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FLOCKING ANALYSIS FOR A GENERALIZED MOTSCH-TADMOR MODEL WITH PIECEWISE INTERACTION FUNCTIONS AND PROCESSING DELAYS

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Abstract. In this paper, a generalized Motsch-Tadmor model with piecewise interaction functions and fixed processing delays is investigated. According to functional differential equation theory and correlation properties of the stochastic matrix, we obtained sufficient conditions for the system achieving flocking, including an upper bound of the time delay parameter. When the parameter is less than the upper bound, the system achieves asymptotic flocking under appropriate assumptions.

Keywords: Motsch-Tadmor model; piecewise interaction function; processing delays; flocking

MSC 2020: 93C15, 93D09, 93C95

1. Introduction

The self-organized collective system is one of the most common phenomena in the natural world, which has appeared in numerous applications and theories, especially in computer science [9], physics [13], biology [25] and social science [1]. It is very important to understand the theoretical mechanisms that lead to collective behaviour. In 2007, the celebrated Cucker-Smale model [7] was proposed:

$$(1.1)\quad \dot{x}_i(t) = v_i(t), \quad \dot{v}_i(t) = \frac{\lambda}{N}\sum_{j=1}^{N}\varphi(|x_j(t)-x_i(t)|)(v_j(t)-v_i(t)), \quad i = 1, 2, \ldots, N,$$

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where $(x_i(t), v_i(t)) \in \mathbb{R}^{2d}$, $i = 1, 2, \ldots, N$ is the position and velocity of the ith agent, λ ($\lambda > 0$) measures the interaction strength, $\varphi(r) = 1/(1+r^2)^\beta$, $\beta \geqslant 0$, represents the interaction function, $|\cdot|$ is the Euclidean vector norm. According to the results of [7], [8], [12], the *unconditional flocking* would occur if $\beta \leqslant \frac{1}{2}$, which means that the system (1.1) achieves flocking without an initial constraint. On the other hand, if $\beta > \frac{1}{2}$, the system (1.1) achieves flocking under some limited initial states and this case is called the *condition flocking*. In 2011, Motsch and Tadmor [23] changed the interaction function from $\varphi(|x_j(t) - x_i(t)|)/N$ into $\psi_{ij}(t)$, where

$$\psi_{ij}(t) = \frac{\varphi(|x_j(t) - x_i(t)|)}{\sum_{k=1}^{N} \varphi(|x_k(t) - x_i(t)|)} \tag{1.2}$$

is an asymmetric function. By the concept of active sets and Lyapunov functional approach, they also proved that $\beta = \frac{1}{2}$ is a critical value for flocking. At the end of [23], Motsch and Tadmor pointed out that the interaction decaying rapidly or cutting off at a finite distance is a more realistic situation.

In 2018, Jin [17] presented a Motsch-Tadmor model with the cut-off interaction function

$$\begin{cases} \dot{x}_i(t) = v_i(t), \quad i = 1, 2, \ldots, N, \\ \dot{v}_i(t) = \dfrac{\lambda}{N_i(t)} \displaystyle\sum_{j \in \mathcal{N}_i(t)} \chi_r(|x_j(t) - x_i(t)|)(v_j(t) - v_i(t)), \end{cases} \tag{1.3}$$

where r is a constant denoting the size of the neighbourhood,

$$\chi_r(s) = \begin{cases} 1, & s < r, \\ 0, & s \geqslant r, \end{cases}$$

is the cut-off interaction function, $\mathcal{N}_i(t) = \{j \colon l_{ij}(t) := |x_j(t) - x_i(t)| < r\}$ is the neighbour set of i and $N_i(t) = \mathrm{Card}(\mathcal{N}_i(t))$ is the number of neighbours for i. Using the algebraic properties of the connected stochastic matrix, Jin obtained a sufficient framework to ensure that the system (1.3) achieves flocking at an exponential rate. Inspired by [17], [3] the proposed a generalized Motsch-Tadmor model with the piecewise interaction function

$$\begin{cases} \dot{x}_i(t) = v_i(t), \quad i = 1, 2, \ldots, N, \\ \dot{v}_i(t) = \dfrac{\lambda}{N_i(t)} \displaystyle\sum_{j \in \mathcal{N}_i(t)} \chi_r^\delta(|x_j(t) - x_i(t)|)(v_j(t) - v_i(t)) \\ \qquad\quad + \dfrac{\lambda}{N - N_i(t)} \displaystyle\sum_{j \notin \mathcal{N}_i(t)} \chi_r^\delta(|x_j(t) - x_i(t)|)(v_j(t) - v_i(t)). \end{cases} \tag{1.4}$$

where

$$\chi_r^\delta(s) = \begin{cases} 1, & s < r, \\ \delta, & s \geqslant r, \end{cases} \qquad \delta \in \mathbb{R},$$

is the piecewise interaction function, which is described by a piecewise constant function. In the above system, if $|x_j(t) - x_i(t)| < r$, then i and j attract to each other. However, if $|x_j(t)-x_i(t)| \geqslant r$, they attract ($\delta > 0$) or repel ($\delta < 0$) each other, and the latter can be understood as the phenomenon of coexistence of cooperation and competition within a population. Their research direction was to get the flocking condition of the system (1.3) with the changes of δ. The authors found that in order for the system (1.4) to achieve flocking, $|\delta|$ has to be small enough. Especially, the result of [3] would degrade into the result of [17] if $|\delta| \to 0$.

The time delay is a non-negligible problem in multi-agent system cluster control and its causes can be roughly divided into two types: information *transmission delay* and information *processing delay.* The transmission delay means that it takes time for agents to receive information from the others limited by the speed of communication, see [20], [4], [5], [6], [10], [11], [15], [2]. The processing delay, also known as reaction delay, refers to the time required for devices to process information, see [26], [16], [19], [18]. In 2020, Liu et al. studied the system (1.3) involving processing delay [21]:

$$(1.5) \qquad \begin{cases} \dot{x}_i(t) = v_i(t), \quad i = 1, 2, \ldots, N, \\ \dot{v}_i(t) = \dfrac{\lambda}{\widetilde{N}_i(t)} \displaystyle\sum_{j \in \widetilde{\mathcal{N}}_i(t)} \chi_r(|\tilde{x}_j(t) - \tilde{x}_i(t)|)(\tilde{v}_j(t) - \tilde{v}_i(t)), \end{cases}$$

where $\tau \in \mathbb{R}_+$ is a fixed delay, $\widetilde{N}_i(t) = N_i(t-\tau)$, $\tilde{v}_i(t) = v_i(t-\tau)$, $\tilde{x}_i(t) = x_i(t-\tau)$. According to functional differential equation theory and correlation properties of matrix eigenvalues, [21] gives sufficient conditions for the system (1.5) achieving flocking, periodic flocking, clustering and periodic clustering, and points that $\frac{1}{2}\pi$ is the critical delay.

In order to generalize the conclusions of [21], we consider a generalized Motsch-Tadmor model with piecewise interaction and processing delay

$$(1.6) \qquad \begin{cases} \dot{x}_i(t) = v_i(t), \quad i = 1, 2, \ldots, N, \\ \dot{v}_i(t) = \dfrac{\lambda}{\widetilde{N}_i(t)} \displaystyle\sum_{j \in \widetilde{\mathcal{N}}_i(t)} \chi_r^\delta(|\tilde{x}_j(t) - \tilde{x}_i(t)|)(\tilde{v}_j(t) - \tilde{v}_i(t)) \\ \qquad\qquad + \dfrac{\lambda}{N - \widetilde{N}_i(t)} \displaystyle\sum_{j \notin \widetilde{\mathcal{N}}_i(t)} \chi_r^\delta(|\tilde{x}_j(t) - \tilde{x}_i(t)|)(\tilde{v}_j(t) - \tilde{v}_i(t)). \end{cases}$$

The initial conditions of the system (1.6) are

$$(1.7) \qquad x_i(\theta) = f_i(\theta), v_i(\theta) = g_i(\theta), \quad f_i(\theta), g_i(\theta) \in C\left([-\tau, 0], \mathbb{R}^d\right), \quad 1 \leqslant i \leqslant N.$$

The main purpose of this paper is to analyse the influence of the processing delay on the system (1.4), obtain a sufficient condition for the system (1.6) achieving flocking or periodic flocking, and compare our results with those of [21].

This paper is organized as follows: In Section 2, we introduce some preliminaries and assumptions for model analysis. In Section 3, we analyse flocking behaviour of the system (1.6) in two cases and obtain sufficient conditions for the system (1.6) achieving flocking. Section 4 is the numerical simulation. The conclusion is in the last section.

2. Modelling formulation and preliminaries

Firstly, we present the mathematical definition of periodic flocking.

Definition 2.1 ([21]). Suppose $(x_i(t), v_i(t)) \in \mathbb{R}^n \times \mathbb{R}^n$, $i = 1, 2, \ldots, N$ is a solution to (1.6) with the initial data (1.7). The system (1.6) is said to achieve periodic flocking, if there are periodic functions $\varphi_{pi}(t)$ with the same period such that

$$\sup_{0\leqslant t<\infty} \max_{1\leqslant i,j\leqslant N} |x_i(t) - x_j(t)| < \infty, \quad \lim_{t\to\infty} (v_i(t) - \varphi_{pi}(t)) = v_\infty, \quad i = 1, 2, \ldots, N,$$

where $v_\infty \in \mathbb{R}^d$ is a constant vector. Especially, if all $\varphi_{pi}(t) = 0$, the system is said to achieve flocking.

In this section, some concepts from graph theory and matrix theory are introduced to analyse the topological structure of the system (1.6). Define the *neighbour graph* of the system (1.6) $G(t) = (\mathcal{V}, \mathcal{E}(t))$, where $\mathcal{V} = \{1, 2, \ldots, N\}$, $\mathcal{E}(t) = \{(i, j)\colon l_{ij(t)} = |x_i(t) - x_j(t)| < r,\ i, j \in \mathcal{V}\}$. A path in $G(t)$ from i to j is a sequence of distinct vertexes $k_0 = i$, $k_1, \ldots, k_q = j \in \mathcal{V}$ such that $(k_{p-1}, k_p) \in \mathcal{E}(t)$ for every $1 \leqslant p \leqslant q$. A graph is said to be connected at time t if there is a path between any two vertices of the graph at t. Denote the *adjacency matrix* and the *average matrix* of $G(t)$ by $A(t) = (a_{ij}(t))_{N\times N}$ and $P(t) = (p_{ij}(t))_{N\times N}$, respectively, where

$$a_{ij}(t) = \begin{cases} 1, & (i,j) \in \mathcal{E}(t), \\ 0, & (i,j) \notin \mathcal{E}(t), \end{cases} \quad \text{and} \quad p_{ij}(t) = \frac{a_{ij}(t)}{N_i(t)}. \tag{2.1}$$

Similarly, define the *distant relative graph* of (1.6): $G^c(t) = (\mathcal{V}, \mathcal{E}^c(t))$, $\mathcal{E}^c(t) = \{(i,j)\colon l_{ij}(t) \geqslant r,\ i, j \in \mathcal{V}\}$, the distant relative set of i: $\mathcal{N}_i^c(t) = \{j\colon l_{ij}(t) \geqslant r\}$, the number of distant relatives for i: $N_i^c(t) = \mathrm{Card}(\mathcal{N}_i^c(t))$, adjacency matrix of $G^c(t)$: $A^c(t) = (a^c_{ij}(t))_{N\times N}$, average matrix of $G^c(t)$: $P^c(t) = (p^c_{ij}(t))_{N\times N}$, where

$$a^c_{ij}(t) = \begin{cases} 0, & (i,j) \in \mathcal{E}(t), \\ 1, & (i,j) \notin \mathcal{E}(t), \end{cases} \quad \text{and} \quad p^c_{ij}(t) = \begin{cases} \dfrac{a^c_{ij}(t)}{N^c_i(t)}, & N_i(t) \neq N, \\ 0, & N_i(t) = N. \end{cases} \tag{2.2}$$

From (2.1) and (2.2), we have the following lemma.

Lemma 2.1. *$P(t)$ and $P^c(t)$ are diagonalizable matrices and all their eigenvalues are real.*

P r o o f. By (2.1) we have

$$p_{ij}(t) = \frac{a_{ij}(t)}{N_i(t)} = \frac{1}{\sqrt{N_i(t)}} \frac{a_{ij}(t)}{\sqrt{N_i(t)N_j(t)}} \sqrt{N_j(t)}.$$

Introduce the symmetric matrix

$$S = \left(\frac{a_{ij(t)}}{\sqrt{N_i(t)N_j(t)}} \right)_{N\times N}.$$

Then

$$P(t) = \operatorname{diag}\left(\frac{1}{\sqrt{N_1(t)}}, \dots, \frac{1}{\sqrt{N_N(t)}} \right) S(t) \operatorname{diag}\big(\sqrt{N_1(t)}, \dots, \sqrt{N_N(t)}\big),$$

which means that $P(t)$ is similar to $S(t)$. On the other hand, $S(t)$ is a real symmetric matrix, so $S(t)$ is diagonalizable and its all eigenvalues are real. Then $P(t)$ is diagonalizable and its all eigenvalues are real.

Set $N^c_{\min}(t) = \min\{N^c_1(t), N^c_2(t), \dots, N^c_N(t)\}$. From (2.2), if $N^c_{\min}(t) > 0$,

$$p^c_{ij}(t) = \frac{a^c_{ij}(t)}{N^c_i(t)} = \frac{1}{\sqrt{N^c_i(t)}} \frac{a^c_{ij}(t)}{\sqrt{N^c_i(t)N^c_j(t)}} \sqrt{N^c_j(t)}.$$

Put

$$S^c(t) = \left(\frac{a^c_{ij}(t)}{\sqrt{N^c_i(t)N^c_j(t)}} \right)_{N\times N}.$$

Then according to the previous argument, $P^c(t)$ is diagonalizable and its all eigenvalues are real. When $N^c_{\min}(t) = 0$, using the the symmetry of $A^c(t)$ and elementary transformations of matrix, there exists a non-singular matrix F such that $P^c(t) = FQ(t)F^{-1}$, where

$$Q(t) = \begin{pmatrix} Q_1(t) & \\ & \mathbf{0} \end{pmatrix}, \quad Q_1(t) = (q_{ij}(t))_{n_1\times n_1} (q_{ij} \neq 0), \quad \mathbf{0} = \{0\}_{n_2\times n_2},$$

$n_1 + n_2 = N$, and the form of $Q_1(t)$ is the same as (2.1). Then $Q_1(t)$ is diagonalizable and its all eigenvalues are real, thus $P^c(t)$ is diagonalizable and its all eigenvalues are real. □

Throughout the paper, we use G_0 and G_0^c to represent the initial neighbour graph and initial distant relative graph related to the system (1.6), with their average matrix given by P_0 and P_0^c, respectively. Owing to Lemma 2.1, we denote all the eigenvalues of P_0 and P_0^c by μ_i, $i = 1, 2, \dots, n_0$ and ν_j, $j = 1, 2, \dots, m_0$, with the algebraic multiplicity p_i and q_j, respectively. Without loss of generality, assume that

$$1 = \mu_1 > \mu_2 > \dots > \mu_{n_0} \quad \text{and} \quad 1 = \nu_1 > \nu_2 > \dots > \nu_{m_0}.$$

By matrix theory, $M = (m_{ij})_{N\times N}$ is a stochastic matrix if $\sum\limits_{j=1}^{N} m_{ij} = 1$ and $m_{ij} \geqslant 0$ for all $i = 1, 2, \dots, N$. For any i, j $(0 \leqslant i, j \leqslant N)$, if there always exists a sequence of integers $k_1, k_2, \dots, k_q$ $(1 \leqslant q \leqslant N-2)$ such that the entries $m_{ik_1}, m_{k_1k_2}, \dots, m_{k_qj}$ of the matrix are all non-zero, then the matrix is called connected. Thus, from (2.1) and (2.2) we know that P_0 and P_0^c are both connected stochastic matrices if and only if $N_{\max} = \max\{N_1(0), \dots, N_N(0)\} < N$ and $N_{\min} = \min\{N_1(0), \dots, N_N(0)\} > 1$ (i.e., $\mathcal{G}_0$ is connected but not complete).

Lemma 2.2 ([3]). *Let $\mathcal{G}_0$ be the initial neighbour graph of the system* (1.6) *and its average matrix be $P_0 = (p_{ij})_{N\times N}$. $P_0^c = (p_{ij}^c)_{N\times N}$ is the average matrix of the distant relative graph. Assume $\mathcal{G}_0$ is connected but not complete. Then there are matrices T_1, T_2 such that $P_0 = T_1 J_1 T_1^{-1}, P_0^c = T_2 J_2 T_2^{-1}$, where*

$$J_1 = \begin{pmatrix} 1 & & & \\ & \mu_2 I_{p_2} & & \\ & & \ddots & \\ & & & \mu_{n_0} I_{p_{n_0}} \end{pmatrix}, \quad J_2 = \begin{pmatrix} 1 & & & \\ & \nu_2 I_{q_2} & & \\ & & \ddots & \\ & & & \nu_{m_0} I_{q_{m_0}} \end{pmatrix}$$

satisfy $\sum\limits_{i=2}^{n_0} p_i = N - 1$, $\sum\limits_{j=2}^{m_0} q_j = N-1$, $0 < |\mu_i| < 1$, $|\nu_j| < 1$, and $T_1 = (t_{ij})_{N\times N}$ satisfies $\det(T_1) \neq 0$, $t_{i1} = a \neq 0$, $1 \leqslant i \leqslant N$.

In the following, we use $|\cdot|$, $\|\cdot\|$ and $\|\cdot\|_F$ to represent the Euclidean vector norm, the spectral norm and the Frobenius norm, respectively. Set $X(t) = (x_1(t), x_2(t), \dots, x_N(t))^\top \in \mathbb{R}^{N\times d}$, $V(t) = (v_1(t), v_2(t), \dots, v_N(t))^\top \in \mathbb{R}^{N\times d}$. From matrix theory, we have

$$\|V\| = \sup_{|\alpha|\neq 0} \frac{|V\alpha|}{|\alpha|} = \sqrt{\sup_{|\alpha|\neq 0} \frac{\alpha^\top V^\top V \alpha}{|\alpha|^2}} = \sqrt{\varrho_{\max}(V^\top V)}, \quad \|V\|_F = \Big(\sum_{i=1}^{N} |v_i|^2\Big)^{1/2},$$

where $\alpha \in \mathbb{R}^d$ and $\varrho_{\max}(V^\top V)$ is the largest eigenvalue of $V^\top V$. Using the Cauchy-Schwarz inequality, it is easy to show that

$$\|V\| \leqslant \|V\|_F \leqslant \sqrt{d}\|V\|. \tag{2.3}$$

Finally, two conclusions about functional differential equations will be presented. Let J_1 is the matrix from Lemma 2.2, consider the equation

$$\dot{u}(t) = -\lambda(I - J_1)\tilde{u}(t)$$

and its characteristic equation

$$h_0(z) = \det(zI + \lambda\mathrm{e}^{-z\tau}(I - J_1)) = z\prod_{i=2}^{n_0}\bigl(z + \lambda(1-\mu_i)\mathrm{e}^{-z\tau}\bigr)^{p_i} = 0.$$

Lemma 2.3 ([21]). *If* $0 \leqslant \lambda\tau(1-\mu_{n_0}) < \frac{1}{2}\pi$, *then all other roots of the equation* $z = -\lambda(1-\mu_i)\mathrm{e}^{-z\tau}$ *have real parts and*

$$a_0 = \max_{2\leqslant i\leqslant n_0} \sup\{\operatorname{Re} z\colon\ z = -\lambda(1-\mu_i)\mathrm{e}^{-z\tau}\} < 0.$$

Lemma 2.4 ([14]). *If* $a_0 = \max\{\operatorname{Re} z\colon\ h_0(z) = 0\}$, *then, for any* $c > a_0$, *there is a constant* $K = K(c)$ *such that the fundamental solution* $S_u(t)$ *of the equation* $\dot{u}(t) = -\lambda(I - J_1)\tilde{u}(t)$ *satisfies the inequality* $\|S_u(t)\| \leqslant K\mathrm{e}^{ct}$.

3. Analysis of the flocking behaviour

To quantize the sensitiveness of the neighbour graph of the system (1.6) when the distance of two particles is near r, we use the following variable of time t:

$$\Gamma(t) = \min\Bigl\{r - \max_{(i,j)\in\mathcal{E}(t)} l_{ij}(t),\ \min_{(i,j)\in\mathcal{E}^c(t)} l_{ij}(t) - r\Bigr\}.$$

By definition, $\Gamma(t) \geqslant 0$. If $\Gamma(0) > 0$, we call this case the non-critical neighbourhood situation. If $\Gamma(0) = 0$, we call it a general neighbourhood situation.

3.1. Flocking behaviour in the non-critical neighbourhood situation. In this case, by the continuity of $l_{ij}(t)$, there exists $t_1 > 0$ such that the average matrix $P(t)$ keeps unchanged on $[0, t_1)$. Thus we obtain the following result.

Theorem 3.1. *Let* $0 \leqslant \lambda\tau(1-\mu_{n_0}) < \frac{1}{2}\pi$. *Assume* $\Gamma(0) > 0$, $N_{\max} < N$, $N_{\min} > 1$ *and*

$$\sqrt{2d}D_1 g_0 K < (|c_1| - \lambda|\delta| D_1(1-\nu_{m_0})D_2K)\Gamma(0),$$

where

$$g_0 = \sup_{\theta\in[-\tau,0]}\|g(\theta)\|,\quad c_1 = \frac{1}{2}\max_{2\leqslant i\leqslant n_0}\sup\{\operatorname{Re} z\colon\ z = -\lambda(1-\mu_i)\mathrm{e}^{-z\tau}\},$$

$$D_1 = \sqrt{\frac{N_{\max}}{N_{\min}}},\quad D_2 = \sqrt{\frac{N - N_{\min}}{N - N_{\max}}},$$

and $K = K(c_1)$ is a constant satisfying Lemma 2.4. Then the system (1.6) will achieve flocking, and there is a constant

$$c \in \Big(\frac{\sqrt{2n}D_1 g_0 K}{\Gamma(0)} + \lambda|\delta|D_1(1-\nu_{m_0})D_2K, -c_1\Big)$$

such that

$$\|V(t) - V_\infty\| \leqslant \frac{c-\Omega(K-1)}{c-\Omega K} D_1 K g_0 \mathrm{e}^{-(c-\Omega K)t},$$

$$\sup_{0\leqslant t\leqslant\infty}\max_{1\leqslant i,j\leqslant N}|x_i(t)-x_j(t)| \leqslant \sqrt{2d}\Big(\|f(0)\| + \frac{c-\Omega(K-1)}{(c-\Omega K)^2}D_1 K g_0\Big) < \infty,$$

where

$$\Omega = \lambda|\delta|D_1(1-\nu_{m_0})D_2,$$

$$V_\infty = T_1\operatorname{diag}(1,0,\dots,0)T_1^{-1}V(0) - \lambda\delta\int_0^\infty T_1\operatorname{diag}(1,0,\dots,0)T_1^{-1}(I-P_0^c)\widetilde{V}(s)\,\mathrm{d}s.$$

P r o o f. The proof is divided into three steps.

The first step: use assumptions to obtain an exponential estimation of velocities on $[0,t_1)$. Owing to $\Gamma(0) > 0$, the average matrix $P(t)$ keeps unchanged on $[0,t_1)$. Then the system (1.6) can be rewritten in the matrix form on $[0,t_1)$ to read as

$$\begin{cases} \dot{V}(t) = -\lambda(I-P_0)\widetilde{V}(t) - \lambda\delta(I-P_0^c)\widetilde{V}(t), \\ V(\theta) = g(\theta), \theta\in[-\tau,0]. \end{cases} \tag{3.1}$$

Let $S_1(t)$ be the fundamental solution operator of the equation

$$\dot{u}(t) = -\lambda(I-J_1)\tilde{u}(t).$$

By using the variation-of-constant formula, the general solution of (3.1) is given as

$$\begin{aligned} V(t+\theta) = {}& T_1\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & S_1(t)\end{pmatrix} T_1^{-1}g(\theta) \\ & - \lambda\delta\int_0^t T_1\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & S_1(t-s)\end{pmatrix} T_1^{-1}(I-P_0^c)\widetilde{V}(s)\,\mathrm{d}s. \end{aligned} \tag{3.2}$$

To get an exponential estimate of velocities, we estimate $\|S_1(t)\|$, $\|T_1\|\cdot\|T_1^{-1}\|$ and $\|I-P_0^c\|$, in turn. Put $a_0 = \max\limits_{2\leqslant i\leqslant n_0}\sup\{\operatorname{Re} z\colon z = -\lambda(1-\mu_i)\mathrm{e}^{-z\tau}\}$, then we have $a_0 < 0$ from Lemma 2.3. Let $c_1 = -\frac{1}{2}a_0$, Lemma 2.4 claims that there is a constant $K = K(c_1)$ such that

$$\|S_1(t)\| \leqslant K\mathrm{e}^{c_1 t}, \quad t \geqslant 0. \tag{3.3}$$

$N_{\max} < N$ and $N_{\min} > 1$, so both P_0 and P_0^c are connected stochastic matrices. Hence, using Lemma 2.2 and Lemma 2.1, direct computation yields

$$\|T_1\| \cdot \|T_1^{-1}\| \leqslant \frac{1}{\sqrt{N_{\min}}}\|O\| \cdot \sqrt{N_{\max}}\|O^{-1}\| \leqslant \sqrt{\frac{N_{\max}}{N_{\min}}} =: D_1, \tag{3.4}$$

where O is the orthogonal matrix to diagonalize the symmetric matrix

$$\left(\frac{a_{ij}(0)}{\sqrt{N_i(0)N_j(0)}}\right)_{N\times N}.$$

Similarly, we have

$$\|T_2\|\cdot\|T_2^{-1}\| \leqslant \sqrt{\frac{N-N_{\min}}{N-N_{\max}}} =: D_2. \tag{3.5}$$

Then from Lemma 2.1 we obtain

$$\|I - P_0^c\| \leqslant (1-\nu_{m_0})D_2. \tag{3.6}$$

According to the last assumption $\sqrt{2d}D_1 g_0 K < (|c_1| - \lambda|\delta|D_1(1-\nu_{m_0})D_2K)\Gamma(0)$ and (3.3), there exists a constant

$$c \in \left(\frac{\sqrt{2n}D_1 g_0 K}{\Gamma(0)} + \lambda|\delta|D_1(1-\nu_{m_0})D_2K, -c_1\right)$$

such that

$$\|S_1(t)\| \leqslant K\mathrm{e}^{-ct}. \tag{3.7}$$

Next, we get an exponential estimate of velocities. Put

$$V_a(t+\theta) = T_1\,\mathrm{diag}(1,0,\dots,0)T_1^{-1}g(\theta) - \lambda\delta\int_0^t T_1\,\mathrm{diag}(1,0,\dots,0)T_1^{-1}(I-P_0^c)\widetilde{V}(s)\,\mathrm{d}s.$$

Using Lemma 2.1 again, we know the entries in the first column of T_1 are the same. Direct calculating the above equality shows that all the rows of $V_a(t+\theta)$ are the same. Then $(I-P_0^c)\widetilde{V}_a(s) = \mathbf{0}$ holds and we have

$$\begin{aligned}
V(t+\theta) - V_a(t+\theta) &= T_1\begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & S_1(t)\end{pmatrix}T_1^{-1}g(\theta) \\
&\quad - \lambda\delta\int_0^t T_1\begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & S_1(t-s)\end{pmatrix}T_1^{-1}(I-P_0^c)\widetilde{V}(s)\,\mathrm{d}s \\
&= T_1\begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & S_1(t)\end{pmatrix}T_1^{-1}g(\theta) \\
&\quad - \lambda\delta\int_0^t T_1\begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & S_1(t-s)\end{pmatrix}T_1^{-1}(I-P_0^c)(\widetilde{V}(s)-\widetilde{V}_a(s))\,\mathrm{d}s.
\end{aligned}$$

Taking the norm of the above inequality and using (3.4), (3.6) and (3.7) yield

$$\sup_{\theta\in[-\tau,0]}\|V(t+\theta)-V_a(t+\theta)\| \leqslant D_1Kg_0\mathrm{e}^{-ct}+\lambda|\delta|D_1(1-\nu_{m_0})D_2K \times\int_0^t \mathrm{e}^{-c(t-s)}\sup_{\theta\in[-\tau,0]}\|V(s+\theta)-V_a(s+\theta)\|\,\mathrm{d}s.$$

By solving the above Gronwall inequality, we obtain

$$\sup_{\theta\in[-\tau,0]}\|V(t+\theta)-V_a(t+\theta)\| \leqslant D_1Kg_0\mathrm{e}^{-(c-\lambda|\delta|D_1(1-\nu_{m_0})D_2K)t},\quad t\in[0,t_1). \tag{3.8}$$

The second step: prove that $P(t)$ remains unchanged for all time, i.e., $t_1=\infty$. If $t_1<\infty$, then there exists (i_0,j_0) such that

$$\tilde{l}_{i_0j_0}(t_1)=|\tilde{x}_{i_0}(t_1)-\tilde{x}_{j_0}(t_1)|=r.$$

Recalling the first equation of (1.6), we have $\dot{x}_{i_0}(t)=v_{i_0}(t)$ and $\dot{x}_{j_0}(t)=v_{j_0}(t)$, then

$$x_{i_0}(t)-x_{j_0}(t)=x_{i_0}(0)-x_{j_0}(0)+\int_0^t(v_{i_0}(s)-v_{j_0}(s))\,\mathrm{d}s.$$

Combining (2.3) and (3.8) yields

$$\begin{aligned}|v_{i_0}(s)-v_{j_0}(s)| &\leqslant \sqrt{2}\sup_{\theta\in[-\tau,0]}\|V(t+\theta)-V_a(t+\theta)\|_F\\ &\leqslant \sqrt{2d}\sup_{\theta\in[-\tau,0]}\|V(t+\theta)-V_a(t+\theta)\|\\ &\leqslant \sqrt{2d}D_1Kg_0\mathrm{e}^{-(c-\lambda|\delta|D_1(1-\nu_{m_0})D_2K)t}.\end{aligned}$$

Then by the fact that

$$c<\frac{\sqrt{2n}D_1g_0K}{\Gamma(0)}+\lambda|\delta|D_1(1-\nu_{m_0})D_2K$$

we obtain

$$\begin{aligned}l_{i_0j_0}(t_1+\theta)&=|x_{i_0}(t_1+\theta)-x_{j_0}(t_1+\theta)|\\ &\leqslant l_{i_0j_0}(0)+\sqrt{2d}D_1g_0K\int_0^\infty \mathrm{e}^{-(c-\lambda|\delta|D_1(1-\nu_{m_0})D_2K)s}\,\mathrm{d}s\\ &=l_{i_0j_0}(0)+\frac{\sqrt{2d}D_1g_0K}{(c-\lambda|\delta|D_1(1-\nu_{m_0})D_2K)}\\ &<l_{i_0j_0}(0)+\Gamma(0)\leqslant r,\quad (i_0,j_0)\in\mathcal{E}(0)\quad\forall\,\theta\in[-\tau,0],\end{aligned}$$

and

$$
\begin{aligned}
l_{i_0j_0}(t_1+\theta) &= |x_{i_0}(t_1+\theta) - x_{j_0}(t_1+\theta)| \\
&\geqslant l_{i_0j_0}(0) - \sqrt{2d}D_1g_0K\int_0^\infty \mathrm{e}^{-(c-\lambda|\delta|D_1(1-\nu_{m_0})D_2K)s}\,\mathrm{d}s \\
&= l_{i_0j_0}(0) - \frac{\sqrt{2d}D_1g_0K}{(c-\lambda|\delta|D_1(1-\nu_{m_0})D_2K)} \\
&> l_{i_0j_0}(0) - \Gamma(0) \geqslant r, \quad (i_0,j_0)\notin\mathcal{E}(0) \quad \forall\,\theta\in[-\tau,0].
\end{aligned}
$$

This implies that

$$
\tilde{l}_{i_0j_0}(t_1) < r, \quad (i_0,j_0)\in\mathcal{E}(0), \quad \text{and} \quad \tilde{l}_{i_0j_0}(t_1) > r, \quad (i_0,j_0)\notin\mathcal{E}(0).
$$

Obviously, the above inequalities contradict the existence of (i_0,j_0) such that $\tilde{l}_{i_0j_0}(t_1) = r$. Thus $t_1 = \infty$ and $P(t)\equiv P_0$ for all time.

The final step: use the exponential estimation of velocities to show that the system achieves flocking. From conclusions of the first step and the second step, we have

$$
\sup_{\theta\in[-\tau,0]} \|V(t+\theta) - V_a(t+\theta)\| \leqslant D_1Kg_0\mathrm{e}^{-(c-\lambda|\delta|D_1(1-\nu_{m_0})D_2K)t}, \quad t\geqslant 0.
$$

Set $\Omega = \lambda|\delta|D_1(1-\nu_{m_0})D_2$ for convenience. Then

$$
\sup_{\theta\in[-\tau,0]} \|V(t+\theta) - V_a(t+\theta)\| \leqslant D_1Kg_0\mathrm{e}^{-(c-\Omega K)t}, \quad t\geqslant 0. \tag{3.9}
$$

Combining

$$
\begin{aligned}
V_a(t+\theta) &= T_1\operatorname{diag}(1,0,\ldots,0)T_1^{-1}g(\theta) \\
&\quad - \lambda\delta\int_0^t T_1\operatorname{diag}(1,0,\ldots,0)T_1^{-1}(I-P_0^c)\widetilde{V}(s)\,\mathrm{d}s \\
&= T_1\operatorname{diag}(1,0,\ldots,0)T_1^{-1}g(\theta) \\
&\quad - \lambda\delta\int_0^t T_1\operatorname{diag}(1,0,\ldots,0)T_1^{-1}(I-P_0^c)(\widetilde{V}(s)-\widetilde{V}_a(s))\,\mathrm{d}s
\end{aligned}
$$

with (3.9), we deduce that

$$
T_1\operatorname{diag}(1,0,\ldots,0)T_1^{-1}g(\theta) = T_1\operatorname{diag}(1,0,\ldots,0)T_1^{-1}V(0)
$$

and there exists V_∞ given by

$$
\begin{aligned}
V_\infty &= T_1\operatorname{diag}(1,0,\ldots,0)T_1^{-1}V(0) \\
&\quad - \lambda\delta\int_0^\infty T_1\operatorname{diag}(1,0,\ldots,0)T_1^{-1}(I-P_0^c)\widetilde{V}(s)\,\mathrm{d}s
\end{aligned} \tag{3.10}
$$

such that

$$\begin{aligned}\|V_a(t)-V_\infty\| &\leqslant \lambda\delta D_1(1-\nu_{m_0})D_2\int_t^\infty \|\widetilde{V}(s)-\widetilde{V}_a(s)\|\,\mathrm{d}s\\ &\leqslant \Omega D_1 K g_0\int_t^\infty \mathrm{e}^{-(c-\Omega K)t}\,\mathrm{d}s=\frac{\Omega D_1Kg_0}{c-\Omega K}\mathrm{e}^{-(c-\Omega K)t}.\end{aligned}$$

Using (3.9) again, we conclude

$$\begin{aligned}\|V(t)-V_\infty\| &\leqslant \|V(t)-V_a(t)\|+\|V_a(t)-V_\infty\|\\ &\leqslant \frac{c-\Omega(K-1)}{c-\Omega K}D_1Kg_0\mathrm{e}^{-(c-\Omega K)t},\quad t\geqslant 0.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|X(t)-tV_\infty\| &= \left\|X(0)+\int_0^t V(s)-V_\infty\,\mathrm{d}s\right\|\\ &\leqslant \|f(0)\|+\frac{c-\Omega(K-1)}{c-\Omega K}D_1Kg_0\int_0^t \mathrm{e}^{-(c-\Omega K)s}\,\mathrm{d}s\\ &\leqslant \|f(0)\|+\frac{c-\Omega(K-1)}{(c-\Omega K)^2}D_1Kg_0,\quad t\geqslant 0,\end{aligned}$$

and thus

$$\begin{aligned}\sup_{0\leqslant t\leqslant\infty}\max_{1\leqslant i,j\leqslant N}|x_i(t)-x_j(t)| &\leqslant \sqrt{2}\|X(t)-tV_\infty\|_2\leqslant\sqrt{2d}\|X(t)-tV_\infty\|\\ &\leqslant \sqrt{2d}\Big(\|f(0)\|+\frac{c-\Omega(K-1)}{(c-\Omega K)^2}D_1Kg_0\Big)<\infty.\end{aligned}$$

□

R e m a r k 3.1. Here we discuss the satisfiability of the assumptions of Theorem 3.1. There are five assumptions in Theorem 3.1: $0\leqslant\lambda\tau(1-\mu_{n_0})<\frac{1}{2}\pi$ comes from Lemma 2.3, $\Gamma(0)>0$ means the non-critical neighbourhood situation, $N_{\max}<N$ and $N_{\min}>1$ are the conditions for Lemma 2.2, $\sqrt{2d}D_1g_0K<(|c_1|-\lambda|\delta|D_1(1-\nu_{m_0})D_2K)\Gamma(0)$ is a technical condition related to the initial conditions. In the last assumption, c_1, $K(c_1)$, $\Gamma(0)$, ν_{m_0}, D_1 and D_2 are all determined by the initial neighbour graph G_0. Although we have only the existence result for $K(c_1)$ (from Lemma 2.4), $(|c_1|-\lambda|\delta|D_1(1-\nu_{m_0})D_2K)\Gamma(0)>0$ can be guaranteed when $|\delta|$ is sufficiently small. On the other hand, we can change g_0 by adjusting the initial velocities without affecting G_0, because G_0 is determined only by the initial positions, and the selection of the initial positions, and the initial velocities are independent of each other. Then the last assumption holds when g_0 is sufficiently small. And the last assumption does not conflict with the first four assumptions, so the assumptions of Theorem 3.1 are actually satisfiable.

3.2. Flocking behaviour in the general neighbourhood situation. In the general case, we put t_n to be the switching moments at the nth time. Then t_n is called the switching time sequence, which could be finite or infinity. Since the average matrix keeps unchanged at each interval (t_n, t_{n+1}), $n = 0, 1, 2, \ldots$ ($t_0 = 0$, $t_1 \geqslant 0$), the matrix $P(t)$ is a constant matrix on (t_n, t_{n+1}), say $P(t_n)$. Assume the initial average matrix keeps unchanged, say $P(\theta) = P_0$ for $\theta \in [-\tau, 0]$.

To understand well the dynamics of the system (1.6) in the general case, we assume that the adjacency matrix does not change frequently and sharply, and consider the following assumptions.

(H_1) There exist positive constants δ, γ and a sequence $t_n^* \in (t_n, t_{n+1})$ such that

$$t_{n+1} - t_n \geqslant \delta, \quad t_{n+1} - t_n^* \geqslant \tau \quad \text{and} \quad \Gamma(t_n^*) \geqslant \gamma \quad \forall\, n.$$

(H_2) Assume that the amplitudes $\|P(t) - P_0\|$ and $\|P^c(t) - P_0^c\|$ are bounded uniformly on t,

$$\eta_1 = \sup_{t \geqslant 0} \|P(t) - P_0\| \quad \text{and} \quad \eta_2 = \sup_{t \geqslant 0} \|P^c(t) - P_0^c\|.$$

The above assumptions ensure that the system is a controllable switching system. (H_1) guarantees that switches are not too frequent and (H_2) guarantees that switches are not too drastic. The following theorem shows that under the assumptions (H_1)–(H_2) and appropriate initial conditions, the topology of the system (1.6) stops changing after a certain time point.

Theorem 3.2. *Let* $0 \leqslant \lambda\tau(1 - \mu_{n_0}) < \frac{1}{2}\pi$. *Assume*

$$N_{\max} < N, \quad N_{\min} > 1, \quad \Lambda D_1(K-1) < |c_1|,$$

and the assumptions (H_1)–(H_2) *hold. Then the system* (1.6) *will achieve flocking, and there is a constant* $c \in (\Lambda D_1 K, -c_1)$ *such that*

$$\sup_{0 \leqslant t \leqslant \infty} \max_{1 \leqslant i,j \leqslant N} |x_i(t) - x_j(t)| \leqslant \sqrt{2d}\Big(\|f(0)\| + \frac{c - \Lambda D_1(K-1)}{(c - \Lambda D_1 K)^2} D_1 K g_0\Big) < \infty,$$

$$\|V(t) - V_\infty\| \leqslant \frac{c - \Lambda D_1(K-1)}{c - \Lambda D_1 K} D_1 K g_0 \mathrm{e}^{-(c - \Lambda D_1 K)t},$$

where

$$\Lambda = \lambda\eta_1 + \lambda|\delta|(1 - \nu_{m_0})D_2 + \lambda|\delta|\eta_2,$$

$$V_\infty = T_1 \operatorname{diag}(1, 0, \ldots, 0) T_1^{-1} V(0) + \int_0^\infty T_1 \operatorname{diag}(1, 0, \ldots, 0) T_1^{-1} [\lambda(P(t) - P_0) - \lambda\delta(I - P^c(t))] \widetilde{V}(s)\, \mathrm{d}s.$$

P r o o f. The proof is also divided into three steps.

The first step: use assumptions to obtain the exponential estimation of velocities. Rewrite the system (1.6) as

$$\begin{cases} \dot{V}(t) = -\lambda(I-P_0)\widetilde{V}(t) + \lambda(P(t)-P_0)\widetilde{V}(t) - \lambda\delta(I-P^c(t))\widetilde{V}(t), \\ V(\theta) = g(\theta), \quad \theta \in [-\tau, 0]. \end{cases} \tag{3.11}$$

By using the variation-of-constant formula, the general solution of (3.11) is given as

$$\begin{aligned} V(t+\theta) &= T_1 \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & S_1(t) \end{pmatrix} T_1^{-1} g(\theta) \\ &\quad + \int_0^t T_1 \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & S_1(t-s) \end{pmatrix} T_1^{-1}[\lambda(P(t)-P_0) - \lambda\delta(I-P^c(t))]\widetilde{V}(s)\,\mathrm{d}s. \end{aligned}$$

Taking

$$\begin{aligned} V_a(t+\theta) &= T_1 \operatorname{diag}(1,0,\dots,0) T_1^{-1} g(\theta) \\ &\quad + \int_0^t T_1 \operatorname{diag}(1,0,\dots,0) T_1^{-1}[\lambda(P(t)-P_0) - \lambda\delta(I-P^c(t))]\widetilde{V}(s)\,\mathrm{d}s \end{aligned}$$

and using the fact that

$$T_1 \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & S_1(t-s) \end{pmatrix} T_1^{-1}[\lambda(P(t)-P_0) - \lambda\delta(I-P^c(t))]\widetilde{V}_a(s) = \mathbf{0},$$

we have

$$\begin{aligned} &V(t+\theta) - V_a(t+\theta) \\ &= T_1 \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & S_1(t) \end{pmatrix} T_1^{-1} g(\theta) \\ &\quad + \int_0^t T_1 \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & S_1(t-s) \end{pmatrix} T_1^{-1}[\lambda(P(t)-P_0) - \lambda\delta(I-P^c(t))]\widetilde{V}(s)\,\mathrm{d}s \\ &= T_1 \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & S_1(t) \end{pmatrix} T_1^{-1} g(\theta) \\ &\quad + \int_0^t T_1 \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & S_1(t-s) \end{pmatrix} T_1^{-1}[\lambda(P(t)-P_0) - \lambda\delta(I-P^c(t))](\widetilde{V}(s) - \widetilde{V}_a(s))\,\mathrm{d}s. \end{aligned}$$

To get an exponential estimate of velocities, we estimate

$$\|\lambda(P(t)-P_0) - \lambda\delta(I-P^c(t))\|.$$

Using (3.5) and (H_2), we obtain

$$\begin{aligned} &\|\lambda(P(t)-P_0) - \lambda\delta(I-P^c(t))\| \\ &\quad\leqslant \lambda\|P(t)-P_0\| + \lambda|\delta|\|I-P_0^c\| + \lambda|\delta|\|P^c(t)-P_0^c\| \\ &\quad\leqslant \lambda\eta_1 + \lambda|\delta|(1-\nu_{m_0})D_2 + \lambda|\delta|\eta_2 := \Lambda. \end{aligned} \tag{3.12}$$

According to the assumption $\Lambda D_1(K-1) < |c_1|$ and (3.3), there is a constant $c \in (\Lambda D_1 K, -c_1)$ such that

$$\|S_1(t)\| \leqslant K\mathrm{e}^{-ct}.$$

Thus

$$\sup_{\theta\in[-\tau,0]} \|V(t+\theta) - V_a(t+\theta)\| \\ \leqslant D_1 K g_0 \mathrm{e}^{-ct} + \Lambda D_1 K \int_0^t \mathrm{e}^{-c(t-s)} \sup_{\theta\in[-\tau,0]} \|V(s+\theta) - V_a(s+\theta)\| \,\mathrm{d}s.$$

Then

$$\mathrm{e}^{ct} \sup_{\theta\in[-\tau,0]} \|V(t+\theta) - V_a(t+\theta)\| \\ \leqslant D_1 K g_0 + \Lambda D_1 K \int_0^t \mathrm{e}^{cs} \sup_{\theta\in[-\tau,0]} \|V(s+\theta) - V_a(s+\theta)\| \,\mathrm{d}s.$$

By solving the above Gronwall inequality, we get

$$\sup_{\theta\in[-\tau,0]} \|V(t+\theta) - V_a(t+\theta)\| \leqslant D_1 K g_0 \mathrm{e}^{-(c-\Lambda D_1 K)t}. \tag{3.13}$$

The second step: prove that there exists $t_{k_1} > 0$ such that $P(t)$ remains unchanged when $t > t_{k_1}$. From (H_1) and the inequality $\Lambda D_1 K < c$, there are a positive integer k_1, positive constants δ, γ and $t^*_{k_1} \in (t_{k_1}, t_{k_1+1})$ such that

$$\sqrt{2n} D_1 K g_0 \mathrm{e}^{-(c-\Lambda D_1 K) k_1 \delta} < (c - \Lambda D_1 K)\gamma,$$

and

$$t_{k_1+1} - t_{k_1} \geqslant \delta, \quad t_{k_1+1} - t^*_{k_1} \geqslant \tau, \quad \Gamma(t^*_{k_1}) \geqslant \gamma.$$

Next, we claim that $t_{k_1+1} = \infty$. If $t_{k_1+1} < \infty$, then there exists (i_0, j_0) such that

$$\tilde{l}_{i_0 j_0}(t_{k_1+1}) = |\tilde{x}_{i_0}(t_{k_1+1}) - \tilde{x}_{j_0}(t_{k_1+1})| = r.$$

Recalling the first equation of (1.6), we have $\dot{x}_{i_0}(t) = v_{i_0}(t)$ and

$$x_{i_0}(t) - x_{j_0}(t) = x_{i_0}(t^*_{k_1}) - x_{j_0}(t^*_{k_1}) + \int_{t^*_{k_1}}^t (v_{i_0}(s) - v_{j_0}(s)) \,\mathrm{d}s.$$

Combining (2.3) and (3.13), we get

$$\begin{aligned} |v_{i_0}(s) - v_{j_0}(s)| &\leqslant \sqrt{2} \sup_{\theta\in[-\tau,0]} \|V(t+\theta) - V_a(t+\theta)\|_F \\ &\leqslant \sqrt{2d} \sup_{\theta\in[-\tau,0]} \|V(t+\theta) - V_a(t+\theta)\| \\ &\leqslant \sqrt{2d} D_1 K g_0 \mathrm{e}^{-(c-\Lambda D_1 K)t}. \end{aligned}$$

Then, when $\theta \in [-\tau, 0]$, we have

$$\begin{aligned} l_{i_0 j_0}(t_{k_1+1}+\theta) &= |x_{i_0}(t_{k_1+1}+\theta) - x_{j_0}(t_{k_1+1}+\theta)| \\ &\leqslant l_{i_0 j_0}(t^*_{k_1}) + \sqrt{2d} D_1 g_0 K \int_{t^*_{k_1}}^{\infty} \mathrm{e}^{-(c-\Lambda D_1 K)s} \,\mathrm{d}s \\ &= l_{i_0 j_0}(t^*_{k_1}) + \frac{\sqrt{2d} D_1 g_0 K \mathrm{e}^{-(c-\Lambda D_1 K) t^*_{k_1}}}{c - \Lambda D_1 K} \\ &< l_{i_0 j_0}(t^*_{k_1}) + \gamma \leqslant l_{i_0 j_0}(t^*_{k_1}) + \Gamma(t^*_{k_1}) \leqslant r, \quad (i_0, j_0) \in \mathcal{E}(t^*_{k_1}), \end{aligned}$$

and

$$\begin{aligned} l_{i_0 j_0}(t_{k_1+1}+\theta) &= |x_{i_0}(t_{k_1+1}+\theta) - x_{j_0}(t_{k_1+1}+\theta)| \\ &\geqslant l_{i_0 j_0}(t^*_{k_1}) - \sqrt{2d} D_1 g_0 K \int_{t^*_{k_1}}^{\infty} \mathrm{e}^{-(c-\Lambda D_1 K)s} \,\mathrm{d}s \\ &= l_{i_0 j_0}(t^*_{k_1}) - \frac{\sqrt{2d} D_1 g_0 K \mathrm{e}^{-(c-\Lambda D_1 K) t^*_{k_1}}}{c - \Lambda D_1 K} \\ &> l_{i_0 j_0}(t^*_{k_1}) - \gamma \geqslant l_{i_0 j_0}(t^*_{k_1}) - \Gamma(t^*_{k_1}) \geqslant r, \quad (i_0, j_0) \notin \mathcal{E}(t^*_{k_1}). \end{aligned}$$

This implies that

$$\tilde{l}_{i_0 j_0}(t_{k_1+1}) = l_{i_0 j_0}(t_{k_1+1} - \tau) < r$$

and

$$\bar{l}_{i_0 j_0}(t_{k_1+1}) = l_{i_0 j_0}(t_{k_1+1} - \tau) > r.$$

Obviously, the above inequalities contradict the existence of (i_0, j_0) such that $\tilde{l}_{i_0 j_0}(t_{k_1+1}) = r$. Thus $t_{k_1+1} = \infty$ and $P(t) \equiv P(t_{k_1})$ for all time $t > t_{k_1}$.

The final step: use exponential estimation of velocities to show that the system achieves flocking. Through the procedure similar to Theorem 3.1 we obtain

$$\begin{aligned} \sup_{t \geqslant 0} \max_{1 \leqslant i,j \leqslant N} |x_i(t) - x_j(t)| &\leqslant \sqrt{2d} \Big(\|f(0)\| + \frac{c - \Lambda D_1 (K-1)}{(c - \Lambda D_1 K)^2} D_1 K g_0 \Big) < \infty, \\ \|V(t) - V_\infty\| &\leqslant \frac{c - \Lambda D_1 (K-1)}{c - \Lambda D_1 K} D_1 K g_0 \mathrm{e}^{-(c - \Lambda D_1 K)t}, \end{aligned}$$

where

$$\begin{aligned} V_\infty = {} & T_1 \operatorname{diag}(1, 0, \ldots, 0) T_1^{-1} V(0) \\ & + \int_0^\infty T_1 \operatorname{diag}(1, 0, \ldots, 0) T_1^{-1} [\lambda (P(t) - P_0) - \lambda \delta (I - P^c(t))] \widetilde{V}(s) \,\mathrm{d}s. \end{aligned}$$

□

R e m a r k 3.2. By a discussion analogous to Remark 3.1, we know that the assumptions of Theorem 3.2 are actually satisfiable. Although Theorems 3.1 and 3.2 give a sufficient framework for the system achieving flocking, the framework is not sharp. We emphasize two points here: (1) When $\delta > 0$, the unconditional flocking would occur if $0 \leqslant \lambda\tau(1 - \mu_{n_0}) < \frac{1}{2}\pi$. (2) When $\delta \leqslant 0$, in addition to the delay constraint, the most important thing is that the neighbour graph needs to be connected [24] or at least remains connected [22].

3.3. Is $\frac{1}{2}\pi$ the critical threshold? The effects of the transmission delay and the processing delay on flocking behaviour are quite different. For the transmission delay, in [15] the author proved that under the assumption that the interaction function of the system decays slowly enough, the sufficient condition for the system to achieve flocking does not contain the limitation on the size of the delay. In particular, for the Cucker-Smale type interaction function $\varphi(r) = 1/(1 + r^2)^\beta$, $\beta \geqslant 0$, when $\beta < \frac{1}{2}$, the system achieves the unconditional flocking regardless of the value of the transmission delay. In the same year, a more brilliant conclusion was obtained in [2]. The author showed that the same unconditional flocking result for the non-delayed case is valid in the delayed case, and for Cucker-Smale type interaction function, the unconditional flocking would occur if $\beta \leqslant \frac{1}{2}$. However, the effect of the processing delay on the system cannot be ignored. In [26] the authors proved that a sufficient condition for flocking of the system with the processing delay should include a constraint on the size of the delay.

In [21] the authors studied the flocking and periodic flocking behaviour of the system (1.6) when $\delta = 0$. They showed that if $\lambda\tau(1 - \mu_{n_0}) = \frac{1}{2}\pi$, the system velocity converges to a periodic velocity plus a constant value under appropriate initial conditions, that is, the system achieves the periodic flocking. In this paper, Theorems 3.1 and 3.2 give a sufficient framework for the system achieving flocking, in which the constraint of the processing delay $0 \leqslant \lambda\tau(1 - \mu_{n_0}) < \frac{1}{2}\pi$ is worth thinking about. Is $\frac{1}{2}\pi$ a critical value? If $\lambda\tau(1 - \mu_{n_0}) = \frac{1}{2}\pi$, does the system (1.6) also achieve periodic flocking when $\delta \neq 0$? Instead of proving a conclusion, let us briefly analyse the velocity change of the system when $\lambda\tau(1 - \mu_{n_0}) = \frac{1}{2}\pi$.

First, consider the equation

$$\dot{u}(t) = -\lambda(1 - \mu_{n_0})u(t - \tau)$$

and its characteristic equation $z = -\lambda(1 - \mu_{n_0})\mathrm{e}^{-\tau z}$. For $\lambda\tau(1 - \mu_{n_0}) = \frac{1}{2}\pi$, the characteristic equation has pure imaginary roots $\pm\frac{1}{2}\pi\mathrm{i}/\tau$. Thus the solution of the above equation is given as

$$u(t) = \cos\Big(\frac{\pi t}{2\tau}\Big)u(0) - \sin\Big(\frac{\pi t}{2\tau}\Big)u(-\tau), \quad t \in (0, t_1).$$

Then we can define the fundamental solution operator as

$$S_0(t)\varphi(\theta) = \cos\Big(\frac{\pi t}{2\tau}\Big)\varphi(0) - \sin\Big(\frac{\pi t}{2\tau}\Big)\varphi(-\tau).$$

Rewrite

$$P_0 = T_1 \begin{pmatrix} 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J_1^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mu_{n_0} I_{p_{n_0}} \end{pmatrix} T_1^{-1}$$

and let $S_1^*(t)$ be a fundamental solution operator of the equation

$$\dot{u}^*(t) = -\lambda(I - J_1^*)\tilde{u}^*(t).$$

By using the variation-of-constant formula, the general solution of (3.1) is given as

$$\begin{aligned} V(t+\theta) = T_1 &\begin{pmatrix} 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & S_1^*(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & S_0(t) I_{p_{n_0}} \end{pmatrix} T_1^{-1} g(\theta) \\ &- \lambda\delta \int_0^t T_1 \begin{pmatrix} 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & S_1^*(t-s) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & S_0(t-s) I_{p_{n_0}} \end{pmatrix} T_1^{-1}(I - P_0^c)\widetilde{V}(s)\,\mathrm{d}s. \end{aligned}$$

Put

$$\begin{aligned} V_a(t+\theta) &= T_1 \operatorname{diag}(1,0,\dots,0) T_1^{-1} g(\theta) \\ &\quad - \lambda\delta \int_0^t T_1 \operatorname{diag}(1,0,\dots,0) T_1^{-1}(I - P_0^c)\widetilde{V}(s)\,\mathrm{d}s, \\ V_b(t+\theta) &= T_1 \begin{pmatrix} 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & S_1^*(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} T_1^{-1} g(\theta) \\ &\quad - \lambda\delta \int_0^t T_1 \begin{pmatrix} 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & S_1^*(t-s) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} T_1^{-1}(I - P_0^c)\widetilde{V}(s)\,\mathrm{d}s, \\ V_p(t) &= \cos\Big(\frac{\pi t}{2\tau}\Big) T_1 \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{p_{n_0}} \end{pmatrix} T_1^{-1} g(0) - \sin\Big(\frac{\pi t}{2\tau}\Big) T_1 \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{p_{n_0}} \end{pmatrix} T_1^{-1} g(-\tau) \\ &= T_1 \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & S_0(t) I_{p_{n_0}} \end{pmatrix} T_1^{-1} g(\theta) \end{aligned}$$

and

$$V_p^*(t) = -\lambda\delta \int_0^t T_1 \begin{pmatrix} 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & S_0(t-s) I_{p_{n_0}} \end{pmatrix} T_1^{-1}(I - P_0^c)\widetilde{V}(s)\,\mathrm{d}s.$$

Then we obtain

$$V(t+\theta) = V_a(t+\theta) + V_b(t+\theta) + V_p(t) + V_p^*(t).$$

When $\delta = 0$, the results in [21] showed that $V(t)$ converges to the periodic velocity $V_p(t)$ plus a constant value under appropriate assumptions. However, numerical examples in Section 4 show that when $\delta \neq 0$, $\lambda\tau(1-\mu_{n_0}) = \frac{1}{2}\pi$ is not the critical condition for the system to achieve flocking, nor it does make the system achieve periodic flocking. We speculate that the integral term $V_p^*(t)$ may be the main reason for the above results.

4. Numerical simulation

We consider 4-agents in the system (1.6), and put $g_i(\theta) = x_i(0)$ and $f_i(\theta) = v_i(0)$, $\theta \in [-\frac{1}{2}, 0]$. Because of the importance of the connectedness of P_0 and P_0^c, for convenience, we choose the initial positions as

$$x_1(0) = 0, \quad x_2(0) = 3, \quad x_3(0) = 6, \quad x_4(0) = 9.$$

The initial velocities are produced randomly in $(0, 1)$,

$$v_1(0) = 0.3299, \quad v_2(0) = 0.6705, \quad v_3(0) = 0.6165, \quad v_4(0) = 0.8559.$$

Case 1: Set $r = 4$. Then we have

$$P_0 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad P_0^c = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}$$

and $\mu_1 = 1$, $\mu_2 = \frac{1}{4} + \sqrt{\frac{11}{48}}$, $\mu_3 = \frac{1}{6}$, $\mu_4 = \frac{1}{4} - \sqrt{\frac{11}{48}}$. Consider the case where there is no interaction between distant relatives, i.e., $\delta = 0$. Simulation of two examples (i) $\lambda = 3$, $\tau = \frac{1}{2}$, (ii) $\lambda = \pi/\big(\frac{3}{4} + \sqrt{\frac{11}{48}}\big)$, $\tau = \frac{1}{2}$, follow (see Fig. 1).

Case 2: Set $\lambda = 3$, $\delta = \pm 0.01$, $\tau = \frac{1}{2}$ and $r = 4$. In this case, the initial neighbour graph is same as in Case 1. There is a slight attraction (repulsion) between distant relatives, i.e., $\delta = 0.01$ ($\delta = -0.01$) (see Fig. 2).

Case 3: $\lambda = 3$, $\delta = \pm 1$, $\tau = \frac{1}{2}$ and $r = 4$. In this case, the initial neighbour graph is the same as in Case 1 and there is a strong attraction (repulsion) between distant relatives (see Fig. 3).

Case 4: $\lambda = \pi/\big(\frac{3}{4} + \sqrt{\frac{11}{48}}\big)$, $\delta = \pm 0.01$, $\tau = \frac{1}{2}$ and $r = 4$. In this case, the initial neighbour graph is the same as in Case 1 and $\lambda\tau(1-\mu_{n_0}) = \frac{1}{2}\pi$ (see Fig. 4).

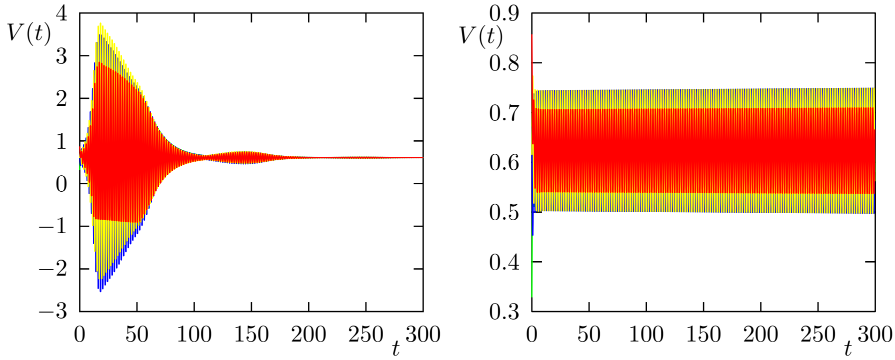

Figure 1. The left one is the case $\lambda = 3$, $\tau = \frac{1}{2}$, $\delta = 0$ and $r = 4$. Then $0 \leqslant \lambda\tau(1-\mu_{n_0}) < \frac{1}{2}\pi$ and the system achieves flocking. The right one is the case $\lambda = \pi/\left(\frac{3}{4} + \sqrt{\frac{11}{48}}\right)$, $\tau = \frac{1}{2}$, $\delta = 0$ and $r = 4$. Then $\lambda\tau(1-\mu_{n_0}) = \frac{1}{2}\pi$ and the system achieves periodic flocking. The above simulations are consistent with the results in [21].

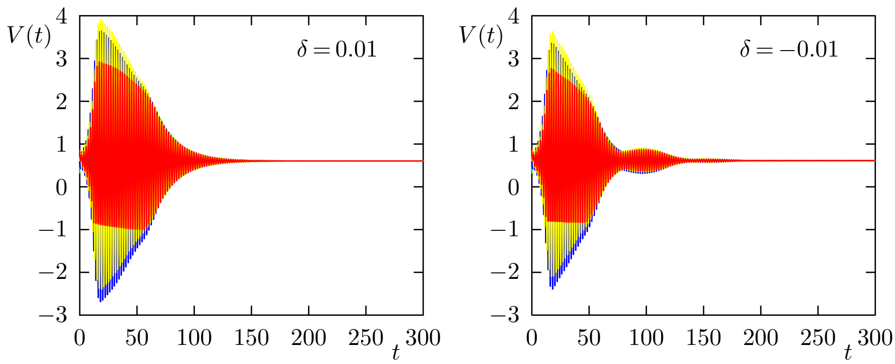

Figure 2. $\lambda = 3$, $\delta = \pm 0.01$, $\tau = \frac{1}{2}$ and $r = 4$. The system also achieves flocking, which is consistent with Theorem 3.1.

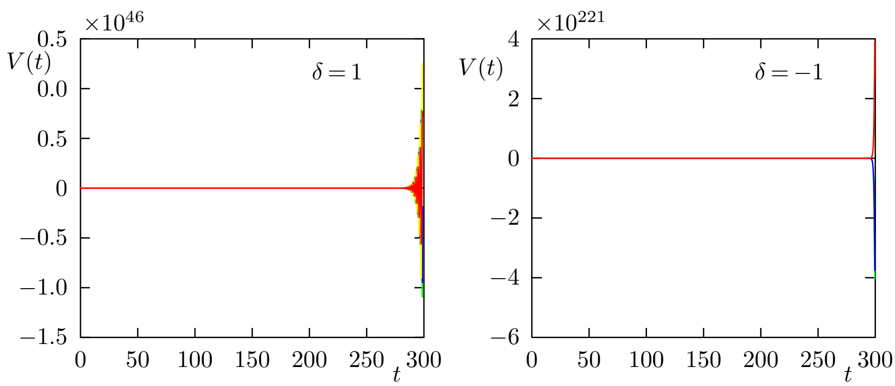

Figure 3. $\lambda = 3$, $\delta = \pm 1$, $\tau = \frac{1}{2}$ and $r = 4$. The system does not achieve flocking. In fact, in order for the system (1.4) to achieve flocking, $|\delta|$ has to be small enough, which is consistent with Theorems 3.1 and 3.2.

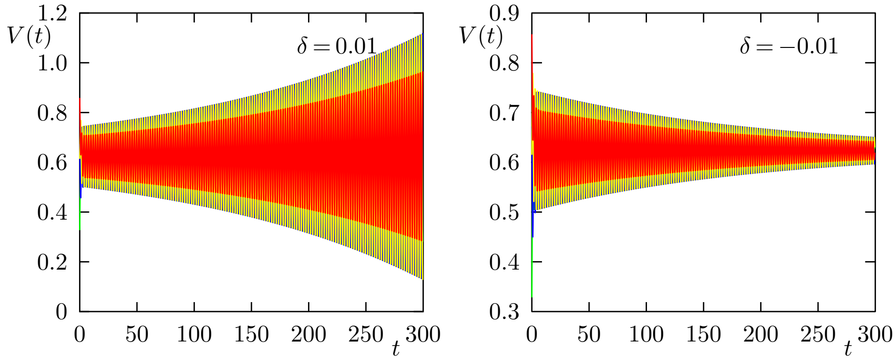

Figure 4. $\lambda = \pi/\left(\frac{3}{4} + \sqrt{\frac{11}{48}}\right)$, $\delta = \pm 0.01$, $\tau = \frac{1}{2}$ and $r = 4$. $\lambda\tau(1-\mu_{n_0}) = \frac{1}{2}\pi$, but the system does not achieve periodic flocking. When $\delta = 0.01$, the velocities of the system diverge. When $\delta = -0.01$, the velocities of the system converge. Hence, when $\delta \neq 0$, $\lambda\tau(1-\mu_{n_0}) = \frac{1}{2}\pi$ is not the critical condition for the system to achieve flocking, nor does it make the system achieve periodic flocking.

Case 5: $\lambda = 3$, $\delta = 0.01$, $\tau = \frac{1}{2}$ and $r = 2$. In this case, we have

$$P_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P_0^c = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

and $\mu_1 = 1$ ($p_1 = 4$). The initial neighbour graph is not connected and there is a slight attraction between distant relatives (see Fig. 5).

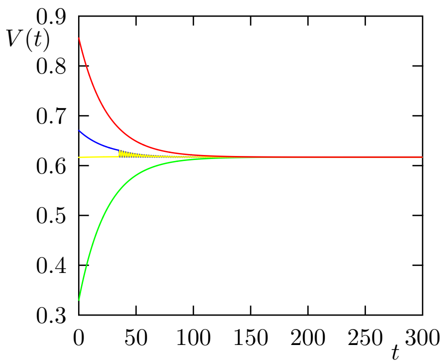

Figure 5. $\lambda = 3$, $\delta = 0.01$, $\tau = \frac{1}{2}$ and $r = 2$. The system also achieves flocking, which means that when $\delta > 0$ the connectedness of P_0 is not necessary for the system achieving flocking. In fact, the unconditional flocking would occur if $0 \leqslant \lambda\tau(1-\mu_{n_0}) < \frac{1}{2}\pi$ and $\delta > 0$.

5. Conclusion

In this paper, we investigated a generalized Motsch-Tadmor model with the piecewise interaction function $\chi_r^\delta(s)$ and a fixed processing delay τ. According to functional differential equation theory and the correlation properties of stochastic matrix, we showed that if $0 \leqslant \lambda\tau(1-\mu_{n_0}) < \frac{1}{2}\pi$, the system would achieve flocking under appropriate assumptions. However, because of the presence of $\delta \neq 0$, the system does not achieve periodic flocking when $\lambda\tau(1-\mu_{n_0}) = \frac{1}{2}\pi$, which is different from the results in [21].

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