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ON SUPERCHARACTER THEORETIC GENERALIZATIONS OF MONOMIAL GROUPS AND ARTIN'S CONJECTURE

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Abstract. We extend the notions of quasi-monomial groups and almost monomial groups in the framework of supercharacter theories, and we study their connection with Artin's conjecture regarding the holomorphy of Artin L-functions.

Keywords: Artin L-function; monomial group; almost monomial group; supercharacter theory

MSC 2020: 11R42, 20C15

1. Introduction

A group G is called *monomial* if every complex irreducible character χ of G is induced by a linear character λ of a subgroup H of G, that is, $\chi = \lambda^G$. A group G is called *quasi-monomial* if for every irreducible character χ of G, there exists a subgroup H of G and a linear character λ of H such that $\lambda^G = d\chi$, where d is a positive integer. A finite group G is called *almost monomial* if for all distinct complex irreducible characters χ and ψ of G there exists a subgroup H of G and a linear character λ of H such that the induced character λ^G contains χ and does not contain ψ. This definition, which generalizes quasi-monomial groups, appears [14] in connection with the study of the holomorphy of Artin L-functions associated to a finite Galois extension of $\mathbb{Q}$ at a point in the complex plane. An equivalent characterization for almost monomial groups is given in Proposition 2.3.

Let $K/\mathbb{Q}$ be a finite Galois extension with Galois group G. For any character χ of G, let $L(s,\chi) := L(s,\chi,K/\mathbb{Q})$ be the corresponding Artin L-function, see [3],

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page 296. Artin's conjecture states that $L(s,\chi)$ is holomorphic in $\mathbb{C}\setminus\{1\}$. If the group G is monomial (or quasi-monomial), then Artin's conjecture holds.

Let $\mathrm{Hol}(s_0)$ be the semigroup of Artin L-functions, holomorphic at $s_0\in\mathbb{C}\setminus\{1\}$. Nicolae in [14] proved that if G is almost monomial, then Artin's conjecture holds at s_0 if and only if $\mathrm{Hol}(s_0)$ is factorial. Let $\chi_1,\ldots,\chi_r$ be the complex irreducible characters of G, $f_1=L(s,\chi_1),\ldots,f_r=L(s,\chi_r)$ the corresponding Artin L-functions. In [7] it was proved that if G is almost monomial and s_0 is not a common zero for any two distinct L-functions f_k and f_l then all Artin L-functions of $K/\mathbb{Q}$ are holomorphic at s_0. Also in [7], some basic properties of almost monomial groups were stated.

The notion of a supercharacter theory for a finite group was introduced in 2008, by Diaconis and Isaacs (see [8]), as follows: A supercharacter theory of a finite group G, is a pair $C=(\mathcal{X},\mathcal{K})$, where $\mathcal{X}=\{X_1,\ldots,X_r\}$ is a partition of $\mathrm{Irr}(G)$, the set of irreducible characters of G, and $\mathcal{K}$ is a partition of G, such that: (1) $\{1\}\in\mathcal{K}$, (2) $|\mathcal{X}|=|\mathcal{K}|$ and (3) $\sigma_X:=\sum\limits_{\psi\in X}\psi(1)\psi$ is constant for each $X\in\mathcal{X}$ and $K\in\mathcal{K}$.

The aim of our paper is to extend the notions of quasi-monomial and almost monomial groups in the framework of supercharacter theories, and to discuss the connections with the supercharacter theoretic Artin conjecture, introduced by Wong (see [16]), which states that $L(s,\sigma_X)$ are holomorphic in $\mathbb{C}\setminus\{1\}$ for each $X\in\mathcal{X}$.

We say that G is C-quasi-monomial if for each $X\in\mathcal{X}$, there exist some subgroups $H_1,\ldots,H_t$ of G and some linear characters $\lambda_1,\ldots,\lambda_t$ such that $\lambda_1^G+\ldots+\lambda_t^G=d\sigma_X$, see Definition 3.3. We prove that the class of C-quasi-monomial groups is closed under factorization and taking direct products, see Theorems 3.6 and 3.8. In Proposition 4.1, we note that a C-quasi-monomial group satisfies the supercharacter theoretic Artin conjecture.

We say that G is *C-almost monomial* if for any $k\neq l$, there exist some subgroups $H_1,\ldots,H_t\leqslant G$ and linear characters $\lambda_1,\ldots,\lambda_t$ of $H_1,\ldots,H_t$ such that: $\lambda_1^G+\ldots+\lambda_t^G=\sum\limits_{i=1}^{m}\alpha_i\sigma_{X_i}$, where $\alpha_i\geqslant 0$ are integers with $\alpha_k>0$ and $\alpha_l=0$, see Definition 3.9. We prove that the class of C-almost monomial groups is closed under factorization and taking direct products, see Theorems 3.14 and 3.15.

Let $F_1:=L(s,\sigma_{X_1}),\ldots,F_m:=L(s,\sigma_{X_m})$ and let $\mathrm{Hol}(C,s_0)$ be the semigroup of the functions of the form $F:=F_1^{a_1}\ldots F_m^{a_m}$, where $a_i\geqslant 0$ are integers which are holomorphic at s_0. In Theorem 4.3, we prove that if G is C-almost monomial, then the supercharacter theoretic Artin conjecture holds at s_0 if and only if $\mathrm{Hol}(C,s_0)$ is factorial. Also, in Theorem 4.4, we prove that if G is C-almost monomial and s_0 is not a common zero for any two distinct L-functions F_l and F_k, where $k\neq l\in\{1,\ldots,m\}$, then the supercharacter theoretic Artin conjecture holds at s_0. These results generalize the aforementioned results on almost-monomial groups.

2. Preliminaries

We recall that a finite group G is monomial (or an M-group) if for any $\chi \in \mathrm{Irr}(G)$ there exists a subgroup $H \leqslant G$ and a linear character λ of H such that $\lambda^G = \chi$. Any Abelian group G is monomial, since all the irreducible characters of G are linear, but the converse is not true. According to Taketa's Theorem (see [15]), every monomial group is solvable, but there are solvable groups which are not monomial, the smallest example being $\mathrm{SL}(2,3)$. A slight generalization of monomial groups is the following:

Definition 2.1. A finite group G is called *quasi-monomial* (or an QM-group) if for any $\chi \in \mathrm{Irr}(G)$ there exists a subgroup $H \leqslant G$ and a linear character λ of H such that $\lambda^G = d\chi$, where d is a positive integer.

It is not known if there are quasi-monomial groups which are not monomial.

For a character ψ of G, we denote by $\mathrm{Cons}(\psi)$ the set of constituents of ψ. We recall the following definition, which generalizes quasi-monomial groups:

Definition 2.2 ([14])**.** A finite group G is called *almost monomial* (or AM-group) if for every two distinct characters $\chi \neq \psi \in \mathrm{Irr}(G)$ there exists a subgroup H of G and a linear character λ of H such that $\chi \in \mathrm{Cons}(\lambda^G)$ and $\psi \notin \mathrm{Cons}(\lambda^G)$.

An important class of almost monomial groups are the symmetric groups, S_n, see [7], Theorem 1.1. If G is an almost monomial group and $N \trianglelefteq G$ is a normal subgroup, then G/N is almost monomial, see [7], Theorem 2.2. Also, if G, G' are finite groups, then $G \times G'$ is almost monomial if and only if G and G' are almost monomial, see [7], Theorem 2.3.

The following result provides an equivalent form of Definition 2.2 and shows that there is a kind of "duality" between the notions of quasi-monomial and almost monomial groups.

Proposition 2.3. *Let G be a finite group and assume that* $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_r\}$. *Then, the following are equivalent:*

(1) *G is almost monomial.*

(2) *For any $k \in \{1, \ldots, r\}$, there exist some subgroups $H_1, \ldots, H_m$ of G and some linear characters $\lambda_1, \ldots, \lambda_m$ of $H_1, \ldots, H_m$ such that:*

$$\mathrm{Cons}(\lambda_1^G + \ldots + \lambda_m^G) = \mathrm{Irr}(G) \setminus \{\chi_k\}.$$

P r o o f. (1) $\Rightarrow$ (2). Without loss of generality, we can assume that $k = r$. According to Definition 2.2, for any $1 \leqslant j \leqslant r-1 =: m$, there exists a subgroup $H_j \leqslant G$ and a linear character λ_j of H_j such that $\chi_j \in \mathrm{Cons}(\lambda_j^G)$ and $\chi_r \notin \mathrm{Cons}(\lambda_j^G)$. It follows that $\mathrm{Cons}(\lambda_1^G + \ldots + \lambda_m^G) = \{\chi_1, \ldots, \chi_{r-1}\}$, as required.

(2) $\Rightarrow$ (1). We fix $1 \leqslant i, k \leqslant r$ with $k \neq i$ and assume Condition (2) is satisfied for k. It follows that there exists $1 \leqslant j \leqslant m$ such that $\chi_i \in \mathrm{Cons}(\lambda_j^G)$. On the other hand, $\chi_k \notin \mathrm{Cons}(\lambda_j^G)$, hence G is almost monomial. □

Example 2.4. Let A_5 be the alternating group of order 5. Since A_5 is a simple non-Abelian group, it is not solvable. Therefore, A_5 is not monomial. However, A_5 is almost monomial: We have that $\mathrm{Irr}(A_5) = \{\chi_1, \chi_2, \chi_3, \chi_4, \chi_5\}$, where χ_1 is the trivial character, χ_2 and χ_3 are conjugated characters of degree 3, χ_4 has degree 4 and χ_5 has degree 5. Obviously, χ_1 is linear. Also, one can check that χ_5 is monomial. Let $H = \langle(12345)\rangle \subset A_5$, which is isomorphic to the cyclic group of order 5. The characters induced from the irreducible (linear) characters of H are $\chi_1+\chi_2+\chi_3+\chi_5$, $\chi_2+\chi_4+\chi_5$ and $\chi_3+\chi_4+\chi_5$.

Let $U = \langle(12)(45), (345)\rangle \subset A_5$, which is isomorphic to S_3. Let $\psi\colon U \to \{\pm 1\}$ be the sign function on U, which is a linear character. We have that $\psi^{A_5} = \chi_2+\chi_3+\chi_4$. From Proposition 2.3 it follows that A_5 is almost monomial.

Let $K/\mathbb{Q}$ be a finite Galois extension. For the character χ of a representation of the Galois group $G := \mathrm{Gal}(K/\mathbb{Q})$ on a finite-dimensional complex vector space, let $L(s,\chi) := L(s,\chi,K/\mathbb{Q})$ be the corresponding Artin L-function, see [3], page 296. Artin conjectured that $L(s,\chi)$ is holomorphic in $\mathbb{C}\setminus\{1\}$. Brauer proved that $L(s,\chi)$ is meromorphic in $\mathbb{C}$. Let $\chi_1,\dots,\chi_r$ be the irreducible characters of G, and $f_1 = L(s,\chi_1),\dots,f_r = L(s,\chi_r)$ the corresponding Artin L-functions.

For two characters φ and ψ of G, $L(s,\varphi+\psi) = L(s,\varphi)\cdot L(s,\psi)$, so the set of L-functions corresponding to all characters of G is a multiplicative semigroup, denoted by Ar.

Since any character of G is a linear combination with positive integer coefficients of irreducible characters, the semigroup Ar is generated by $f_1,\dots,f_r$, that is

$$\mathrm{Ar} := \{f_1^{k_1}\cdot\ldots\cdot f_r^{k_r} :\ k_1 \geqslant 0,\dots,k_r \geqslant 0\}.$$

Since $f_1,\dots,f_r$ are multiplicatively independent, see [2], Satz 5, page 106, it follows that Ar is factorial of rank r; in other words, Ar is isomorphic to $\mathbb{Z}^r_{\geqslant 0}$. Moreoever, Nicolae in [12] proved that $f_1,\dots,f_r$ are algebraically independent over $\mathbb{C}$, a result extended later in [6], where it was proved that $f_1,\dots,f_r$ are algebraically independent over the field of meromorphic functions of order < 1.

For $s_0 \in \mathbb{C}, s_0 \neq 1$ let $\mathrm{Hol}(s_0)$ be the subsemigroup of Ar consisting of the L-functions which are holomorphic at s_0. Nicolae in [13] proved that $\mathrm{Hol}(s_0)$ is an affine subsemigroup of Ar, isomorphic to an affine subsemigroup of $\mathbb{Z}^r_{\geqslant 0}$. Artin's conjecture at s_0 can be stated as: $\mathrm{Hol}(s_0) = \mathrm{Ar}$. We end this section by recalling the following results:

Theorem 2.5 ([14])**.** *If* $G = \mathrm{Gal}(K/\mathbb{Q})$ *is almost monomial and* $s_0 \in \mathbb{C} \setminus \{1\}$, *then the following assertions are equivalent:*

(1) *Artin's conjecture is true at* s_0: $\mathrm{Hol}(s_0) = \mathrm{Ar}$.

(2) *The semigroup* $\mathrm{Hol}(s_0)$ *is factorial.*

Theorem 2.6 ([7])**.** *If* $G = \mathrm{Gal}(K/\mathbb{Q})$ *is almost monomial, and* s_0 *is not a common zero for any two distinct L-functions* f_k *and* f_l, *then all Artin L-functions of* $K/\mathbb{Q}$ *are holomorphic at* s_0.

3. Supercharacter theoretic quasi and almost monomial groups

Diaconis and Isaacs in [8] introduced the theory of supercharacters as follows:

Definition 3.1. Let G be a finite group. Let $\mathcal{K}$ be a partition of G and let $\mathcal{X}$ be a partition of $\mathrm{Irr}(G)$. The ordered pair $C := (\mathcal{X}, \mathcal{K})$ is a *supercharacter theory* if:

(1) $\{1\} \in \mathcal{K}$,

(2) $|\mathcal{X}| = |\mathcal{K}|$, and

(3) for each $X \in \mathcal{X}$, the character $\sigma_X = \sum_{\psi \in X} \psi(1)\psi$ is constant on each $K \in \mathcal{K}$.

The characters σ_X are called *supercharacters*, and the elements K in $\mathcal{K}$ are called *superclasses*. We denote by $\mathrm{Sup}(G)$ the set of supercharacter theories of G.

Diaconis and Isaacs showed that their theory enjoys properties similar to the classical character theory. For example, every superclass is a union of conjugacy classes in G, see [8], Theorem 2.2. The irreducible characters and conjugacy classes of G give a supercharacter theory of G, which will be referred to as the *classical theory* of G.

Also, as noted in [8], every group G admits a non-classical theory with only two supercharacters 1_G and $\mathrm{Reg}(G) - 1_G$ and superclasses $\{1\}$ and $G \setminus \{1\}$, where 1_G denotes the trivial character of G and

$$\mathrm{Reg}(G) = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)\chi$$

is the regular character of G. This theory will be called the *maximal theory* of G.

Let $C = (\mathcal{X}, \mathcal{K})$ and $C' = (\mathcal{X}', \mathcal{K}')$ be two supercharacter theories of G. We write $\mathcal{X} \preceq \mathcal{X}'$ if every $X \in \mathcal{X}$ is a subset of some $X' \in \mathcal{X}'$. This is equivalent to saying that any $X' \in \mathcal{X}'$ is a union of parts of $\mathcal{X}$. According to [10], Corollary 3.4, $\mathcal{X} \preceq \mathcal{X}'$ if and only if $\mathcal{K} \preceq \mathcal{K}'$.

Definition 3.2 ([10], Definition 3.4)**.** We say that $C \preceq C'$ if $\mathcal{X} \preceq \mathcal{X}'$.

The set $(\operatorname{Sup}(G), \preceq)$ forms a poset with the minimal element being the classical theory of G and the maximal element being the maximal theory of G.

We introduce the following generalization of Definition 2.1:

Definition 3.3. Let G be a finite group and let $C := (\mathcal{X}, \mathcal{K})$ be a supercharacter theory on G. Assume that $\mathcal{X} = \{X_1, \ldots, X_m\}$. We say that G is *C-quasi-monomial* (or a C-QM-group), if for any $k \in \{1, \ldots, m\}$, there exists some subgroups $H_1, \ldots, H_t \leqslant G$ (not necessarily distinct) and linear characters $\lambda_1, \ldots, \lambda_t$ of $H_1, \ldots, H_t$ such that:

$$\lambda_1^G + \ldots + \lambda_t^G = d\sigma_{X_k},$$

where d is a positive integer.

Proposition 3.4. *Let G be a finite group. Then the following hold:*

(1) *If $(\mathcal{X}, \mathcal{K})$ is the classical theory of G, then G is quasi-monomial in the sense of Definition 2.1 if and only if G is C-quasi-monomial in the sense of Definition 3.3.*

(2) *If $C, C' \in \operatorname{Sup}(G)$ with $C \preceq C'$ and G is C-quasi-monomial, then G is C'-quasi-monomial.*

(3) *If C is the maximal theory of G, then G is C-quasi-monomial.*

Proof. (1) and (2) are obvious.

(3) According to the Aramata-Brauer Theorem (see [1] and [4]) $\operatorname{Reg}(G) - 1_G$ can be written as a positive rational linear combination of induced linear characters. It follows that there exist some subgroups $H_1, \ldots, H_t \leqslant G$ (not necessarily distinct) and linear characters $\lambda_1, \ldots, \lambda_t$ of $H_1, \ldots, H_t$ such that

$$\lambda_1^G + \ldots + \lambda_t^G = d(\operatorname{Reg}(G) - 1_G),$$

where d is a positive integer. On the other hand, $(1_G)^G = 1_G$. Thus, we get the required result. □

Let G be a finite group and let $N \trianglelefteq G$ be a normal subgroup of G. It is well known that $\operatorname{Irr}(G/N)$ is in bijection with the set

$$\{\chi \in \operatorname{Irr}(G)\colon N \subset \operatorname{Ker}(\chi)\}.$$

For a character $\widetilde{\chi} \in \operatorname{Irr}(G/N)$, we denote by χ the corresponding character in $\operatorname{Irr}(G)$, that is $\chi(g) := \widetilde{\chi}(\hat{g})$ for all $g \in G$, where $\hat{g}$ is the class of g in G/N.

Lemma 3.5. *Let G be a finite group, $H \leqslant G$ a subgroup and $N \trianglelefteq G$ a normal subgroup. Let λ be a linear character of H such that $N \subset \operatorname{Ker}(\lambda^G)$. Then:*

(1) *$H \cap N \subset \operatorname{Ker}(\lambda)$, hence $\tilde{\lambda}\colon HN/N \to \mathbb{C}^*$, $\tilde{\lambda}(hN) := \lambda(h)N$, is a linear character of HN/N.*

(2) *For any character χ of G with $N \subset \operatorname{Ker}(\chi)$, we have that $\langle \tilde{\lambda}^{G/N}, \widetilde{\chi} \rangle = \langle \lambda^G, \chi \rangle$.*

P r o o f. (1) We assume that $H \cap N \not\subseteq \operatorname{Ker}(\lambda)$. Then $\lambda_{H\cap N}$ is a nontrivial linear character of $H \cap N$. On the other hand, since $N \subset \operatorname{Ker}(\lambda^G)$, it follows that $(\lambda^G)_{H\cap N} = |G : H| 1_{H\cap N}$. Therefore, by Frobenius reciprocity, we have that:

$$\langle (\lambda_{H\cap N})^H, (\lambda^G)_H \rangle = \langle \lambda_{H\cap N}, (\lambda^G)_{H\cap N} \rangle = 0. \tag{3.1}$$

On the other hand, we have that:

$$\langle (\lambda_{H\cap N})^H, \lambda \rangle = \langle \lambda_{H\cap N}, \lambda_{H\cap N} \rangle = 1. \tag{3.2}$$

From (3.1) and (3.2) it follows that

$$0 = \langle \lambda, (\lambda^G)_H \rangle = \langle \lambda^G, \lambda^G \rangle,$$

and we get a contradiction.

(2) By Frobenius reciprocity, we have that:

$$\begin{aligned}\langle \tilde{\lambda}^{G/N}, \widetilde{\chi} \rangle = \langle \tilde{\lambda}, \widetilde{\chi}|_{HN/N} \rangle = \frac{|H\cap N|}{|H|} \sum_{\widetilde{h} \in HN/N} \tilde{\lambda}(\widetilde{h}) \overline{\widetilde{\chi}(\widetilde{h})} = \frac{1}{|H|} \sum_{h\in H} \lambda(h) \overline{\chi(h)} \\ = \langle \lambda, \chi_H \rangle = \langle \lambda^G, \chi \rangle,\end{aligned}$$

hence, we are done. □

Let G be a finite group and let $C := (\mathcal{X}, \mathcal{K})$ be a supercharacter theory of G. Let $N \trianglelefteq G$ be a normal subgroup of G. The group N is called *C-normal* or *supernormal* if N is a union of superclasses from C, see [10] and [11]. Let $X \in \mathcal{X}$. By abuse of notation, we write $X \subset \operatorname{Irr}(G/N)$ if $N \subset \operatorname{Ker}(\chi)$ for all $\chi \in X$. If $X \subset \operatorname{Irr}(G/N)$, we denote $\widetilde{X} = \{\widetilde{\chi} \colon \chi \in X\}$. Let $K \in \mathcal{K}$. We denote $\widetilde{K} := KN/N \subset G/N$.

Now, assume that N is C-normal. Without loss of generality, we can assume that $X_i \subset \operatorname{Irr}(G/N)$ for $1 \leqslant i \leqslant p$ and $X_i \subsetneq \operatorname{Irr}(G/N)$ for $p+1 \leqslant i \leqslant m$. Let $\widetilde{\mathcal{X}} := \{\widetilde{X_1}, \ldots, \widetilde{X_p}\}$ and $\widetilde{\mathcal{K}} := \{\widetilde{K}_1, \ldots, \widetilde{K}_p\}$. According to [10], Proposition 6.4, the pair $\widetilde{C} := C^{G/N} = (\widetilde{\mathcal{X}}, \widetilde{\mathcal{K}})$ is a supercharacter theory of G/N.

Theorem 3.6. *With the above notations, if G is C-quasi-monomial and $N \trianglelefteq G$ is a C-normal subgroup of G, then G/N is $C^{G/N}$-quasi-monomial.*

P r o o f. Let $\widetilde{X_k} \in \widetilde{\mathcal{X}}$. Since G is C-quasi-monomial, there exist some subgroups $H_1, \ldots, H_t \leqslant G$ and some linear characters $\lambda_1, \ldots, \lambda_t$ of $H_1, \ldots, H_t$ such that

$$\lambda_1^G + \ldots + \lambda_t^G = d\sigma_{X_k},$$

where d is a positive integer. We fix an index i with $1 \leqslant i \leqslant t$. Since $\operatorname{Cons}(\lambda_i^G) \subset X_k$, and for any $\chi \in X_k$, we have $N \subset \operatorname{Ker}(\chi)$, it follows that $N \subset \operatorname{Ker}(\lambda_i^G)$. From

Lemma 3.5 (1), it follows that $H_i \cap N \subset \mathrm{Ker}(\lambda_i)$ and thus $\widetilde{\lambda_i}\colon H_iN/N \to \mathbb{C}^*$, $\widetilde{\lambda_i}(h_iN) = \lambda_i(h_i)$, is a linear character of the subgroup H_iN/N of G/N. From Lemma 3.5 (2) and straightforward computations, it follows that:

$$\widetilde{\lambda_1}^{G/N} + \ldots + \widetilde{\lambda_t}^{G/N} = d\sigma_{\widetilde{X_k}},$$

and thus G/N is C-quasi-monomial. □

We recall the following result:

Lemma 3.7 ([10], Proposition 8.1). *Let G and G' be two finite groups and let $C = (\mathcal{X}, \mathcal{K})$ and $C' = (\mathcal{X}', \mathcal{K}')$ be supercharacter theories of G and G', respectively. Then $C \times C' = (\mathcal{X} \times \mathcal{X}', \mathcal{K} \times \mathcal{K}')$ is a supercharacter theory of the direct product $G \times G'$.*

Theorem 3.8. *Let G and G' be two finite groups and let $C = (\mathcal{X}, \mathcal{K})$ and $C' = (\mathcal{X}', \mathcal{K}')$ be supercharacter theories of G and G', respectively. Then the following are equivalent:*

(1) *G is C-quasi-monomial and G' is C'-quasi-monomial.*

(2) *$G \times G'$ is $C \times C'$-quasi-monomial.*

Proof. (1) $\Rightarrow$ (2) Let $X \in \mathcal{X}$ and $X' \in \mathcal{X}'$. From hypothesis, there exist $H_1, \ldots, H_t \leqslant G$, $\lambda_1, \ldots, \lambda_t$, linear characters of $H_1, \ldots, H_t$ such that

$$\lambda_1^G + \ldots + \lambda_t^G = d\sigma_{X_k},$$

where $d \geqslant 1$ is an integer. Also, there exist $H'_1, \ldots, H'_{t'} \leqslant G'$, $\mu_1, \ldots, \mu_{t'}$, linear characters of $H'_1, \ldots, H'_{t'}$ such that

$$\mu_1^{G'} + \ldots + \mu_{t'}^{G'} = d'\sigma_{X'},$$

where $d' \geqslant 1$ is an integer. We consider the subgroups $H_i \times H'_{i'}$ of $G \times G'$ and the linear characters $\lambda_i \times \mu_{i'}$ of $H_i \times H'_{i'}$, where $1 \leqslant i \leqslant t$ and $1 \leqslant i' \leqslant t'$. A straightforward computation shows that

$$\sum_{i=1}^{t} \sum_{i'=1}^{t'} (\lambda_i \times \mu_{i'})^{G \times G'} = dd'\sigma_{X \times X'},$$

thus, $G \times G'$ is $C \times C'$-quasi-monomial.

(2) $\Rightarrow$ (1) The group G' can be seen as a $C \times C'$-normal subgroup of $G \times G'$, hence, by Theorem 3.6, it follows that G is C-quasi-monomial. □

We introduce the following generalization of both Definitions 2.2 and 3.3:

Definition 3.9. Let G be a finite group and let $C = (\mathcal{X}, \mathcal{K})$ be a supercharacter theory on G. Assume that $\mathcal{X} = \{X_1, \ldots, X_m\}$. We say that G is *C-almost monomial* if for any $k \neq l$ there exist some subgroups $H_1, \ldots, H_t \leqslant G$ (not necessarily distinct) and linear characters $\lambda_1, \ldots, \lambda_t$ of $H_1, \ldots, H_t$ such that:

$$\lambda_1^G + \ldots + \lambda_t^G = \sum_{i=1}^{m} \alpha_i \sigma_{X_i},$$

where $\alpha_i \geqslant 0$ are integers with $\alpha_k > 0$ and $\alpha_l = 0$.

Proposition 3.10. *If G is a finite group and $C = (\mathcal{X}, \mathcal{K})$ is the classical theory on G, then G is C-almost monomial, in the sense of Definition* 3.9, *if and only if G is almost monomial in the sense of Definition* 2.2.

Proof. Assume that $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_r\}$, $d_j = \chi_j(1)$ for $j \in \{1, \ldots, r\}$, and let $d = \mathrm{lcm}(d_1, \ldots, d_r)$. If G is almost monomial, then for any $k \neq l$, there exists a subgroup H of G and a linear character λ of H such $\chi_k \in \mathrm{Cons}(\lambda^G)$ and $\chi_l \notin \mathrm{Cons}(\lambda^G)$. Then

$$d\lambda^G = \alpha_1(d_1\chi_1) + \ldots + \alpha_r(d_r\chi_r)$$

for some integers $\alpha_j \geqslant 0$ with $\alpha_k > 0$ and $\alpha_l = 0$.

Conversely, if G is $(\mathcal{X}, \mathcal{K})$-almost monomial, then there exist $H_1, \ldots, H_t \leqslant G$, subgroups of G, and linear characters $\lambda_1, \ldots, \lambda_t$ of $H_1, \ldots, H_t$ such that:

$$\lambda_1^G + \ldots + \lambda_t^G = \alpha_1(d_1\chi_1) + \ldots + \alpha_r(d_r\chi_r),$$

where $\alpha_j \geqslant 0$ are integers with $\alpha_k > 0$ and $\alpha_l = 0$. In particular, $\chi_l \notin \mathrm{Cons}(\lambda_j^G)$ for any $j \in \{1, \ldots, r\}$ and there exists $j_0 \in \{1, \ldots, r\}$ with $\chi_k \in \mathrm{Cons}(\lambda_{j_0}^G)$. We choose $H = H_{j_0}$ and $\lambda = \lambda_{j_0}$ and we note that $\chi_k \in \mathrm{Cons}(\lambda^G)$ and $\chi_l \notin \mathrm{Cons}(\lambda^G)$. Thus, G is almost monomial. □

The following result generalizes Proposition 2.3 and its proof is similar to the proof of Proposition 2.3, so we skip it.

Proposition 3.11. *Let G be a finite group and let $C = (\mathcal{X}, \mathcal{K})$ be a supercharacter theory on G, where $\mathcal{X} = \{X_1, \ldots, X_m\}$. Then the following are equivalent:*

(1) *G is C-almost monomial.*

(2) *For any $k \in \{1, \ldots, m\}$, there exist some subgroups $H_1, \ldots, H_s$ of G and some linear characters $\lambda_1, \ldots, \lambda_s$ of $H_1, \ldots, H_s$ such that:*

$$\lambda_1^G + \ldots + \lambda_m^G = \alpha_1\sigma_{X_1} + \ldots + \alpha_{k-1}\sigma_{X_{k-1}} + \alpha_{k+1}\sigma_{X_{k-1}} + \ldots + \alpha_m\sigma_{X_m},$$

where $\alpha_i > 0$ are some integers.

For a finite group G, we may ask if $C, C' \in \mathrm{Sup}(G)$ with $C \preceq C'$ and G is C-almost monomial then G is C'-almost monomial also, as in the quasi-monomial case, see Proposition 3.4 (2). The following example shows that this phenomenon is not always true:

Example 3.12. Let $G = \mathrm{SL}(2,3)$ be the special linear group of degree two over a field of three elements. It is well known that G is solvable, but it is not monomial. However, G is almost monomial. The group G has 7 irreducible characters: $\chi_1 = 1_G$, χ_2, χ_3 are linear, χ_4, χ_5, χ_6 have the degree 2 and χ_7 has the degree 3. The characters χ_1, χ_2, χ_3 and χ_7 are monomial, but χ_4, χ_5 and χ_6 are not. Also, $\chi_5 = \chi_2\chi_4$ and $\chi_6 = \chi_3\chi_4$. Moreover, $\chi_{45} := \chi_4 + \chi_5$, $\chi_{46} := \chi_4 + \chi_6$ and $\chi_{456} := \chi_4 + \chi_5 + \chi_6$ are monomial, and any monomial character of G is a positive linear combination of χ_1, χ_2, χ_3, χ_7, χ_{45}, χ_{46} and χ_{456}. We let:

$$\mathcal{X} := \{X_1 := \{\chi_1\},\ X_2 := \{\chi_2, \chi_3\},\ X_3 := \{\chi_4\},\ X_4 := \{\chi_5, \chi_6\}, X_5 := \{\chi_7\}\}.$$

One can easily check that there exists a partition $\mathcal{K}$ of G such that $C = (\mathcal{X}, \mathcal{K})$ is a supercharacter theory of G ($\mathcal{K}$ is the set of classes for the equivalence relation $g \sim g'$ if and only if $\sigma_{X_i}(g) = \sigma_{X_j}(g')$ for all $1 \leqslant i, j \leqslant 5$).

We claim that G is not C-almost monomial. Indeed, we cannot find subgroups $H_1, \ldots, H_t$ and linear characters $\lambda_1, \ldots, \lambda_t$ of $H_1, \ldots, H_t$ such that

$$\lambda_1^G + \ldots + \lambda_t^G = \alpha_1\sigma_{X_1} + \alpha_2\sigma_{X_2} + \alpha_3\sigma_{X_3} + \alpha_5\sigma_{X_5},$$

with $\alpha_3 > 0$, since for any $H \leqslant G$ and $\lambda \in \mathrm{Lin}(H)$ with $\chi_4 \in \mathrm{Cons}(\lambda^G)$, one has $\chi_5 \in \mathrm{Cons}(\lambda^G)$ or $\chi_6 \in \mathrm{Cons}(\lambda^G)$. Contradiction by Proposition 3.11.

We let: $\mathcal{X}' := \{X_1' := \{\chi_1\},\ X_2' := \{\chi_2, \chi_3\},\ X_3' := \{\chi_4, \chi_5, \chi_6\},\ X_4' := \{\chi_7\}\}$. Then, there exists a partition $\mathcal{K}'$ of G such that $C' = (\mathcal{X}', \mathcal{K}')$ is a supercharacter theory of G. Since χ_1, χ_2, χ_3, χ_{456} and χ_7 are monomial, it follows that G is C'-quasi-monomial.

Lemma 3.13. *Let G be a finite group, $H \leqslant G$ a subgroup and $N \trianglelefteq G$ a normal subgroup. Let λ be a linear character of H and χ an irreducible character of G with $N \subseteq \mathrm{Ker}(\chi)$. If $H \cap N \nsubseteq \mathrm{Ker}(\lambda)$, then $\chi \notin \mathrm{Cons}(\lambda^G)$.*

Proof. Since $H \cap N \nsubseteq \mathrm{Ker}(\lambda)$, it follows that $\lambda_{H\cap N}$ is a nontrivial linear character of $H \cap N$. Since $N \subseteq \mathrm{Ker}(\chi)$, it follows that $\chi_{H\cap N} = \chi(1)1_{H\cap N}$. Therefore,

$$0 = \langle \lambda_{H\cap N}, \chi_{H\cap N} \rangle = \langle (\lambda_{H\cap N})^H, \chi_H \rangle. \tag{3.3}$$

Since $\lambda \in \mathrm{Cons}((\lambda_{H\cap N})^H)$, from (3.3) it follows that

$$0 = \langle \lambda, \chi_H \rangle = \langle \lambda^G, \chi \rangle,$$

thus, $\chi \notin \mathrm{Cons}(\lambda^G)$, as required. □

The following result generalizes Theorem 2.2 of [7] and Theorem 3.6:

Theorem 3.14. *Let G be a finite group and let $C = (\mathcal{X}, \mathcal{K})$ be a supercharacter theory of G. Let $N \trianglelefteq G$ be a C-normal subgroup of G. If G is C-almost monomial, then G/N is $C^{G/N}$-almost monomial.*

Proof. Let $\widetilde{X_k} \in \widetilde{\mathcal{X}}$. Since G is C-almost monomial, from Proposition 3.11, it follows that there exist some subgroups $H_1, \ldots, H_s \leqslant G$ and some linear characters $\lambda_1, \ldots, \lambda_s$ of $H_1, \ldots, H_s$ such that

$$\lambda_1^G + \ldots + \lambda_s^G = \alpha_1 \sigma_{X_1} + \ldots + \alpha_{k-1} \sigma_{X_{k-1}} + \alpha_{k+1} \sigma_{X_{k+1}} + \ldots + \alpha_m \sigma_{X_m},$$

where $\alpha_i > 0$ are some integers.

Without loss of generality, from Lemma 3.13, we can assume that there exists $1 \leqslant t \leqslant s$ such that $H_i \cap N \subseteq \mathrm{Ker}(\lambda_j)$ for all $1 \leqslant j \leqslant t$ and $H_i \cap N \nsubseteq \mathrm{Ker}(\lambda_j)$ for all $t+1 \leqslant j \leqslant s$. From Lemma 3.5 (2), we can define the linear characters $\widetilde{\lambda_j}$ of $H_j N/N$ for $1 \leqslant j \leqslant t$, and, applying Lemma 3.13, we have:

$$\widetilde{\lambda_1}^G + \ldots + \widetilde{\lambda_s}^G = \alpha_1 \sigma_{\widetilde{X_1}} + \ldots + \alpha_{k-1} \sigma_{\widetilde{X_{k-1}}} + \alpha_{k+1} \sigma_{\widetilde{X_{k+1}}} + \ldots + \alpha_p \sigma_{\widetilde{X_p}},$$

and thus G/N is $C^{G/N}$-almost monomial. □

The following result generalizes Theorem 2.3 of [7] and Theorem 3.8:

Theorem 3.15. *Let G and G' be two finite groups and let $C = (\mathcal{X}, \mathcal{K})$ and $C' = (\mathcal{X}', \mathcal{K}')$ be supercharacter theories of G and G', respectively. Then the following are equivalent:*

(1) *G is C-almost monomial and G' is C'-almost monomial.*
(2) *$G \times G'$ is $C \times C'$-almost monomial.*

Proof. (1) ⇒ (2). Assume that $\mathcal{X} = \{X_1, \ldots, X_m\}$ and $\mathcal{X}' = \{X'_1, \ldots, X'_{m'}\}$. We fix

$$(k, k') \in \{1, \ldots, m\} \times \{1, \ldots, m'\}.$$

Since G is C-almost monomial, from Proposition 3.11 it follows that there exist some subgroups $H_1, \ldots, H_s$ of G, some linear characters $\lambda_1, \ldots, \lambda_s$ of $H_1, \ldots, H_s$, and some positive integers α_i such that:

$$\lambda_1^G + \ldots + \lambda_s^G = \alpha_1 \sigma_{X_1} + \ldots + \alpha_{k-1} \sigma_{X_{k-1}} + \alpha_{k+1} \sigma_{X_{k+1}} + \ldots + \alpha_m \sigma_{X_m}. \tag{3.4}$$

Similarly, there exists some subgroups $H'_1, \ldots, H'_{s'}$ of G', some linear characters $\lambda_1, \ldots, \lambda_{s'}$ of $H'_1, \ldots, H'_{s'}$, and some positive integers α'_i such that:

$$\lambda_1'^{G'} + \ldots + \lambda_{s'}'^{G'} = \alpha'_1 \sigma_{X'_1} + \ldots + \alpha'_{k'-1} \sigma_{X'_{k'-1}} + \alpha'_{k'+1} \sigma_{X'_{k'+1}} + \ldots + \alpha_m \sigma_{X'_m}. \tag{3.5}$$

If $\mathbf{1}$ is the unique character of the trivial subgroup of G, and $\mathbf{1}'$ is the unique character of the trivial subgroup of G', then

$$\text{(3.6)} \qquad \mathbf{1}^G = \operatorname{Reg}(G) = \sigma_{X_1} + \ldots + \sigma_{X_m}, \quad \mathbf{1}'^{G'} = \operatorname{Reg}(G') = \sigma_{X'_1} + \ldots + \sigma_{X'_{m'}}.$$

By straightforward computations, from (3.4), (3.5) and (3.6), it follows that:

$$\begin{aligned}(\lambda_1 \times 1_{G'})^{G\times G'} &+ \ldots + (\lambda_s \times 1_{G'})^{G\times G'} + (1_G \times \lambda'_1)^{G\times G'} + \ldots + (1_G \times \lambda'_{s'})^{G\times G'} \\ &= \sum_{i=1}^{m} \sum_{\substack{i'=1,\\ i'\neq k'}}^{m'} \alpha_{i'} \sigma_{X_i \times X'_{i'}} + \sum_{\substack{i=1,\\ i\neq k}}^{m} \sum_{i'=1}^{m'} \alpha_i \sigma_{X_i \times X'_{i'}} = \sum_{i=1}^{m} \sum_{i'=1}^{m'} a_{ii'} \sigma_{X_i \times X'_{i'}}.\end{aligned}$$

Note that $a_{ii'} > 0$ for all $(i,i') \in \{1,\ldots,m\} \times \{1,\ldots,m'\}$ with $(i,i') \neq (k,k')$ and $a_{kk'} = 0$. Therefore, from Proposition 3.11, it follows that $G \times G'$ is $C \times C'$-almost monomial.

(2) $\Rightarrow$ (1). Follows from Theorem 3.14, using a similar argument as in the proof of Theorem 3.8. □

4. Supercharacter theoretic Artin conjecture

Let G be a finite group. Let $C = (\mathcal{X}, \mathcal{K}) \in \operatorname{Sup}(G)$ be a supercharacter theory of G. We consider the multiplicative semigroup $\operatorname{Ar}(C)$ generated by $\{L(s,\sigma_X)\colon\ X \in \mathcal{X}\}$. Obviously, $\operatorname{Ar}(C)$ is a subsemigroup of Ar. Also, we consider

$$\operatorname{Hol}(C, s_0) = \operatorname{Hol}(s_0) \cap \operatorname{Ar}(C),$$

the semigroup of the L-functions associated to C, which are holomorphic at s_0.

Assume that $\mathcal{X} = \{X_1, \ldots, X_m\}$. For $1 \leqslant i \leqslant m$, we have that:

$$F_i := L(s, \sigma_{X_i}) = \prod_{\chi_j \in X_i} f_j^{d_j},$$

where $d_j := \chi_j(1)$, $1 \leqslant j \leqslant r$. The semigroup $\operatorname{Ar}(C)$ is generated by $F_1, \ldots, F_m$. It follows that $F_1, \ldots, F_m$ are also multiplicatively independent, hence the semigroup $\operatorname{Ar}(C)$ is factorial of rank m, i.e., it is isomorphic to $\mathbb{Z}^m_{\geqslant 0}$.

For $1 \leqslant i \leqslant m$, let $l_i = \operatorname{ord}_{s_0}(F_i)$, where $\operatorname{ord}_{s_0}(F_i)$ denotes the order of the meromorphic function F_i at s_0. We have that:

$$\operatorname{Hol}(C, s_0) = \{F_1^{a_1} \ldots F_m^{a_m}\colon\ a_1 l_1 + \ldots + a_m l_m \geqslant 0\}.$$

Hence, by Gordan's lemma, see for instance [5], Lemma 2.9, the semigroup $\operatorname{Hol}(C, s_0)$ is finitely generated. See also the proof of Theorem 1 of [13].

The supercharacter-theoretic variant of Artin's conjecture (or C-Artin conjecture) at s_0, see [16], Conjecture 1, can be stated as: $\operatorname{Hol}(C, s_0) = \operatorname{Ar}(C)$.

Proposition 4.1. *Let G be a finite group which is C-quasi-monomial. Then G satisfies the C-Artin conjecture for every $s_0 \in \mathbb{C} \setminus \{1\}$.*

Proof. Since G is C-quasi-monomial, for any $k \in \{1, \ldots, m\}$, there exist some subgroups $H_1, \ldots, H_t \leqslant G$ and linear characters $\lambda_1, \ldots, \lambda_t$ of $H_1, \ldots, H_t$ such that:

$$\lambda_1^G + \ldots + \lambda_t^G = d\sigma_{X_k},$$

where d is a positive integer. It follows that

$$F_k^d = \prod_{i=1}^{t} L(\lambda_i^G, s)$$

is holomorphic at s_0, hence F_k is holomorphic at s_0. □

Remark 4.2. If $G = \mathrm{Gal}(K/\mathbb{Q})$ is equipped with the maximal theory C, then, according to Proposition 3.4 (2), G is C-quasi-monomial. Hence, from Proposition 4.1, it follows that G satisfies the C-Artin conjecture at s_0. Note that the Artin L-functions attached to supercharacters with respect to the maximal theory are:

$$L(s, 1_G) = \zeta(s) \quad \text{and} \quad L(s, \mathrm{Reg}(G) - 1_G) = \zeta_K(s)/\zeta(s).$$

By a result of Aramata and Brauer (see [4]), we know that $\zeta_K(s)/\zeta(s)$ is holomorphic at s_0 and, of course, the Riemann-zeta function $\zeta(s)$ is holomorphic on $\mathbb{C} \setminus \{1\}$.

The following result generalizes Theorem 2.5:

Theorem 4.3. *Let $G = \mathrm{Gal}(K/\mathbb{Q})$, let $C = (\mathcal{X}, \mathcal{K})$ be a supercharacter theory of G, and let $s_0 \in \mathbb{C} \setminus \{1\}$. If G is C-almost monomial, then the following are equivalent:*

(1) *The supercharacter theoretic Artin conjecture is true at s_0:* $\mathrm{Hol}(C, s_0) = \mathrm{Ar}(C)$.

(2) *The semigroup* $\mathrm{Hol}(C, s_0)$ *is factorial.*

Proof. (1) ⇒ (2) Since the semigroup $\mathrm{Ar}(C)$ is factorial, there is nothing to prove.

(2) ⇒ (1) Suppose that the supercharacter theoretic Artin conjecture at s_0 is not true. Then, there exists $1 \leqslant k \leqslant m$ such that

$$\mathrm{ord}_{s_0}(F_k) < 0. \tag{4.1}$$

The Dedekind zeta function ζ_K of K can be decomposed as

$$\zeta_K = \prod_{i=1}^{r} f_i^{d_i} = F_1 \ldots F_m. \tag{4.2}$$

Since ζ_K is holomorphic in $\mathbb{C} \setminus \{1\}$, it follows that

$$\mathrm{ord}_{s_0}(\zeta_k) \geqslant 0. \tag{4.3}$$

From (4.1), (4.2) and (4.3) it follows that there exists $l \in \{1,\dots,m\}$ such that $\mathrm{ord}_{s_0}(F_l) > 0$. For $i \in \{1,\dots,m\}$, let $n_i := \min\{t\colon\ t \geqslant 0 \text{ and } \mathrm{ord}_{s_0}(F_l^t F_i) \geqslant 0\}$.

Since $f_1,\dots,f_r$ are multiplicatively independent, the functions $F_l^{n_1}F_1,\dots,F_l^{n_m}F_m$ are irreducible in $\mathrm{Hol}(C,s_0)$. The Hilbert basis $\mathcal{H}$ of $\mathrm{Hol}(C,s_0)$ is the uniquely determined minimal system of generators of $\mathrm{Hol}(C,s_0)$, hence $\mathrm{Hol}(C,s_0)$ is factorial if and only if $\mathcal{H}$ has m elements. It follows that $\mathcal{H} = \{F_l^{n_1}F_1,\dots,F_l^{n_m}F_m\}$.

From (4.1) it follows that $n_k > 0$. Since G is C-almost monomial, there exist some subgroups $H_1,\dots,H_t$ of G and linear characters $\lambda_1,\dots,\lambda_t$ of $H_1,\dots,H_t$ such that

$$\lambda_1^G + \ldots + \lambda_t^G = \alpha_1\sigma_{X_1} + \ldots + \alpha_m\sigma_{X_m}, \tag{4.4}$$

where $\alpha_i \geqslant 0$ are integers, $\alpha_k > 0$ and $\alpha_l = 0$. By Class Field Theory, for any $i \in \{1,\dots,n\}$, the Artin L-function $L(s,\lambda_i^G)$ is a Hecke L-function, so it is holomorphic at s_0. Hence, the function

$$F := \prod_{i=1}^{t} L(s,\lambda_i^G) = L(s,\lambda_1^G + \ldots + \lambda_t^G), \tag{4.5}$$

is holomorphic at s_0.

From (4.4) and (4.5) it follows that $F = F_1^{\alpha_1}\ldots F_m^{\alpha_m} \in \mathrm{Hol}(C,s_0)$. Since $\alpha_k > 0$ and $\alpha_l = 0$ this contradicts the fact that F is a product of elements of $\mathcal{H}$. $\square$

The following result generalizes Theorem 2.6:

Theorem 4.4. *Let $G = \mathrm{Gal}(K/\mathbb{Q})$ and let $C = (\mathcal{X},\mathcal{K})$ be a supercharacter theory of G with $\mathcal{X} = \{X_1,\dots,X_m\}$. If G is C-almost monomial and s_0 is not a common zero for any two distinct L-functions $L(s,\sigma_{X_l})$ and $L(s,\sigma_{X_k})$, where $k \neq l \in \{1,\dots,m\}$, then all Artin L-functions from $\mathrm{Ar}(C)$ are holomorphic at s_0, i.e., the supercharacter theoretic Artin conjecture is true at s_0.*

Proof. We assume that s_0 is a pole of F_j, that is, $\mathrm{ord}_{s_0}(F_j) < 0$. Since the Dedekind zeta function $\zeta_K = F_1\ldots F_m$ is holomorphic at s_0, there is an index $k \neq j$ such that $F_k(s_0) = 0$. Since G is C-almost monomial, there exist some subgroups $H_1,\dots,H_t \leqslant G$ and $\lambda_1,\dots,\lambda_t$ some linear characters on $H_1,\dots,H_t$ such that

$$\lambda_1^G + \ldots + \lambda_t^G = \alpha_1\sigma_{X_1} + \ldots + \alpha_m\sigma_{X_m},$$

with $\alpha_j > 0$ and $\alpha_k = 0$. The L-function

$$L(s,\lambda_1^G + \ldots + \lambda_t^G) = F_1^{\alpha_1}\ldots F_{k-1}^{\alpha_{k-1}}\cdot F_{k+1}^{\alpha_{k+1}}\ldots F_m^{\alpha_m},$$

is holomorphic at s_0. Since $\alpha_j > 0$ and $\mathrm{ord}_{s_0}(F_j) < 0$, it follows that there exists some index $l \notin \{j,k\}$ such that $F_l(s_0) = 0$, which contradicts the hypothesis. $\square$

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