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CARLESON MEASURES FOR WEIGHTED HARMONIC MIXED NORM SPACES ON BOUNDED DOMAINS IN $\mathbb{R}^n$

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Abstract. We study weighted mixed norm spaces of harmonic functions defined on smoothly bounded domains in $\mathbb{R}^n$. Our principal result is a characterization of Carleson measures for these spaces. First, we obtain an equivalence of norms on these spaces. Then we give a necessary and sufficient condition for the embedding of the weighted harmonic mixed norm space into the corresponding mixed norm space.

Keywords: harmonic function; mixed norm space; Carleson measure

MSC 2020: 42B35, 31B05

1. Introduction

Let X be a topological vector space of functions defined on a domain Ω. The problem of characterizing measures μ on Ω such that X continuously embeds into $L^p(\mu)$ is a classical one. Such measures are called *Carleson measures for the space* X, the characterization always involves a geometric condition on μ.

The case of spaces of analytic functions, of one or several complex variables, has been extensively studied by many authors. We note recent results on the weighted Bergman spaces with Békollé weights on the unit ball in [14], on the spaces induced by two-side doubling weights in [11], and on the mixed norm weighted Bergman spaces on homogeneous Siegel domains in [3].

The case when X consists of harmonic functions has also been studied. For example, a Carleson type embedding theorem for weighted Bergman spaces of harmonic functions on $\Omega \subset\subset \mathbb{R}^n$ can be found in [9]. For results on weighted mixed norm spaces in $\mathbb{R}^{n+1}_+$, see [2].

In this paper, we prove a Carleson type embedding theorem for weighted mixed norm spaces $B^{p,q}_\alpha(\Omega)$ defined on smoothly bounded domains $\Omega \subset\subset \mathbb{R}^n$. This scale

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of spaces includes weighted Bergman spaces on Ω. We alert the reader that here we are looking for embeddings into mixed norm spaces $L^{p,q}(\Omega,\mu)$, in distinction with the usual case of embeddings into $L^p(\Omega,\mu)$. A different type of embedding of one space $B^{p,q}_\alpha(\Omega)$ into another $B^{p_1,q_1}_{\alpha_1}(\Omega)$ was proved in [1] under certain conditions on the parameters.

Our results are motivated by paper [8]. One of the results obtained in that paper is a characterization of Carleson measures for (unweighted) mixed norm spaces. We generalize that result by allowing power-type weights. The novelty here is the use of representation of the weighted Bergman kernel obtained by Engliš after the publication of [8], see [5].

Throughout this paper, we apply the convention of using C to denote any positive constant which may change from one occurrence to the next. Given two positive quantities A and B, we write $A \asymp B$ if there exist constants $0 < c \leqslant C$ such that $cA \leqslant B \leqslant CA$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^∞ boundary and let $\varrho(x)$ be a defining function for Ω. This means ϱ is a real valued function on $\mathbb{R}^n$ which is C^∞ in a neighborhood of the boundary $\partial\Omega$ of Ω such that $\Omega = \{x \in \mathbb{R}^n \colon \varrho(x) > 0\}$ is bounded and $|\nabla\varrho(x)| \neq 0$ on $\partial\Omega$. Throughout this paper such a domain Ω is fixed. It is convenient to work with a particular defining function, namely the distance function $r(x)$ defined by $r(x) = d(x,\partial\Omega)$ for $x \in \overline{\Omega}$ and $r(x) = -d(x,\partial\Omega)$ for $x \notin \overline{\Omega}$. Indeed, there is an $\varepsilon > 0$ such that for all $0 < r \leqslant \varepsilon$ the set $\Omega_r = \{x \in \mathbb{R}^n \colon r(x) > r\}$ is a smoothly bounded subdomain of Ω with the defining function $r(x) - r$. We fix such $\varepsilon > 0$. We denote by Γ_r the boundary $\partial\Omega_r = \{x \in \mathbb{R}^n \colon r(x) = r\}$.

We denote by $\mathrm{d}\sigma_r$ the induced surface measure on $\partial\Omega_r$. The symbol $\mathrm{d}m$ denotes the Lebesgue volume measure on $\mathbb{R}^n$. We also use weighted measures $\mathrm{d}m_\gamma(x) = r(x)^\gamma \mathrm{d}m(x)$ on Ω, where $\gamma \in \mathbb{R}$. Furthermore, $h(\Omega)$ denotes the space of all harmonic functions in Ω.

For $0 < p < \infty$ and $0 < r \leqslant \varepsilon$, we set

$$M_p(f,r) = \left\{ \int_{\Gamma_r} |f(\zeta)|^p \,\mathrm{d}\sigma_r(\zeta) \right\}^{1/p}. \tag{1.1}$$

Now let $0 < p, q < \infty$ and $\alpha > 0$. We define a mixed norm space $B^{p,q}_\alpha(\Omega)$ as the space of all $f \in h(\Omega)$ such that the (quasi-)norm

$$\|f\|_{B^{p,q}_\alpha} = \left\{ \int_0^\varepsilon r^{\alpha q - 1} M^q_p(f,r) \,\mathrm{d}r \right\}^{1/q} \tag{1.2}$$

is finite. $B^{p,q}_\alpha(\Omega)$ is a Banach space for $1 \leqslant p < \infty$ and $1 \leqslant q < \infty$.

Let us set $r_0 = \max\{r(x) : x \in \Omega\} > 0$ and put $r_j = r_0/2^j$ for all $j \in \mathbb{N}$. For $j \in \mathbb{N}$, we set

$$S_j = \{x \in \Omega\colon\ r_j < r(x) \leqslant r_{j-1}\}.$$

It is obvious that $\Omega = \bigcup\limits_{j=1}^{\infty} S_j$.

For a given Borel measure μ on Ω, and positive fixed p, and q, we define $L^{p,q}(\Omega,\, \mathrm{d}\mu)$ as the set of all Borel measurable functions f on Ω such that

$$\|f\|_{L^{p,q}(\mu)} = \biggl\{\sum_{j=1}^{\infty}\biggl[\int_{S_j} |f(x)|^p \,\mathrm{d}\mu(x)\biggr]^{q/p} 2^{j(q/p-1)}\biggr\}^{1/q} < \infty.$$

In other words, if we set

$$a_j(f) = \biggl(\int_{S_j} |f(x)|^p \,\mathrm{d}\mu(x)\biggr)^{1/p} 2^{j(1/p-1/q)}$$

then $\|f\|_{L^{p,q}(\mu)} = \|a_j(f)\|_{l^q}$. Also, we define $L^{p,q}_{\alpha}(\Omega)$ for $\alpha > 0$ as the set of all Lebesgue measurable functions f on Ω such that

$$\|f\|_{L^{p,q}_{\alpha}} = \biggl\{\sum_{j=1}^{\infty}\biggl[\int_{S_j} |f(x)|^p \,\mathrm{d}m(x)\biggr]^{q/p} 2^{j(q/p-\alpha q)}\biggr\}^{1/q} < \infty.$$

For $\gamma > -1$, let $R_\gamma(x,y)$ be the reproducing kernel of the harmonic Bergman space $b^2_\gamma(\Omega) = h(\Omega) \cap L^2(\Omega,\, \mathrm{d}m_\gamma)$. Note that $b^2_\gamma(\Omega) = B^{2,2}_{(\gamma+1)/2}(\Omega)$. For every function $f \in b^2_\gamma(\Omega)$ we have a reproducing formula

$$f(x) = \int_{\Omega} R_\gamma(x,y) f(y) \,\mathrm{d}m_\gamma(y), \quad x \in \Omega.$$

The kernel $R_\gamma(x,y)$ is symmetric and real-valued.

2. Carleson type measures for weighted harmonic mixed norm spaces

First, we state and prove a theorem about the equivalence of certain norms.

Theorem 2.1. *Let $1 < p, q < \infty$, and $\alpha > 0$. Then, we have*

$$\|f\|^q_{B^{p,q}_{\alpha}} \asymp \sum_{j=1}^{\infty}\biggl[\int_{S_j} |f(x)|^p \,\mathrm{d}m(x)\biggr]^{q/p} 2^{j(q/p-\beta-1)}, \quad f \in h(\Omega),$$

where $\beta = \alpha q - 1$.

P r o o f. In the proof we will use the fact that there is a positive constant C, depending only on the defining function of Ω, such that

$$M_p(f,r) \leqslant CM_p(f,s) \quad \text{for } 0 < s < r \leqslant \varepsilon, \tag{2.1}$$

see Theorem 3.1 of [12]. Let J be a positive integer such that $r_J > \varepsilon$ and $r_{J+1} \leqslant \varepsilon$, where, as before, $r_j = r_0/2^j$ for $j = 1, 2, \ldots$ We set $\Omega_J = \{x \in \Omega\colon\ r(x) \geqslant r_{J+1}\}$. Note that $r_j \to 0$ as $j \to \infty$. Now using (1.2) we have

$$\|f\|^q_{B^{p,q}_\alpha} = \int_0^\varepsilon r^\beta M^q_p(f,r)\,\mathrm{d}r = \int_{r_{J+1}}^\varepsilon r^\beta M^q_p(f,r)\,\mathrm{d}r + \sum_{j=J+2}^\infty \int_{r_j}^{r_{j-1}} r^\beta M^q_p(f,r)\,\mathrm{d}r. \tag{2.2}$$

We first estimate the infinite sum $\sum\limits_{j=J+2}^\infty \int_{r_j}^{r_{j-1}} r^\beta M^q_p(f,r)\,\mathrm{d}r$ by estimating each term in the sum. Using (2.1) twice we have, for $j \geqslant J+2$,

$$\begin{aligned}
\int_{r_j}^{r_{j-1}} r^\beta M^q_p(f,r)\,\mathrm{d}r &\leqslant CM^q_p(f,r_j)\int_{r_j}^{r_{j-1}} r^\beta\,\mathrm{d}r \\
&= C\Big(M^p_p(f,r_j)\frac{r_0^{(\beta+1)p/q}}{2^{j(\beta+1)p/q}}\Big)^{q/p} \\
&\leqslant C\bigg(\int_{r_{j+1}}^{r_j} M^p_p(f,r)r^{(\beta+1)p/q-1}\,\mathrm{d}r\bigg)^{q/p} \\
&\leqslant C2^{(j+1)(q/p-(\beta+1))}\bigg(\int_{r_{j+1}}^{r_j}\int_{\Gamma_r} |f(\zeta)|^p\,\mathrm{d}\sigma_r(\zeta)\,\mathrm{d}r\bigg)^{q/p} \\
&\leqslant C\bigg[\int_{S_{j+1}} |f(x)|^p\,\mathrm{d}m(x)\bigg]^{q/p} 2^{(j+1)(q/p-1-\beta)}.
\end{aligned}$$

Hence, summation over j gives

$$\sum_{j=J+2}^\infty \int_{r_j}^{r_{j-1}} r^\beta M^q_p(f,r)\,\mathrm{d}r \leqslant C\sum_{j=J+3}^\infty \bigg[\int_{S_j} |f(x)|^p\,\mathrm{d}m(x)\bigg]^{q/p} 2^{j(q/p-1-\beta)}.$$

Also, using (2.1) twice, we similarly get

$$\begin{aligned}
\int_{r_j}^{r_{j-1}} r^\beta M^q_p(f,r)\,\mathrm{d}r &\geqslant CM^q_p(f,r_{j-1})\int_{r_j}^{r_{j-1}} r^\beta\,\mathrm{d}r \\
&= C\Big(M^p_p(f,r_{j-1})\frac{r_0^{(\beta+1)p/q}}{2^{(j-1)(\beta+1)p/q}}\Big)^{q/p} \\
&\geqslant C\bigg(\int_{r_{j-1}}^{r_{j-2}} M^p_p(f,r)r^{(\beta+1)p/q-1}\,\mathrm{d}r\bigg)^{q/p} \\
&\geqslant C\bigg[\int_{S_{j-1}} |f(x)|^p\,\mathrm{d}m(x)\bigg]^{q/p} 2^{(j-1)(q/p-\beta-1)},
\end{aligned}$$

where the above inequalities hold for $j = J+3, J+4, \dots$ Hence, we have

$$\sum_{j=J+2}^{\infty} \left[\int_{S_j} |f(x)|^p \,\mathrm{d}m(x) \right]^{q/p} 2^{j(q/p-1-\beta)} \leqslant C \sum_{j=J+3}^{\infty} \int_{r_j}^{r_{j-1}} r^{\beta} M_p^q(f,r)\,\mathrm{d}r.$$

Now we estimate $\int_{r_{J+1}}^{\varepsilon} r^{\beta} M_p^q(f,r)\,\mathrm{d}r$. We obtain

$$\begin{aligned}
\int_{r_{J+1}}^{\varepsilon} r^{\beta} M_p^q(f,r)\,\mathrm{d}r &\leqslant C r_{J+1}^{\beta} \sup_{\Omega_J} |f(x)|^q \\
&\leqslant C r_{J+1}^{\beta} \sum_{j=1}^{J+1} \left[\int_{S_j} |f(x)|^p \,\mathrm{d}m(x) \right]^{q/p} 2^{j(q/p-1)} \\
&\leqslant C \sum_{j=1}^{J+1} \left[\int_{S_j} |f(x)|^p \,\mathrm{d}m(x) \right]^{q/p} 2^{j(q/p-1-\beta)}
\end{aligned}$$

using definition of the sets Ω_J and S_j, $j = 1, \dots, J+1$, and subharmonicity of $|f|^p$. On the other hand, using the maximum modulus principle, we have

$$\begin{aligned}
\sum_{j=1}^{J+1} \left[\int_{S_j} |f(x)|^p \,\mathrm{d}m(x) \right]^{q/p} 2^{j(q/p-1-\beta)} &\leqslant C \sup_{\Omega_J} |f(x)|^q \leqslant C \sup_{\Omega_\varepsilon} |f(x)|^q \\
&\leqslant C \sup_{\Gamma_\varepsilon} |f(x)|^q = C M_\infty^q(f,\varepsilon) \\
&\leqslant C \int_0^{\varepsilon} r^{\beta} M_p^q(f,r)\,\mathrm{d}r.
\end{aligned}$$

We have used Lemma 4 of [7] to obtain the last inequality. □

Notice that $r(x)^{\gamma} \asymp 2^{-j\gamma}$ on S_j for $j \in \mathbb{N}$ and $\gamma \in \mathbb{R}$, hence the sum in Theorem 2.1 is equivalent to

$$\sum_{j=1}^{\infty} \left[\int_{S_j} |f(x)|^p \,\mathrm{d}m_{\gamma}(x) \right]^{q/p} 2^{j(q/p-1)},$$

where $\gamma = \beta p/q = p(\alpha - 1/q)$ and $\mathrm{d}m_{\gamma}(x) = r(x)^{\gamma}\mathrm{d}m(x)$, as before. Note that we will use the letter β to denote $\alpha q - 1$ in the whole paper.

Note that the sum appearing in Theorem 2.1 is actually $\|f\|_{L_{\alpha}^{p,q}}^q$. The theorem is also valid for $0 < p, q < \infty$ and it can be proved in this more general case using the inequality

$$M_p^q(f,r) \leqslant \frac{C}{r} \int_{c_1 r}^{c_2 r} M_p^q(f,s)\,\mathrm{d}s$$

which holds for all $f \in h(\Omega)$, $0 < r < \varepsilon/c_2$ and constants $0 < c_1 < 1 < c_2$, instead of (2.1). A proof of the above inequality can be found in [7]. A further modification of the proof, needed in the case $0 < p, q < \infty$, is the use of subharmonic behaviour of $|f|^p$ as a substitute for subharmonicity, which is no longer available for $0 < p < 1$. This is a classical result proved in [6], Lemma 2 of IV. 9.

For $x, y \in \Omega$, let $D(x,y) = r(x) + r(y) + |x - y|$. The boundary behavior of the harmonic Bergman kernel can be most efficiently decribed by this quasi-distance function. Let

$$E_\delta(x) = \{y \in \Omega\colon\ |y - x| < \delta r(x)\} \quad \text{for } x \in \Omega \text{ and } 0 < \delta < 1.$$

The next lemma was proven in [8]. We use it to obtain Proposition 2.5 which, in turn, is needed to get vital estimates of the test functions f_x, see Lemma 2.6 below.

Lemma 2.2 (see [8]). *For any $s > n - 1$, there exists some C (depending on s) such that for all $x \in \Omega$ and $0 < r \leqslant \varepsilon$,*

$$\int_{\Gamma_r} \frac{\mathrm{d}\sigma(y)}{D(x,y)^s} \leqslant \frac{C}{(r(x) + r)^{s-(n-1)}}.$$

Estimates of the harmonic Bergman kernel on a smoothly bounded domain in $\mathbb{R}^n$ are given in [10]. However, we need more general estimates of the weighted harmonic Bergman kernel on a bounded domain in $\mathbb{R}^n$ with smooth boundary. These can be derived from general results on Schwartz kernels for the so-called *singular Green operators*, which can be found in [5].

Specifically, Corollary 12 of [5] describes the boundary behavior of the Schwartz kernel of the singular Green operator. Theorem 14 of [5] is a slight strengthening of Corollary 12 for the particular case of the singular Green operator, namely the Bergman projection, and it gives the form of the Bergman kernel which is the Schwartz kernel of the Bergman projection. The theorem describes the Bergman kernel in the term of $|x - \tilde{y}|^n$, where $\tilde{y}$ is the "reflection" of y with respect to the boundary of Ω. An expression for the Bergman kernel valid on all of $\Omega \times \Omega$ is given as a remark after the theorem. Since the weighted Bergman kernel $R_\gamma(x, y)$ is the Schwartz kernel of a particular Green operator (see Subsection 7.1 of [5]), it can be described using Corollary 12, similarly as the Bergman kernel in Theorem 14, in the term of $|x - \tilde{y}|^{n+\gamma}$. Description can be obtained even for the more general case of the weight of the Bergman kernel but we need description only for power-type weight. Notice that $D(x,y) \asymp |x - \tilde{y}|$ so we can estimate the weighted Bergman kernel by the quasi-distance function $D(x, y)$. The relevant estimates are given in the following proposition.

Proposition 2.3 (see [5]). *For $\gamma > -1$, $x, y \in \Omega$, there is a positive constant C such that*

$$|R_\gamma(x,y)| \leqslant C\frac{1}{D(x,y)^{n+\gamma}} \quad \text{and} \quad \Bigl|\frac{\partial R_\gamma(y,x)}{\partial y}\Bigr| \leqslant C\frac{1}{D(x,y)^{n+\gamma+1}}.$$

Moreover, for some positive constant C

$$|R_\gamma(x,x)| \geqslant C\frac{1}{r(x)^{n+\gamma}}.$$

In particular, for $x = y$ we get that

$$|R_\gamma(x,x)| \asymp \frac{1}{r(x)^{n+\gamma}}. \tag{2.3}$$

Also, notice that if $\delta \in (0,1)$, then we have

$$(1-\delta)r(x) < r(y) < (1+\delta)r(x) \tag{2.4}$$

for $x \in \Omega$ and $y \in E_\delta(x)$. Now, we state and prove a weighted analogue of Lemma 2.3 of [4].

Lemma 2.4. *There exists a $\delta_0 \in (0,1)$ such that $|R_\gamma(x,y)| \asymp 1/r(x)^{n+\gamma}$ for $x \in \Omega$ and $y \in E_{\delta_0}(x)$.*

Proof. For $y \in E_\delta(x)$, $0 < \delta < 1$ we have

$$\begin{aligned} |R_\gamma(y,x) - R_\gamma(x,x)| &\leqslant |y-x| \max\Bigl\{\Bigl|\frac{\partial R_\gamma(y,x)}{\partial y}\Bigr| \colon\ y \in \overline{E_\delta(x)}\Bigr\} \\ &\leqslant C\delta r(x) \max\Bigl\{\frac{1}{D(x,y)^{n+\gamma+1}} \colon\ y \in \overline{E_\delta(x)}\Bigr\}, \end{aligned}$$

where we used Proposition 2.3. Note that $D(x,y) = r(x) + r(y) + |x-y| \geqslant r(x)$ so we have

$$|R_\gamma(y,x) - R_\gamma(x,x)| \leqslant \frac{C\delta}{r(x)^{n+\gamma}}. \tag{2.5}$$

Therefore, we can choose $\delta = \delta_0 > 0$ such that, using the opposite triangle inequality and (2.3), we have

$$\begin{aligned} |R_\gamma(y,x)| &= |R_\gamma(x,x) - (R_\gamma(x,x) - R_\gamma(y,x))| \\ &\geqslant |R_\gamma(x,x)| - \frac{C\delta_0}{r(x)^{n+\gamma}} \geqslant \frac{C}{r(x)^{n+\gamma}}, \quad y \in E_{\delta_0}(x). \end{aligned}$$

The estimate from above is a consequence of Proposition 2.3 and (2.4). □

The next proposition is about the sharp estimate for the integral mean of the Bergman kernel. The unweighted version of this theorem is given in [8].

Proposition 2.5. *For $p > (n-1)/(n+\gamma)$, we have*

$$M_p(R_\gamma(\cdot, x), r) \leqslant C(r(x)+r)^{(n-1)/p-(n+\gamma)}$$

and the exponent on the right-hand side is the best possible.

Proof. We have $|R_\gamma(y,x)| \leqslant C/D(x,y)^{n+\gamma}$ and $|R_\gamma(x,x)| \asymp r(x)^{-(n+\gamma)}$ for all $x, y \in \Omega$. Then, since $p(n+\gamma) > n-1$, we can use Lemma 2.2 with $s = p(n+\gamma)$ to get

$$\begin{aligned} M_p^p(R_\gamma(\cdot, x), r) &= \int_{\Gamma_r} |R_\gamma(y,x)|^p \, \mathrm{d}\sigma_r(y) \\ &\leqslant C \int_{\Gamma_r} \frac{\mathrm{d}\sigma_r(y)}{D(x,y)^{p(n+\gamma)}} \leqslant \frac{C}{(r(x)+r)^{p(n+\gamma)-(n-1)}}. \end{aligned} \tag{2.6}$$

Taking pth root we get the desired inequality. Let us prove that the exponent on the right-hand side is the best possible.

We have $|R_\gamma(y,x)| \geqslant C/r(x)^{n+\gamma}$ for $y \in E_\delta(x)$ and some δ fixed from Lemma 2.4. Now for $x \in \Gamma_r$ we have

$$\begin{aligned} M_p^p(R_\gamma(\cdot, x), r) &\geqslant \int_{\Gamma_r \cap E_\delta(x)} |R_\gamma(y,x)|^p \, \mathrm{d}\sigma_r(y) \\ &\geqslant \int_{\Gamma_r \cap E_\delta(x)} \frac{C}{r(x)^{p(n+\gamma)}} \, \mathrm{d}\sigma_r(y) \\ &\geqslant \frac{C}{r(x)^{p(n+\gamma)-(n-1)}}, \end{aligned}$$

which means that the exponent is the best possible. □

The following lemma is about the norm estimation of a test function which will be used in proving our main result.

Lemma 2.6. *Let $1 < p, q < \infty$, $\alpha > 0$, $\alpha > 1/q - 1/p$ and $\gamma = p(\alpha - 1/q)$. Let, for $x \in \Omega$, a function $f_x \colon \Omega \to \mathbb{R}$ be defined by*

$$f_x(y) = \frac{R_\gamma(y,x)}{R_\gamma(x,x)^{1-1/p+1/p(n+\gamma)-1/q(n+\gamma)}}.$$

Then f_x belongs to the function space $B^{p,q}_\alpha(\Omega)$, and moreover, $\|f_x\|^q_{B^{p,q}_\alpha} \leqslant C$, where C is independent of $x \in \Omega$.

Proof. Notice that for $1 < p, q < \infty$, $n \geqslant 2$ we have

$$1 - \frac{1}{p} + \frac{1}{p(n+\gamma)} - \frac{1}{q(n+\gamma)} > 0.$$

From the definition of f_x we see that $f_x \in h(\Omega)$ and

$$|f_x(y)|^p = \frac{|R_\gamma(y,x)|^p}{|R_\gamma(x,x)|^{p-1+1/(n+\gamma)-p/q(n+\gamma)}}.$$

Since $p > 1$, we have $p > 1 - (\gamma+1)/(n+\gamma) = (n-1)/(n+\gamma)$ so we can use Proposition 2.5 and the estimate (2.3) to conclude

$$M_p(f_x, t) \leqslant \frac{Cr(x)^{n+\gamma-(n+\gamma)/p+1/p-1/q}}{(r(x)+t)^{n+\gamma-n/p+1/p}}$$

and

$$\|f_x\|^q_{B^{p,q}_\alpha} \leqslant C \int_0^\varepsilon \frac{Cr(x)^{(n+\gamma)q-(n+\gamma)q/p+q/p-1}}{(r(x)+t)^{(n+\gamma)q-qn/p+q/p}} t^\beta \,\mathrm{d}t \leqslant C,$$

where C is independent of x. □

We will need some more lemmas. The first one is about covering of Ω and essentially comes from [13], and the second one can be found in [8].

Lemma 2.7 (see [13]). *If $0 < \delta < 1$, then there exists a sequence $\{a_k\}$ in Ω satisfying the following conditions:*

(1) $\Omega = \bigcup\limits_{k=1}^{\infty} E_{\delta/3}(a_k)$.

(2) *There exists a positive integer N such that every point in Ω belongs to at most N of the sets $E_\delta(a_k)$.*

Notice that $a_k \to \partial\Omega$ as $k \to \infty$.

Lemma 2.8 (see [8]). *If $0 < \delta < 1$, then there exists a positive integer $N = N(\delta)$ such that for $x \in S_k$, $k = 1, 2, \ldots,$*

$$E_\delta(x) \subset \bigcup_{j=k-N}^{k+N} S_j,$$

where $S_j = \emptyset$ if $j \leqslant 0$.

Notice that $N = 1$ if δ is small enough.

Let $\{a_k\}$ be a sequence from Lemma 2.7. Then we have for $f \in h(\Omega)$,

$$r(a_k)^\gamma \max_{x \in \overline{E}_{\delta/3}(a_k)} |f(x)|^p m(E_{\delta/3}(a_k)) \leqslant C \int_{E_\delta(a_k)} |f(y)|^p r(y)^\gamma \, \mathrm{d}m(y), \tag{2.7}$$

where we used subharmonicity of $|f|^p$ and (2.4).

Now, we can prove our main theorem which relates Carleson type condition for function space $B^{p,q}_\alpha(\Omega)$ to the embedding of $B^{p,q}_\alpha(\Omega)$ in $L^{p,q}(\Omega, \mathrm{d}\mu)$.

Theorem 2.9. *Let $1 < p, q < \infty$, $\alpha > 0$, $\alpha > 1/q - 1/p$, $0 < \delta < 1$ and let μ be a Borel measure on Ω. Then the following conditions are equivalent.*

(1) *The measure μ satisfies a Carleson type condition*

$$\mu(E_\delta(x)) \leqslant C r(x)^{n+\gamma}, \quad x \in \Omega, \tag{2.8}$$

where $\gamma = \beta p/q = p(\alpha - 1/q)$.

(2) *We have continuous embedding $B^{p,q}_\alpha(\Omega) \hookrightarrow L^{p,q}(\Omega, \mathrm{d}\mu)$.*

P r o o f. Suppose $B^{p,q}_\alpha(\Omega) \hookrightarrow L^{p,q}(\Omega, \mathrm{d}\mu)$. Let us fix $x \in \Omega$ and choose the test function

$$f_x(y) = \frac{R_\gamma(y,x)}{R_\gamma(x,x)^{1-1/p+1/p(n+\gamma)-1/q(n+\gamma)}}, \quad y \in \Omega,$$

from Lemma 2.6. By that lemma, there exists a constant C independent of x, such that $\|f_x\|^q_{B^{p,q}_\alpha} \leqslant C$. By Lemma 2.4, there exists a $\delta_0 \in (0,1)$ such that $|R_\gamma(x,y)| \asymp 1/r(x)^{n+\gamma}$ for $x \in \Omega$ and $y \in E_{\delta_0}(x)$. Also, we may assume that $x \in S_k$ and we use (2.4) and Lemma 2.8 to obtain

$$\begin{aligned}
&\Big[\frac{\mu(E_{\delta_0}(x))}{r(x)^{n+\gamma}}\Big]^{q/p} \\
&\quad = \Big[\int_{E_{\delta_0}(x)} r(x)^{-(n+\gamma)} \, \mathrm{d}\mu(y)\Big]^{q/p} \\
&\quad = \Big[\int_{E_{\delta_0}(x)} r(x)^{-(n+\gamma)p} r(x)^{(n+\gamma)p} r(x)^{p/q-1} r(x)^{1-p/q} r(x)^{-(n+\gamma)} \, \mathrm{d}\mu(y)\Big]^{q/p} \\
&\quad \asymp \Big[\int_{E_{\delta_0}(x)} |f_x(y)|^p r(y)^{p/q-1} \, \mathrm{d}\mu(y)\Big]^{q/p} \\
&\quad \leqslant C \sum_{j=k-N}^{k+N} \Big[\int_{S_j} |f_x(y)|^p \, \mathrm{d}\mu(y)\Big]^{q/p} 2^{j(q/p-1)} \\
&\quad \leqslant C \sum_{j=1}^{\infty} \Big[\int_{S_j} |f_x(y)|^p \, \mathrm{d}\mu(y)\Big]^{q/p} 2^{j(q/p-1)} \leqslant C \|f_x\|^q_{B^{p,q}_\alpha} \leqslant C.
\end{aligned}$$

Thus, $\mu(E_{\delta_0}(x))/r(x)^{n+\gamma}$ is bounded by a constant, therefore, $\mu(E_\delta(x))/r(x)^{n+\gamma}$ is also bounded for all $0<\delta<1$ (see Corollary 3.3 from [4]) which means that (2.8) holds.

Conversely, suppose μ satisfies the Carleson type condition (2.8). Without loss of generality, we may assume that δ is small enough so that $N=1$ in Lemma 2.8. For $j\in\mathbb{N}$, set $K_j=\{k\in\mathbb{N}\colon E_{\delta/3}(a_k)\cap S_j\neq\emptyset\}$, where $\{a_k\}$ is a sequence from Lemma 2.7. Then $S_j\subset\bigcup_{k\in K_j}E_{\delta/3}(a_k)$, the last union we denote by E_j. We have $m(E_\delta(x))\asymp r(x)^n$. Using this together with (2.7), we obtain

$$\begin{aligned}
\|f\|^q_{L^{p,q}(\mu)} &= \sum_{j=1}^{\infty}\biggl[\int_{S_j}|f(x)|^p\,\mathrm{d}\mu(x)\biggr]^{q/p}2^{j(q/p-1)} \leqslant \sum_{j=1}^{\infty}\biggl[\int_{E_j}|f(y)|^p\,\mathrm{d}\mu(y)\biggr]^{q/p}2^{j(q/p-1)}\\
&\leqslant \sum_{j=1}^{\infty}\biggl[\sum_{K_j}\max_{y\in\overline{E}_{\delta/3}(a_k)}|f(y)|^p\mu(E_{\delta/3}(a_k))\biggr]^{q/p}2^{j(q/p-1)}\\
&\leqslant C\sum_{j=1}^{\infty}\biggl[\sum_{K_j}\max_{y\in\overline{E}_{\delta/3}(a_k)}|f(y)|^p r(a_k)^\gamma m(E_{\delta/3}(a_k))\biggr]^{q/p}2^{j(q/p-1)}\\
&\leqslant C\sum_{j=1}^{\infty}\biggl[\sum_{K_j}\int_{E_\delta(a_k)}|f(y)|^p r(y)^\gamma\,\mathrm{d}m(y)\biggr]^{q/p}2^{j(q/p-1)}\\
&\leqslant C\sum_{j=2}^{\infty}\biggl[\int_{\{y\in\Omega\colon r_{j+2}<r(y)\leqslant r_{j-2}\}}|f(y)|^p\,\mathrm{d}m_\gamma(y)\biggr]^{q/p}2^{j(q/p-1)}\\
&\leqslant C\sum_{j=1}^{\infty}\biggl[\int_{S_j}|f(y)|^p\,\mathrm{d}m_\gamma(y)\biggr]^{q/p}2^{j(q/p-1)}\asymp\|f\|^q_{B^{p,q}_\alpha},
\end{aligned}$$

where the last relation relies on Theorem 2.1 and the remark following it. $\square$

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References

[1] *M. Arsenović, T. Jovanović*: Embedding of harmonic mixed norm spaces on smoothly bounded domains in $\mathbb{R}^n$. Open Math. *17* (2019), 1260–1268. MR doi

[2] *M. Arsenović, R. F. Shamoyan*: On embeddings, traces and multipliers in harmonic function spaces. Kragujevac J. Math. *37* (2013), 45–64. zbl MR

[3] *M. Calzi, M. M. Peloso*: Carleson and reverse Carleson measures on homogeneous Siegel domains. Available at `https://arxiv.org/abs/2105.06342v2` (2021), 40 pages.

[4] *B. R. Choe, Y. J. Lee, K. Na*: Toeplitz operators on harmonic Bergman spaces. Nagoya Math. J. *174* (2004), 165–186. zbl MR doi

[5] *M. Engliš*: Boundary singularity of Poisson and harmonic Bergman kernels. J. Math. Anal. Appl. *429* (2015), 233–272. zbl MR doi
[6] *C. L. Fefferman, E. M. Stein*: H^p spaces of several variables. Acta Math. *129* (1972), 137–193. zbl MR doi
[7] *Z. Hu*: Estimate for the integral mean of harmonic functions on bounded domains in $\mathbb{R}^n$. Sci. China, Ser. A *38* (1995), 36–46. zbl MR
[8] *Z. Hu, X. Lv*: Carleson type measures for harmonic mixed norm spaces with application to Toeplitz operators. Chin. Ann. Math., Ser. B *34* (2013), 623–638. zbl MR doi
[9] *T. Jovanović*: On Carleson-type embeddings for Bergman spaces of harmonic functions. Anal. Math. *44* (2018), 493–499. zbl MR doi
[10] *H. Kang, H. Koo*: Estimates of the harmonic Bergman kernel on smooth domains. J. Funct. Anal. *185* (2001), 220–239. zbl MR doi
[11] *H. Keshavarzi*: Characterization of forward, vanishing and reverse Bergman Carleson measures using sparse domination. Available at `https://arxiv.org/abs/2110.08926v1` (2021), 23 pages.
[12] *K. Nam, I. Park*: Volume integral means of harmonic functions on smooth boundary domains. Bull. Korean Math. Soc. *51* (2014), 1195–1204. zbl MR doi
[13] *V. L. Oleinik*: Embedding theorems for weighted classes of harmonic and analytic functions. J. Sov. Math. *9* (1978), 228–243. zbl doi
[14] *C. Tong, J. Li*: Carleson measures on the weighted Bergman spaces with Békollé weights. Chin. Ann. Math., Ser. B *42* (2021), 583–600. zbl MR doi

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