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RINGS GENERALIZED BY TRIPOTENTS AND NILPOTENTS

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Abstract. We present new characterizations of the rings for which every element is a sum of two tripotents and a nilpotent that commute. These extend the results of Z. L. Ying, M. T. Koşan, Y. Zhou (2016) and Y. Zhou (2018).

Keywords: nilpotent; tripotent; 2-idempotent; exchange ring

MSC 2020: 16E50, 13B99, 16U99

1. Introduction

Throughout, all rings are associative with the identity. A ring R is a Zhou nil-clean if every element in R is the sum of a nilpotent and two tripotents that commute, see [10]. Here, an element $p \in R$ is a tripotent if $p^3 = p$. Recently, many authors studied such rings generalized by tripotents and nilpotents, see e.g., [1], [4], [5], [6], [7], [8], [9], [10]. The purpose of this paper is to completely characterize Zhou nil-clean rings in terms of element-wise conditions.

Recall that a ring R is strongly nil f-clean if for any $a \in R$ there exist a root e of the polynomial f and a nilpotent w such that $a = e + w$ that commute, see [2]. In Section 2, we prove that a ring R is a Zhou nil-clean ring if and only if it is strongly nil $(x-2)(x-1)x(x+1)(x+2)$-clean. An element $e \in R$ is a 2-idempotent if e^2 is an idempotent, i.e., $e^2 = e^4$. Moreover, we establish the connection between Zhou nil-clean ring and its 2-idempotents. We prove that a ring R is a Zhou nil-clean if and only if every element a in R is the sum of two 2-idempotents and a nilpotent in $\mathbb{Z}[a]$.

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A ring R is an exchange ring provided that for any $a \in R$, there exists an idempotent $e \in R$ such that $e \in aR$ and $1 - e \in (1 - a)R$, see [2]. An element u in a ring is unipotent if $1 - u$ is nilpotent. Finally, in Section 3 we characterize Zhou nil-clean rings by their exchange property. We prove that a ring R is a Zhou nil-clean ring if and only if R is an exchange in which u^4 is unipotent for any $u \in R^{-1}$.

We use $N(R)$ to denote the set of all nilpotents in R and $J(R)$ the Jacobson radical of R. The symbol R^{-1} stands for the set of all units in R, $\mathbb{N}$ is the set of all natural numbers, and $\mathbb{Z}[x] = \{f(x)\colon\ f(t)$ is a polynomial with integral coefficients$\}$.

2. Clean-like characterization

In this section we present the clean-like characterization for a Zhou nil-clean ring.

Lemma 2.1. *Let R be a Zhou nil-clean ring with $5 \in N(R)$. Then R is strongly nil $(x-2)(x-1)x(x+1)(x+2)$-clean.*

P r o o f. Let $a \in R$. Then $a - a^5 \in N(R)$. Set $x = 3a + a^2 + a^4$ and $y = 3a - a^2 - a^4$. Then $x - x^3, y - y^3 \in N(R)$. Set

$$b = \frac{x^2 + x}{2}, \quad c = \frac{x^2 - x}{2}; \quad p = \frac{y^2 + y}{2}, \quad q = \frac{y^2 - y}{2}.$$

Then $b - b^2, c - c^2; p - p^2, q - q^2 \in N(R)$. Thus, we can find the idempotents $e, f; g, h \in \mathbb{Z}[a]$ such that

$$b - e, c - f; p - g, q - h \in N(R).$$

Since $x = b - c$ and $y = p - q$, we see that $x - (e - f); y - (g - h) \in N(R)$. Therefore,

$$a = (x + y) - 5a = (e - f) + (g - h) + w \quad \text{for some } w \in N(R).$$

As $bc = \frac{1}{4}(x^4 - x^2) \in N(R)$, we see that $ef \in N(R)$ and so $ef = 0$. As $pq \in N(R)$, likewise $gh = 0$.

Since $5 \in R$ is nilpotent, we directly verify that

$$x^2 + x \equiv -a + 2a^2 + a^3 - 2a^4, \quad y^2 + y \equiv 2a - a^3 + a^4,$$
$$x^2 - x \equiv -2a + a^3 + a^4, \quad y^2 - y \equiv a + 2a^2 - a^3 - 2a^4 \pmod{N(R)}.$$

Moreover, we have

$$bp \equiv 2a + a^2 - 2a^3 - a^4, \quad bq \equiv 0;$$
$$cp \equiv 0, \quad cq \equiv -2a + a^2 + 2a^3 - a^4 \pmod{N(R)}.$$

Thus, we see that $eh, fg \in N(R)$; hence, $eh = fg = 0$. Accordingly, we see that $ef = gh = eh = fg = 0$, and then we check that

$$\begin{aligned}(e-f+g-h)^5 &= (e+f+g+h+2eg+2fh)^2(e-f+g-h)\\ &= (e+f+g+h+14eg+14fh)(e-f+g-h)\\ &= e-f+g-h+30(eg-fh).\end{aligned}$$

Moreover, we have

$$\begin{aligned}(e-f+g-h)^3 &= (e+f+g+h+2eg+2fh)(e-f+g-h)\\ &= e-f+g-h+6(eg-fh).\end{aligned}$$

Let $\alpha = e-f+g-h$, then $\alpha^5 = \alpha + 5(\alpha^3 - \alpha)$. Hence, $\alpha^5 = 5\alpha^3 - 4\alpha$. Let $f(x) = x^5 - 5x^3 + 4x$, then $f(x) = (x-2)(x-1)x(x+1)(x+2)$ as required. □

We are ready to present a new characterization of a Zhou nil-clean ring.

Theorem 2.2. *Let R be a ring. Then the following are equivalent:*
(1) *R is Zhou nil-clean.*
(2) *$a^5 - 5a^3 + 4a \in R$ is nilpotent for all $a \in R$.*
(3) *R is strongly nil $(x-2)(x-1)x(x+1)(x+2)$-clean.*

Proof. (1) ⇒ (3) In view of [10], Theorem 2.11, R is isomorphic to R_1, R_2, R_3 or the product of these rings, where R_1 is strongly nil-clean and $2 \in N(R_1)$; R_2 is strongly 2-nil-clean and $3 \in N(R_2)$; R_3 is Zhou nil-clean and $5 \in N(R_3)$.

Case 1: Let $a \in R_1$. In view of [10], Lemma 2.4, there exists an idempotent $e \in \mathbb{Z}[a]$ such that $a - e \in N(R_1)$. We easily check that $e^5 = 5e^3 - 4e$.

Case 2: Let $a \in R_2$. By virtue of [10], Lemma 2.6, there exists a tripotent $e \in \mathbb{Z}[a]$ such that $a - e \in N(R_1)$. We easily check that $e^5 = 5e^3 - 4e$.

Case 3: Let $a \in R_3$. According to Lemma 2.1, there exists an idempotent $e \in \mathbb{Z}[a]$ such that $a - e \in N(R_1)$. We easily check that $e^5 = 5e^3 - 4e$.

Let $a \in R$. Combining the preceding steps, we can find $e \in \mathbb{Z}[a]$ such that $a - e \in R$ is nilpotent and $e^5 = 5e^3 - 4e$. Let $f(x) = x^5 - 5x^3 + 4x$. Then $f(x) = (x-2)(x-1)x(x+1)(x+2)$, and so R is strongly nil f-clean, as desired.

(3) ⇒ (2) Let $a \in R$. Then there exists $e \in \mathbb{Z}[a]$ such that $w := a - e \in R$ is nilpotent and $e^5 = 5e^3 - 4e$. Hence, $(a-w)^5 = 5(a-w)^3 - 4(a-w)$. Thus, $a^5 - 5a^3 + 4a \in N(R)$, as required.

(2) ⇒ (1) By hypothesis, $2^3 \times 3 \times 5 = 3^5 - 5 \times 3^3 + 4 \times 3 \in N(R)$; hence, $2 \times 3 \times 5 \in N(R)$. Write $2^n \times 3^n \times 5^n = 0$ $(n \in \mathbb{N})$. Thus, $R \cong R_1 \times R_2 \times R_3$, where $R_1 = R/2^nR$, $R_2 = R/3^nR$ and $R_3 = R/3^nR$.

Case 1: Let $x \in R_1$. As $2 \in N(R_1)$, we see that $x^4(x-1) = x^5 - x^4 \in N(R_1)$, and so $x - x^2 \in N(R_1)$. This shows that R_1 is strongly nil-clean. Hence, R_1 is Zhou nil-clean, by [10], Proposition 2.5.

Case 2: Let $x \in R_2$. As $3 \in N(R_2)$, we see that $x(x^2-1)^2 = x^5 - 2x^3 + x \in N(R_2)$. This shows that $x - x^3 \in N(R_2)$. Hence, R_2 is strongly 2-nil-clean. In view of [10], Proposition 2.5, R_2 is Zhou nil-clean.

Case 3: Let $x \in R_3$. As $5 \in N(R_3)$, we have $x - x^5 \in N(R_3)$. In light of [10], Proposition 2.10, R_3 is Zhou nil-clean.

Therefore, we conclude that R is Zhou nil-clean. □

Corollary 2.3. *A ring R is Zhou nil-clean if and only if*

(1) *$J(R)$ is nil;*

(2) *$R/J(R)$ has the identity $(x-2)(x-1)x(x+1)(x+2) = 0$.*

Proof. This is obvious by Theorem 2.2. □

We now turn to establish the connection between Zhou nil-clean rings and their 2-idempotents. For future use, we now derive the following lemma.

Lemma 2.4. *If every element in a ring R is the sum of two 2-idempotents and a nilpotent that commute, then $30 \in N(R)$.*

Proof. Write $3 = g + h + w$, where $g^2 = g^4$, $h^2 = h^4$, $w \in N(R)$ and e, f, w commute with each other. Let $e = g^3$ and $f = h^3$. Then $3 = e + f + b$, where $e^3 = e$, $f^3 = f$, $b = w + (e - e^3) + (f - f^3)$. As $(e-e^3)^2 = (f - f^3)^2 = 0$, we see that $b \in N(R)$. Hence, $3 - e = f + b$, and so $(3-e)^3 = f^3 + 3bf(b+f)$. This implies that $(3-e)^3 - (3-e) \in N(R)$, i.e., $24 - 9e(3-e) = 24 - 27e + 9e^2 \in N(R)$. We infer that $2^3 \times 3 \times (3 - 2e) = (3+e)(24 - 27e + 9e^2) \in N(R)$. Hence, we can find $w \in N(R)$ such that $2^3 \times 3^2 = 2^4 \times 3e + w$, and so $2^9 \times 3^6 = 2^{12} \times 3^3 e^3 + w'$ for $w' \in N(R)$. It follows that $2^9 \times 3^4(2^2 - 3^2) \in N(R)$, i.e., $2^9 \times 3^4 \times 5 \in N(R)$. Therefore, $30 \in N(R)$, as asserted. □

Theorem 2.5. *The following are equivalent for a ring R:*

(1) *R is Zhou nil-clean.*

(2) *For any $a \in R$, there exist 2-idempotents $e, f \in \mathbb{Z}[a]$ and a nilpotent $w \in R$ such that $a = e + f + w$.*

(3) *Every element in R is the sum of two 2-idempotents and a nilpotent that commute.*

Proof. (1) $\Rightarrow$ (2) This is obvious by [3], Theorem 2.11, as every tripotent in R is a 2-idempotent.

(2) $\Rightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (1) In view of Lemma 2.4, $30 \in N(R)$. Write $2^n \times 3^n \times 5^n = 0$ ($n \in \mathbb{N}$).

Then $R \cong R_1 \times R_2 \times R_3$, where $R_1 = R/2^nR$, $R_2 = R/3^nR$ and $R_3 = R/5^nR$. Set $S = R_2 \times R_3$. Then $R \cong R_1 \times S$.

Step 1: Let $a^2 = a^4$ in R_1. Then $a^2(1-a)(1+a) = 0$. As $2 \in N(R_1)$, we see that $(a-a^2)^2 \in N(R)$; hence, $a - a^2 \in N(R_1)$. In light of [10], Proposition 2.5, there exists an idempotent $e \in \mathbb{Z}[a]$ such that $a - e \in N(R_1)$. Thus, every element in R_1 is the sum of two idempotents and a nilpotent that commute. This shows that R_1 is strongly 2-nil-clean; hence, it is Zhou nil-clean.

Step 2: Let $a^2 = a^4 \in S$. Then $a(a-a^3) = 0$, and so $(a-a^3)^2 = a(a-a^3)(1-a^2) \in N(R)$. Hence, $a - a^3 \in N(S)$. As $2 \in U(S)$, it follows by [10], Lemma 2.6, that there exists $e^3 = e \in S$ such that $a - e \in N(S)$.

Let $c \in S$. Then we can find 2-idempotents $b, c \in S$ such that $a - b - c \in N(S)$ and $a, b, c \in S$ commute. By the preceding discussion, we have tripotents $g \in \mathbb{Z}[b]$ and $h \in \mathbb{Z}[c]$ such that $b - g, c - h \in N(S)$. Therefore, $a - g - h = (a - b - c) + (b - g) + (c - h) \in N(S)$, where a, g, h commute. Therefore, every element in S is the sum of two tripotents and a nilpotent in S. That is, S is Zhou nil-clean.

Therefore, R is Zhou nil-clean, as asserted. □

Corollary 2.6. *The following are equivalent for a ring R:*

(1) *R is strongly 2-nil-clean.*

(2) *Every element in R is the sum of a 2-idempotent and a nilpotent that commute.*

P r o o f. (1) ⇒ (2) This is obvious by [3], Theorem 2.8.

(2) ⇒ (1) In view of Theorem 2.5, R is Zhou nil-clean. Write $2 = e + w$, where $e^3 = e \in R$, $w \in N(R)$. Then $2^3 - 2 \in N(R)$, and so $6 \in N(R)$. In light of [9], Lemma 3.5, R is strongly nil-clean. □

3. Exchange properties

The class of exchange rings is very large, see [2]. We now characterize Zhou nil-clean rings in terms of their exchange properties. We need an elementary lemma.

Lemma 3.1. *Let R be an exchange ring in which u^4 is unipotent for any unit $u \in R$. Then $30 \in N(R)$.*

P r o o f. Since R is an exchange ring, there exists an idempotent $f \in 3R$ such that $1 - f \in (1-3)R$. Write $f = 3a$ for some $a \in R$. We may assume that $a = fa$, then $3a^2 = 3a$. Then $1 - f = (1-3)b$ for some $b \in R$ with $b(1-f) = b$, therefore $(-2)b^2 = b$. Now we have $3 = 3 - (1-f) + (1-f)$. It is obvious that $3 - (1-f)$ is a unit with the inverse $a - b$.

Set $e = 1 - f$ and $u = 3 - (1 - f)$. Then $3 = e + u$. By hypothesis, $u^4 = 1 + w$, so $(3 - e)^4 = 1 + w$, then $81 - 65e = 1 + w$ which implies that $80 = 65e + w$. Thus, $80e = 65e + we$, hence, $15e = we$. Also $80 = 65e + w = 60e + 5e + w = 5e + (4e + 1)w$. Then $80^2 = 5^2 e + 10e(4e + 1)w + (4e + 1)^2 w^2$ and $5 \times 80 = 5^2 e + 5(4e + 1)w$; whence $80^2 - 5 \times 80 \in N(R)$. This implies $2^4 \times 3 \times 5^3 \in N(R)$ and so $2 \times 3 \times 5 \in N(R)$. □

Lemma 3.2. *Let R be an exchange ring in which u^4 is unipotent for any unit $u \in R$. Then $J(R)$ is nil.*

Proof. By virtue of Lemma 3.1, $30 \in N(R)$. Write $30^n = 0$ for some $n \in \mathbb{N}$. Then $R \cong R_1 \times R_2 \times R_3$, where $R_1 = R/2^n R, R_2 = R/3^n R$ and $R_3 = R/5^n R$. Let $x \in J(R_1)$, so $1 - x \in U(R_1)$. Let $u = 1 - x$, as R_1 is a Kosan ring, $u^4 + 1$ is in $N(R_1)$. We have $(1 - u)^4 = 1 + u^4 + 2(3u^2 - 2u - 2u^3) \in N(R_1)$, as $2 \in N(R_1)$, then $(1 - u)^4 \in N(R_1)$ which implies that $1 - u \in N(R_1)$ and so $x \in N(R_1)$. Now let $x \in N(R_2)$, then $(1 + x)^4 = 1 + w$ for some $w \in N(R_2)$, so $x^4 + 2(x^3 + 3x^2 + 2x) = x(x^3 + 2(2x^2 + 3x + 2))$ since $3 \in N(R_2)$, $2 \in U(R_2)$, and we have $2x^2 + 3x + 2 \in U(R_2)$. Therefore, $x^3 + 2(2x^2 + 3x + 2) \in U(R_2)$, which implies that $x \in N(R_2)$. For R_3, as $5 \in N(R_3)$, we deduce that $2 \in U(R_3)$, so in a similar way, we can prove that $J(R_3)$ is nil. Therefore, $J(R)$ is nil. □

Lemma 3.3. *Let R be an exchange ring in which u^4 is unipotent for any unit $u \in R$. If $J(R) = 0$, then R is reduced.*

Proof. We claim that $N(R) = 0$. If not, there exists some $a \in R$ such that $a^2 = 0$. Since R is an exchange ring with $J(R) = 0$, we have $eRe \cong M_2(T)$ for some idempotent $e \in R$ and some ring T. By hypothesis, we deduce that u^4 is unipotent for any unit $u \in eRe$. Then we easily check that U^4 is unipotent for any invertible $U \in M_2(T)$. Since $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \in GL_2(T)$, we have

$$A := \begin{pmatrix} -2 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^4 - I_4 \in N(M_2(T)).$$

Let $S = \{m \cdot 1_R \colon m \in \mathbb{Z}\}$. Then S is a commutative subring of T. As $A \in N(M_2(S))$, we see that $\det(A) = 3 \in N(S)$, and so $3 \in N(T)$. Since $2 = 3 - 1 \in U(T)$, $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2^{-1} & -2^{-1} \\ 2^{-1} & 2^{-1} \end{pmatrix}^{-1} \in GL_2(T)$, and then

$$\begin{pmatrix} -5 & 0 \\ 0 & -5 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^4 - I_4 \in GL_2(T).$$

This implies that $5 \in N(T)$; hence, $1_T = (3 \cdot 2 - 5) \cdot 1_T \in N(T)$, a contradiction. Therefore, $N(R) = 0$, i.e., R is reduced. □

We have accumulated all the information necessary to prove the following theorem.

Theorem 3.4. *A ring R is Zhou nil-clean if and only if*

(1) *R is an exchange ring;*

(2) *u^4 is unipotent for any unit $u \in R$.*

P r o o f. $\Rightarrow$ It is clear that every Zhou nil-clean ring is periodic and so it is strongly clean. Now let $u \in U(R)$, then $u^5 - u \in N(R)$, $u(u^4 - 1) \in N(R)$, and since u is a unit then $u^4 - 1 \in N(R)$. Thus, u^4 is unipotent for any unit $u \in R$.

$\Leftarrow$ By virtue of Lemma 3.2, $J(R)$ is nil. Set $S = R/J(R)$. Then S is an exchange ring in which u^4 is unipotent for any unit $u \in R$. Obviously, $J(S) = 0$. In view of Lemma 3.3, S is reduced. Thus, S is isomorphic to a subdirect product of some domains S_i. We see that each S_i is a homomorphic image of S; hence, S_i is an exchange Koşan ring with trivial idempotents. In light of [2], Lemma 17.2.1 and Lemma 3.2, S_i is local and $J(S_i)$ is nil. For any $x \in S_i$, $x - x^5 \in J(S_i) \subseteq N(S_i)$. Let $\bar{a} \in S$. Then $a - a^5 \in N(S) = 0$, and so $R/J(R)$ has the identity $x^5 = x$. Therefore, R is Zhou nil-clean by [10], Theorem 2.11. $\square$

Corollary 3.5. *A ring R is Zhou nil-clean if and only if R is a clean ring in which u^4 is unipotent for any unit $u \in R$.*

P r o o f. Since every clean ring is an exchange ring, we complete the proof by Theorem 3.4. $\square$

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