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A DIOPHANTINE EQUATION INVOLVING SPECIAL PRIME NUMBERS

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Abstract. Let $[\cdot]$ be the floor function. In this paper, we prove by asymptotic formula that when $1 < c < \frac{3441}{2539}$, then every sufficiently large positive integer N can be represented in the form

$$N = [p_1^c] + [p_2^c] + [p_3^c] + [p_4^c] + [p_5^c],$$

where p_1, p_2, p_3, p_4, p_5 are primes such that $p_1 = x^2 + y^2 + 1$.

Keywords: Diophantine equation; prime; exponential sum; asymptotic formula

MSC 2020: 11L07, 11L20, 11P32

1. Introduction and main result

The celebrated Waring's problem states that for every integer $n \geqslant 2$ there is a positive integer $G(n)$ such that if $s \geqslant G(n)$, then every sufficiently large positive integer N can be represented as

$$N = x_1^n + x_2^n + \ldots + x_s^n,$$

where $x_1, x_2, \ldots, x_s$ are nonnegative integers. The first proof of Waring's problem belongs to Hilbert, see [13]. In 1933–1934, Segal in [22], [23] considered Waring's problem with noninteger powers. He showed that when $c > 1$ is not an integer, there exists $k_0(c) > 0$ such that every sufficiently large positive integer N can be represented in the form

$$N = [x_1^c] + [x_2^c] + \ldots + [x_k^c],$$

where $x_1, x_2, \ldots, x_k$ are nonnegative integers and $k > k_0(c)$. Subsequently, the result of Segal was sharpened by Deshouillers (see [4]) and by Arkhipov and Zhitkov, see [1].

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In the interesting case for $k = 2$ we find papers by Deshouillers (see [5]), Gritsenko (see [11]) and most recently Konyagin (see [18]), where it is shown that the range $1 < c < \frac{3}{2}$ is permissible.

Another important problem in number theory is Hua's theorem on five squares of primes. In 1938, Hua in [15] proved that every sufficiently large positive integer N such that $N \equiv 5 \pmod{24}$, can be represented in the form

$$N = p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2,$$

where p_1, p_2, p_3, p_4, p_5 are prime numbers. As an analogue of Hua's five square theorem, in 2019, Zhang and Li in [25] investigated the Diophantine equation

$$[p_1^c] + [p_2^c] + [p_3^c] + [p_4^c] + [p_5^c] = N, \tag{1.1}$$

where p_1, p_2, p_3, p_4, p_5 are primes, $c > 1$ and N is a positive integer. For $1 < c < \frac{4109054}{1999527}$, $c \neq 2$ they showed that for the sum

$$R(N) = \sum_{[p_1^c]+[p_2^c]+[p_3^c]+[p_4^c]+[p_5^c]=N} \log p_1 \log p_2 \log p_3 \log p_4 \log p_5,$$

the asymptotic formula

$$R(N) = \frac{\Gamma^5(1+1/c)}{\Gamma(5/c)} N^{5/c-1} + \mathcal{O}(N^{5/c-1} \exp(-(\log N)^{1/4})) \tag{1.2}$$

holds.

Afterwards, the result of Zhang and Li was sharpened by Li (see [19]) to $2 < c < \frac{408}{197}$ and by Baker (see [2]) to $1 < c < \frac{609}{293}$, $c \neq 2$ and this is the best result up to now.

On the other hand, in 1960 Linnik in [20] showed that there exist infinitely many prime numbers of the form $p = x^2 + y^2 + 1$, where x and y are integers. More precisely, he proved the asymptotic formula

$$\sum_{p \leqslant X} r(p-1) = \pi \prod_{p>2} \Big(1 + \frac{\chi_4(p)}{p(p-1)}\Big) \frac{X}{\log X} + \mathcal{O}\Big(\frac{X (\log\log X)^7}{(\log X)^{1+\theta_0}}\Big),$$

where $r(k)$ is the number of solutions of the equation $k = x^2 + y^2$ in integers, $\chi_4(k)$ is the nonprincipal character modulo 4 and

$$\theta_0 = \frac{1}{2} - \frac{1}{4} \mathrm{e} \log 2 = 0.0289\ldots \tag{1.3}$$

Recently, the author in [9] showed that for any fixed $1 < c < \frac{5363}{3900}$, every sufficiently large positive number N and a small constant $\varepsilon > 0$, the Diophantine inequality

$$|p_1^c + p_2^c + p_3^c + p_4^c + p_5^c - N| < \varepsilon$$

has a solution in primes p_1, p_2, p_3, p_4, p_5, such that $p_1 = x^2 + y^2 + 1$.

These results raise the question of the solvability of the Diophantine equation (1.1) in primes p_1, p_2, p_3, p_4, p_5, with $p_1 = x^2 + y^2 + 1$. Let N be a sufficiently large positive integer and

$$X = N^{1/c}. \tag{1.4}$$

Define

$$\widetilde{\Gamma} = \sum_{\substack{X/2<p_1,p_2,p_3,p_4,p_5\leqslant X\\ [p_1^c]+[p_2^c]+[p_3^c]+[p_4^c]+[p_5^c]=N}} r(p_1-1)\log p_1 \log p_2 \log p_3 \log p_4 \log p_5. \tag{1.5}$$

We establish the following theorem.

Theorem 1.1. *Let $1 < c < \frac{3441}{2539}$. Then for every sufficiently large positive integer N, the asymptotic formula*

$$\begin{aligned}\widetilde{\Gamma} = \pi\prod_p\Big(1+\frac{\chi_4(p)}{p(p-1)}\Big)\frac{\Gamma^5(1+1/c)}{\Gamma(5/c)}\Big(1-\frac{1}{2^{5-c}}\Big)N^{5/c-1}\\ +\mathcal{O}\Big(\frac{N^{5/c-1}(\log\log N)^5}{(\log N)^{\theta_0}}\Big)\end{aligned} \tag{1.6}$$

holds. Here θ_0 is defined by (1.3).

Conjecture 1.2. *There exists $c_0 > 1$ such that for any fixed $1 < c < c_0$ and every sufficiently large positive integer N, the Diophantine equation*

$$[p_1^c] + [p_2^c] + [p_3^c] + [p_4^c] + [p_5^c] = N$$

has a solution in prime numbers p_1, p_2, p_3, p_4, p_5, such that $p_i = x_i^2 + y_i^2 + 1$ for $1 \leqslant i \leqslant 5$.

2. Notations

Let N be a sufficiently large positive integer. The letter p with or without subscript will always denote a prime number. The notation $m \sim M$ means that m runs through the interval $(\frac{1}{2}M, M]$. As usual, $\varphi(n)$ is Euler's function and $\Lambda(n)$ is von Mangoldt's function. Instead of $m \equiv n \pmod d$ we write for simplicity $m \equiv n\,(d)$. By ε we denote an arbitrary small positive number, not the same in all appearances. Moreover, $e(y) = \mathrm{e}^{2\pi \mathrm{i} y}$. As usual, $[t]$, $\{t\}$ and $\|t\|$ denote the integer part of t, the fractional part of t and the distance from t to the nearest integer, respectively. Throughout this paper unless something else is said, we suppose that $1 < c < \frac{3441}{2539}$. Denote

$$\text{(2.1)}\quad D = \frac{X^{1/2}}{(\log N)^{(6A+34)/3}}, \quad A > 3;$$

$$\text{(2.2)}\quad \Delta = X^{1/4-c};$$

$$\text{(2.3)}\quad S_{l,d;J}(t) = \sum_{\substack{p\in J\\ p\equiv l(d)}} e(t[p^c])\log p;$$

$$\text{(2.4)}\quad S(t) = S_{1,1;(X/2,X]}(t);$$

$$\text{(2.5)}\quad \overline{S}_{l,d;J}(t) = \sum_{\substack{p\in J\\ p\equiv l(d)}} e(tp^c)\log p;$$

$$\text{(2.6)}\quad \overline{S}(t) = \overline{S}_{1,1;(X/2,X]}(t);$$

$$\text{(2.7)}\quad I_J(t) = \int_J e(ty^c)\,\mathrm{d}y;$$

$$\text{(2.8)}\quad I(t) = I_{(X/2,X]}(t);$$

$$\text{(2.9)}\quad E(y,t,d,a) = \sum_{\substack{\mu y<n\leqslant y\\ n\equiv a(d)}} \Lambda(n)e(tn^c) - \frac{1}{\varphi(d)}\int_{\mu y}^{y} e(tx^c)\,\mathrm{d}x, \quad \text{where } 0<\mu<1.$$

3. Preliminary lemmas

Lemma 3.1. *For any complex numbers $a(n)$ we have*

$$\Big|\sum_{a<n\leqslant b} a(n)\Big|^2 \leqslant \Big(1+\frac{b-a}{Q}\Big)\sum_{|q|\leqslant Q}\Big(1-\frac{|q|}{Q}\Big)\sum_{\substack{a<n\\ n+q\leqslant b}} a(n+q)\overline{a(n)},$$

where Q is any positive integer.

P r o o f. See [17], Lemma 8.17. □

Lemma 3.2. *Let $|f^{(m)}(u)| \asymp YX^{1-m}$ for $1\leqslant X<u<X_0\leqslant 2X$ and $m\geqslant 1$. Then*

$$\Big|\sum_{X<n\leqslant X_0} e(f(n))\Big| \ll Y^{\varkappa}X^{\lambda}+Y^{-1},$$

where $(\varkappa,\lambda)$ is any exponent pair.

P r o o f. See [10], equation (3.3.4). □

Lemma 3.3. *For every $\varepsilon > 0$, the pair $\left(\frac{32}{205}+\varepsilon, \frac{269}{410}+\varepsilon\right)$ is an exponent pair.*

Proof. See [16], Corollary of Theorem 1. □

Lemma 3.4. *Let α, β be real numbers such that*

$$\alpha\beta(\alpha-1)(\beta-1)(\alpha-2)(\beta-2) \neq 0.$$

Set

$$\Sigma_{II} = \sum_{m\sim M} a(m) \sum_{l\sim L} b(l) e\Big(F\frac{m^\alpha l^\beta}{M^\alpha L^\beta}\Big),$$

where

$$F > 0, \quad M \geqslant 1, \quad L \geqslant 1, \quad |a(m)| \leqslant 1, \quad |b(l)| \leqslant 1.$$

Then

$$\begin{aligned}\Sigma_{II}(FML)^{-\varepsilon} \ll (F^4M^{31}L^{34})^{1/42} &+ (F^6M^{53}L^{51})^{1/66} + (F^6M^{46}L^{41})^{1/56} \\ &+ (F^2M^{38}L^{29})^{1/40} + (F^3M^{43}L^{32})^{1/46} + (FM^9L^6)^{1/10} \\ &+ (F^2M^7L^6)^{1/10} + (FM^6L^6)^{1/8} + M^{1/2}L + ML^{1/2} + F^{-1/2}ML.\end{aligned}$$

Proof. See [21], Theorem 9. □

Lemma 3.5. *Let $x, y \in \mathbb{R}$ and $H \geqslant 3$. Then the formula*

$$e(-x\{y\}) = \sum_{|h|\leqslant H} c_h(x)e(hy) + \mathcal{O}\Big(\min\Big(1, \frac{1}{H\|y\|}\Big)\Big)$$

holds. Here

$$c_h(x) = \frac{1-e(-x)}{2\pi \mathrm{i}(h+x)}.$$

Proof. See [3], Lemma 12. □

Lemma 3.6. *Let $3 < U < V < Z < X$ and suppose that $Z - \frac{1}{2} \in \mathbb{N}$, $X \gg Z^2U$, $Z \gg U^2$, $V^3 \gg X$. Assume further that $F(n)$ is a complex valued function such that $|F(n)| \leqslant 1$. Then the sum*

$$\sum_{n\sim X} \Lambda(n)F(n)$$

can be decomposed into $O(\log^{10} X)$ sums, each of which is either of Type I:

$$\sum_{m\sim M} a(m) \sum_{l\sim L} F(ml), \quad \text{where } L \gg Z,\ LM \asymp X,\ |a(m)| \ll m^\varepsilon,$$

or of Type II:

$$\sum_{m\sim M} a(m) \sum_{l\sim L} b(l)F(ml),$$

where

$$U \ll L \ll V, \quad LM \asymp X, \quad |a(m)| \ll m^{\varepsilon}, \quad |b(l)| \ll l^{\varepsilon}.$$

P r o o f. See [12], Lemma 3. □

Lemma 3.7. *Let* $1 < c < 3$, $c \neq 2$ *and* $|t| \leqslant \Delta$. *Then the asymptotic formula*

$$\sum_{X/2<p\leqslant X} e(tp^c)\log p = \int_{X/2}^{X} e(ty^c)\,\mathrm{d}y + \mathcal{O}\Big(\frac{X}{\mathrm{e}^{(\log X)^{1/5}}}\Big)$$

holds.

P r o o f. See [24], Lemma 14. □

Lemma 3.8. *Let* $1 < c < 3$, $c \neq 2$, $|t| \leqslant \Delta$ *and* $A > 0$ *be fixed. Then the inequality*

$$\sum_{d\leqslant\sqrt{X}/(\log N)^{(6A+34)/3}} \max_{y\leqslant X} \max_{(a,d)=1} |E(y,t,d,a)| \ll \frac{X}{\log^A X}$$

holds. Here Δ *and* $E(y,t,d,a)$ *are denoted by* (2.2) *and* (2.9).

P r o o f. See [8], Lemma 18. □

Lemma 3.9. *For the sum denoted by* (2.4) *and the integral denoted by* (2.8) *we have*

(i) $$\int_{-\Delta}^{\Delta} |S(t)|^2\,\mathrm{d}t \ll X^{2-c}\log^2 X,$$

(ii) $$\int_{-\Delta}^{\Delta} |I(t)|^2\,\mathrm{d}t \ll X^{2-c}\log X,$$

(iii) $$\int_{0}^{1} |S(t)|^2\,\mathrm{d}t \ll X\log X.$$

P r o o f. It follows from the arguments used in [24], Lemma 7, pages 293–294. □

Lemma 3.10. *For the sum denoted by* (2.3) *we have*

$$\int_{-\Delta}^{\Delta} |S_{l,d;J}(t)|^2\,\mathrm{d}t \ll \frac{X^{2-c}\log^3 X}{d^2}.$$

P r o o f. It follows by the arguments used in [6], Lemma 6 (i), pages 344–345. □

Lemma 3.11. *For any real number t and $H \geqslant 1$, there holds*

$$\min\Bigl(1, \frac{1}{H\|t\|}\Bigr) = \sum_{h=-\infty}^{\infty} a_h e(ht), \quad \text{where } a_h \ll \min\Bigl(\frac{\log 2H}{H}, \frac{1}{|h|}, \frac{H}{|h|^2}\Bigr).$$

P r o o f. See [12], page 245. □

The next two lemmas are due to Hooley.

Lemma 3.12. *For any constant $\omega > 0$ we have*

$$\sum_{p \leqslant X} \Bigl| \sum_{\substack{d \mid p-1 \\ \sqrt{X}(\log X)^{-\omega} < d < \sqrt{X}(\log X)^{\omega}}} \chi_4(d) \Bigr|^2 \ll \frac{X(\log\log X)^7}{\log X},$$

where the constant in Vinogradov's symbol depends on $\omega > 0$.

Lemma 3.13. *Suppose that $\omega > 0$ is a constant and let $\mathcal{F}_\omega(X)$ be the number of primes $p \leqslant X$ such that $p-1$ has a divisor in the interval $(\sqrt{X}(\log X)^{-\omega}, \sqrt{X}(\log X)^{\omega})$. Then*

$$\mathcal{F}_\omega(X) \ll \frac{X(\log\log X)^3}{(\log X)^{1+2\theta_0}},$$

where θ_0 is defined by (1.3) and the constant in Vinogradov's symbol depends only on $\omega > 0$.

The proofs of very similar results are available in [14], Chapter 5, pages 89–105.

Lemma 3.14. *For the sum denoted by (2.4) we have*

$$\int_0^1 |S(t)|^4 \,\mathrm{d}t \ll (X^{4-c} + X^2)X^{\varepsilon}.$$

P r o o f. See [25], Lemma 2.6. □

Lemma 3.15. *Let $0 < t < 1$. Set*

$$A(t) = \sum_{n \sim X} e(t[n^c]). \tag{3.1}$$

Then

$$A(t) \ll X^{(269c+538)/1217} \log X + \frac{X^{1-c}}{t}.$$

P r o o f. See [26], Lemma 11. □

4. Outline of the proof

From (1.5) and well-known identity

$$r(n) = 4\sum_{d|n} \chi_4(d)$$

we write

$$\widetilde{\Gamma} = 4(\Gamma_1 + \Gamma_2 + \Gamma_3), \tag{4.1}$$

where

$$\Gamma_1 = \sum_{\substack{X/2<p_1,p_2,p_3,p_4,p_5\leqslant X\\ [p_1^c]+[p_2^c]+[p_3^c]+[p_4^c]+[p_5^c]=N}} \Bigg(\sum_{\substack{d|p_1-1\\ d\leqslant D}} \chi_4(d)\Bigg) \prod_{k=1}^{5} \log p_k, \tag{4.2}$$

$$\Gamma_2 = \sum_{\substack{X/2<p_1,p_2,p_3,p_4,p_5\leqslant X\\ [p_1^c]+[p_2^c]+[p_3^c]+[p_4^c]+[p_5^c]=N}} \Bigg(\sum_{\substack{d|p_1-1\\ D<d<X/D}} \chi_4(d)\Bigg) \prod_{k=1}^{5} \log p_k, \tag{4.3}$$

$$\Gamma_3 = \sum_{\substack{X/2<p_1,p_2,p_3,p_4,p_5\leqslant X\\ [p_1^c]+[p_2^c]+[p_3^c]+[p_4^c]+[p_5^c]=N}} \Bigg(\sum_{\substack{d|p_1-1\\ d\geqslant X/D}} \chi_4(d)\Bigg) \prod_{k=1}^{5} \log p_k. \tag{4.4}$$

In order to estimate Γ_1 and Γ_3 we have to consider the sum

$$I_{l,d;J}(N) = \sum_{\substack{X/2<p_2,p_3,p_4,p_5\leqslant X\\ [p_1^c]+[p_2^c]+[p_3^c]+[p_4^c]+[p_5^c]=N\\ p_1\equiv l(d)\\ p_1\in J}} \prod_{k=1}^{5} \log p_k, \tag{4.5}$$

where d and l are coprime natural numbers, and $J \subset (\frac{1}{2}X, X]$-interval. If $J = (\frac{1}{2}X, X]$, then we write for simplicity $I_{l,d}(N)$. Clearly

$$I_{l,d;J}(N) = \int_{-\Delta}^{1-\Delta} S_{l,d;J}(t)S^4(t)e(-tN)\,\mathrm{d}t = I^{(1)}_{l,d;J}(N) + I^{(2)}_{l,d;J}(N), \tag{4.6}$$

where

$$I^{(1)}_{l,d;J}(N) = \int_{-\Delta}^{\Delta} S_{l,d;J}(t)S^4(t)e(-tN)\,\mathrm{d}t, \tag{4.7}$$

$$I^{(2)}_{l,d;J}(N) = \int_{\Delta}^{1-\Delta} S_{l,d;J}(t)S^4(t)e(-tN)\,\mathrm{d}t. \tag{4.8}$$

We shall estimate $I^{(1)}_{l,d;J}(N)$, Γ_3, Γ_2 and Γ_1, respectively, in Sections 5, 6, 7 and 8. In Section 9 we shall finalize the proof of Theorem 1.1.

5. Asymptotic formula for $I^{(1)}_{l,d;J}(N)$

Using (2.3), (2.5) and $|t| \leqslant \Delta$ we write

$$
\begin{aligned}
(5.1)\qquad S_{l,d;J}(t) &= \sum_{\substack{p\in J\\ p\equiv l(d)}} e(tp^c + \mathcal{O}(|t|))\log p = \sum_{\substack{p\in J\\ p\equiv l(d)}} e(tp^c)(1+\mathcal{O}(|t|))\log p\\
&= \overline{S}_{l,d;J}(t) + \mathcal{O}\Big(\frac{\Delta X\log X}{d}\Big).
\end{aligned}
$$

Put

$$
\begin{aligned}
(5.2)\qquad S_1 &= S(t),\\
(5.3)\qquad S_2 &= S_{l,d;J}(t),\\
(5.4)\qquad I_1 &= I(t),\\
(5.5)\qquad I_2 &= \frac{I_J(t)}{\varphi(d)}.
\end{aligned}
$$

We use the identity

$$
\begin{aligned}
(5.6)\qquad S_1^4S_2 = {}& I_1^4I_2 + (S_2-I_2)I_1^4 + S_2(S_1-I_1)I_1^3 + S_1S_2(S_1-I_1)I_1^2\\
&+ S_1^2S_2(S_1-I_1)I_1 + S_1^3S_2(S_1-I_1).
\end{aligned}
$$

Define

$$
(5.7)\qquad \Phi_{\Delta,J}(X,d) = \frac{1}{\varphi(d)}\int_{-\Delta}^{\Delta} I^4(t)I_J(t)e(-Nt)\,\mathrm{d}t.
$$

From (2.2)–(2.8), (4.7), (5.1)–(5.7), Lemmas 3.7, 3.9, 3.10 and Cauchy's inequality it follows

$$
\begin{aligned}
(5.8)\qquad & I^{(1)}_{l,d;J}(X) - \Phi_{\Delta,J}(X,d)\\
&= \int_{-\Delta}^{\Delta}\Big(S_{l,d;J}(t) - \frac{I_J(t)}{\varphi(d)}\Big)I^4(t)e(-Nt)\,\mathrm{d}t\\
&\quad + \int_{-\Delta}^{\Delta} S_{l,d;J}(t)(S(t)-I(t))I^3(t)e(-Nt)\,\mathrm{d}t\\
&\quad + \int_{-\Delta}^{\Delta} S(t)S_{l,d;J}(t)(S(t)-I(t))I^2(t)e(-Nt)\,\mathrm{d}t\\
&\quad + \int_{-\Delta}^{\Delta} S^2(t)S_{l,d;J}(t)(S(t)-I(t))I(t)e(-Nt)\,\mathrm{d}t\\
&\quad + \int_{-\Delta}^{\Delta} S^3(t)S_{l,d;J}(t)(S(t)-I(t))e(-Nt)\,\mathrm{d}t
\end{aligned}
$$

$$\ll X^2\Big(\max_{|t|\leqslant\Delta}\Big|\overline{S}_{l,d;J}(t)-\frac{I_J(t)}{\varphi(d)}\Big|+\frac{\Delta X\log X}{d}\Big)\int_{-\Delta}^{\Delta}|I(t)|^2\,\mathrm{d}t$$
$$+\frac{X^2\log X}{d}\Big(\frac{X}{\mathrm{e}^{(\log X)^{1/5}}}+\Delta X\Big)\int_{-\Delta}^{\Delta}|I(t)|^2\,\mathrm{d}t$$
$$+\frac{X^2\log X}{d}\Big(\frac{X}{\mathrm{e}^{(\log X)^{1/5}}}+\Delta X\Big)\int_{-\Delta}^{\Delta}|S(t)|^2\,\mathrm{d}t$$
$$\ll X^{4-c}(\log X)\max_{|t|\leqslant\Delta}\Big|\overline{S}_{l,d;J}(t)-\frac{I_J(t)}{\varphi(d)}\Big|+\frac{X^{5-c}}{d\mathrm{e}^{(\log X)^{1/6}}}.$$

Put

$$\Phi_J(X,d)=\frac{1}{\varphi(d)}\int_{-\infty}^{\infty}I^4(t)I_J(t)e(-Nt)\,\mathrm{d}t. \tag{5.10}$$

Using (2.7), (2.8), (5.7), (5.9) and the estimates

$$I_J(t)\ll\min\Big(X,\frac{X^{1-c}}{|t|}\Big),\quad I(t)\ll\min\Big(X,\frac{X^{1-c}}{|t|}\Big), \tag{5.11}$$

we deduce

$$\Phi_{\Delta,J}(X,d)-\Phi_J(X,d)\ll\frac{1}{\varphi(d)}\int_{\Delta}^{\infty}|I(t)|^4|I_J(t)|\,\mathrm{d}t\ll\frac{X^{5-5c}}{\varphi(d)}\int_{\Delta}^{\infty}\frac{\mathrm{d}t}{t^5}\ll\frac{X^{5-5c}}{\varphi(d)\Delta^4}$$

and therefore

$$\Phi_{\Delta,J}(X,d)=\Phi_J(X,d)+\mathcal{O}\Big(\frac{X^{5-5c}}{\varphi(d)\Delta^4}\Big). \tag{5.12}$$

Finally (2.2), (5.8), (5.11) and the identity

$$I^{(1)}_{l,d;J}(X)=I^{(1)}_{l,d;J}(X)-\Phi_{\Delta,J}(X,d)+\Phi_{\Delta,J}(X,d)-\Phi_J(X,d)+\Phi_J(X,d)$$

imply

$$I^{(1)}_{l,d;J}(X)=\Phi_J(X,d)+\mathcal{O}\Big(X^{4-c}(\log X)\max_{|t|\leqslant\Delta}\Big|\overline{S}_{l,d;J}(t)-\frac{I_J(t)}{\varphi(d)}\Big|\Big)+\mathcal{O}\Big(\frac{X^{5-c}}{d\mathrm{e}^{(\log X)^{1/6}}}\Big). \tag{5.13}$$

We are now in a good position to estimate the sum Γ_3.

6. Upper bound of Γ_3

Consider the sum Γ_3. Since

$$\sum_{\substack{d|p_1-1\\ d\geqslant X/D}} \chi_4(d) = \sum_{\substack{m|p_1-1\\ m\leqslant(p_1-1)D/X}} \chi_4\Big(\frac{p_1-1}{m}\Big) = \sum_{j=\pm1} \chi_4(j) \sum_{\substack{m|p_1-1\\ m\leqslant(p_1-1)D/X\\ (p_1-1)/m\equiv j\,(4)}} 1,$$

from (4.4) and (4.5) we obtain

$$\Gamma_3 = \sum_{\substack{m<D\\ 2|m}} \sum_{j=\pm1} \chi_4(j) I_{1+jm,4m;J_m}(N),$$

where $J_m = (\max\{1+mX/D, \frac{1}{2}X\}, X]$. The last formula and (4.6) give us

$$\Gamma_3 = \Gamma_3^{(1)} + \Gamma_3^{(2)}, \tag{6.1}$$

where

$$\Gamma_3^{(i)} = \sum_{\substack{m<D\\ 2|m}} \sum_{j=\pm1} \chi_4(j) I^{(i)}_{1+jm,4m;J_m}(N), \quad i=1,2. \tag{6.2}$$

6.1. Estimation of $\Gamma_3^{(1)}$. From (5.12) and (6.2) we get

$$\Gamma_3^{(1)} = \Gamma^* + \mathcal{O}(X^{4-c}(\log X)\Sigma_1) + \mathcal{O}\Big(\frac{X^{5-c}}{\mathrm{e}^{(\log X)^{1/6}}}\Sigma_2\Big), \tag{6.3}$$

where

$$\Gamma^* = \sum_{\substack{m<D\\ 2|m}} \Phi_J(X,4m) \sum_{j=\pm1} \chi_4(j), \tag{6.4}$$

$$\Sigma_1 = \sum_{\substack{m<D\\ 2|m}} \max_{|t|\leqslant\Delta} \Big|\overline{S}_{1+jm,4m;J}(t) - \frac{I_J(t)}{\varphi(4m)}\Big|, \tag{6.5}$$

$$\Sigma_2 = \sum_{m<D} \frac{1}{4m}. \tag{6.6}$$

From the properties of $\chi(k)$ we have that

$$\Gamma^* = 0. \tag{6.7}$$

By (2.1), (2.3), (2.7), (6.5) and Lemma 3.8 we deduce

$$\Sigma_1 \ll \frac{X}{\log^A X}. \tag{6.8}$$

It is well known that

$$\Sigma_2 \ll \log X. \tag{6.9}$$

Bearing in mind (6.3), (6.7), (6.8) and (6.9) we find

$$\Gamma_3^{(1)} \ll \frac{X^{5-c}}{\log X}. \tag{6.10}$$

6.2. Estimation of $\Gamma_3^{(2)}$. Now we consider $\Gamma_3^{(2)}$. The formulas (4.8) and (6.2) yield

$$\Gamma_3^{(2)} = \int_{\Delta}^{1-\Delta} S^4(t)K(t)e(-Nt)\,\mathrm{d}t, \tag{6.11}$$

where

$$K(t) = \sum_{\substack{m<D \\ 2|m}} \sum_{j=\pm 1} \chi_4(j) S_{1+jm,4m;J_m}(t). \tag{6.12}$$

Lemma 6.1. *For the sum denoted by* (6.12) *we have*

$$\int_0^1 |K(t)|^2\,\mathrm{d}t \ll X \log^6 X.$$

P r o o f. It follows by the arguments used in [8], Lemma 22. □

Lemma 6.2. *Assume that*

$$\Delta \leqslant |t| \leqslant 1-\Delta, \quad |a(m)| \ll m^{\varepsilon}, \quad LM \asymp X, \quad L \gg X^{51/103}, \quad H = X^{164/2539} \tag{6.13}$$

and $c_h(t)$ *denote complex numbers such that* $|c_h(t)| \ll (1+|h|)^{-1}$. *Set*

$$S_I = \sum_{|h|\leqslant H} c_h(t) \sum_{m\sim M} a(m) \sum_{l\sim L} e((h+t)m^c l^c). \tag{6.14}$$

Then

$$S_I \ll X^{2375/2539+\varepsilon}.$$

Proof. By (6.13) and (6.14) we write

$$S_I \ll X^{\varepsilon} \max_{|\eta|\in(\Delta,H+1)} \sum_{m\sim M} \left| \sum_{l\sim L} e(\eta m^c l^c) \right|. \tag{6.15}$$

We first consider the case when

$$M \ll X^{7273/18414}. \tag{6.16}$$

From (2.2), (6.13), (6.15), (6.16) and Lemma 3.2 with the exponent pair $\left(\frac{2}{40}, \frac{33}{40}\right)$ it follows

$$\begin{aligned} S_I &\ll X^{\varepsilon} \max_{|\eta|\in(\Delta,H+1)} \sum_{m\sim M} \left((|\eta| X^c L^{-1})^{2/40} L^{33/40} + \frac{1}{|\eta| X^c L^{-1}} \right) \\ &\ll X^{\varepsilon} \max_{|\eta|\in(\Delta,H+1)} \left(|\eta|^{2/40} X^{2c/40} M L^{31/40} + \frac{LM}{|\eta| X^c} \right) \\ &\ll X^{\varepsilon} (H^{1/20} X^{2c+31/40} M^{9/40} + \Delta^{-1} X^{1-c}) \ll X^{2375/2539+\varepsilon}. \end{aligned} \tag{6.17}$$

Next we consider the case when

$$X^{7273/18414} \ll M \ll X^{52/103}. \tag{6.18}$$

Using (6.15), (6.18) and Lemma 3.4 we deduce

$$S_I \ll X^{2375/2539+\varepsilon}. \tag{6.19}$$

Bearing in mind (6.17) and (6.19) we establish the statement of the lemma. □

Lemma 6.3. *Assume that*

$$\begin{gathered} \Delta \leqslant |t| \leqslant 1-\Delta, \quad |a(m)| \ll m^{\varepsilon}, \quad |b(l)| \ll l^{\varepsilon}, \quad LM \asymp X, \\ X^{1/9} \ll L \ll X^{1/3}, \quad H = X^{164/2539} \end{gathered} \tag{6.20}$$

and $c_h(t)$ *denote complex numbers such that* $|c_h(t)| \ll (1+|h|)^{-1}$.

$$S_{II} = \sum_{|h|\leqslant H} c_h(t) \sum_{m\sim M} a(m) \sum_{l\sim L} b(l) e((h+t)m^c l^c). \tag{6.21}$$

Then

$$S_{II} \ll X^{2375/2539+\varepsilon}.$$

P r o o f. Using (6.20), (6.21), Cauchy's inequality and Lemma 3.1 with $Q = X^{6475/30468}$ we obtain

$$
\begin{aligned}
(6.22)\quad S_{II} &\ll \sum_{|h|\leqslant H} |c_h(\alpha)| \Big| \sum_{m\sim M} a(m) \sum_{l\sim L} b(l) e((h+t)m^c l^c)) \Big| \\
&\ll \sum_{|h|\leqslant H} |c_h(t)| \Big(\sum_{m\sim M} |a(m)|^2 \Big)^{1/2} \Big(\sum_{m\sim M} \Big| \sum_{l\sim L} b(l) e((h+t)m^c l^c) \Big|^2 \Big)^{1/2} \\
&\ll M^{1/2+\varepsilon} \sum_{|h|\leqslant H} |c_h(t)| \Big(\sum_{m\sim M} \frac{L}{Q} \sum_{|q|<Q} \Big(1 - \frac{q}{Q}\Big) \\
&\quad \times \sum_{\substack{l\sim L\\ l+q\sim L}} b(l+q)\overline{b(l)} e(f_h(l,m,q)) \Big)^{1/2} \\
&\ll M^{1/2+\varepsilon} \sum_{|h|\leqslant H} |c_h(t)| \frac{L}{Q} \sum_{M<m\leqslant M_1} \Big(L^{1+\varepsilon} + \sum_{1\leqslant |q|<Q} \Big(1 - \frac{q}{Q}\Big) \\
&\quad \times \sum_{\substack{l\sim L\\ l+q\sim L}} b(l+q)\overline{b(l)} e(f_h(l,m,q)) \Big)^{1/2} \\
&\ll X^{\varepsilon} \sum_{|h|\leqslant H} |c_h(t)| \Big(\frac{X^2}{Q} + \frac{X}{Q} \sum_{1\leqslant q\leqslant Q} \sum_{l\sim L} \Big| \sum_{m\sim M} e(f(l,m,q)) \Big| \Big)^{1/2},
\end{aligned}
$$

where $f_h(l,m,q) = (h+t)m^c((l+q)^c - l^c)$. Now (2.2), (6.20), (6.22) and Lemma 3.2 with the exponent pair $\left(\frac{32}{205}+\varepsilon, \frac{269}{410}+\varepsilon\right)$ imply

$$
\begin{aligned}
S_{II} &\ll X^{\varepsilon} \sum_{|h|\leqslant H} |c_h(t)| \\
&\quad \times \Big(\frac{X^2}{Q} + \frac{X}{Q} \sum_{1\leqslant q\leqslant Q} \sum_{l\sim L} \Big((|h+t|qX^{c-1})^{32/205} M^{269/410} + \frac{1}{|h+t|qX^{c-1}} \Big) \Big)^{1/2} \\
&\ll X^{\varepsilon} \sum_{|h|\leqslant H} |c_h(t)| \\
&\quad \times \Big(\frac{X^2}{Q} + \frac{X}{Q} (H^{32/205} X^{32(c-1)/205} M^{269/410} Q^{237/205} L + \Delta^{-1} X^{1-c} L \log Q) \Big)^{1/2} \\
&\ll X^{2375/2539+\varepsilon} \sum_{|h|\leqslant H} |c_h(t)| \ll X^{2375/2539+\varepsilon} \sum_{|h|\leqslant H} \frac{1}{1+|h|} \ll X^{2375/2539+\varepsilon},
\end{aligned}
$$

which proves the statement of the lemma. □

Lemma 6.4. *Let $\Delta \leqslant |t| \leqslant 1-\Delta$. Then for the exponential sum denoted by* (2.4) *we have*

$$S(t) \ll X^{2375/2539+\varepsilon}.$$

P r o o f. In order to prove the lemma we will use the formula

$$S(t) = S^*(t) + \mathcal{O}(X^{1/2}), \tag{6.23}$$

where

$$S^*(t) = \sum_{X/2<n\leqslant X} \Lambda(n)e(t[n^c]). \tag{6.24}$$

By (6.24) and Lemma 3.5 with $x = t$, $y = n^c$ and $H = X^{164/2539}$ we get

$$\begin{aligned} S^*(t) &= \sum_{X/2<n\leqslant X} \Lambda(n)e(tn^c - t\{n^c\}) = \sum_{X/2<n\leqslant X} \Lambda(n)e(tn^c)e(-t\{n^c\}) \\ &= \sum_{X/2<n\leqslant X} \Lambda(n)e(tn^c)\Big(\sum_{|h|\leqslant H} c_h(t)e(hn^c) + \mathcal{O}\Big(\min\Big(1, \frac{1}{H\|n^c\|}\Big)\Big)\Big) \\ &= \sum_{|h|\leqslant H} c_h(t) \sum_{X/2<n\leqslant X} \Lambda(n)e((h+t)n^c) \\ &\quad + \mathcal{O}\Big((\log X) \sum_{X/2<n\leqslant X} \min\Big(1, \frac{1}{H\|n^c\|}\Big)\Big) \\ &= S_0^*(t) + \mathcal{O}\Big((\log X) \sum_{X/2<n\leqslant X} \min\Big(1, \frac{1}{H\|n^c\|}\Big)\Big), \end{aligned} \tag{6.25}$$

where

$$S_0^*(t) = \sum_{|h|\leqslant H} c_h(t) \sum_{X/2<n\leqslant X} \Lambda(n)e((h+t)n^c). \tag{6.26}$$

Let

$$U = X^{1/103}, \quad V = X^{1/3}, \quad Z = [X^{51/103}] + \frac{1}{2}.$$

According to Lemma 3.6, the sum $S_0^*(t)$ can be decomposed into $O(\log^{10} X)$ sums, each of which is either of Type I:

$$\sum_{|h|\leqslant H} c_h(t) \sum_{m\sim M} a(m) \sum_{l\sim L} e((h+t)m^c l^c), \quad \text{where } L \gg Z,\ LM \asymp X,\ |a(m)| \ll m^{\varepsilon},$$

or of Type II:

$$\sum_{|h|\leqslant H} c_h(t) \sum_{m\sim M} a(m) \sum_{l\sim L} b(l) e((h+t)m^c l^c),$$

where

$$U \ll L \ll V, \quad LM \asymp X, \quad |a(m)| \ll m^{\varepsilon}, \quad |b(l)| \ll l^{\varepsilon}.$$

Using (6.26), Lemma 6.2 and Lemma 6.3 we obtain

$$S_0^*(t) \ll X^{2375/2539+\varepsilon}. \tag{6.27}$$

On the other hand, Lemma 3.2 with the exponent pair $\left(\frac{1}{2}, \frac{1}{2}\right)$ and Lemma 3.11 yield

(6.28)

$$\begin{aligned}
&\sum_{X/2<n\leqslant X} \min\Big(1, \frac{1}{H\|n^c\|}\Big) \\
&\quad = \sum_{X/2<n\leqslant X} \sum_{k=-\infty}^{\infty} a_k e(kn^c) \ll \sum_{k=-\infty}^{\infty} |a_k| \Big| \sum_{X/2<n\leqslant X} e(kn^c) \Big| \\
&\quad \ll \frac{X\log 2H}{H} + \sum_{1\leqslant k\leqslant H} \frac{1}{k} \Big| \sum_{X/2<n\leqslant X} e(kn^c) \Big| + \sum_{k>H} \frac{H}{k^2} \Big| \sum_{X/2<n\leqslant X} e(kn^c) \Big| \\
&\quad \ll \frac{X\log 2H}{H} + \sum_{1\leqslant k\leqslant H} \frac{1}{k} (k^{1/2} X^{c/2} + k^{-1} X^{1-c}) + \sum_{k>H} \frac{H}{k^2} (k^{1/2} X^{c/2} + k^{-1} X^{1-c}) \\
&\quad \ll X^{\varepsilon}(H^{-1}X + H^{1/2}X^{c/2} + X^{1-c}) \ll X^{2375/2539+\varepsilon}.
\end{aligned}$$

Taking into account (6.23), (6.25), (6.27) and (6.28) we establish the statement of the lemma. □

Lemma 6.5. *For the sum denoted by* (2.4) *we have*

$$\int_{\Delta}^{1-\Delta} |S(t)|^6 \, \mathrm{d}t \ll X^{14660/2539-c+\varepsilon}.$$

Proof. For any continuous function $\mathfrak{S}(t)$ defined in the interval $[\Delta-1, 1-\Delta]$ we have

$$\begin{aligned}
\Big| \int_{\Delta}^{1-\Delta} S(t)\mathfrak{S}(t) \, \mathrm{d}t \Big| &= \Big| \sum_{p\sim X} (\log p) \int_{\Delta}^{1-\Delta} e(tp^c)\mathfrak{S}(t) \, \mathrm{d}t \Big| \\
&\leqslant \sum_{p\sim X} (\log p) \Big| \int_{\Delta}^{1-\Delta} e(tp^c)\mathfrak{S}(t) \, \mathrm{d}t \Big| \\
&\leqslant (\log X) \sum_{n\sim X} \Big| \int_{\Delta}^{1-\Delta} e(tn^c)\mathfrak{S}(t) \, \mathrm{d}t \Big|.
\end{aligned} \tag{6.29}$$

By (6.29) and Cauchy's inequality we obtain

$$(6.30)\quad \left|\int_{\Delta}^{1-\Delta} S(t)\mathfrak{S}(t)\,\mathrm{d}t\right|^2 \leqslant X(\log X)^2 \sum_{n\sim X}\left|\int_{\Delta}^{1-\Delta} e(tn^c)\mathfrak{S}(t)\,\mathrm{d}t\right|^2$$
$$= X(\log X)^2 \int_{\Delta}^{1-\Delta} \overline{\mathfrak{S}(y)}\,\mathrm{d}y \int_{\Delta}^{1-\Delta} \mathfrak{S}(t)A(t-y)\,\mathrm{d}t$$
$$\leqslant X(\log X)^2 \int_{\Delta}^{1-\Delta} |\mathfrak{S}(y)|\,\mathrm{d}y \int_{\Delta}^{1-\Delta} |\mathfrak{S}(t)||A(t-y)|\,\mathrm{d}t,$$

where $A(t)$ is defined by (3.1). Using Lemma 3.15 we write

(6.31)
$$\int_{\Delta}^{1-\Delta} |\mathfrak{S}(t)||A(t-y)|\,\mathrm{d}t$$
$$\ll \int_{\substack{\Delta\leqslant|t|\leqslant1-\Delta\\ |t-y|\leqslant X^{-c}}} |\mathfrak{S}(t)||A(t-y)|\,\mathrm{d}t + \int_{\substack{\Delta\leqslant|t|\leqslant1-\Delta\\ X^{-c}<|t-y|\leqslant2(1-\Delta)}} |\mathfrak{S}(t)||A(t-y)|\,\mathrm{d}t$$
$$\ll X \int_{\substack{\Delta\leqslant|t|\leqslant1-\Delta\\ |t-y|\leqslant X^{-c}}} |\mathfrak{S}(t)|\,\mathrm{d}t$$
$$+ \int_{\substack{\Delta\leqslant|t|\leqslant1-\Delta\\ X^{-c}<|t-y|\leqslant2(1-\Delta)}} |\mathfrak{S}(t)|\Big(X^{(269c+538)/1217}\log X + \frac{X^{1-c}}{|t-y|}\Big)\,\mathrm{d}t$$
$$\ll X \max_{\Delta\leqslant|t|\leqslant1-\Delta} |\mathfrak{S}(t)| \int_{|t-y|\leqslant X^{-c}} \mathrm{d}t + X^{(269c+538)/1217}(\log X) \int_{\Delta}^{1-\Delta} |\mathfrak{S}(t)|\,\mathrm{d}t$$
$$+ X^{1-c} \max_{\Delta\leqslant|t|\leqslant1-\Delta} |\mathfrak{S}(t)| \int_{X^{-c}<|t-y|\leqslant2(1-\Delta)} \frac{1}{|t-y|}\,\mathrm{d}t$$
$$\ll X^{1-c}(\log X) \max_{\Delta\leqslant|t|\leqslant1-\Delta} |\mathfrak{S}(t)| + X^{(269c+538)/1217}(\log X) \int_{\Delta}^{1-\Delta} |\mathfrak{S}(t)|\,\mathrm{d}t.$$

Now (6.30) and (6.31) imply

$$(6.32)\quad \left|\int_{\Delta}^{1-\Delta} S(t)\mathfrak{S}(t)\,\mathrm{d}t\right|^2 \leqslant X^{2-c+\varepsilon} \max_{\Delta\leqslant|t|\leqslant1-\Delta} |\mathfrak{S}(t)| \int_{\Delta}^{1-\Delta} |\mathfrak{S}(t)|\,\mathrm{d}t$$
$$+ X^{(269c+1755)/1217+\varepsilon}\Big(\int_{\Delta}^{1-\Delta} |\mathfrak{S}(t)|\,\mathrm{d}t\Big)^2.$$

Let us put first

$$(6.33)\quad \mathfrak{S}_1(t) = \overline{S(t)}|S(t)|^3.$$

Bearing in mind (6.32), (6.33), Lemma 3.14 and Lemma 6.4 we get
(6.34)

$$\begin{aligned}\int_\Delta^{1-\Delta}|S(t)|^5\,\mathrm{d}t&=\int_\Delta^{1-\Delta}S(t)\mathfrak{S}_1(t)\,\mathrm{d}t\\&\ll X^{7289/2539-c/2+\varepsilon}\bigg(\int_\Delta^{1-\Delta}|S(t)|^4\,\mathrm{d}t\bigg)^{1/2}+X^{(269c+1755)/2434+\varepsilon}\int_\Delta^{1-\Delta}|S(t)|^4\,\mathrm{d}t\\&\ll X^{7289/2539-c/2+\varepsilon}\bigg(\int_0^{1}|S(t)|^4\,\mathrm{d}t\bigg)^{1/2}+X^{(269c+1755)/2434+\varepsilon}\int_0^{1}|S(t)|^4\,\mathrm{d}t\\&\ll X^{12367/2539-c+\varepsilon}+X^{(11491-2165c)/2434+\varepsilon}\\&\ll X^{12367/2539-c+\varepsilon}.\end{aligned}$$

Next we put

$$\mathfrak{S}_2(t)=\overline{S(t)}|S(t)|^4. \tag{6.35}$$

Taking into account (6.32), (6.34), (6.35) and Lemma 6.4 we find

$$\begin{aligned}\int_\Delta^{1-\Delta}|S(t)|^6\,\mathrm{d}t&=\int_\Delta^{1-\Delta}S(t)\mathfrak{S}_2(t)\,\mathrm{d}t\\&\ll X^{16953/5078-c/2+\eta}\bigg(\int_\Delta^{1-\Delta}|S(t)|^5\,\mathrm{d}t\bigg)^{1/2}+X^{269c+1755/2434+\varepsilon}\int_\Delta^{1-\Delta}|S(t)|^5\,\mathrm{d}t\\&\ll X^{14660/2539-c+\varepsilon}+X^{34557223/6179926-2165c/2434+\varepsilon}\\&\ll X^{14660/2539-c+\varepsilon}.\end{aligned}$$

The lemma is proved. □

Bearing in mind (6.11), Cauchy's inequality, Lemmas 6.1, 6.4 and 6.5, we deduce

$$\begin{aligned}\Gamma_3^{(2)}&\ll\max_{\Delta\leqslant t\leqslant 1-\Delta}|S(t)|\bigg(\int_\Delta^{1-\Delta}|S(t)|^6\,\mathrm{d}t\bigg)^{1/2}\bigg(\int_\Delta^{1-\Delta}|K(t)|^2\,\mathrm{d}t\bigg)^{1/2}\\&\ll\max_{\Delta\leqslant t\leqslant 1-\Delta}|S(t)|\bigg(\int_\Delta^{1-\Delta}|S(t)|^6\,\mathrm{d}t\bigg)^{1/2}\bigg(\int_0^{1}|K(t)|^2\,\mathrm{d}t\bigg)^{1/2}\\&\ll\frac{X^{5-c}}{\log X}.\end{aligned}\tag{6.36}$$

6.3. Estimation of Γ_3. Summarizing (6.1), (6.10) and (6.36) we obtain

$$\Gamma_3\ll\frac{X^{5-c}}{\log X}. \tag{6.37}$$

7. Upper bound of Γ_2

In this section we need a lemma that gives us information about the upper bound of the number of solutions of the binary equation corresponding to (1.1).

Lemma 7.1. *Let $1 < c < 3$, $c \neq 2$ and N_0 be a sufficiently large positive integer. Then for the number of solutions $B_0(N_0)$ of the Diophantine equation*

$$[p_1^c] + [p_2^c] + [p_3^c] + [p_4^c] = N_0 \tag{7.1}$$

in prime numbers p_1, p_2, p_3, p_4 we have that

$$B_0(N_0) \ll \frac{N_0^{4/c-1}}{\log^4 N_0}.$$

P r o o f. Define

$$B(X_0) = \sum_{\substack{X_0/2<p_1,p_2,p_3,p_4\leqslant X_0\\ [p_1^c]+[p_2^c]+[p_3^c]+[p_4^c]=N_0}} \log p_1 \log p_2 \log p_3 \log p_4, \tag{7.2}$$

where

$$X_0 = N_0^{1/c}. \tag{7.3}$$

From (7.2) we write

$$B(X_0) = \int_{-\Delta_0}^{1-\Delta_0} S_0^4(t) e(-N_0 t)\,\mathrm{d}t = B_1(X_0) + B_2(X_0), \tag{7.4}$$

where

$$S_0(t) = \sum_{X_0/2<p\leqslant X_0} e(t[p^c]) \log p, \tag{7.5}$$

$$\overline{S}_0(t) = \sum_{X_0/2<p\leqslant X_0} e(tp^c) \log p, \tag{7.6}$$

$$\Delta_0 = \frac{(\log X_0)^{A_0}}{X_0^c}, \quad A_0 > 10, \tag{7.7}$$

$$B_1(X_0) = \int_{-\Delta_0}^{\Delta_0} S_0^4(t) e(-N_0 t)\,\mathrm{d}t, \tag{7.8}$$

$$B_2(X_0) = \int_{\Delta_0}^{1-\Delta_0} S_0^4(t) e(-N_0 t)\,\mathrm{d}t. \tag{7.9}$$

First we estimate $B_1(X_0)$. Put

$$I_0(t) = \int_{X_0/2}^{X_0} e(ty^c)\,\mathrm{d}y, \tag{7.10}$$

$$\Psi_{\Delta_0}(X_0) = \int_{-\Delta_0}^{\Delta_0} I_0^4(t) e(-N_0 t)\,\mathrm{d}t, \tag{7.11}$$

$$\Psi(X_0) = \int_{-\infty}^{\infty} I_0^4(t) e(-N_0 t)\,\mathrm{d}t. \tag{7.12}$$

Using (5.10), (7.10) and (7.12) we obtain

$$\begin{aligned} \Psi(X_0) &= \int_{-X_0^{-c}}^{X_0^{-c}} I_0^4(t) e(-N_0 t)\,\mathrm{d}t + \int_{|t|>X_0^{-c}} I_0^4(t) e(-N_0 t)\,\mathrm{d}t \\ &\ll \int_{-X_0^{-c}}^{X_0^{-c}} X_0^4\,\mathrm{d}t + \int_{X_0^{-c}}^{\infty} \Big(\frac{X_0^{1-c}}{t}\Big)^4 \mathrm{d}t \ll X_0^{4-c}. \end{aligned} \tag{7.13}$$

On the other hand (5.1), (7.5)–(7.8), (7.11), Lemma 3.7 and the trivial estimations

$$S_0(t) \ll X_0, \quad I_0(t) \ll X_0 \tag{7.14}$$

give us

$$\begin{aligned} B_1(X_0) - \Psi_{\Delta_0}(X_0) &\ll \int_{-\Delta_0}^{\Delta_0} |S_0^4(t) - I_0^4(t)|\,\mathrm{d}t \\ &\ll \int_{-\Delta_0}^{\Delta_0} |S_0(t) - I_0(t)|(|S_0(t)|^3 + |I_0(t)|^3)\,\mathrm{d}t \\ &\ll \Big(\max_{|t|\leqslant\Delta_0} |\overline{S}_0(t) - I_0(t)| + \Delta_0 X_0\Big) \\ &\quad \times \Big(\int_{-\Delta_0}^{\Delta_0} |S_0(t)|^3\,\mathrm{d}t + \int_{-\Delta_0}^{\Delta_0} |I_0(t)|^3\,\mathrm{d}t\Big) \\ &\ll \Big(\frac{X_0}{\mathrm{e}^{(\log X_0)^{1/5}}} + \Delta_0 X_0\Big)\Delta_0 X_0^3 \ll \frac{X_0^{4-c}}{\mathrm{e}^{(\log X_0)^{1/6}}}. \end{aligned} \tag{7.15}$$

By (5.10), (7.7), (7.11) and (7.12) we deduce

$$|\Psi(X_0) - \Psi_{\Delta_0}(X_0)| \ll \int_{\Delta_0}^{\infty} |I_0(t)|^4\,\mathrm{d}t \ll \frac{1}{X_0^{4(c-1)}} \int_{\Delta_0}^{\infty} \frac{\mathrm{d}t}{t^4} \ll \frac{1}{X_0^{4(c-1)}\Delta_0^3} \ll \frac{X_0^{4-c}}{\log X_0}. \tag{7.16}$$

Now (7.13), (7.15) and (7.16) and the identity

$$B_1(X_0) = B_1(X_0) - \Psi_{\Delta_0}(X_0) + \Psi_{\Delta_0}(X_0) - \Psi(X_0) + \Psi(X_0)$$

imply

$$B_1(X_0) \ll X_0^{4-c}. \tag{7.17}$$

It remains to estimate $B_2(X_0)$. By (7.3), (7.9), (7.14) and partial integration it follows

$$B_2(X_0) = -\frac{1}{2\pi i}\int_{\Delta_0}^{1-\Delta_0} \frac{S_0^4(t)}{N_0} de(-N_0 t) \tag{7.18}$$
$$= -\frac{S_0^4(t)e(-N_0 t)}{2\pi i N_0}\Big|_{\Delta_0}^{1-\Delta_0} + \frac{1}{2\pi i N_0}\int_{\Delta_0}^{1-\Delta_0} e(-N_0 t)d(S_0^4(t))$$
$$\ll X_0^{4-c} + X_0^{-c}|\Omega|,$$

where

$$\Omega = \int_{\Delta_0}^{1-\Delta_0} e(-N_0 t)d(S_0^4(t)). \tag{7.19}$$

Next we consider Ω. Put

$$\Gamma:\ z = f(t) = S_0^4(t), \quad \Delta_0 \leqslant t \leqslant 1-\Delta_0. \tag{7.20}$$

Using (7.14), (7.19) and (7.20) we obtain

$$\Omega = \int_\Gamma e(-N_0 f^{-1}(z))\,\mathrm{d}z \ll \int_\Gamma |\mathrm{d}z| \ll (|f(\Delta_0)| + |f(1-\Delta_0)|) \ll X_0^4. \tag{7.21}$$

Bearing in mind (7.18) and (7.21) we find

$$B_2(X_0) \ll X_0^{4-c}. \tag{7.22}$$

Summarizing (7.4), (7.17) and (7.22) we deduce

$$B(X_0) \ll X_0^{4-c}. \tag{7.23}$$

Taking into account (7.2), (7.3) and (7.23), for the number of solutions $B_0(N_0)$ of the Diophantine equation (7.1) we get

$$B_0(N_0) \ll \frac{N_0^{4/c-1}}{\log^4 N_0}.$$

The lemma is proved. □

We are now ready to estimate the sum Γ_2. We denote by $\mathcal{F}(X)$ the set of all primes $X/2 < p \leqslant X$ such that $p-1$ has a divisor belonging to the interval $(D, X/D)$. The inequality $xy \leqslant x^2 + y^2$ and (4.3) give us

$$\Gamma_2^2 \ll (\log X)^{10} \sum_{\substack{X/2<p_1,\dots,p_{10}\leqslant X \\ [p_1^c]+[p_2^c]+[p_3^c]+[p_4^c]+[p_5^c]=N \\ [p_6^c]+[p_7^c]+[p_8^c]+[p_9^c]+[p_{10}^c]=N}} \Bigg| \sum_{\substack{d|p_1-1 \\ D<d<X/D}} \chi_4(d) \Bigg| \Bigg| \sum_{\substack{t|p_6-1 \\ D<t<X/D}} \chi_4(t) \Bigg|$$

$$\ll (\log X)^{10} \sum_{\substack{X/2<p_1,\dots,p_{10}\leqslant X \\ [p_1^c]+[p_2^c]+[p_3^c]+[p_4^c]+[p_5^c]=N \\ [p_6^c]+[p_7^c]+[p_8^c]+[p_9^c]+[p_{10}^c]=N \\ p_6\in\mathcal{F}(X)}} \Bigg| \sum_{\substack{d|p_1-1 \\ D<d<X/D}} \chi_4(d) \Bigg|^2 .$$

The summands in the last sum for which $p_1 = p_6$ can be estimated with $\mathcal{O}(X^{7+\varepsilon})$. Thus,

$$\Gamma_2^2 \ll (\log X)^{10} \Sigma_0 + X^{7+\varepsilon}, \tag{7.24}$$

where

$$\Sigma_0 = \sum_{X/2<p_1\leqslant X} \Bigg| \sum_{\substack{d|p_1-1 \\ D<d<X/D}} \chi_4(d) \Bigg|^2 \sum_{\substack{X/2<p_6\leqslant X \\ p_6\in\mathcal{F}(X) \\ p_6\neq p_1}} \sum_{\substack{X/2<p_2,p_3,p_5,p_{10}\leqslant X \\ [p_1^c]+[p_2^c]+[p_3^c]+[p_4^c]+[p_5^c]=N \\ [p_6^c]+[p_7^c]+[p_8^c]+[p_9^c]+[p_{10}^c]=N}} 1. \tag{7.25}$$

Now (7.25) and Lemma 7.1 yield

$$\Sigma_0 \ll \frac{X^{8-2c}}{\log^8 X} \Sigma_0' \Sigma_0'', \tag{7.26}$$

where

$$\Sigma_0' = \sum_{X/2<p\leqslant X} \Bigg| \sum_{\substack{d|p-1 \\ D<d<X/D}} \chi_4(d) \Bigg|^2, \quad \Sigma_0'' = \sum_{\substack{X/2<p\leqslant X \\ p\in\mathcal{F}(X)}} 1.$$

Applying Lemma 3.12 we obtain

$$\Sigma_0' \ll \frac{X(\log\log X)^7}{\log X}. \tag{7.27}$$

Using Lemma 3.13 we get

$$\Sigma_0'' \ll \frac{X(\log\log X)^3}{(\log X)^{1+2\theta_0}}, \tag{7.28}$$

where θ_0 is denoted by (1.3). Finally (7.24), (7.26), (7.27) and (7.28) imply

$$\Gamma_2 \ll \frac{X^{5-c}(\log\log X)^5}{(\log X)^{\theta_0}}. \tag{7.29}$$

Consider the sum Γ_1. From (4.2), (4.5) and (4.6) we deduce

$$\Gamma_1 = \Gamma_1^{(1)} + \Gamma_1^{(2)}, \tag{8.1}$$

where

$$\Gamma_1^{(i)} = \sum_{d\leqslant D} \chi_4(d) I_{1,d}^{(i)}(N), \quad i = 1, 2. \tag{8.2}$$

8.1. Estimation of $\Gamma_1^{(1)}$. First we consider $\Gamma_1^{(1)}$. Using formula (5.12) for $J = (X/2, X]$, (8.2) and treating the remainder term in the same way as for $\Gamma_3^{(1)}$ we find

$$\Gamma_1^{(1)} = \Phi(X) \sum_{d\leqslant D} \frac{\chi_4(d)}{\varphi(d)} + \mathcal{O}\Big(\frac{X^{5-c}}{\log X}\Big), \tag{8.3}$$

where

$$\Phi(X) = \int_{-\infty}^{\infty} I^5(t) e(-Nt)\,\mathrm{d}t. \tag{8.4}$$

Lemma 8.1. *For the integral denoted by* (8.4), *the asymptotic formula*

$$\Phi(X) = \frac{\Gamma^5(1+1/c)}{\Gamma(5/c)}\Big(1 - \frac{1}{2^{5-c}}\Big) X^{5-c} + \mathcal{O}\Big(\frac{X^{5-c}}{\mathrm{e}^{(\log X)^{1/6}}}\Big)$$

holds.

P r o o f. From (8.4) write

$$\Phi(X) = \Theta_1 - \Omega_S + \Omega_S + \Theta_2, \tag{8.5}$$

where

$$\Theta_1 = \int_{-\Delta}^{\Delta} I^5(t) e(-Nt)\,\mathrm{d}t, \tag{8.6}$$

$$\Theta_2 = \int_{|t|>\Delta} I^5(t) e(-Nt)\,\mathrm{d}t, \tag{8.7}$$

$$\Omega_S = \int_{-\Delta}^{\Delta} S^5(t) e(-Nt)\,\mathrm{d}t. \tag{8.8}$$

By (2.2), (2.4), (2.5), (5.1), (8.6), (8.8), Lemmas 3.7 and 3.9 we obtain

$$
\begin{aligned}
(8.9)\quad \Theta_1 - \Omega_S &\ll \int_{-\Delta}^{\Delta} |S^5(t) - I^5(t)|\,\mathrm{d}t \\
&\ll \max_{|t|\leqslant\Delta} |S(t) - I(t)| \Big(\int_{-\Delta}^{\Delta} |S(t)|^4\,\mathrm{d}t + \int_{-\Delta}^{\Delta} |I(t)|^4\,\mathrm{d}t \Big) \\
&\ll \Big(\max_{|t|\leqslant\Delta} |\overline{S}(t) - I(t)| + \Delta X \Big) X^2 \Big(\int_{-\Delta}^{\Delta} |S(t)|^2\,\mathrm{d}t + \int_{-\Delta}^{\Delta} |I(t)|^2\,\mathrm{d}t \Big) \\
&\ll \Big(\frac{X}{\mathrm{e}^{(\log X)^{1/5}}} + \Delta X \Big) X^{4-c} \log^2 X \ll \frac{X^{5-c}}{\mathrm{e}^{(\log X)^{1/6}}}.
\end{aligned}
$$

Arguing as in Proposition 3.1 of [25] for the integral denoted by (8.8) we get

$$
(8.10)\qquad \Omega_S = \frac{\Gamma^5(1+1/c)}{\Gamma(5/c)} \Big(1 - \frac{1}{2^{5-c}}\Big) X^{5-c} + \mathcal{O}\Big(\frac{X^{5-c}}{\mathrm{e}^{(\log X)^{1/4}}}\Big).
$$

From (2.2), (5.10) and (8.7) it follows

$$
(8.11)\qquad \Theta_2 \ll \int_{\Delta}^{\infty} |I(t)|^5\,\mathrm{d}t \ll \frac{1}{X^{5(c-1)}} \int_{\Delta}^{\infty} \frac{\mathrm{d}t}{t^5} \ll \frac{1}{X^{5(c-1)}\Delta^4} = X^{4-c}.
$$

Now the lemma follows from (8.5), (8.9), (8.10) and (8.11). □

According to [7] we have

$$
(8.12)\qquad \sum_{d\leqslant D} \frac{\chi_4(d)}{\varphi(d)} = \frac{\pi}{4} \prod_p \Big(1 + \frac{\chi_4(p)}{p(p-1)}\Big) + \mathcal{O}\Big(X^{-1/20}\Big).
$$

From (8.3) and (8.12) we obtain

$$
(8.13)\qquad \Gamma_1^{(1)} = \frac{\pi}{4} \prod_p \Big(1 + \frac{\chi_4(p)}{p(p-1)}\Big) \Phi(X) + \mathcal{O}\Big(\frac{X^{5-c}}{\log X}\Big) + \mathcal{O}(\Phi(X) X^{-1/20}).
$$

Now (8.13) and Lemma 8.1 yield

$$
(8.14)\qquad \Gamma_1^{(1)} = \frac{\pi}{4} \prod_p \Big(1 + \frac{\chi_4(p)}{p(p-1)}\Big) \frac{\Gamma^5(1+1/c)}{\Gamma(5/c)} (1 - 1/2^{5-c}) X^{5-c} + \mathcal{O}\Big(\frac{X^{5-c}}{\log X}\Big).
$$

8.2. Estimation of $\Gamma_1^{(2)}$. Arguing as in the estimation of $\Gamma_3^{(2)}$ we get

$$
(8.15)\qquad \Gamma_1^{(2)} \ll \frac{X^{5-c}}{\log X}.
$$

8.3. Estimation of Γ_1. Summarizing (8.1), (8.14) and (8.15) we deduce

$$
(8.16)\qquad \Gamma_1 = \frac{\pi}{4} \prod_p \Big(1 + \frac{\chi_4(p)}{p(p-1)}\Big) \frac{\Gamma^5(1+1/c)}{\Gamma(5/c)} (1 - 1/2^{5-c}) X^{5-c} + \mathcal{O}\Big(\frac{X^{5-c}}{\log X}\Big).
$$

9. Proof of the theorem

Bearing in mind (1.4), (4.1), (6.37), (7.29) and (8.16) we establish asymptotic formula (1.6). This completes the proof of the theorem.

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