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# LOW MACH NUMBER LIMIT OF A COMPRESSIBLE EULER-KORTEWEG MODEL

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*Abstract.* This article deals with the low Mach number limit of the compressible Euler-Korteweg equations. It is justified rigorously that solutions of the compressible Euler-Korteweg equations converge to those of the incompressible Euler equations as the Mach number tends to zero. Furthermore, the desired convergence rates are also obtained.

*Keywords*: Euler-Korteweg equation; compressible flow; low Mach number limit; modulated energy function

*MSC 2020*: 35B40, 35Q35, 35Q31

## 1. Introduction

The aim of this paper is to prove the incompressible limit for a three-dimensional Euler-Korteweg model, which arises in modeling capillary fluids: these comprise liquid-vapor mixtures [16], superfluid [14], or even regular fluids at sufficiently small scales [21]. The most general form of the Euler-Korteweg system takes the form [5], [22]:

$$\partial_t n + \operatorname{div}(n\mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(n\mathbf{u}) + \operatorname{div}(n\mathbf{u}\otimes\mathbf{u} + p\mathbb{I}) = \operatorname{div}\mathcal{K} \tag{1.2}$$

for $(x,t)\in\mathbb{R}^3\times[0,\infty)$. Here, $n>0$ denotes the fluid density, $\mathbf{u}$ the fluid velocity, $p$ the fluid pressure, and $\mathcal{K}$ the so-called Korteweg stress tensor defined as

$$\mathcal{K}(n,\nabla n) = \Big(n\operatorname{div}(\kappa(n)\nabla n) + \frac{1}{2}(\kappa(n) - n\kappa'(n))|\nabla n|^2\Big)\mathbb{I} - \kappa(n)\nabla n\otimes\nabla n \tag{1.3}$$

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with $\kappa(n)$ being the capillary coefficient. The symbol $\mathbb{I}$ denotes the unit matrix in $\mathbb{R}^{3\times 3}$. We assume that the fluid pressure $p(n)$ is a smooth function of $n$ and satisfies $p'(n) > 0$. Define the function $H(n)$ by

$$H'(n) = h(n) = \int_1^n \frac{p'(\tau)}{\tau}\,\mathrm{d}\tau \tag{1.4}$$

and

$$H(1) = 0.$$

In fact, $H(n)$ can be written as

$$H(n) = n\int_1^n \frac{p(\tau)}{\tau^2}\,\mathrm{d}\tau - p(1)(n-1).$$

Then we have $H'(1) = 0$, $H''(n) > 0$. Therefore, in the neighborhood of 1, we have

$$H(x) \geqslant C|x-1|^2 \tag{1.5}$$

for some constant $C$. Furthermore, $\kappa(n)$ is assumed to be a given smooth function of $n$ with $\kappa(n)$ positive and bounded away from zero on some open range for the density $J_n = (J_n^-, J_n^+)$, see [3]. In quantum hydrodynamics, the capillary coefficient is chosen so that $\kappa(n) = \hbar/n$ with the Planck constant $\hbar$ (see [1], [9]), whereas for classical fluid mechanics, it is often chosen to be constant [8]. The Euler-Korteweg model results from a modification of the standard Euler equations governing the motion of compressible inviscid fluids through the adjunction of the Korteweg stress tensor, which takes into account capillarity effects. In paper [5], Benzoni-Gavage, Danchin and Descombes obtained the well-posedness of the Cauchy problem for the Euler-Korteweg model (1.1)–(1.2) in the one-dimensional case by reformulating the equations in Lagrangian coordinates. For the multi-dimensional case, they obtained the (local) well-posedness for the model (1.1)–(1.2) in the Eulerian formulation [6]. The existence and non-uniqueness of global non-dissipative weak solutions was obtained by Donatelli, Feireisl, Marcati [11]. The global well-posedness for the model (1.1)–(1.2) was very recently proved by Audiard, Haspot for small irrotational initial data in $\mathbb{R}^3$, see [4].

When the capillary effect does not appear, (1.7)–(1.8) or (1.9)–(1.10) reduce to the compressible Euler equations, whose low Mach number limit (or incompressible limit) problem has attracted much attention [19], [23], [2], [15], [13], [25], [10], and so on. We also remark that there are many related references on this topic, for example [20], [17], [18], [12], [24]. To the authors' best knowledge, there is no result on the incompressible limit of the Euler-Korteweg model (1.1)–(1.2).

In this paper, our main purpose is to analyze the incompressible limit of smooth solutions for the compressible Euler-Korteweg equation (1.1)–(1.2). From the physical point of view, the compressible flow behaves asymptotically like an incompressible flow, when the density is almost constant and the velocity is small, in a large time scale. In order to study this dynamics on the model (1.1)–(1.2) we perform the incompressible scaling given by

$$n = n^\varepsilon(x, \varepsilon t), \quad \mathbf{u} = \varepsilon \mathbf{u}^\varepsilon(x, \varepsilon t), \tag{1.6}$$

where $\varepsilon \in (0,1)$ is the Mach number. With the scaling (1.6), the model (1.1)–(1.2) becomes

$$\partial_t n^\varepsilon + \operatorname{div}(n^\varepsilon \mathbf{u}^\varepsilon) = 0, \tag{1.7}$$

$$\partial_t (n^\varepsilon \mathbf{u}^\varepsilon) + \operatorname{div}(n^\varepsilon \mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon) = \frac{1}{\varepsilon^2} \operatorname{div}(\mathcal{K}(n^\varepsilon, \nabla n^\varepsilon) - p(n^\varepsilon)\mathbb{I}). \tag{1.8}$$

In view of the equality

$$\begin{aligned}
&\operatorname{div}(\mathcal{K}(n^\varepsilon, \nabla n^\varepsilon)) \\
&= \underbrace{\nabla(n^\varepsilon \operatorname{div}(\kappa(n^\varepsilon)\nabla n^\varepsilon))}_{A_1} + \underbrace{\frac{1}{2}\nabla(\kappa(n^\varepsilon) - n^\varepsilon \kappa'(n^\varepsilon)|\nabla n^\varepsilon|^2)}_{A_2} \underbrace{- \operatorname{div}(\kappa(n^\varepsilon)\nabla n^\varepsilon \otimes \nabla n^\varepsilon)}_{A_3} \\
&= \underbrace{\nabla n^\varepsilon \operatorname{div}(\kappa(n^\varepsilon)\nabla n^\varepsilon) + n^\varepsilon \nabla(\kappa(n^\varepsilon)\Delta n^\varepsilon + \kappa'(n^\varepsilon)|\nabla n^\varepsilon|^2)}_{=A_1} \\
&\quad + \underbrace{\frac{1}{2}\nabla(\kappa(n^\varepsilon))|\nabla n^\varepsilon|^2 + \frac{1}{2}\kappa(n^\varepsilon)\nabla(|\nabla n^\varepsilon|^2)}_{=A_{21}} \\
&\quad + \underbrace{\Big(-\frac{1}{2} n^\varepsilon \nabla(\kappa'(n^\varepsilon)|\nabla n^\varepsilon|^2) - \frac{1}{2}\kappa'(n^\varepsilon)|\nabla n^\varepsilon|^2 \nabla n^\varepsilon\Big)}_{=A_{22}} \quad (\text{where } A_{21} + A_{22} = A_2) \\
&\quad + \underbrace{\Big(-\nabla n^\varepsilon \operatorname{div}(\kappa(n^\varepsilon)\nabla n^\varepsilon) - \frac{1}{2}\kappa(n^\varepsilon)\nabla(|\nabla n^\varepsilon|^2)\Big)}_{=A_3} \\
&= n^\varepsilon \nabla\Big(\kappa(n^\varepsilon)\Delta n^\varepsilon + \frac{1}{2}\kappa'(n^\varepsilon)|\nabla n^\varepsilon|^2\Big),
\end{aligned}$$

substituting (1.7) into (1.8) and recalling (1.4), we may rewrite the above system (1.1)–(1.2) as

$$\partial_t n^\varepsilon + \operatorname{div}(n^\varepsilon \mathbf{u}^\varepsilon) = 0, \tag{1.9}$$

$$\partial_t \mathbf{u}^\varepsilon + (\mathbf{u}^\varepsilon \cdot \nabla)\mathbf{u}^\varepsilon = \frac{1}{\varepsilon^2}\nabla\Big(\kappa(n^\varepsilon)\Delta n^\varepsilon + \frac{1}{2}\kappa'(n^\varepsilon)|\nabla n^\varepsilon|^2 - h(n^\varepsilon)\Big). \tag{1.10}$$

We set the initial data of (1.7)–(1.8) or (1.9)–(1.10) as

$$n^\varepsilon(\cdot, 0) = n_0^\varepsilon(x), \quad \mathbf{u}^\varepsilon(\cdot, 0) = \mathbf{u}_0^\varepsilon(x). \tag{1.11}$$

From the mathematical point of view, it is reasonable to expect that, as $\varepsilon \to 0^+$, $n^\varepsilon \to 1$ in an appropriate function space and the systems (1.7)–(1.8) or (1.9)–(1.10) with the initial data (1.11) become the incompressible Euler equations (limit system)

$$\operatorname{div} \mathbf{u} = 0, \tag{1.12}$$

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla \pi = 0, \tag{1.13}$$

$$\mathbf{u}(x, 0) = \mathbf{u}_0, \tag{1.14}$$

where the hydrostatic pressure $\pi$ is the "limit" of

$$-\frac{1}{\varepsilon^2}\Big(\kappa(n^\varepsilon)\Delta n^\varepsilon + \frac{1}{2}\kappa'(n^\varepsilon)|\nabla n^\varepsilon|^2 - h(n^\varepsilon)\Big)$$

in some sense. In other words, we recover the incompressible Euler equations (1.12)–(1.13) and the hydrostatic pressure appears as the limit of the "renormalized" pressure

$$-\frac{1}{\varepsilon^2}\Big(\kappa(n^\varepsilon)\Delta n^\varepsilon + \frac{1}{2}\kappa'(n^\varepsilon)|\nabla n^\varepsilon|^2 - h(n^\varepsilon)\Big).$$

This paper is devoted to the rigorous justification of the convergence of the above incompressible limit (i.e., the low Mach number limit) for smooth solution of the compressible Euler-Korteweg model in periodic domains $\mathbb{R}^3$.

Throughout this paper, $C > 0$ is a generic constant independent of $\varepsilon$.

## 2. Main result

Before stating the main results, we first describe the existence of the classical solution to the compressible Euler-Korteweg system.

**Proposition 2.1** ([6]). *For the initial data $(n_0^\varepsilon, \mathbf{u}_0^\varepsilon)$ of the system* (1.9)–(1.11) *satisfying $(n_0^\varepsilon - 1, \mathbf{u}_0^\varepsilon) \in H^4(\mathbb{R}^3) \times H^3(\mathbb{R}^3)$ and $n_0^\varepsilon$ taking its values in a compact subset, there exists $T > 0$ and the unique solution $(n^\varepsilon, \mathbf{u}^\varepsilon)$ of the system* (1.9)–(1.11) *such that*

$$(n^\varepsilon - 1, \mathbf{u}^\varepsilon) \in C([0,T];\ H^4(\mathbb{R}^3) \times H^3(\mathbb{R}^3)) \cap C^1([0,T];\ H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)). \tag{2.1}$$

Now we state the main result of our paper.

**Theorem 2.1.** *We suppose that the initial data $(n_0^\varepsilon, \mathbf{u}_0^\varepsilon) \in H^4(\mathbb{R}^3) \times H^3(\mathbb{R}^3)$ satisfy the conditions of Proposition* 2.1. *We also assume that the initial data satisfy*

$$\int_{\mathbb{R}^3} \Big\{\frac{1}{2}n_0^\varepsilon|\mathbf{u}_0^\varepsilon - \mathbf{u}_0|^2 + \frac{1}{\varepsilon^2}H(n_0^\varepsilon) + \frac{1}{2\varepsilon^2}\kappa(n_0^\varepsilon)|\nabla n_0^\varepsilon|^2\Big\}\,\mathrm{d}x \leqslant C\varepsilon. \tag{2.2}$$

*Furthermore, let* $(n^\varepsilon, \mathbf{u}^\varepsilon)$ *satisfying* (2.1) *be the classical solution to* (1.9)–(1.10) *with the initial value* $(n_0^\varepsilon, \mathbf{u}_0^\varepsilon)$ *and let* $\mathbf{u}$ *be the classical solution to* (1.12)–(1.14)*, both on the time interval* $(0,T)$*. Then, as* $\varepsilon \to 0$,

$$\|n^\varepsilon - 1\|_{L^\infty([0,T],L^2(\mathbb{R}^3))} \leqslant C\varepsilon^{3/2}, \tag{2.3}$$

$$\|\sqrt{n^\varepsilon}\mathbf{u}^\varepsilon - \mathbf{u}\|_{L^\infty([0,T],L^2(\mathbb{R}^3))} \leqslant C\varepsilon. \tag{2.4}$$

R e m a r k 2.1. In fact, the weak solutions of the compressible system (1.9)–(1.11) also converge to the classical solution of the limit incompressible model and the proof can be done in the same way though appropriate adjustments.

The proof of our result is based on the modulated energy method introduced by Brenier [7]. The idea of the modulated energy method is to modulate the energy of the given system by test functions and to obtain a stability inequality when these test functions are the solution to the limit system. We introduce the following form of modulated energy:

$$\mathcal{H}^\varepsilon(t) = \int_{\mathbb{R}^3} \Big\{ \frac{1}{2} n^\varepsilon |\mathbf{u}^\varepsilon - \mathbf{u}|^2 + \frac{1}{\varepsilon^2} H(n^\varepsilon) + \frac{1}{2\varepsilon^2}\kappa(n^\varepsilon)|\nabla n^\varepsilon|^2 \Big\} \,\mathrm{d}x, \tag{2.5}$$

where $\mathbf{u}$ is the smooth solution of the incompressible Euler equations (1.12)–(1.14). These terms in the right-hand side of the equation (2.5) express the differences of the kinetic, internal and Korteweg energies. We employ the evolution equations and elaborated computations to prove the inequality

$$\mathcal{H}^\varepsilon(t) \leqslant C\int_0^t \mathcal{H}^\varepsilon(s)\,\mathrm{d}s + \varepsilon^\alpha \tag{2.6}$$

for some positive constant $\alpha > 0$. Gronwall's inequality then implies the result.

## 3. Proof of Theorem 2.1

In this section we give the proof of Theorem 2.1 by the modulated energy method. First, we derive the energy estimates for (1.7)–(1.8) or (1.9)–(1.10). The energy identity for (1.7)–(1.8) or (1.9)–(1.10) is given as follows.

**Lemma 3.1.** *For the smooth solution to the problem* (1.7)–(1.8) *or* (1.9)–(1.10)*, we have the energy identity*

$$\frac{\mathrm{d}}{\mathrm{d}t} E^\varepsilon(t) = 0, \tag{3.1}$$

*where the energy $E^\varepsilon(t)$ is defined by*

$$E^\varepsilon(t) = \int_{\mathbb{R}^3} \Big\{ \frac{1}{2} n^\varepsilon |\mathbf{u}^\varepsilon|^2 + \frac{1}{\varepsilon^2} H(n^\varepsilon) + \frac{1}{2\varepsilon^2} \kappa(n^\varepsilon) |\nabla n^\varepsilon|^2 \Big\} \,\mathrm{d}x.$$

P r o o f. Taking the inner product of the equation (1.10) with $n^\varepsilon \mathbf{u}^\varepsilon$ in the space $L^2(\mathbb{T}^3)$, we have

$$\begin{aligned} (3.2) \qquad & (\partial_t \mathbf{u}^\varepsilon + (\mathbf{u}^\varepsilon \cdot \nabla)\mathbf{u}^\varepsilon, n^\varepsilon \mathbf{u}^\varepsilon) - \frac{1}{\varepsilon^2} (\nabla(\kappa(n^\varepsilon)\Delta n^\varepsilon), n^\varepsilon \mathbf{u}^\varepsilon) \\ & - \frac{1}{2\varepsilon^2} (\nabla(\kappa'(n^\varepsilon)|\nabla n^\varepsilon|^2), n^\varepsilon \mathbf{u}^\varepsilon) + \frac{1}{\varepsilon^2} (\nabla h(n^\varepsilon), n^\varepsilon \mathbf{u}^\varepsilon) \\ & = \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3 + \mathbb{I}_4 = 0. \end{aligned}$$

The four terms $\mathbb{I}_i$ $(i = 1, 2, 3, 4)$ are estimated separately as follows.

▷ Estimate of $\mathbb{I}_1$: Using the continuity equation (1.9) and integrating by parts, we get

$$\begin{aligned} (3.3) \qquad \mathbb{I}_1 &= \int_{\mathbb{R}^3} n^\varepsilon \mathbf{u}^\varepsilon \cdot \partial_t \mathbf{u}^\varepsilon \,\mathrm{d}x + \int_{\mathbb{R}^3} (n^\varepsilon \mathbf{u}^\varepsilon \cdot \nabla)\mathbf{u}^\varepsilon \cdot \mathbf{u}^\varepsilon \,\mathrm{d}x \\ &= \frac{1}{2} \int_{\mathbb{R}^3} n^\varepsilon \partial_t |\mathbf{u}^\varepsilon|^2 \,\mathrm{d}x + \int_{\mathbb{R}^3} \sum_{i,j=1}^{3} n^\varepsilon \mathbf{u}_i^\varepsilon \partial_{x_i} \mathbf{u}_j^\varepsilon \mathbf{u}_j^\varepsilon \,\mathrm{d}x \\ &= \frac{1}{2} \int_{\mathbb{R}^3} n^\varepsilon \partial_t |\mathbf{u}^\varepsilon|^2 \,\mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^3} n^\varepsilon \mathbf{u}^\varepsilon \cdot \nabla |\mathbf{u}^\varepsilon|^2 \,\mathrm{d}x \\ &= \frac{1}{2} \int_{\mathbb{R}^3} n^\varepsilon \partial_t |\mathbf{u}^\varepsilon|^2 \,\mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^3} \partial_t n^\varepsilon |\mathbf{u}^\varepsilon|^2 \,\mathrm{d}x \,\mathrm{d}x = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} n^\varepsilon |\mathbf{u}^\varepsilon|^2 \,\mathrm{d}x. \end{aligned}$$

▷ Estimate of $\mathbb{I}_2$: Integrating by parts and using the continuity equation (1.9), we get

$$\begin{aligned} (3.4) \qquad \mathbb{I}_2 &= -\frac{1}{\varepsilon^2} (\Delta n^\varepsilon, \kappa(n^\varepsilon)\partial_t n^\varepsilon) \\ &= \frac{1}{\varepsilon^2} (\nabla n^\varepsilon, \kappa(n^\varepsilon)\partial_t \nabla n^\varepsilon) + \frac{1}{\varepsilon^2} (\nabla n^\varepsilon, \kappa'(n^\varepsilon)\partial_t n^\varepsilon \nabla n^\varepsilon) \\ &= \frac{1}{2\varepsilon^2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \kappa(n^\varepsilon) |\nabla n^\varepsilon|^2 \,\mathrm{d}x + \frac{1}{2\varepsilon^2} (\nabla n^\varepsilon, \partial_t \kappa(n^\varepsilon) \nabla n^\varepsilon). \end{aligned}$$

▷ Estimate of $\mathbb{I}_3$: Integration by parts gives

$$(3.5) \qquad \mathbb{I}_3 = -\frac{1}{2\varepsilon^2} (\kappa'(n^\varepsilon)|\nabla n^\varepsilon|^2, \partial_t n^\varepsilon) = -\frac{1}{2\varepsilon^2} (\nabla n^\varepsilon, \partial_t \kappa(n^\varepsilon) \nabla n^\varepsilon).$$

▷ Estimate of $\mathbb{I}_4$: From (1.4), we can derive

$$(3.6) \qquad \mathbb{I}_4 = \frac{1}{\varepsilon^2} (h(n^\varepsilon), \partial_t n^\varepsilon) = \frac{1}{\varepsilon^2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} H(n^\varepsilon) \,\mathrm{d}x.$$

Combining (3.3)–(3.6), we obtain the energy identity (3.1). □

From (1.5) and the energy identity (3.1), it is easy to obtain the following estimate.

**Lemma 3.2.** *Let $(n^\varepsilon, \mathbf{u}^\varepsilon)$ be the weak solution to the Euler-Korteweg equations* (1.9)–(1.11) *on* $[0,T]$. *Then there exists a constant $C>0$ such that for all $\varepsilon \in (0,1)$,*

$$\|n^\varepsilon - 1\|_{L^\infty([0,T];L^2(\mathbb{R}^3))} \leqslant C\varepsilon. \tag{3.7}$$

Next, we begin to derive the uniform estimates of the modulated energy functional.

**Lemma 3.3.** *Let $T>0$. Then $\mathcal{H}^\varepsilon(t) \leqslant C\varepsilon$ uniformly in* $[0,T]$.

P r o o f. By using (3.1) and a direct computation, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^\varepsilon(t) &= \frac{\mathrm{d}}{\mathrm{d}t}E^\varepsilon(t) - \int_{\mathbb{R}^3} \partial_t(n^\varepsilon \mathbf{u}^\varepsilon \cdot \mathbf{u})\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} \partial_t(n^\varepsilon |\mathbf{u}|^2)\,\mathrm{d}x \\ &= -\int_{\mathbb{R}^3} \partial_t(n^\varepsilon \mathbf{u}^\varepsilon)\cdot \mathbf{u}\,\mathrm{d}x - \int_{\mathbb{R}^3} (n^\varepsilon \mathbf{u}^\varepsilon)\cdot \partial_t \mathbf{u}\,\mathrm{d}x \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^3} \partial_t n^\varepsilon |\mathbf{u}|^2\,\mathrm{d}x + \int_{\mathbb{R}^3} n^\varepsilon \mathbf{u}\cdot \partial_t \mathbf{u}\,\mathrm{d}x \\ &= \sum_{i=1}^4 \mathbb{J}_i.\end{aligned} \tag{3.8}$$

The four terms in (3.8) are estimated as follows.

▷ Estimate of $\mathbb{J}_1$: Taking the inner product of the equation (1.8) with $\mathbf{u}$ in the space $L^2(\mathbb{T}^3)$ and using the equations (1.3), (1.12), we have

$$\begin{aligned}\mathbb{J}_1 =& -(n^\varepsilon \mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon, \nabla \mathbf{u}) - \frac{1}{\varepsilon^2}(\operatorname{div}\mathcal{K}, \mathbf{u}) + \frac{1}{\varepsilon^2}(\nabla p, \mathbf{u}) \\ =& -(n^\varepsilon(\mathbf{u}^\varepsilon - \mathbf{u})\otimes(\mathbf{u}^\varepsilon - \mathbf{u}), \nabla\mathbf{u}) - (n^\varepsilon \mathbf{u}^\varepsilon \otimes \mathbf{u}, \nabla \mathbf{u}) \\ & - (n^\varepsilon \mathbf{u}\otimes \mathbf{u}^\varepsilon, \nabla\mathbf{u}) + (n^\varepsilon \mathbf{u}\otimes\mathbf{u}, \nabla\mathbf{u}) \\ & + \frac{1}{\varepsilon^2}(n\operatorname{div}(\kappa(n)\nabla n), \operatorname{div}\mathbf{u}) - \frac{1}{\varepsilon^2}(\kappa(n)\nabla n\otimes\nabla n, \nabla\mathbf{u}) - \frac{1}{\varepsilon^2}(p, \operatorname{div}\mathbf{u}) \\ =& -\int_{\mathbb{R}^3} n^\varepsilon(\mathbf{u}^\varepsilon - \mathbf{u})\otimes(\mathbf{u}^\varepsilon - \mathbf{u}) : \nabla\mathbf{u}\,\mathrm{d}x - \int_{\mathbb{R}^3}(n^\varepsilon \mathbf{u}^\varepsilon\otimes\mathbf{u}) : \nabla\mathbf{u}\,\mathrm{d}x \\ & - \int_{\mathbb{R}^3}(n^\varepsilon\mathbf{u}\otimes\mathbf{u}^\varepsilon) : \nabla\mathbf{u}\,\mathrm{d}x + \int_{\mathbb{R}^3}(n^\varepsilon\mathbf{u}\otimes\mathbf{u}) : \nabla\mathbf{u}\,\mathrm{d}x \\ & - \frac{1}{\varepsilon^2}\int_{\mathbb{R}^3}\kappa(n)\nabla n\otimes\nabla n : \nabla\mathbf{u}\,\mathrm{d}x \\ \leqslant& \, C\int_{\mathbb{R}^3} n^\varepsilon|\mathbf{u}^\varepsilon - \mathbf{u}|^2\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3} n^\varepsilon\mathbf{u}^\varepsilon\cdot\nabla|\mathbf{u}|^2\,\mathrm{d}x - \int_{\mathbb{R}^3} n^\varepsilon\mathbf{u}^\varepsilon\cdot((\mathbf{u}\cdot\nabla)\mathbf{u})\,\mathrm{d}x \\ & + \int_{\mathbb{R}^3} n^\varepsilon(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{u}\,\mathrm{d}x + \frac{C}{\varepsilon^2}\int_{\mathbb{R}^3}\kappa(n)|\nabla n|^2\,\mathrm{d}x \\ \leqslant& \, C\mathcal{H}^\varepsilon(t) - \mathbb{J}_3 - \int_{\mathbb{R}^3} n^\varepsilon\mathbf{u}^\varepsilon\cdot((\mathbf{u}\cdot\nabla)\mathbf{u})\,\mathrm{d}x + \int_{\mathbb{R}^3} n^\varepsilon\mathbf{u}\cdot((\mathbf{u}\cdot\nabla)\mathbf{u})\,\mathrm{d}x.\end{aligned} \tag{3.9}$$

▷ Estimate of $\mathbb{J}_2$: Multiplying (1.13) by $n^\varepsilon \mathbf{u}^\varepsilon$, we obtain

$$
\begin{aligned}
\mathbb{J}_2 &= (n^\varepsilon \mathbf{u}^\varepsilon, (\mathbf{u}\cdot\nabla)\mathbf{u}) + (n^\varepsilon \mathbf{u}^\varepsilon, \nabla\pi) \\
&= \int_{\mathbb{R}^3} n^\varepsilon \mathbf{u}^\varepsilon \cdot ((\mathbf{u}\cdot\nabla)\mathbf{u})\,\mathrm{d}x - \int_{\mathbb{R}^3} \operatorname{div}(n^\varepsilon \mathbf{u}^\varepsilon)\pi\,\mathrm{d}x \\
&= \int_{\mathbb{R}^3} n^\varepsilon \mathbf{u}^\varepsilon \cdot ((\mathbf{u}\cdot\nabla)\mathbf{u})\,\mathrm{d}x + \int_{\mathbb{R}^3} \partial_t(n^\varepsilon - 1)\pi\,\mathrm{d}x \\
&= \int_{\mathbb{R}^3} n^\varepsilon \mathbf{u}^\varepsilon \cdot ((\mathbf{u}\cdot\nabla)\mathbf{u})\,\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} (n^\varepsilon - 1)\pi\,\mathrm{d}x + \int_{\mathbb{R}^3} (n^\varepsilon - 1)\partial_t\pi\,\mathrm{d}x \\
&\leqslant \int_{\mathbb{R}^3} n^\varepsilon \mathbf{u}^\varepsilon \cdot ((\mathbf{u}\cdot\nabla)\mathbf{u})\,\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} (n^\varepsilon - 1)\pi\,\mathrm{d}x + C\varepsilon.
\end{aligned} \tag{3.10}
$$

The integral $\mathbb{J}_3$ cancels with a contribution originating from $\mathbb{J}_1$.

▷ Estimate of $\mathbb{J}_4$: Using the equations (1.12)–(1.13), Lemma 3.2 and integrating by parts, we obtain

$$
\begin{aligned}
\mathbb{J}_4 &= -\int_{\mathbb{R}^3} n^\varepsilon \mathbf{u} \cdot ((\mathbf{u}\cdot\nabla)\mathbf{u})\,\mathrm{d}x - \int_{\mathbb{R}^3} n^\varepsilon \mathbf{u}\cdot\nabla\pi\,\mathrm{d}x \\
&= -\int_{\mathbb{R}^3} n^\varepsilon \mathbf{u} \cdot ((\mathbf{u}\cdot\nabla)\mathbf{u})\,\mathrm{d}x - \int_{\mathbb{R}^3} (n^\varepsilon - 1)\mathbf{u}\cdot\nabla\pi\,\mathrm{d}x \\
&\leqslant -\int_{\mathbb{R}^3} n^\varepsilon \mathbf{u} \cdot ((\mathbf{u}\cdot\nabla)\mathbf{u})\,\mathrm{d}x + C\varepsilon.
\end{aligned} \tag{3.11}
$$

Combining (3.9)–(3.11) with (3.8) we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathcal{H}^\varepsilon(t) + \int_{\mathbb{R}^3} (n^\varepsilon - 1)\pi\,\mathrm{d}x\Big) \leqslant C\mathcal{H}^\varepsilon(t) + C\varepsilon. \tag{3.12}
$$

Integrating over $[0,t]$ gives

$$
\mathcal{H}^\varepsilon(t) + \int_{\mathbb{R}^3} (n^\varepsilon - 1)\pi\,\mathrm{d}x \leqslant \mathcal{H}^\varepsilon(0) + C\int_0^t \mathcal{H}^\varepsilon(s)\,\mathrm{d}s + \int_{\mathbb{R}^3} (n_0^\varepsilon - 1)\pi_0\,\mathrm{d}x + C\varepsilon, \tag{3.13}
$$

where $\pi_0 = \pi|_{t=0}$. By the equations (1.5), (2.2) and Lemma 3.1, we have

$$
\mathcal{H}^\varepsilon(t) \leqslant C\int_0^t \mathcal{H}^\varepsilon(s)\,\mathrm{d}s + C\varepsilon, \tag{3.14}
$$

which implies the proof of Lemma 3.3 is completed by the Gronwall inequality. □

We are now in the position to prove Theorem 2.1 which is a consequence of Lemma 3.3. By the definition of $\mathcal{H}^\varepsilon(t)$, the inequality (1.5) and Lemma 3.3, we claim that the estimate (2.3) holds. Using Lemma 3.3 and the Hölder inequality, we have that

$$\begin{aligned}\|\sqrt{n^\varepsilon}\mathbf{u}^\varepsilon-\mathbf{u}\|^2_{L^2(\mathbb{R}^3)}&\leqslant 2\|\sqrt{n^\varepsilon}(\mathbf{u}^\varepsilon-\mathbf{u})\|^2_{L^2(\mathbb{R}^3)}+2\|(1-\sqrt{n^\varepsilon})\mathbf{u}\|^2_{L^2(\mathbb{R}^3)}\\&\leqslant C\varepsilon+C\|1-\sqrt{n^\varepsilon}\|^2_{L^2(\mathbb{R}^3)}\\&\leqslant C\varepsilon+C\|1-n^\varepsilon\|^2_{L^2(\mathbb{R}^3)}\\&\leqslant C\varepsilon\end{aligned}$$

for any $t\in[0,T]$. Here, we have used the elementary inequality

$$|1-\sqrt{x}|^2\leqslant|1-x|^2$$

for any $x\geqslant 0$. Therefore, we conclude that (2.4) holds.

Thus the proof of Theorem 2.1 is finished. □

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