Czechoslovak Mathematical Journal

Min Hu; Dinghuai Wang

The John-Nirenberg inequality for functions of bounded mean oscillation with bounded negative part

Czechoslovak Mathematical Journal, Vol. 72 (2022), No. 4, 1121–1131

Persistent URL: http://dml.cz/dmlcz/151134

Czechoslovak Mathematical Journal, 72 (147) (2022), 1121–1131

THE JOHN-NIRENBERG INEQUALITY FOR FUNCTIONS OF BOUNDED MEAN OSCILLATION WITH BOUNDED NEGATIVE PART

Min Hu, Dinghuai Wang, Wuhu

Received September 28, 2021. Published online May 5, 2022.

Abstract. A version of the John-Nirenberg inequality suitable for the functions $b \in \mathrm{BMO}$ with $b^- \in L^\infty$ is established. Then, equivalent definitions of this space via the norm of weighted Lebesgue space are given. As an application, some characterizations of this function space are given by the weighted boundedness of the commutator with the Hardy-Littlewood maximal operator.

Keywords: bounded mean oscillation; commutator; Hardy-Littlewood maximal operator, John-Nirenberg inequality

MSC 2020: 42B35, 42B25

1. Introduction

The bounded mean oscillation space (BMO) was introduced by John and Nirenberg in 1961 (see [10]), it is defined by the semi-norm

$$\|b\|_{\mathrm{BMO}} = \sup_Q \frac{1}{|Q|} \int_Q |b(x) - b_Q| \,\mathrm{d}x < \infty,$$

where $b_Q = (1/|Q|) \int_Q b(x) \,\mathrm{d}x$. The authors also established the John-Nirenberg inequality for a BMO function as follows. If $b \in \mathrm{BMO}$, there exist positive constants a_1 and a_2 such that for any cube Q and $t > 0$, we get

$$|\{x \in Q\colon\ |b(x) - b_Q| > t\}| \leqslant a_1 \mathrm{e}^{-a_2 t/\|b\|_{\mathrm{BMO}}} |Q|.$$

This project has been supported by the National Natural Science Foundation of China (Nos. 11971237, 12071223, 12101010) and Natural Science Foundation of China of Anhui Province (No. 2108085QA19)

DOI: 10.21136/CMJ.2022.0362-21

Subsequently, the theory of BMO has received wide attention. It is known that functions in a Sobolev space with critical exponent are embedded into the space of functions of bounded mean oscillation, and therefore satisfy the John-Nirenberg inequality and a corresponding exponential integrability estimate. In 2020, Martínez and Spector in [11] established an improvement to the John-Nirenberg inequality for functions in critical Sobolev spaces. Moreover, in the theory of boundedness of commutators, many results show that BMO function is the right set, see [3], [4], [5], [9]. The foundational paper of Coifman, Rochberg and Weiss (see [5]) proved that the commutator $[b,T](f) = bT(f) - T(bf)$ is bounded on some Lebesgue spaces if and only if b belongs to BMO, where T is the Riesz transform. The theory was then generalized and extended in several directions. For instance, Janson in [9] extended the result to the commutators of Calderón-Zygmund operators with smooth homogeneous kernels; Bloom in [3] investigated the same result in the two-weight case; and Uchiyama in [14] extended the boundedness result to compactness.

On the other hand, Bastero, Milman and Ruiz in 2000 (see [1]) studied the class of functions for which the commutator with the Hardy-Littlewood maximal function $[b,M]$ is bounded on the Lebesgue space, where

$$M(f)(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|\,\mathrm{d}y.$$

It is proved that the commutator $[b,M](f)(x) = b(x)M(f)(x) - M(bf)(x)$ is bounded on L^p, $1 < p < \infty$, if and only if $b \in \mathrm{BMO}$ with $b^- \in L^\infty$, where $b^-(x) = -\min\{b(x),0\}$. They also showed that $b \in \mathrm{BMO}$ with $b^- \in L^\infty$ if and only if

$$\|b\|_{\mathrm{BMO}_p^-} = \sup_Q \biggl(\frac{1}{|Q|}\int_Q |b(x) - M_Q(b)(x)|^p\,\mathrm{d}x\biggr)^{1/p} < \infty,$$

where

$$M_Q(b)(x) = \sup_{Q\supseteq Q'\ni x} \frac{1}{|Q'|}\int_{Q'} |b(y)|\,\mathrm{d}y.$$

Here, $1 \leqslant p < \infty$, and, for brevity, we put $\mathrm{BMO}^- = \mathrm{BMO}_1^-$.

An interesting question arises: can we establish the John-Nirenberg inequality suitable for those functions $b \in \mathrm{BMO}$ with $b^- \in L^\infty$? In this paper, we will give a positive answer as follows.

Theorem 1.1. *Suppose that $\|b\|_{\mathrm{BMO}^-} = 1$, then for any cube Q and $t > 0$, we have*

$$|\{x \in Q\colon\ |b(x) - M_Q(b)(x)| > t\}| \leqslant c_1 \mathrm{e}^{-c_2 t}|Q|,$$

where c_1 and c_2 are positive constants.

As an application, some characterizations of this function class will be given. We first recall the definition of the Muckenhoupt class. For a nonnegative locally integrable function ω on $\mathbb{R}^n$, if it satisfies the condition

$$\sup_Q \frac{1}{|Q|}\int_Q \omega(x)\,\mathrm{d}x\left(\frac{1}{|Q|}\int_Q \omega(x)^{-p'/p}\,\mathrm{d}x\right)^{p-1} < \infty, \quad 1 < p < \infty,$$

then the function ω is in the Muckenhoupt A_p class, see [12]. And ω belongs to the class A_1 if

$$\frac{1}{|Q|}\int_Q \omega(x)\,\mathrm{d}x\Big(\operatorname*{ess\,sup}_{x\in Q}\omega(x)^{-1}\Big) < \infty.$$

We write $A_\infty = \bigcup\limits_{1\leqslant p<\infty} A_p$. For a nonnegative locally integrable function ω on $\mathbb{R}^n$, ω is in the Muckenhoupt $A_{p,q}$ class if

$$\sup_Q\left(\frac{1}{|Q|}\int_Q \omega(x)^q\,\mathrm{d}x\right)^{1/q}\left(\frac{1}{|Q|}\int_Q \omega(x)^{-p'}\,\mathrm{d}x\right)^{1/p'} < \infty, \quad 1 < p, q < \infty.$$

From Theorem 1.1, it follows that BMO_p^- spaces are independent of the scale $p \in (1,\infty)$ in the sense of norm. We further consider the characterization connections with the weighted Lebesgue spaces. In 1975, Muckenhoupt and Wheeden in [13] proved that for all ω-locally integrable functions b such that

$$\|b\|_{\mathrm{BMO}_\omega} = \sup_Q \frac{1}{\omega(Q)}\int_Q |b(x) - b_{\omega,Q}|\omega(x)\,\mathrm{d}x < \infty,$$

where

$$\omega \in A_\infty, \quad \omega(Q) = \int_Q \omega(x)\,\mathrm{d}x \quad \text{and} \quad b_{\omega,Q} = \frac{1}{\omega(Q)}\int_Q b(x)\omega(x)\,\mathrm{d}x,$$
$$\mathrm{BMO} = \mathrm{BMO}_\omega \quad \text{and} \quad \|b\|_{\mathrm{BMO}} \approx \|b\|_{\mathrm{BMO}_\omega}.$$

Specifically, Ho in [7] also proved that

$$\|b\|_{\mathrm{BMO}} \approx \sup_Q\left(\frac{1}{\omega(Q)}\int_Q |b(x) - b_Q|^p\omega(x)\,\mathrm{d}x\right)^{1/p} < \infty.$$

Furthermore, Wang, Zhou and Teng in [17] proved a similar result in terms of $A_{p,q}$ weights. Inspired by the results above, we obtain the following conclusions.

Theorem 1.2. *Let $1 < p < \infty$ and $\omega \in A_p$. Then $b \in \mathrm{BMO}^-$ if and only if there exists a constant $C > 0$ such that for any Q,*

$$\frac{1}{\omega(Q)}\int_Q |b(x) - M_Q(b)(x)|^p\omega(x)\,\mathrm{d}x \leqslant C.$$

Theorem 1.3. *Let $1 < p < q < \infty$ and $\omega \in A_{p,q}$. Then $b \in \mathrm{BMO}^-$ if and only if there exists a constant $C > 0$ such that for any Q,*

$$\left(\int_Q |b(x) - M_Q(b)(x)|^q \omega(x)^q \, \mathrm{d}x\right)^{1/q} \leqslant C|Q|^{1/q-1/p}\left(\int_Q \omega(x)^p \, \mathrm{d}x\right)^{1/p}.$$

In fact, in the definition of BMO^- space, the (weighted) Lebesgue spaces can be replaced by variable Lebesgue spaces (see [15], [16]) or a ball Banach function space X, see [7], [8]. We leave the details to the interested reader.

Finally, applying Theorem 1.2, we can characterize BMO^- by the boundedness of commutators on weighted Lebesgue spaces.

Theorem 1.4. *Let $1 < p < \infty$ and $\omega \in A_p$. Then, the following are equivalent:*
(1) $b \in \mathrm{BMO}^-$;
(2) $[b, M]$ *is a bounded operator from $L^p(\omega)$ to $L^p(\omega)$.*

In this paper, we write $A \lesssim B$ if $A \leqslant CB$ for some constant C that can depend on the dimension, Lebesgue exponents, weight constants, and on various other constants appearing in the assumptions. In the above we do not track the dependence on the weight constants and we encourage the interested readers to do so.

2. Main lemma and proof of Theorem 1.1

Before proceeding with the proof of Theorem 1.1, we recall the following result. For a fixed cube Q, the BMO seminorm relative to Q is defined by

$$\|b\|_{\mathrm{BMO}(Q)} = \sup_{Q' \subset Q} \frac{1}{|Q'|}\int_{Q'} |b(x) - b_{Q'}| \, \mathrm{d}x.$$

Bennett, Devore and Sharpley in [2], Theorem 4.2, showed that there exists some $C_0 > 0$ such that for all $b \in \mathrm{BMO}(Q)$,

$$\|M_Q(b)\|_{\mathrm{BMO}(Q)} \leqslant C_0 \|b\|_{\mathrm{BMO}(Q)}. \tag{2.1}$$

In order to prove the John-Nirenberg inequality suitable for a BMO^- function, we need the following auxiliary lemma. The approach to this part is similar to that in [6] but we need to carefully use the properties of the local maximal function M_Q.

Lemma 2.1. *Suppose that $\|b\|_{\mathrm{BMO}^-} = 1$, then for any cube Q and $t > 0$, we have*

$$|\{x \in Q\colon |b(x) - (M_Q(b))_Q| > t\}| \leqslant b_1 \mathrm{e}^{-b_2 t}|Q|,$$

where b_1 and b_2 are positive constants.

P r o o f. Let $t > 0$ and Q be a fixed cube. Then

$$0 \leqslant (M_Q(b))_Q - b_Q \leqslant \|b\|_{\mathrm{BMO}^-} = 1,$$

which implies that $(M_Q(b))_Q$ is a well-defined constant depending on Q and b. Meanwhile, it follows from Proposition 4 in [1] and the assumption that $\|b\|_{\mathrm{BMO}^-} = 1$, that $\|b\|_{\mathrm{BMO}} \leqslant 2$, and the inequality (2.1) gives

$$\begin{aligned}\int_Q |b(x) - (M_Q(b))_Q|\,\mathrm{d}x &\leqslant \int_Q |b(x) - M_Q(b)(x)|\,\mathrm{d}x \\ &\quad + \int_Q |M_Q(b)(x) - (M_Q(b))_Q|\,\mathrm{d}x \\ &\leqslant \|b\|_{\mathrm{BMO}^-}|Q| + \|M_Q(b)\|_{\mathrm{BMO}(Q)}|Q| \\ &\leqslant (1 + 2C_0)|Q|.\end{aligned} \tag{2.2}$$

We write $E_Q = \{x \in Q \colon |b(x) - (M_Q(b))_Q| > t\}$, then

$$|E_Q| \leqslant \int_{E_Q} \frac{|b(x) - (M_Q(b))_Q|}{t}\,\mathrm{d}x \leqslant \frac{(1 + 2C_0)|Q|}{t}.$$

Write $F_1(t) = (1 + 2C_0)/t$, then $|E_Q| \leqslant F_1(t)|Q|$.

For a fixed $s > 1 + 2C_0$, applying the Calderón-Zygmund decomposition to $|b(x) - (M_Q(b))_Q|$, we obtain countably many disjoint cubes $\{Q_j\}$ such that $Q_j \subset Q$ and

(i) $s < (1/|Q_j|)\int_{Q_j} |b(x) - (M_Q(b))_Q|\,\mathrm{d}x \leqslant 2^n s$;

(ii) $|b(x) - (M_Q(b))_Q| \leqslant s$ for almost every $x \in Q \setminus \bigcup_j Q_j$.

Therefore, by (i), we have

$$\begin{aligned}|(M_{Q_j}(b))_{Q_j} - (M_Q(b))_Q| &= \frac{1}{|Q_j|}\int_{Q_j} |M_{Q_j}(b)(y) - (M_Q(b))_Q|\,\mathrm{d}y \\ &\leqslant \frac{1}{|Q_j|}\int_{Q_j} |b(y) - M_{Q_j}(b)(y)|\,\mathrm{d}y \\ &\quad + \frac{1}{|Q_j|}\int_{Q_j} |b(y) - (M_Q(b))_Q|\,\mathrm{d}y \\ &\leqslant 1 + 2^n s \leqslant 2^{n+1}s.\end{aligned}$$

We deal with the cases $t \leqslant 2^{n+1}s$ and $t > 2^{n+1}s$ separately.

Case 1: $t \leqslant 2^{n+1}s$. Then use the trivial estimate

$$|E_Q| \leqslant |Q| \leqslant \mathrm{e}^{-t}\mathrm{e}^{2^{n+1}s}|Q|. \tag{2.3}$$

Case 2: $t > 2^{n+1}s$. It follows from (ii) that $E_Q \subset \bigcup_j Q_j$ up to a set of measure zero. From this, (i) and (2.2), it follows that

$$\begin{aligned}
|E_Q| &\leqslant \sum_j |\{x \in Q_j \colon\ |b(x) - (M_Q(b))_Q| > t\}| \\
&\leqslant \sum_j |\{x \in Q_j \colon\ |b(x) - (M_{Q_j}(b))_{Q_j}| + |(M_{Q_j}(b))_{Q_j} - (M_Q(b))_Q| > t\}| \\
&\leqslant \sum_j |\{x \in Q_j \colon\ |b(x) - (M_{Q_j}(b))_{Q_j}| > t - 2^{n+1}s\}| \\
&\leqslant \sum_j F_1(t - 2^{n+1}s) \cdot |Q_j| \\
&\leqslant F_1(t - 2^{n+1}s) \sum_j \frac{1}{s} \int_{Q_j} |b(x) - (M_Q(b))_Q| \, \mathrm{d}x \\
&\leqslant \frac{F_1(t - 2^{n+1}s)(1 + 2C_0)}{s} |Q|.
\end{aligned}$$

Put $F_2(t) = F_1(t - 2^{n+1}s)(1 + 2C_0)/s$, then we have $|E_Q| \leqslant F_2(t)|Q|$.

Continuing this process infinitely, we can obtain for any $k \geqslant 2$, that $|E_Q| \leqslant F_k(t)|Q|$ with

$$F_k(t) = \frac{F_{k-1}(t - 2^{n+1}s)(1 + 2C_0)}{s} = \frac{(1 + 2C_0)^k}{s^{k-1}(t - (k-1)2^{n+1}s)}.$$

We now return to the estimate for $|E_Q|$. For any $t > 2^{n+1}s$, there exists some $k \in \mathbb{N}$ such that

$$k \cdot 2^{n+1}s < t \leqslant (k+1) \cdot 2^{n+1}s,$$

which shows that

$$\begin{aligned}
|E_Q| &\leqslant |\{x \in Q \colon\ |b(x) - (M_Q(b))_Q| > t\}| \\
&\leqslant |\{x \in Q \colon\ |b(x) - (M_Q(b))_Q| > k \cdot 2^{n+1}s\}| \\
&\leqslant F_k(k \cdot 2^{n+1}s)|Q| = \frac{1}{2^{n+1}} \Big(\frac{s}{1 + 2C_0}\Big)^{-k} |Q| \\
&\leqslant \frac{s}{2^{n+1}(1 + 2C_0)} \mathrm{e}^{-(t/2^{n+1}s)\ln(s/(1+2C_0))} |Q|,
\end{aligned} \tag{2.4}$$

since $-k \leqslant 1 - t/2^{n+1}s$.

Combining estimates (2.3) and (2.4), and setting $s = \mathrm{e}(1 + 2C_0)$, we see that

$$|\{x \in Q \colon\ |b(x) - (M_Q(b))_Q| > t\}| \leqslant b_1 \mathrm{e}^{-b_2 t} |Q|$$

for some positive constants b_1 and b_2, which proves the inequality of Lemma 2.1. □

We now proceed with the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $t > 0$, Q be a fixed cube and $\|b\|_{\mathrm{BMO}^-} = 1$. The inequality (2.1) shows that $M_Q(b) \in \mathrm{BMO}(Q)$ with $\|M_Q(b)\|_{\mathrm{BMO}(Q)} \leqslant 2C_0$. Combining the John-Nirenberg inequality applied to $M_Q(b)$ on the cube Q with the result of Bennett, Devore and Sharpley gives

$$|\{x \in Q\colon\ |M_Q(b)(x) - (M_Q(b))_Q| > t\}| \leqslant a_1 \mathrm{e}^{-a_2/2C_0 t}|Q|. \tag{2.5}$$

Let $c = \max\{a_1, b_1\}$ and $c_2 = \min\{a_2/2C_0, b_2\}$. We arrive at

$$\begin{aligned}\frac{1}{c|Q|}&\mathrm{e}^{c_2 t}|\{x \in Q\colon\ |b(x) - M_Q(b)(x)| > t\}|\\ &\leqslant \frac{1}{c|Q|}\mathrm{e}^{c_2 t}|\{x \in Q\colon\ |b(x) - (M_Q(b))_Q| > t\}|\\ &\quad+ \frac{1}{c|Q|}\mathrm{e}^{c_2 t}|\{x \in Q\colon\ |(M_Q(b))_Q - M_Q(b)(x)| > t\}|.\end{aligned}$$

By Lemma 2.1 and (2.5), there exist constants $c_1 = 2c$ and c_2 such that

$$|\{x \in Q\colon\ |b(x) - M_Q(b)(x)| > t\}| \leqslant c_1 \mathrm{e}^{-c_2 t}|Q|.$$

Thus, we have completed the proof of Theorem 1.1. □

3. Proofs of Theorems 1.2 to 1.4

Using the result of Theorem 1.1, we now give the proofs of Theorems 1.2 to 1.4.

Proof of Theorem 1.2. Suppose that $b \in \mathrm{BMO}^-$ with $\|b\|_{\mathrm{BMO}^-} = 1$. According to Theorem 1.1, there are two constants $c_1, c_2 > 0$ such that for any cube Q and $\lambda > 0$,

$$|\{x \in Q\colon\ |b(x) - M_Q(b)(x)| > \lambda\}| \leqslant c_1 \mathrm{e}^{-c_2\lambda}|Q|.$$

Since $\omega \in A_p \subset A_\infty$, for any measurable set $S \subset Q$ there exists a positive constant ε such that

$$\frac{\omega(S)}{\omega(Q)} \lesssim \Big(\frac{|S|}{|Q|}\Big)^{\varepsilon}.$$

This implies that for $S = \{x \in Q\colon\ |b(x) - M_Q(b)(x)| > \lambda\}$, we have

$$\frac{\omega(\{x \in Q\colon\ |b(x) - M_Q(b)(x)| > \lambda\})}{\omega(Q)} \lesssim \Big(\frac{|\{x \in Q\colon\ |b(x) - M_Q(b)(x)| > \lambda\}|}{|Q|}\Big)^{\varepsilon} \lesssim \mathrm{e}^{-c_2\varepsilon\lambda}.$$

Hence, for any cube Q,

(3.1)
$$\int_Q |b(x)-M_Q(b)(x)|^p\omega(x)\,\mathrm{d}x = p\int_0^\infty \lambda^{p-1}\omega(\{x\in Q\colon |b(x)-M_Q(b)(x)|>\lambda\})\,\mathrm{d}\lambda \lesssim \int_0^\infty \lambda^{p-1}\mathrm{e}^{-c_2\varepsilon\lambda}\omega(Q)\,\mathrm{d}\lambda \lesssim \omega(Q).$$

Similarly to the proof process in [1], we can obtain that $b\in\mathrm{BMO}$ with $b^-\in L^\infty$ is necessary for inequality (3.1) as follows.

Let $Q_1=\{x\in Q\colon b(x)\leqslant b_Q\}$ and $Q_2=Q\setminus Q_1$. Then
$$\int_{Q_1}|b(x)-b_Q|\,\mathrm{d}x=\int_{Q_2}|b(x)-b_Q|\,\mathrm{d}x.$$
It follows from Hölder's inequality, (3.1) and $\omega\in A_p$ that
$$\begin{aligned}\int_Q|b(x)-b_Q|\,\mathrm{d}x &= 2\int_{Q_1}|b(x)-b_Q|\,\mathrm{d}x\leqslant 2\int_{Q_1}|b(x)-M_Q(b)(x)|\,\mathrm{d}x\\ &\leqslant 2\Big(\int_Q|b(x)-M_Q(b)(x)|^p\omega(x)\,\mathrm{d}x\Big)^{1/p}\Big(\int_Q\omega(x)^{1-p'}\,\mathrm{d}x\Big)^{1/p'}\\ &\lesssim\Big(\int_Q\omega(x)\,\mathrm{d}x\Big)^{1/p}\Big(\int_Q\omega(x)^{1-p'}\,\mathrm{d}x\Big)^{1/p'}\lesssim|Q|.\end{aligned}$$
Therefore, we can conclude that $b\in\mathrm{BMO}$. Meanwhile,
$$\begin{aligned}\int_Q b^-(x)\,\mathrm{d}x &\leqslant \int_Q|M_Q(b)(x)-b(x)|\,\mathrm{d}x\\ &\leqslant\Big(\int_Q|b(x)-M_Q(b)(x)|^p\omega(x)\,\mathrm{d}x\Big)^{1/p}\Big(\int_Q\omega(x)^{1-p'}\,\mathrm{d}x\Big)^{1/p'}\\ &\lesssim\Big(\int_Q\omega(x)\,\mathrm{d}x\Big)^{1/p}\Big(\int_Q\omega(x)^{1-p'}\,\mathrm{d}x\Big)^{1/p'}\lesssim|Q|,\end{aligned}$$
which implies that $b^-\in L^\infty$ by Lebesgue's differentiation theorem, and the result follows from here. □

Proof of Theorem 1.3. Suppose that $\|b\|_{\mathrm{BMO}^-}=1$. Since $\omega\in A_{p,q}$, we have $\nu=\omega^q\in A_q\subset A_\infty$. Then for $S=\{x\in Q\colon |b(x)-M_Q(b)(x)|>\lambda\}$, there exists a positive constant ε such that
$$\frac{\nu(S)}{\nu(Q)}\lesssim\Big(\frac{|S|}{|Q|}\Big)^\varepsilon,$$
which shows that
$$\nu(\{x\in Q\colon |b(x)-M_Q(b)(x)|>\lambda\})\lesssim\mathrm{e}^{-c_2\varepsilon\lambda}\nu(Q).$$

Meanwhile, we conclude that

$$\begin{aligned}\|(b-M_Q(b))\chi_Q\|_{L^q(\nu)}^q &= q\int_0^\infty \lambda^{q-1}\nu(\{x\in Q\colon\ |b(x)-M_Q(b)(x)|>\lambda\})\,\mathrm{d}\lambda\\ &\lesssim \int_0^\infty \lambda^{q-1}\mathrm{e}^{-c_2\varepsilon\lambda}\nu(Q)\,\mathrm{d}\lambda\lesssim \nu(Q).\end{aligned}$$

By Hölder's inequality, we arrive at

$$|Q|\leqslant\left(\int_Q \omega(x)^p\,\mathrm{d}x\right)^{1/p}\left(\int_Q \omega(x)^{-p'}\,\mathrm{d}x\right)^{1/p'}.$$

It follows from $\omega\in A_{p,q}$ that

$$\begin{aligned}|Q|^{1/p-1/q}\frac{\nu(Q)^{1/q}}{\|\chi_Q\|_{L^p(\omega^p)}} &\leqslant |Q|^{1/p-1/q-1}\left(\int_Q \omega(x)^q\,\mathrm{d}x\right)^{1/q}\left(\int_Q \omega(x)^{-p'}\,\mathrm{d}x\right)^{1/p'}\\ &\leqslant\left(\frac{1}{|Q|}\int_Q \omega(x)^q\,\mathrm{d}x\right)^{1/q}\left(\frac{1}{|Q|}\int_Q \omega(x)^{-p'}\,\mathrm{d}x\right)^{1/p'}.\end{aligned}$$

Thus, $b\in \mathrm{BMO}^-$ shows that

$$\frac{\|(b-M_Q(b))\chi_Q\|_{L^q(\omega^q)}}{\|\chi_Q\|_{L^p(\omega^p)}}\lesssim |Q|^{1/q-1/p}.$$

Now let's show the proof of sufficiency. From the definition of $\omega\in A_{p,q}$, it follows that

$$\begin{aligned}\int_Q |b(x)-M_Q(b)(x)|\,\mathrm{d}x &\leqslant\left(\int_Q |b(x)-M_Q(b)(x)|^p\omega(x)^p\,\mathrm{d}x\right)^{1/p}\left(\int_Q \omega(x)^{-p'}\,\mathrm{d}x\right)^{1/p'}\\ &\leqslant\left(\int_Q |b(x)-M_Q(b)(x)|^q\omega(x)^q\,\mathrm{d}x\right)^{1/q}\\ &\quad\times\left(\int_Q \omega(x)^{-p'}\,\mathrm{d}x\right)^{1/p'}|Q|^{1/p-1/q}\\ &\leqslant\|(b-M_Q(b)(x))\chi_Q\|_{L^q(\omega^q)}\left(\int_Q \omega(x)^{-p'}\,\mathrm{d}x\right)^{1/p'}|Q|^{1/p-1/q}\\ &\lesssim\left(\int_Q \omega(x)^{-p'}\,\mathrm{d}x\right)^{1/p'}\left(\int_Q \omega(x)^p\,\mathrm{d}x\right)^{1/p}\\ &\lesssim |Q|\left(\frac{1}{|Q|}\int_Q \omega(x)^{-p'}\,\mathrm{d}x\right)^{1/p'}\left(\frac{1}{|Q|}\int_Q \omega(x)^q\,\mathrm{d}x\right)^{1/q}\lesssim |Q|.\end{aligned}$$

Therefore, we have completed the proof of Theorem 1.3. □

P r o o f of Theorem 1.4. We only need to show that $b \in \mathrm{BMO}^-$ is necessary for the boundedness of the commutator $[b, M]$ on $L^p(\omega)$, since the weighted boundedness of the commutator $[b, M]$ had been shown by Zhang in [18]. For the necessity, he proved only the unweighted case.

Let Q be any fixed cube. For all $x \in Q$, it is obvious to find out that there holds $M(\chi_Q)(x) \equiv 1$ and that $M(b\chi_Q)(x) = M_Q(b)(x)$. Therefore, we have finished the proof of the equality

$$[b, M](\chi_Q)(x) = b(x) - M_Q(b)(x), \quad x \in Q.$$

From the boundedness of $[b, M]$ from $L^p(\omega)$ to $L^p(\omega)$, we arrive at

$$\frac{\|(b - M_Q(b))\chi_Q\|_{L^p(\omega)}}{\|\chi_Q\|_{L^p(\omega)}} \leqslant \|[b, M]\|_{L^p(\omega)\to L^p(\omega)},$$

which shows that $b \in \mathrm{BMO}^-$ by Theorem 1.2. □

Acknowledgments. We would like to thank the anonymous referee for his/her comments.

References

[1] *J. Bastero, M. Milman, F. J. Ruiz*: Commutators for the maximal and sharp functions. Proc. Am. Math. Soc. *128* (2000), 3329–3334. zbl MR doi
[2] *C. Bennett, R. A. DeVore, R. Sharpley*: Weak-L^∞ and BMO. Ann. Math. (2) *113* (1981), 601–611. zbl MR doi
[3] *S. Bloom*: A commutator theorem and weighted BMO. Trans. Am. Math. Soc. *292* (1985), 103–122. zbl MR doi
[4] *S. Chanillo*: A note on commutators. Indiana Univ. Math. J. *31* (1982), 7–16. zbl MR doi
[5] *R. R. Coifman, R. Rochberg, G. Weiss*: Factorization theorems for Hardy spaces in several variables. Ann. Math. (2) *103* (1976), 611–635. zbl MR doi
[6] *J. García-Cuerva, J. L. Rubio de Francia*: Weighted Norm Inequalities and Related Topics. North-Holland Mathematics Studies 116. North-Holland, Amsterdam, 1985. zbl MR doi
[7] *K.-P. Ho*: Characterizations of BMO by A_p weights and p-convexity. Hiroshima Math. J. *41* (2011), 153–165. zbl MR doi
[8] *M. Izuki, T. Noi, Y. Sawano*: The John-Nirenberg inequality in ball Banach function spaces and application to characterization of BMO. J. Inequal. Appl. *2019* (2019), Article ID 268, 11 pages. zbl MR doi
[9] *S. Janson*: Mean oscillation and commutators of singular integral operators. Ark. Math. *16* (1978), 263–270. zbl MR doi
[10] *F. John, L. Nirenberg*: On functions of bounded mean oscillation. Commun. Pure Appl. Math. *14* (1961), 415–426. zbl MR doi
[11] *Á. D. Martínez, D. Spector*: An improvement to the John-Nirenberg inequality for functions in critical Sobolev spaces. Adv. Nonlinear Anal. *10* (2021), 877–894. zbl MR doi
[12] *B. Muckenhoupt*: Weighted norm inequalities for the Hardy maximal function. Trans. Am. Math. Soc. *165* (1972), 207–226. zbl MR doi

[13] *B. Muckenhoupt, R. L. Wheeden*: Weighted bounded mean oscillation and the Hilbert transform. Stud. Math. *54* (1976), 221–237. zbl MR doi

[14] *A. Uchiyama*: On the compactness of operators of Hankel type. Tohoku Math. J., II. Ser. *30* (1978), 163–171. zbl MR doi

[15] *D. Wang*: Notes on commutator on the variable exponent Lebesgue spaces. Czech. Math. J. *69* (2019), 1029–1037. zbl MR doi

[16] *D. Wang, J. Zhou, Z. Teng*: Some characterizations of BLO space. Math. Nachr. *291* (2018), 1908–1918. zbl MR doi

[17] *D. Wang, J. Zhou, Z. Teng*: Characterizations of BMO and Lipschitz spaces in terms of $A_{p,q}$ weights and their applications. J. Aust. Math. Soc. *107* (2019), 381–391. zbl MR doi

[18] *P. Zhang*: Multiple weighted estimates for commutators of multilinear maximal function. Acta Math. Sin., Engl. Ser. *31* (2015), 973–994. zbl MR doi

Author's address: Min Hu, School of Economics and Management, Anhui Normal University, Wuhu, 241002, P. R. China, email: humin@ahnu.edu.cn; Dinghuai Wang (corresponding author), School of Mathematics and Statistics, Anhui Normal University, Wuhu, 241002, P. R. China, e-mail: Wangdh1990@126.com.