Xin Liang; Baiyan Xu

A note on weakly-supplemented subgroups and the solvability of finite groups

Czechoslovak Mathematical Journal, Vol. 72 (2022), No. 4, 1045–1046

Persistent URL: http://dml.cz/dmlcz/151127

Czechoslovak Mathematical Journal, 72 (147) (2022), 1045–1046

A NOTE ON WEAKLY-SUPPLEMENTED SUBGROUPS AND THE SOLVABILITY OF FINITE GROUPS

Xin Liang, Baiyan Xu, Changchun

Received September 6, 2021. Published online March 18, 2022.

Abstract. Suppose that G is a finite group and H is a subgroup of G. The subgroup H is said to be weakly-supplemented in G if there exists a proper subgroup K of G such that $G = HK$. In this note, by using the weakly-supplemented subgroups, we point out several mistakes in the proof of Theorem 1.2 of Q. Zhou (2019) and give a counterexample.

Keywords: weakly-supplemented subgroup; solvable group; finite group

MSC 2020: 20D10, 20D20

1. Introduction

All groups considered in this paper are finite groups. Our notation and terminology are standard. The reader may refer to [2]. It is well known that a subgroup H of a finite group G is complemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K = 1$. Also, we call a subgroup H of a finite group G weakly-supplemented in G if there exists a proper subgroup K of G such that $G = HK$. It is easy to prove that being weakly-supplemented is a generalization of being complemented. In [1], Hall proved that a finite group G is solvable if and only if every Sylow subgroup of G is complemented. New criteria for the solvability of finite groups was obtained by Zhou in [3]. He proved that a finite group G is solvable if and only if every Sylow subgroup of G of odd order is weakly-supplemented. However, this claim does not hold in general. In this note, we provide a counterexample to Theorem 1.2 in [3].

2. A counterexample

In this section, we point out the mistakes in the proof of Theorem 1.2 of [3] and give a counterexample. At first, we shall need the following two lemmas.

DOI: 10.21136/CMJ.2022.0326-21

Lemma 2.1. *Let G be a finite group and $p \in \pi(G)$. If there exists a subgroup H of G whose index in G is a natural power of p, then every Sylow p-subgroup of G is weakly-supplemented in G.*

P r o o f. Suppose that there exists a subgroup H of G such that $|G : H| = p^n$, where n is a natural number. Let P be an arbitrary Sylow p-subgroup of G, then there exists an element $x \in G$ such that $P^x \cap H$ is a Sylow p-subgroup of H. This implies that $|P^x : P^x \cap H| = p^n$. Thus, we can get $|P^x H| = |P^x||H|/|P^x \cap H| = p^n|H| = |G|$. Hence, $G = P^x H$ which implies $G = PH^{x^{-1}}$. Thus, P is weakly supplemented in G.

Lemma 2.2. *Let G be a finite group such that $G = A \times B$, where A is a non-solvable group and B a solvable group. If $\pi(A) \subseteq \pi(B)$, then all Sylow subgroups of G are weakly-supplemented in G.*

P r o o f. Let p be an arbitrary prime in $\pi(G)$. By Lemma 2.1, it is only to prove that there exists a subgroup of G whose index is a power of p. Since B is solvable and $\pi(A) \subseteq \pi(B)$, B must contain a Hall p'-subgroup K. Thus, again by the condition $\pi(A) \subseteq \pi(B)$, it is easy to prove that $K \times A$ is the subgroup of G which we need. □

Example 2.1. Let $A \cong \mathrm{PSL}(3, 2)$, $B \cong \mathbb{Z}_3$ and $G = A \times B$. Then $|G| = 504$. It is well known that $\mathrm{PSL}(3, 2)$ has a subgroup of order 24, and thus G has a subgroup of order 72. Hence, every Sylow 7-subgroup of G is complemented in G, and thus weakly-supplemented in G. For any $P \in \mathrm{Syl}_3(G)$, since $A < G$ and $G = AB$, by the two lemmas above it follows that P is weakly-supplemented in G. Hence, every Sylow subgroup of G of odd order is weakly-supplemented in G. However, G is not solvable.

Acknowledgements. The authors are very grateful to the referee who read the manuscript carefully and provided a lot of valuable suggestions and useful comments. It should be said that we would not have been able to polish the final version of this paper well without his or her outstanding efforts.

References

[1] *P. Hall*: A characteristic property of soluble groups. J. Lond. Math. Soc. *12* (1937), 198–200. zbl MR doi

[2] *B. Huppert*: Endliche Gruppen. I. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen 134. Springer, Berlin, 1967. (In German.) zbl MR doi

[3] *Q. Zhou*: On weakly-supplemented subgroups and the solvability of finite groups. Czech. Math. J. *69* (2019), 331–335. zbl MR doi

Authors' address: X i n L i a n g (corresponding author), B a i y a n X u Basic Research Section, Changchun Guanghua University, Road Economic & Technological Development Zone, Changchun, Jilin 130033, P. R. China, e-mail: mathxj2012@163.com.