslant |(Q_{j_0})^{2,+}| = 2^n |Q_{j_0}|,$$

since the cubes Q_j are disjoint. Thus, by (3.1) we get

$$\begin{aligned}\sum_{m > m_0} |F_m \cap Q_{j_0}^+| = \sum_{m > m_0} \ \sum_{j \in i_m\colon\ Q_j^+ \subsetneq Q_{j_0}^+} |E_j^+| &\leqslant 2\mu \sum_{m > m_0} \ \sum_{j \in i_m\colon\ Q_j^+ \subsetneq Q_{j_0}^+} |Q_j| \\ &\leqslant 2^{n+1} \mu |Q_{j_0}| \leqslant 2^{n+1} |E_{j_0}^+|.\end{aligned}$$

This last estimate implies that

$$\sum_{m = m_0 + 1}^{m_0 + 2^{n+2}} |F_m \cap Q_{j_0}^+| < 2^{n+1} |E_{j_0}^+|$$

and then there must be an index m, $m_0 + 1 \leqslant m \leqslant m_0 + 2^{n+2}$ such that

$$|F_m \cap Q_{j_0}^+| < \frac{|E_{j_0}^+|}{2}.$$

By (3.2) we get

$$|F_{m_0+2^{n+2}} \cap Q_{j_0}^+| \leqslant |F_m \cap Q_{j_0}^+| < \frac{|E_{j_0}^+|}{2},$$

and consequently,

$$\frac{|Q_{j_0}^+ \cap (E \setminus F_{m_0+2^{n+2}})|}{|Q_{j_0}|} > \frac{1}{2}\frac{|E_{j_0}^+|}{|Q_{j_0}|} > \frac{\mu}{2}.$$

Now, we can estimate

$$\begin{aligned}
\sum_{j\in\Gamma_\mu} w(Q_j) &= \sum_{m=0}^{\infty}\sum_{j\in i_m} w(Q_j) \\
&< \Big(\frac{2}{\mu}\Big)^p \sum_{m=0}^{\infty}\sum_{j\in i_m} w(Q_j)\Big(\frac{|Q_j^+ \cap (E\setminus F_{m_0+2^{n+2}})|}{|Q_j|}\Big)^p \\
&\leqslant \Big(\frac{2}{\mu}\Big)^p \sum_{m=0}^{\infty}\sum_{j\in i_m} [(w,v)]^p_{A_p^{+,d}(\mathcal{R})} w(Q_j)\frac{v(Q_j^+ \cap (E\setminus F_{m_0+2^{n+2}}))}{w(Q_j)} \\
&\leqslant \Big(\frac{2}{\mu}\Big)^p [(w,v)]^p_{A_p^{+,d}(\mathcal{R})} \sum_{m=0}^{\infty} \int_{\sigma_m - \sigma_{m+2^{n+2}}} \mathcal{X}_E v \\
&= \Big(\frac{2}{\mu}\Big)^p [(w,v)]^p_{A_p^{+,d}(\mathcal{R})} \sum_{k=0}^{2^{n+2}-1}\sum_{m=0}^{\infty} \int_{\sigma_{2^{n+2}m+k} - \sigma_{2^{n+2}(m+1)+k}} \mathcal{X}_E v \\
&\leqslant \Big(\frac{2}{\mu}\Big)^p [(w,v)]^p_{A_p^{+,d}(\mathcal{R})} \sum_{k=0}^{2^{n+2}-1} \int_{\sigma_k} \mathcal{X}_E v \\
&\leqslant \frac{2^{n+p+2}[(w,v)]^p_{A_p^{+,d}(\mathcal{R})}}{\mu^p} v(E\cap\sigma_0).
\end{aligned}$$

□

P r o o f of Theorem 1.1. We shall first prove that (a) implies (b). Fix a dyadic cube Q and a measurable subset E of Q^+. Assume that $|E| > 0$, since otherwise the condition follows inmediately. For every x in Q we have that

$$M^{+,d}\mathcal{X}_E(x) \geqslant \frac{1}{|Q|}\int_{Q^+} \mathcal{X}_E = \frac{|E|}{|Q|},$$

which implies that $Q \subset \{x\colon\ M^{+,d}\mathcal{X}_E(x) > |E|/(2|Q|)\}$. By using statement (a) we get

$$w(Q) \leqslant C\Big(\frac{|Q|}{|E|}\Big)^p v(E),$$

which shows that $(w,v) \in A_p^{+,d}(\mathcal{R})$.

Now we prove that (b) implies (a). Fix a measurable set E and assume, without loss of generality, that $0 < |E| < \infty$. For fixed $t > 0$, let $\mathcal{F}$ be the family of dyadic cubes Q such that $|E \cap Q^+|/|Q| > t$ and let $\{Q_j\}_j$ be the family of the maximal cubes of $\mathcal{F}$. It follows that the cubes Q_j are disjoint and

$$\{x\colon\ M^{+,d}\mathcal{X}_E(x) > t\} = \bigcup_j Q_j.$$

We shall consider a partition of this family of cubes. Given $k \geqslant 0$, we set

$$C_k = \Big\{j\colon\ 2^k t < \frac{|E \cap Q_j^+|}{|Q_j|} \leqslant 2^{k+1} t\Big\}$$

and apply Lemma 3.1 to the family C_k with $\mu = 2^k t$ for every k. Therefore,

$$\sum_{j\in C_k} w(Q_j) \leqslant \frac{C[(w,v)]^p_{A_p^{+,d}(\mathcal{R})}}{(2^k t)^p} v\Big(\bigcup_{j\in C_k} E_j^+\Big).$$

This yields

$$\begin{aligned} w(\{x\colon\ M^{+,d}\mathcal{X}_E(x) > t\}) &= \sum_j w(Q_j) = \sum_{k=0}^{\infty}\sum_{j\in C_k} w(Q_j) \\ &\leqslant \sum_{k=0}^{\infty} \frac{C[(w,v)]^p_{A_p^{+,d}(\mathcal{R})}}{(2^k t)^p} v(E) \\ &= \frac{C[(w,v)]^p_{A_p^{+,d}(\mathcal{R})}}{t^p} v(E), \end{aligned}$$

which completes the proof. □

4. Restricted weak (p,p) type of M^+ in $\mathbb{R}^2$

We devote this section to the proof of Theorem 1.2. Along this section we shall assume that the space, where we work, is $\mathbb{R}^2$. We begin by introducing some specifics in this setting.

We say that a square Q has dyadic size if $l(Q) = 2^k$ for an integer k. Let $l(Q)$ denote the length of the sides of Q. Given a square Q, αQ will denote the square with the same center as Q and sides of length $\alpha l(Q)$.

For $h > 0$ and $Q = [a, a+h] \times [b, b+h]$, we set $\widetilde{Q}$ the dilation of Q to the right and to the bottom in $\frac{1}{2} l(Q)$. That is, $\widetilde{Q} = [a, a + \frac{3}{2}h] \times [b - \frac{1}{2}h, b+h]$.

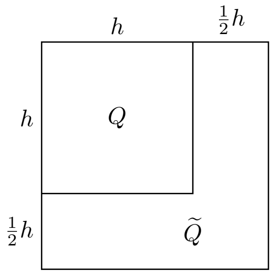

Figure 1. The cubes Q and $\widetilde{Q}$.

Given $x = (x_1, x_2) \in \mathbb{R}^2$ and $h > 0$ recall that $Q_{x,h} = [x_1, x_1 + h] \times [x_2, x_2 + h]$. We shall consider the following partition of a cube $Q_{x,h}$:

$$Q_{x,h} = Q_{x,h/2} \cup Q^1_{x,h} \cup Q^2_{x,h} \cup Q^3_{x,h},$$

where

$$\begin{aligned} Q^1_{x,h} &= [x_1 + \tfrac{1}{2}h, x_1 + h] \times [x_2 + \tfrac{1}{2}h, x_2 + h], \\ Q^2_{x,h} &= [x_1 + \tfrac{1}{2}h, x_1 + h] \times [x_2, x_2 + \tfrac{1}{2}h], \\ Q^3_{x,h} &= [x_1, x_1 + \tfrac{1}{2}h] \times [x_2 + \tfrac{1}{2}h, x_2 + h]. \end{aligned}$$

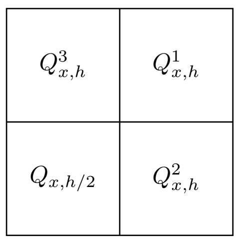

Figure 2. Subsquares of $Q_{x,h}$.

The proof of Theorem 1.2 relies on the following covering lemma, that is, a consequence of Lemma 3.1 stated and proved in [1], when we take $f = \mathcal{X}_E$. This result contains a covering argument that is related to the subcube $Q^2_{x,h}$. For the main proof we will require the corresponding versions for $Q^1_{x,h}$ and $Q^3_{x,h}$, which can be achieved by following similar ideas.

Lemma 4.1. *Let $t > 0$ and E be a measurable set such that $0 < |E| < \infty$. Let $A = \{x_j\}_{j=1}^N$ be a finite set of points in $\mathbb{R}^2$. Suppose that for every $1 \leqslant j \leqslant N$, we have a square of dyadic size Q_j with x_j as its upper right corner and that satisfies*

$$\frac{t}{4} < \frac{|E \cap Q_j^{+2}|}{|Q_j|}.$$

Then there exists a set $\Gamma \subset \{1, \ldots, N\}$ *such that*

$$A \subset \bigcup_{i \in \Gamma} \widetilde{Q}_i \tag{4.1}$$

and

$$\frac{t}{4} < \frac{|E \cap \widetilde{Q}_j^+|}{|Q_j|}. \tag{4.2}$$

Moreover, for every $i, j \in \Gamma$ *with* $i \neq j$ *we have* $\widetilde{Q}_i \not\subseteq \widetilde{Q}_j$ *and the squares* $\widetilde{Q}_i$, $i \in \Gamma$, *of the same size are almost disjoint, that is, there exists a positive constant* C *such that for every* l

$$\sum_{i \in \Gamma, l(Q_i) = l} \mathcal{X}_{\widetilde{Q}_i}(x) \leqslant C.$$

This implies that the squares $(\widetilde{Q}_i)^+$ *are almost disjoint, too. Further, if*

$$\frac{|E \cap (\widetilde{Q}_j)^+|}{|Q_j|} \leqslant 8t,$$

then there exists a family of sets $\{F_j\}_{j \in \Gamma}$ *with* $F_j \subset (\widetilde{Q}_j)^+$ *such that*

$$\frac{t}{8} < \frac{|E \cap F_j|}{|Q_j|}$$

and they are almost disjoint, that is, there exists a positive constant C *(independent of everything) such that*

$$\sum_{j \in \Gamma} \mathcal{X}_{F_j}(x) \leqslant C.$$

P r o o f of Theorem 1.2. The fact that (a) implies (b) can be achieved in a similar way to Theorem 1.1. Let us prove that (b) implies (a). The operator M^+ is pointwise equivalent to the operator

$$\mathcal{M}^+ f(x) = \sup_{k \in \mathbb{Z}} \frac{1}{|Q_{x,2^k}|} \int_{Q_{x,2^k}} |f|,$$

that is, the one-sided maximal operator defined over squares of dyadic size. We shall consider for $i = 1, 2, 3$ the operators

$$\mathcal{M}^{+i} f(x) = \sup_{k \in \mathbb{Z}} \frac{1}{|Q^i_{x,2^k}|} \int_{Q^i_{x,2^k}} |f|,$$

where the cubes $Q^i_{x,2^k}$ are depicted in Figure 2.

Let us fix a measurable set E with $0 < |E| < \infty$. Let (w, v) be a pair of weights in $A_p^+(\mathcal{R})$. We shall prove that

$$w(\{x \in \mathbb{R}^2 \colon \mathcal{M}^+(\mathcal{X}_E)(x) > t\}) \leqslant \frac{C}{t^p} v(E)$$

for every $t > 0$. It will be enough to show that

$$w(\{x \in \mathbb{R}^2 \colon t < \mathcal{M}^+(\mathcal{X}_E)(x) \leqslant 2t\}) \leqslant \frac{C}{t^p} v(E),$$

and this also reduces to proving that

$$w(\{x \in \mathbb{R}^2 \colon t < \mathcal{M}^{+i}(\mathcal{X}_E)(x),\, \mathcal{M}^+(\mathcal{X}_E)(x) \leqslant 2t\}) \leqslant \frac{C}{t^p} v(E) \tag{4.3}$$

for $i = 1, 2, 3$. We show the proof for $i = 2$, being similar for the other indices.

Given a positive number ξ we consider the truncated maximal operator defined by

$$M_\xi^{+2}(\mathcal{X}_E)(x) = \sup_{\substack{h=2^k>\xi \\ k\in\mathbb{Z}}} \frac{4|E \cap Q_{x,h}^2|}{h^2}.$$

Observe that $M_\xi^{+2}(\mathcal{X}_E) \uparrow \mathcal{M}^{+2}(\mathcal{X}_E)$ when $\xi \to 0^+$. Therefore, it will be enough to prove that

$$w(\{x \in \mathbb{R}^2 \colon t < \mathcal{M}_\xi^{+2}(\mathcal{X}_E)(x),\, \mathcal{M}^+(\mathcal{X}_E)(x) \leqslant 2t\}) \leqslant \frac{C}{t^p} v(E) \tag{4.4}$$

for every $t > 0$ and with C independent of ξ, E and t.

By virtue of Lemma 2.1 we can assume that $w \in L^1_{\mathrm{loc}}$ and also that there exists a positive constant γ such that $0 < \gamma \leqslant w(x)$ for every $x \in \mathbb{R}^2$.

Let $\Omega_t = \{x \in \mathbb{R}^2 \colon t < \mathcal{M}_\xi^{+2}(\mathcal{X}_E)(x),\, \mathcal{M}^+(\mathcal{X}_E)(x) \leqslant 2t\}$. The measure $\mathrm{d}\mu(x) = w(x)\,\mathrm{d}x$ is finite over compact sets since we are assuming $w \in L^1_{\mathrm{loc}}$. Therefore, inequality (4.4) follows if we prove that

$$w(K) \leqslant \frac{C}{t^p} v(E)$$

for every compact set $K \subset \Omega_t$ and with C independent of K.

Fix a compact set $K \subset \Omega_t$. For every $x = (x_1, x_2) \in K$ there exists a square $Q_x = [x_1 - l, x_1] \times [x_2 - l, x_2]$ with $\xi \leqslant l$, $l = 2^k$ for some $k \in \mathbb{Z}$ and

$$\frac{t}{4} < \frac{|E \cap Q_x^{+2}|}{|Q_x|}.$$

Let $Q_{x,2l} = [x_1, x_1 + 2l] \times [x_2, x_2 + 2l]$. We have that $(\widetilde{Q}_x)^{+2} \subset Q_{x,2l}$ (see Figure 3) and thus,

$$\frac{|E \cap (\widetilde{Q}_x)^{+2}|}{|Q_x|} \leqslant \frac{|E \cap Q_{x,2l}|}{|Q_x|} = \frac{4|E \cap Q_{x,2l}|}{|Q_{x,2l}|} \leqslant 4\mathcal{M}^+(\mathcal{X}_E)(x) \leqslant 8t.$$

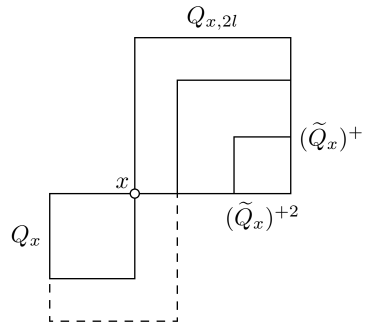

Figure 3. $(\widetilde{Q}_x)^{+2} \subset Q_{x,2l}$.

Therefore, we have that for every $x \in K$ there exists a square $Q_x = [x_1 - l, x_1] \times [x_2 - l, x_2]$ such that $\xi \leqslant l$,

$$\frac{t}{4} < \frac{|E \cap Q_x^{+2}|}{|Q_x|} \quad \text{and} \quad \frac{|E \cap (\widetilde{Q}_x)^{+2}|}{|Q_x|} \leqslant 8t.$$

We have also that there exists a positive constant M, depending on t and E, such that $l \leqslant M$ since

$$|Q_x| \leqslant \frac{4|E \cap Q_x^{+2}|}{t} \leqslant \frac{4|E|}{t} < \infty.$$

This implies that there exists a square R such that $\bigcup_{x \in K} \widetilde{Q}_x \subset R$. We shall consider the square $2R$. Since w is integrable in $2R$, there exists $0 < \varepsilon < 1$ such that if $Q \subset R$ is a square, then

$$w((1+\varepsilon)Q \setminus Q) \leqslant \gamma \xi^2.$$

If $Q \subset R$ verifies $l(Q) \geqslant \xi$, then

$$w((1+\varepsilon)Q \setminus Q) \leqslant \gamma \xi^2 \leqslant \gamma |Q| \leqslant w(Q).$$

This yields

$$w((1+\varepsilon)Q) \leqslant 2w(Q)$$

for every $Q \subset R$ with $l(Q) \geqslant \xi$. Particularly,

$$w((1+\varepsilon)\widetilde{Q}_x) \leqslant 2w(\widetilde{Q}_x) \quad \text{for every } x \in K.$$

Let $B_x(r)$ be the ball of radius r centered at x. We have that $K \subset \bigcup_{x\in K} B_x(\frac{1}{2}\xi\varepsilon)$, and then there exist $x_1, x_2, \ldots, x_s \in K$ such that $K \subset \bigcup_{j=1}^{s} B_{x_j}(\frac{1}{2}\xi\varepsilon)$, since K is compact.

We apply now Lemma 4.1 to the set $A = \{x_j\}_{j=1}^s$ and the squares $\{Q_j\}_{j=1}^s$ associated to the points x_j. Then there exists a set $\Gamma \subset \{1, \ldots, s\}$ that verifies $A \subset \bigcup_{i\in\Gamma} \widetilde{Q}_{x_i}$ and there also exist $\{F_{x_i}\colon\ i \in \Gamma\}$, $F_{x_i} \subset (\widetilde{Q}_{x_i})^+$,

$$\frac{t}{8} < \frac{|E \cap F_{x_i}|}{|Q_{x_i}|} \tag{4.5}$$

and

$$\sum_{i\in\Gamma} \mathcal{X}_{F_{x_i}}(x) \leqslant C.$$

Observe that if $x_j \in A$, there exists $i \in \Gamma$ such that $x_j \in \widetilde{Q}_{x_i}$. Then $B_{x_j}(\frac{1}{2}\xi\varepsilon) \subset (1+\varepsilon)\widetilde{Q}_{x_i}$. In fact, this is straightforward if we assume $0 < \xi < 1$. Consequently, we have that

$$K \subset \bigcup_{j=1}^{s} B_{x_j}\Big(\frac{\xi\varepsilon}{2}\Big) \subset \bigcup_{i\in\Gamma} (1+\varepsilon)\widetilde{Q}_{x_i},$$

which implies that

$$w(K) \leqslant \sum_{i\in\Gamma} w((1+\varepsilon)\widetilde{Q}_{x_i}) \leqslant 2\sum_{i\in\Gamma} w(\widetilde{Q}_{x_i}).$$

Thus, by using (4.5) and the $A_p^+(\mathcal{R})$ condition of (w,v), we obtain

$$\begin{aligned}
w(K) &\leqslant 2\sum_{i\in\Gamma} w(\widetilde{Q}_{x_i}) \leqslant \frac{C}{t^p}\sum_{i\in\Gamma} w(\widetilde{Q}_{x_i})\Big(\frac{|E\cap F_{x_i}|}{|Q_{x_i}|}\Big)^p \\
&= \frac{C}{t^p}\sum_{i\in\Gamma} w(\widetilde{Q}_{x_i})\Big(\frac{|(\widetilde{Q}_{x_i})^+|}{|Q_{x_i}|}\Big)^p\Big(\frac{|E\cap F_{x_i}|}{|(\widetilde{Q}_{x_i})^+|}\Big)^p \\
&\leqslant \frac{C}{t^p}[(w,v)]^p_{A_p^+(\mathcal{R})}\sum_{i\in\Gamma} v(E\cap F_{x_i}) \\
&\leqslant \frac{C}{t^p}[(w,v)]^p_{A_p^+(\mathcal{R})} v\Big(E \cap \Big(\bigcup_{i\in\Gamma} F_{x_i}\Big)\Big) \\
&\leqslant \frac{C}{t^p}[(w,v)]^p_{A_p^+(\mathcal{R})} v(E).
\end{aligned}$$

□

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