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ON n-SUBMODULES AND $G.n$-SUBMODULES

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Abstract. We investigate some properties of n-submodules. More precisely, we find a necessary and sufficient condition for every proper submodule of a module to be an n-submodule. Also, we show that if M is a finitely generated R-module and $\sqrt{\mathrm{Ann}_R(M)}$ is a prime ideal of R, then M has n-submodule. Moreover, we define the notion of $G.n$-submodule, which is a generalization of the notion of n-submodule. We find some characterizations of $G.n$-submodules and we examine the way the aforementioned notions are related to each other.

Keywords: n-ideal; n-submodule; primary submodule

MSC 2020: 13C13, 16D10

1. Introduction and preliminaries

Throughout this paper, R denotes a commutative ring with identity, and all modules are unitary. The concept of prime ideal is important in commutative algebra. As defined by Mohamadian in [12], an r-ideal of R is a proper ideal I with the property that $a, b \in R$, $ab \in I$ and $\mathrm{ann}_M(a) = 0$ imply $b \in I$. Tekir et al. in [14] defined n-ideals and determined some of their properties. According to their results, any n-ideal is an r-ideal.

In module theory, prime submodules are defined similar to prime ideals in ring theory and play an important role. Koc and Tekir in [7] defined r-submodules, while Tekir et al. in [14] defined n-submodules. Ahmadi and Moghaderi in [1] found some fundamental characteristics of n-submodules. For example, each n-submodule is also an r-submodule. Also, if $(N : M) \subseteq \sqrt{\mathrm{Ann}(M)}$ for a submodule N of M, then N is a primary submodule if and only if it is an n-submodule.

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In this paper, we find some additional properties of n-submodules and define the concept of $G.n$-submodule. We prove that

$$n\text{-submodule} \Rightarrow G.n\text{-submodule} \Rightarrow r\text{-submodule}.$$

Also, $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ denote the set of natural numbers, the ring of integers, and the field of rational numbers, respectively. If N is an R-submodule of M, the *annihilator* of the R-module $\frac{M}{N}$ is defined to be $\mathrm{Ann}_R(\frac{M}{N}) = (N :_R M) = \{r \in R\colon\ rM \subseteq N\}$. Furthermore, the *annihilator* of M, denoted by $\mathrm{Ann}_R(M)$, is $(0 :_R M)$. Suppose that I is an ideal of R. We define the *radical* of I by $\sqrt{I} = \{a \in R\colon\ a^n \in I \text{ for some } n \in \mathbb{N}\}$.

A proper submodule N of M is called *prime (primary)* if $rm \in N$ for $r \in R$ and $m \in M$ implies that either $m \in N$ or $r \in (N :_R M)$ ($r^n \in (N :_R M)$ for some $n \in \mathbb{N}$), see [3], [8], [10], [11], and [13].

An R-module M is said to be a *multiplication module* if for each submodule N of M, there exists an ideal I of R such that $N = IM$. Equivalently, M is a multiplication module if and only if $N = (N :_R M)M$ for each submodule N of M. We refer the reader to [4] and [5] for more details.

The concepts of n-ideal and n-submodule were introduced in [14]. A proper ideal I of R is said to be an *n-ideal* if the condition $ab \in I$ with $a \notin \sqrt{0}$ for every $a, b \in R$ implies $b \in I$. Also, a proper submodule N of M is called an *n-submodule* if for $a \in R$ and $m \in M$, $am \in N$ and $a \notin \sqrt{\mathrm{Ann}_R(M)}$ imply $m \in N$.

In Section 2, we investigate some properties of n-submodules. We find a necessary and sufficient condition for every proper submodule of a module to be an n-submodule. Also, we show that if M is a finitely generated R-module and $\sqrt{\mathrm{Ann}_R(M)}$ is a prime ideal of R, then M has an n-submodule.

In Section 3, we define the notion of $G.n$-submodule. We show that any n-submodule is an $G.n$-submodule, and that any $G.n$-submodule is an r-submodule. Also, we find some characterizations of this new notion.

2. n-SUBMODULES

Let M be a module over a commutative ring R. Recall that a proper submodule N of M is said to be an n-submodule if for $a \in R$ and $m \in M$, $am \in N$ and $a \notin \sqrt{\mathrm{Ann}_R(M)}$ imply $m \in N$.

Theorem 2.1. *Let M be a torsion-free R-module, and N be a proper submodule of M. Then, the following statements are equivalent.*

(1) *N is an n-submodule of M.*
(2) *$aN = N \cap aM$ for every $a \in R - \sqrt{\mathrm{Ann}_R(M)}$.*
(3) *$N = (N :_M a)$ for every $a \in R - \sqrt{\mathrm{Ann}_R(M)}$.*

P r o o f. (1) $\Rightarrow$ (2) It is clear that $aN \subseteq N \cap aM$. If $am \in N \cap aM$, where $a \in R - \sqrt{\mathrm{Ann}_R(M)}$ and $m \in M$, then $m \in N$. So, $aN = N \cap aM$.

(2) $\Rightarrow$ (3) We know that $N \subseteq (N :_M a)$. If $m \in (N :_M a)$, then $am \in N$. So, $am \in N \cap aM = aN$. Now, since M is torsion-free, $m \in N$. Hence, $N = (N :_M a)$.

(3) $\Rightarrow$ (1) If $am \in N$ with $a \in R - \sqrt{\mathrm{Ann}_R(M)}$ and $m \in M$, then $m \in (N :_M a) = N$. $\square$

Proposition 2.1. *Let M be an R-module and N be an n-submodule of M. Then,*

$$N = (0 :_M \mathrm{Ann}_R(N)) \quad \text{or} \quad \sqrt{\mathrm{Ann}_R(N)} = \sqrt{\mathrm{Ann}_R(M)}.$$

P r o o f. Let $\sqrt{\mathrm{Ann}_R(N)} \neq \sqrt{\mathrm{Ann}_R(M)}$. It is clear that $N \subseteq (0 :_M \mathrm{Ann}_R(N))$. Since $\sqrt{\mathrm{Ann}_R(M)} \subset \sqrt{\mathrm{Ann}_R(N)}$, there exists some a in $\sqrt{\mathrm{Ann}_R(N)} - \sqrt{\mathrm{Ann}_R(M)}$. Hence, a $k \in \mathbb{N}$ can be found such that $a^k \in \mathrm{Ann}_R(N)$. Let $m \in (0 :_M \mathrm{Ann}_R(N))$. Then, $a^k m = 0 \in N$. Since N is an n-submodule and $a^k \notin \sqrt{\mathrm{Ann}_R(M)}$, $m \in N$. So, $N = (0 :_M \mathrm{Ann}_R(N))$. $\square$

Proposition 2.2. *Let N be a proper submodule of a torsion-free R-module M. If $aN = N$ for any $a \in R - \sqrt{\mathrm{Ann}_R(M)}$, then N is an n-submodule.*

P r o o f. Let $aN = N$ for any $a \in R - \sqrt{\mathrm{Ann}_R(M)}$. Assume that $am \in N$ for $a \in R - \sqrt{\mathrm{Ann}_R(M)}$ and $m \in M$. Hence, $am \in aN$ by the hypothesis. Then, since M is torsion-free, $m \in N$ and, therefore, N is an n-submodule. $\square$

Proposition 2.3. *Let N be an n-submodule of an R-module M, and S be a nonempty subset of R. Then $(N :_M S)$ equals M, or it is an n-submodule.*

P r o o f. Let $(N :_M S)$ be a proper submodule of M. Also, suppose that $am \in (N :_M S)$ for $a \in R - \sqrt{\mathrm{Ann}_R(M)}$ and $m \in M$. Then, $aSm \subseteq N$ and thus $Sm \subseteq N$, because N is an n-submodule. Therefore, $m \in (N :_M S)$ and, consequently, $(N :_M S)$ is an n-submodule. $\square$

Corollary 2.1. *Let N be an n-submodule of an R-module M and S be a nonempty subset of R. Then, $S \nsubseteq (N :_R M)$ if and only if $(N :_M S)$ is an n-submodule.*

P r o o f. $\Rightarrow$) This follows from Proposition 2.3.

$\Leftarrow$) Since $(N :_M S)$ is an n-submodule of M, $(N :_M S)$ is a proper submodule of M. Therefore, $SM \nsubseteq N$. Thus, $S \nsubseteq (N :_R M)$. $\square$

Proposition 2.4 ([1], Proposition 2.3 (i))**.** *If N is an n-submodule of M, then* $(N :_R M) \subseteq \sqrt{\mathrm{Ann}_R(M)}$.

Theorem 2.2 ([1], Theorem 2.22)**.** *Let N be a submodule of M such that* $(N :_R M) \subseteq \sqrt{\mathrm{Ann}_R(M)}$. *Then, the following statements are equivalent.*

(1) *N is an n-submodule.*

(2) *N is a primary submodule of M.*

Corollary 2.2. *Let M be an R-module. Suppose that L is an n-submodule of M and that K is a primary submodule of M. If $K \subseteq L$, then K is an n-submodule of M.*

By using the fact that every irreducible submodule of a Noetherian module is a primary submodule (see [6], Proposition 1–17), we obtain the following corollary.

Corollary 2.3. *Let M be a Noetherian R-module and N be an irreducible submodule of M such that $(N :_R M) \subseteq \sqrt{\mathrm{Ann}_R(M)}$. Then, N is an n-submodule of M.*

Proposition 2.5. *If N is a primary R-submodule of M such that $(N :_R M)$ is maximal in the set of all n-ideals, then N is an n-submodule of M.*

P r o o f. Let $am \in N$ for $a \in R$ and $m \in M$, where $a \notin \sqrt{\mathrm{Ann}_R(M)}$. By [14], Theorem 2.11, $\sqrt{0} = \sqrt{(N :_R M)}$. So, $(N :_R M) \subseteq \sqrt{(N :_R M)} = \sqrt{0} \subseteq \sqrt{\mathrm{Ann}_R(M)}$. Therefore, by Theorem 2.2, N is an n-submodule of M. □

Lemma 2.1. *Let M be an R-module. If M has an n-submodule, then $\sqrt{\mathrm{Ann}_R(M)}$ is a prime ideal.*

P r o o f. Let N be an n-submodule of M. Then, by Proposition 2.4, $(N :_R M) \subseteq \sqrt{\mathrm{Ann}_R(M)}$. Since $\mathrm{Ann}_R(M) \subseteq (N :_R M)$, we conclude that $\sqrt{(N :_R M)} = \sqrt{\mathrm{Ann}_R(M)}$. By Theorem 2.2, N is a primary submodule. So, $\sqrt{(N :_R M)}$ is a prime ideal of R. Therefore, $\sqrt{\mathrm{Ann}_R(M)}$ is a prime ideal of R. □

Proposition 2.6. *Let M be a finitely generated R-module. Then, M has an n-submodule if and only if $\sqrt{\mathrm{Ann}_R(M)}$ is a prime ideal of R.*

P r o o f. $\Rightarrow$) By Lemma 2.1, $\sqrt{\mathrm{Ann}_R(M)}$ is a prime ideal of R.

$\Leftarrow$) Let $\sqrt{\mathrm{Ann}_R(M)}$ be a prime ideal of R. Put

$$\mathfrak{A} = \{L\colon\ L \text{ is a submodule of } M, (L : M) \subseteq \sqrt{\mathrm{Ann}_R(M)}\}.$$

Since $0 \in \mathfrak{A}$, we find that $\mathfrak{A} \neq \emptyset$. Moreover, since M is finitely generated, by using Zorn's lemma we find a maximal element K of $\mathfrak{A}$. Now, we show that K is an

n-submodule of M. Suppose that $rm \in K$ for some $r \in R$ and $m \notin K$. Since K is a maximal element of $\mathfrak{A}$, $(K + \langle m \rangle : M) \nsubseteq \sqrt{\mathrm{Ann}_R(M)}$. For $a \in (K + \langle m \rangle : M) - \sqrt{\mathrm{Ann}_R(M)}$ we obtain $aM \subseteq K + \langle m \rangle$. This implies $raM \subseteq K + \langle rm \rangle \subseteq K$. So, $ra \in (K : M) \subseteq \sqrt{\mathrm{Ann}_R(M)}$. Since $\sqrt{\mathrm{Ann}_R(M)}$ is a prime ideal of R and $a \notin \sqrt{\mathrm{Ann}_R(M)}$, $r \in \sqrt{\mathrm{Ann}_R(M)}$. Therefore, K is an n-submodule. □

Note that an r-submodule is a proper submodule N of M if for $a \in R$, $m \in M$, and whenever $am \in N$ with $\mathrm{ann}_M(a) = 0$, then $m \in N$, see [7].

Theorem 2.3. *Let M be an R-module. Then, the following statements are equivalent.*

(1) *$\langle 0 \rangle$ is an n-submodule.*

(2) *$\langle 0 \rangle$ is a primary submodule.*

(3) *$Z(M) = \sqrt{\mathrm{Ann}_R(M)}$.*

(4) *Every r-submodule is an n-submodule.*

Proof. It is clear that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

(3) $\Rightarrow$ (4) Let N be an r-submodule and $am \in N$ for $a \in R - \sqrt{\mathrm{Ann}_R(M)}$. Since $Z(M) = \sqrt{\mathrm{Ann}_R(M)}$, $\mathrm{ann}_M(a) = 0$. Hence, $m \in N$. Therefore, N is an n-submodule.

(4) $\Rightarrow$ (1) Since $\langle 0 \rangle$ is an r-submodule, it is an n-submodule. □

By [1], Proposition 2.21 and Theorem 2.3 ((3) $\Rightarrow$ (4)), we obtain the following corollary.

Corollary 2.4. *Let M be an R-module such that $Z(M) = \sqrt{\mathrm{Ann}_R(M)}$. Then, the notions of r-submodule and n-submodule coincide.*

Corollary 2.5. *Let M be a torsion-free R-module. Then, the notions of r-submodule and n-submodule coincide.*

Proposition 2.7. *In a finitely generated R-module, every n-submodule is contained in a maximal n-submodule.*

Lemma 2.2. *Let R be an integral domain. Then, $T(M) = M$ or $T(M)$ is an n-submodule.*

Proof. Let $T(M) \neq M$ and $am \in T(M)$ for $a \in R - \sqrt{\mathrm{Ann}_R(M)}$ and $m \in M$. Then, there exists $0 \neq b \in R$ such that $bam = 0$. Therefore, $m \in T(M)$. □

Remember that a nonempty subset S of R is multiplicatively closed precisely when $ab \in S$ for any $a, b \in S$. Assume M is an R-module and S is a multiplicatively closed subset of R. The fraction module at S is thus denoted by M_S. It is worth noting that M_S is both an R_S-module and an R-module. Let $f\colon M \to M_S$ be the natural homomorphism with $f(m) = m/1$ as its definition. If L is a submodule of M_S, then $f^{-1}(L)$ is always a submodule of M, which is known as the L^c of L contraction.

Theorem 2.4. *Let M be an R-module and S be a multiplicative closed subset of R. If $\langle 0\rangle$ is an n-submodule of M, then the kernel of $\varphi\colon M \to M_S$ is either $\langle 0\rangle$ or M.*

Proof. Suppose that there exists $0 \neq y \in \ker(\varphi)$. Then, there exists $s \in S$ such that $sy = 0$. Since $\langle 0\rangle$ is an n-submodule and $0 \neq y$, $s \in S \cap \sqrt{\mathrm{Ann}_R(M)}$. Therefore, $M_S = 0$ and $\ker(\varphi) = M$. □

Corollary 2.6. *Let M be a finitely generated R-module. Then, $\langle 0\rangle$ is an n-submodule of M if and only if the kernel of $\varphi\colon M \to M_S$ is either $\langle 0\rangle$ or M for every multiplicative closed subset S of R.*

Proof. By [1], Lemma 2.32 and Theorem 2.4, the proof is straightforward. □

Theorem 2.5. *Let M be a finitely generated R-module, S be a multiplicative closed subset of R, and $\langle 0\rangle$ be an n-submodule. If L is an n-submodule of M_S, then L^c is an n-submodule of M.*

Proof. Let $rm \in L^c$ for $r \notin \sqrt{\mathrm{Ann}_R(M)}$ and $m \in M$. Assume that $\frac{r}{1} \in \sqrt{\mathrm{Ann}_{R_S}(M_S)}$. Then, there exists $n \in \mathbb{N}$ such that $\left(\frac{r}{1}\right)^n \in \mathrm{Ann}_{R_S}(M_S)$. Since M is a finitely generated R-module, there exists $s \in S$ such that $sr^n M = \langle 0\rangle$. Since $\langle 0\rangle$ is an n-submodule and $r^n \notin \sqrt{\mathrm{Ann}_R(M)}$, $s \in \sqrt{\mathrm{Ann}_R(M)}$. Therefore, $M_S = \langle 0\rangle$, which is a contradiction. Hence, $\frac{r}{1} \notin \sqrt{\mathrm{Ann}_{R_S}(M_S)}$. We conclude that $\frac{r}{1}\frac{m}{1} \in L$ and $\frac{m}{1} \in L$. So, $m \in L^c$ and L^c is an n-submodule of M. □

Proposition 2.8. *Let M be an R-module, S be a multiplicative closed subset of R, and L be a submodule of M_S. If L^c is an n-submodule of M, then L is an n-submodule of M_S.*

Proof. Let $\frac{r}{s_1}\frac{m}{s_2} \in L$ for $\frac{r}{s_1} \notin \sqrt{\mathrm{Ann}_{R_S}(M_S)}$ and $\frac{m}{s_2} \in M_S$. It is clear that $r \notin \sqrt{\mathrm{Ann}_R(M)}$. Since $rm \in L^c$ and L^c is an n-submodule of M, $\frac{m}{s_2} \in L$. Hence, L is an n-submodule of M_S. □

Proposition 2.9. *Let M_1 and M_2 be R-modules, $\sqrt{\mathrm{Ann}_R(M_1)} = \sqrt{\mathrm{Ann}_R(M_2)}$ and $M = M_1 \times M_2$.*

(1) *If L_1 is an n-submodule of M_1, then $L_1 \times M_2$ is an n-submodule of M.*
(2) *If L_1 is an n-submodule of M_1 and L_2 is an n-submodule of M_2, then $L_1 \times L_2$ is an n-submodule of M.*

Proof. (1) Suppose that $r(m_1, m_2) \in L_1 \times M_2$, where $r \notin \sqrt{\mathrm{Ann}_R(M_1 \times M_2)}$ for $r \in R$ and $(m_1, m_2) \in M_1 \times M_2$. Since

$$\sqrt{\mathrm{Ann}_R(M_1)} = \sqrt{\mathrm{Ann}_R(M_2)} = \sqrt{\mathrm{Ann}_R(M_1) \cap \mathrm{Ann}_R(M_2)} = \sqrt{\mathrm{Ann}_R(M_1 \times M_2)},$$

we find that $r \notin \sqrt{\mathrm{Ann}_R(M_1)}$. It follows that $m_1 \in L_1$. So, $(m_1, m_2) \in L_1 \times M_2$.

(2) The proof is similar to that of part (1). □

In [9], Macdonald introduced the notion of secondary module. Recall that a nonzero R-module M is said to be *secondary* if for every $a \in R$, the endomorphism of M given by the multiplication by a is either surjective or nilpotent.

Lemma 2.3. *If every proper submodule of M is an n-submodule, then M is a secondary R-module.*

Proof. Assume that $r \in R$ and that $\varphi_r\colon M \to M$ is defined by $\varphi_r(m) = rm$. If φ_r is not surjective, then $\mathrm{Im}(\varphi_r) \neq M$. So, there exists $m \in M - \mathrm{Im}(\varphi_r)$. Thus, $\varphi_r(m) = rm \in \mathrm{Im}(\varphi_r)$. Since $\mathrm{Im}(\varphi_r)$ is an n-submodule, $r \in \sqrt{\mathrm{Ann}_R(M)}$. This implies that φ_r is nilpotent. □

Proposition 2.10. *Let M be an R-module and $\sqrt{\mathrm{Ann}_R(M)}$ be a finitely generated ideal of R. If every proper submodule of M is an n-submodule, then every ascending chain of its cyclic submodules stops.*

Proof. Let $Rm_1 \subset Rm_2 \subset Rm_3 \subset \ldots \subset Rm_k \subset \ldots$ be a chain of cyclic submodules of M. Then

$$m_1 = r_2 m_2 = r_2 r_3 m_3 = \ldots = r_2 \ldots r_k m_k = \ldots$$

for $r_1, r_2, \ldots \in R$. Since Rm_i is an n-submodule, $r_i \in \sqrt{\mathrm{Ann}_R(M)}$. On the other hand, since $\sqrt{\mathrm{Ann}_R(M)}$ is a finitely generated ideal of R, we conclude the existence of $n \in \mathbb{N}$ such that $(\sqrt{\mathrm{Ann}_R(M)})^n \subseteq \mathrm{Ann}_R(M)$. So, $m_1 = r_2 \ldots r_n r_{n+1} m_{n+1} = 0$. It follows that $m_i = 0$ for all $i \in \mathbb{N}$, which is a contradiction. Therefore, every ascending chain of cyclic submodules of M stops. □

Corollary 2.7. *Let M be an R-module and $\sqrt{\mathrm{Ann}_R(M)}$ be a finitely generated ideal of R. Every proper submodule of M is an n-submodule if and only if every ascending chain of cyclic submodules of M stops, and M is a secondary R-module.*

P r o o f. $\Rightarrow$) This follows from Lemma 2.3 and Proposition 2.10.

$\Leftarrow$) By [1], Proposition 2.39, the proof is straightforward. □

Corollary 2.8. *Let R be a Noetherian ring and M be a finitely generated R-module. If M is a secondary R-module, then every proper submodule of M is an n-submodule.*

Proposition 2.11. *Let M be an R-module. Then, the following statements are equivalent.*

(1) *Every proper submodule of M is an n-submodule.*

(2) *Every proper cyclic submodule of M is an n-submodule.*

P r o o f. (1) $\Rightarrow$ (2) This is clear.

(2) $\Rightarrow$ (1) Assume that K is a proper submodule of M, and that $rm \in K$ for $m \in M$ and $r \in R - \sqrt{\mathrm{Ann}_R(M)}$. Then, there exists $k \in K$ such that $rm \in Rk$. Since Rk is an n-submodule, $m \in Rk \subseteq K$. Thus, K is an n-submodule. □

Proposition 2.12. *Let M be a finitely generated R-module. Then, the following statements are equivalent.*

(1) *Every proper submodule of M is an n-submodule.*

(2) $\sqrt{\mathrm{Ann}_R(M)}$ *is a maximal ideal of R.*

P r o o f. (1) $\Rightarrow$ (2) Since M is a finitely generated R-module, M has a maximal submodule N. So, $(N :_R M)$ is a maximal ideal of R. By Proposition 2.4, $(N :_R M) \subseteq \sqrt{\mathrm{Ann}_R(M)}$. Therefore, $\sqrt{\mathrm{Ann}_R(M)}$ is a maximal ideal of R.

(2) $\Rightarrow$ (1) Let $\sqrt{\mathrm{Ann}_R(M)}$ be a maximal ideal of R and N be a proper submodule of M. Suppose that $rm \in N$, where $r \in R - \sqrt{\mathrm{Ann}_R(M)}$ and $m \in M$. Since $\sqrt{\mathrm{Ann}_R(M)}$ is a maximal ideal of R, $\mathrm{Ann}_R(M) + \langle r \rangle = R$. Hence, there exist $s \in \mathrm{Ann}_R(M)$ and $t \in R$ such that $s + tr = 1$. So, $m = sm + trm = trm \in N$. It follows that N is an n-submodule. □

Corollary 2.9. *Let $\mathrm{Ann}_R(M)$ be a maximal ideal of R. Then, every proper submodule of M is an n-submodule.*

Corollary 2.10. *Let M be a vector space. Then, every proper submodule of M is an n-submodule.*

Theorem 2.6. *Let M be an R-module. Every proper submodule of M is an n-submodule if and only if for every submodule N of M and for every $a \in R - \sqrt{\mathrm{Ann}_R(M)}$, $aN = N$ holds.*

P r o o f. $\Rightarrow$) Suppose that every proper submodule of M is an n-submodule, and $a \in R - \sqrt{\mathrm{Ann}_R(M)}$. We show that $aM = M$. If $aM \neq M$, then aM is an n-submodule. Since $am \in aM$ for all $m \in M$, $m \in aM$ and so $aM = M$, which is a contradiction. Similarly, it can be shown that $aN = N$ for every submodule N of M.

$\Leftarrow$) Let N be a proper submodule of M, and $am \in N$ for $a \in R$ and $m \in M$, where $a \notin \sqrt{\mathrm{Ann}_R(M)}$. Since $Rm = a(Rm)$, there exists $r \in R$ such that $m = ram$. It follows that $m \in N$. $\square$

An R-module M is called a *comultiplication module* if for every submodule N of M, there exists an ideal I of R such that $N = (0 :_M I)$, or equivalently, $N = (0 :_M \mathrm{Ann}_R(N))$, see [2].

Proposition 2.13. *Let M be a comultiplication module and*

$$\sqrt{\mathrm{Ann}_R(M)} = \mathrm{Ann}_R(M).$$

If M has an n-submodule, then the following statements are true.
(1) *Every n-submodule is maximal.*
(2) $\langle 0\rangle$ *is an n-submodule.*
(3) *M is a simple module.*

P r o o f. (1) Let N be an n-submodule and K be a proper submodule of M such that $N \subseteq K$. Then, $\mathrm{Ann}_R(M) \subset \mathrm{Ann}_R(K)$. So, there exists some a in $\mathrm{Ann}_R(K) - \sqrt{\mathrm{Ann}_R(M)}$. Suppose that $k \in K$. Then, $ak = 0 \in N$ and so, $k \in N$. Therefore, N is a maximal submodule of M.

(2) Let N be an n-submodule of M. By (1), N is a maximal submodule of M. So, $(N :_R M)$ is a maximal ideal of R. Hence, $\sqrt{\mathrm{Ann}_R(M)}$ is a maximal ideal of R. It follows that $\langle 0\rangle$ is an n-submodule.

(3) By (1) and (2), $\langle 0\rangle$ is a maximal submodule of M. Therefore, M is simple. $\square$

Proposition 2.14. *Suppose that $N_1, N_2, \ldots, N_n$ are primary submodules of M such that the radicals $\sqrt{(N_i :_R M)}$ are not comparable. Then, $\bigcap_{i=1}^{n} N_i$ is an n-submodule if and only if N_i is an n-submodule for each $i \in \{1, 2, \ldots, n\}$.*

P r o o f. $\Rightarrow$) Let $am \in N_k$ for $a \in R$ and $m \in M$, where $a \notin \sqrt{\mathrm{Ann}_R(M)}$ and $1 \leqslant k \leqslant n$. Since the radicals $\sqrt{(N_i :_R M)}$ are not comparable, there exists some r in $\bigcap_{i=1, i\neq k}^{n} \sqrt{(N_i :_R M)} - \sqrt{(N_k :_R M)}$. So, there exists $t \in \mathbb{N}$ such that $r^t am \in \bigcap_{i=1}^{n} N_i$. It follows that $r^t m \in N_k$ for some k. Thus, $m \in N_k$.

$\Leftarrow$) This follows from [1], Proposition 2.3 (ii). $\square$

Theorem 2.7. *Let $\{P_\alpha\}_{\alpha\in I}$ be a family of prime submodules of M. If $\bigcap_{\alpha\in I} P_\alpha$ is an n-submodule, then $\bigcap_{\alpha\in I} P_\alpha$ is a prime submodule.*

Proof. Let $am \in \bigcap_{\alpha\in I} P_\alpha$, where $a \in R$ and $m \in M$. If $a \notin \left(\bigcap_{\alpha\in I} P_\alpha : M\right)$, then $a \notin \sqrt{\mathrm{Ann}_R(M)}$. Since $\bigcap_{\alpha\in I} P_\alpha$ is an n-submodule, $m \in \bigcap_{\alpha\in I} P_\alpha$. It follows that $\bigcap_{\alpha\in I} P_\alpha$ is a prime submodule. □

Theorem 2.8. *Let $\{P_\alpha\}_{\alpha\in I}$ be a family of primary submodules of M. If $\bigcap_{\alpha\in I} P_\alpha$ is an n-submodule, then $\bigcap_{\alpha\in I} P_\alpha$ is a primary submodule.*

Proof. Let $am \in \bigcap_{\alpha\in I} P_\alpha$, where $a \in R$ and $m \in M$. If $a \notin \sqrt{\left(\bigcap_{\alpha\in I} P_\alpha : M\right)}$, then $a \notin \sqrt{\mathrm{Ann}_R(M)}$. Since $\bigcap_{\alpha\in I} P_\alpha$ is an n-submodule, $m \in \bigcap_{\alpha\in I} P_\alpha$. It follows that $\bigcap_{\alpha\in I} P_\alpha$ is a primary submodule. □

Lemma 2.4. *Let M be a finitely generated R-module and N be an n-submodule. Then,* $\mathrm{rad}(N)$ *is an n-submodule if and only if* $\mathrm{rad}(N)$ *is a prime submodule.*

Proof. $\Rightarrow$) By Theorem 2.7, $\mathrm{rad}(N)$ is a prime submodule.
$\Leftarrow$) Suppose that $am \in \mathrm{rad}(N)$, where $a \notin \sqrt{\mathrm{Ann}_R(M)}$ and $m \in M$. Since N is an n-submodule, by Proposition 2.4, $(N :_R M) \subseteq \sqrt{\mathrm{Ann}_R(M)}$. This implies that $\sqrt{(N :_R M)} = \sqrt{\mathrm{Ann}_R(M)}$. Since $(\mathrm{rad}(N) : M) = \sqrt{(N :_R M)} = \sqrt{\mathrm{Ann}_R(M)}$, $m \in \mathrm{rad}(N)$. Hence, $\mathrm{rad}(N)$ is an n-submodule. □

Proposition 2.15. *Let M and K be R-modules such that $M \subseteq K$ and $\sqrt{\mathrm{Ann}_R(M)} = \sqrt{\mathrm{Ann}_R(K)}$. If N is an n-submodule of M, then there exists an n-submodule L of K such that $N = L \cap M$.*

Proof. Put

$$\mathfrak{A} = \{T\colon\ T \leqslant K \text{ and } T \cap M = N\}.$$

Since $N \in \mathfrak{A}$, $\mathfrak{A}$ is not empty. By using Zorn's lemma, we find a maximal element L of $\mathfrak{A}$. Now, we show that L is an n-submodule of K. Suppose that $rk \in L$ for some $r \in R - \sqrt{\mathrm{Ann}_R(K)}$ and $k \in K$. Assume that $k \notin L$. Since L is a maximal element of $\mathfrak{A}$, $(L + \langle k\rangle) \cap M \nsubseteq N$. So, there exist $l \in L$, $m \in M - N$ and $s \in R$ such that $l + sk = m$. Since $rl + rsk = rm \in N$ and $r \notin \sqrt{\mathrm{Ann}_R(M)}$, $m \in N$, which is a contradiction. Therefore, L is an n-submodule. □

Proposition 2.16. *Let N be an n-submodule, and L be a prime submodule of an R-module M such that $(L :_R M) \subseteq \sqrt{\mathrm{Ann}_R(M)}$. Then, $N \cap L$ is an n-submodule.*

Lemma 2.5. *Let M be an R-module. Suppose that K is an n-submodule of M, L is a primary submodule of M, and $K \nsubseteq L$. Then, $K \cap L$ is an n-submodule of M if and only if $(L :_R M) \subseteq \sqrt{\mathrm{Ann}_R(M)}$.*

Proof. Let $K \cap L$ be an n-submodule of M. Since $K \nsubseteq L$, there exists some k in $K - L$. Assume that $r \in (L :_R M) - \sqrt{\mathrm{Ann}_R(M)}$. Then, $rk \in K \cap L$ and since $K \cap L$ is an n-submodule, $k \in K \cap L \subseteq L$, which is a contradiction. So, $(L :_R M) \subseteq \sqrt{\mathrm{Ann}_R(M)}$. Now, for the converse, let $(L :_R M) \subseteq \sqrt{\mathrm{Ann}_R(M)}$. By Theorem 2.2, L is an n-submodule. By [1], Proposition 2.3 (ii), $K \cap L$ is an n-submodule of M. $\square$

Proposition 2.17. *Let N be an n-submodule and L be an r-submodule of an R-module M. Then, $N \cap L$ is an r-submodule.*

Proposition 2.18. *If $\langle 0\rangle$ is the only r-submodule of an R-module M, then $\langle 0\rangle$ is an n-submodule.*

Proof. Let $rm = 0$ for $r \in R - \sqrt{\mathrm{Ann}_R(M)}$ and $m \in M$. Then, $r \notin \mathrm{Ann}_R(M)$ and hence by [7], Corollary 1, $\mathrm{Ann}_M(r) = \langle 0\rangle$. Since $\langle 0\rangle$ is an r-submodule, it follows that $m = 0$. Therefore, $\langle 0\rangle$ is an n-submodule. $\square$

Proposition 2.19. *Let M be an R-module, and S be a multiplicative closed subset of R such that $R - \sqrt{\mathrm{Ann}_R(M)} \subseteq S$. If S^* is an S-closed subset of M and N is a submodule of M such that $N \cap S^* = \emptyset$, then there exists an n-submodule L of M such that $N \subseteq L$ and $L \cap S^* = \emptyset$.*

Proof. Put $\Omega = \{L\colon\ N \subseteq L \leqslant M;\ L \cap S^* = \emptyset\}$. Since $N \in \Omega$, Ω is a nonempty set and by Zorn's lemma, it has a maximal element like L. Since $L \cap S^* = \emptyset$, L is a proper submodule of M. Assume that L is not an n-submodule. Then, there exist $r \in R - \sqrt{\mathrm{Ann}_R(M)}$ and $m \in M - L$ such that $rm \in L$. Since L is maximal in Ω and $L \subsetneqq (L :_M r)$, we deduce that $(L :_M r) \notin \Omega$. So, there exists $y \in S^*$ such that $ry \in L$. Now, since S^* is S-closed and $r \in S$, $ry \in L \cap S^*$, which is contradiction. Therefore, L is an n-submodule. $\square$

Proposition 2.20. *Let N be a submodule of an R-submodule of M. Then, N is an n-submodule of M if and only if $N[x]$ is an n-submodule of $M[x]$.*

Proof. Let N be an n-submodule of M. By [1], Proposition 2.41, $N[x]$ is an n-submodule of $M[x]$. Now, let $N[x]$ be an n-submodule of $M[x]$ and $rm \in N$ for $r \in R - \sqrt{\mathrm{Ann}_R(M)}$ and $m \in M$. Then, $rm \in N[x]$ and since $N[x]$ is an n-submodule, $m \in N[x]$. Therefore, $m \in N$ and so, N is an n-submodule. $\square$

Proposition 2.21. *Suppose that $R = R_1 \times R_2 \times \ldots \times R_n$ and $M = M_1 \times M_2 \times \ldots \times M_n$, where M_i is a nonzero R_i-module for $1 \leqslant i \leqslant n$ and $n \geqslant 2$. Then, M has no n-submodules.*

Proof. Assume that N is an n-submodule of M. Since $N \neq M$, there exists j, $1 \leqslant j \leqslant n$, such that $(0, \ldots, 0, \underbrace{m}_{j\text{th}}, 0, \ldots, 0) \in M - N$. Then,

$$(1, \ldots, 1, \underbrace{0}_{j\text{th}}, 1, \ldots, 1)(0, \ldots, 0, \underbrace{m}_{j\text{th}}, 0, \ldots, 0) \in N.$$

So, $(1, \ldots, 1, \underbrace{0}_{j\text{th}}, 1, \ldots, 1) \in R - \sqrt{\mathrm{Ann}_R(M)}$ and since N is an n-submodule, $(0, \ldots, 0, \underbrace{m}_{j\text{th}}, 0, \ldots, 0) \in N$, which is a contradiction. □

Let M be a module over a commutative ring R, and N be a proper submodule of M. We say that N has the *star property* if $a \in R$, $m \in M$, $am \in N$ and $a \notin \mathrm{Ann}_R(M)$ imply $m \in N$.

Proposition 2.22. *Let N be a submodule of an R-module M that has the star property. Then, N is an n-submodule.*

Proposition 2.23. *Let N be a submodule of an R-module M that has the star property. Then, $(N :_R M) = \mathrm{Ann}_R(M)$.*

Proof. Assume that $(N :_R M) \nsubseteq \mathrm{Ann}_R(M)$. Then, there exists $r \in (N :_R M)$ such that $r \notin \mathrm{Ann}_R(M)$. Thus, $rM \subseteq N$ and since N is an n-submodule, we conclude that $N = M$, a contradiction. Hence, $(N :_R M) \subseteq \mathrm{Ann}_R(M)$. We have $\mathrm{Ann}_R(M) \subseteq (N :_R M)$. So, $(N :_R M) = \mathrm{Ann}_R(M)$. □

Lemma 2.6. *Let N be a submodule of an R-module M that has the star property. Then, N is a prime submodule.*

Proof. The proof is straightforward. □

Lemma 2.7. *Let N be a submodule of an R-module M that has the star property. Then, $\mathrm{Ann}_R(M)$ is a prime ideal.*

Proof. By Proposition 2.23, $(N :_R M) = \mathrm{Ann}_R(M)$ and by Lemma 2.6, N is a prime submodule. Therefore, $(N :_R M)$ is a prime ideal. Hence, $\mathrm{Ann}_R(M)$ is a prime ideal. □

Lemma 2.8. *Let N be a submodule of an R-module M that has the star property. If K is an n-submodule, then the following statements are true.*

(1) *For every $a \in R$ and $m \in M$, $am \in K$ and $a \notin \operatorname{Ann}_R(M)$ imply $m \in K$.*

(2) *K is a prime submodule.*

(3) $(K : M) = \operatorname{Ann}_R(M)$.

3. Generalization of n-submodules

In this section, we introduce a new class of submodules, namely, the class of $G.n$-submodules. The notion of $G.n$-submodule is a generalization of the notion of n-submodule. We present some characterizations of $G.n$-submodules, and we examine the way the aforementioned notions are related to each other.

Definition 3.1. Let M be a module over a commutative ring R. A proper submodule N of M is said to be a *generalization of n-submodule* (*$G.n$-submodule*) if for $a \in R$ and $m \in M$, $am \in N$ and $a \notin \sqrt{\operatorname{Ann}_R(N)}$ imply $m \in N$.

Example 3.1.

(1) Suppose that R is a ring that has only one prime ideal. Then, every proper submodule of the R-module R is a $G.n$-submodule.

(2) As a $\mathbb{Z}$-module, $\mathbb{Z}_6$ does not have any $G.n$-submodules.

(3) In $\mathbb{Z}_4$ as a $\mathbb{Z}$-module, $\langle\overline{2}\rangle$ is a $G.n$-submodule.

(4) In $\mathbb{Z}_4 \bigoplus \mathbb{Z}$ as a $\mathbb{Z}$-module, $\langle\overline{2}\rangle \bigoplus \langle 0\rangle$ is a $G.n$-submodule.

Proposition 3.1. *Every n-submodule is a $G.n$-submodule.*

It is clear that in general, a $G.n$-submodule is not necessarily an n-submodule, see Example 3.1 (4).

Lemma 3.1. *In a torsion-free R-module, the notions of n-submodule and $G.n$-submodule coincide.*

Lemma 3.2. *Let M be an R-module, and 0 be an n-submodule. Then, the notions of n-submodule and $G.n$-submodule coincide.*

Proof. Let N be a $G.n$-submodule and $am \in N$ for $a \in R$ and $m \in M$ with $a \notin \sqrt{\operatorname{Ann}_R(M)}$. If $a \notin \sqrt{\operatorname{Ann}_R(N)}$, then $m \in N$. If $a \in \sqrt{\operatorname{Ann}_R(N)}$, then there exists $k \in \mathbb{N}$ such that $a^k \in \operatorname{Ann}_R(N)$. So, $a^{k+1}m = 0$. Since 0 is an n-submodule, $m = 0$. Therefore, $m \in N$. This implies that N is an n-submodule. $\square$

Theorem 3.1. *Let M be an R-module, and N be a proper submodule of M. Then, the following statements are equivalent.*

(1) *N is a $G.n$-submodule of M.*

(2) *$N = (N :_M a)$ for every $a \notin \sqrt{\mathrm{Ann}_R(N)}$.*

(3) *For any ideal I of R and any submodule K of M, $IK \subseteq N$ with $I \not\subseteq \sqrt{\mathrm{Ann}_R(N)}$ implies $K \subseteq N$.*

Proof. (1) $\Rightarrow$ (2) Let N be a $G.n$-submodule of M. For every $a \in R$, the inclusion $N \subseteq (N :_M a)$ always holds. Let $a \notin \sqrt{\mathrm{Ann}_R(N)}$ and $m \in (N :_M a)$. Then, $am \in N$. Since N is a $G.n$-submodule, we conclude that $m \in N$ and thus, $N = (N :_M a)$.

(2) $\Rightarrow$ (3) Suppose that $IK \subseteq N$ for an ideal I of R and a submodule K of M, where $I \not\subseteq \sqrt{\mathrm{Ann}_R(N)}$. Since $I \not\subseteq \sqrt{\mathrm{Ann}_R(N)}$, there exists $a \in I$ such that $a \notin \sqrt{\mathrm{Ann}_R(N)}$. Then, $aK \subseteq N$ and so $K \subseteq (N :_M a) = N$, by (2).

(3) $\Rightarrow$ (1) Let $am \in N$ for $a \in R$ and $m \in M$, where $a \notin \sqrt{\mathrm{Ann}_R(N)}$. To complete the proof of the desired result, it is sufficient to take $I := Ra$ and $K := Rm$. □

Proposition 3.2.

(1) *If N is a $G.n$-submodule of M, then $(N :_R M) \subseteq \sqrt{\mathrm{Ann}_R(N)}$.*

(2) *If N is a $G.n$-submodule of M, then $(N :_R M) \subseteq \sqrt{\mathrm{Ann}_R(M)}$.*

(3) *Let $\{N_i\}_{i \in I}$ be a nonempty set of $G.n$-submodules of an R-module M. Then, $\bigcap_{i \in I} N_i$ is a $G.n$-submodule.*

(4) *Let $\{N_i\}_{i \in I}$ be a finite chain of $G.n$-submodules of a finitely generated R-module M. Then, $\bigcup_{i \in I} N_i$ is a $G.n$-submodule of M.*

Proof. (1) Assume that N is a $G.n$-submodule, but $(N :_R M) \not\subseteq \sqrt{\mathrm{Ann}_R(N)}$. Then, there exists $r \in (N :_R M)$ such that $r \notin \sqrt{\mathrm{Ann}_R(N)}$. Thus, $rM \subseteq N$ and since N is a $G.n$-submodule, we conclude that $N = M$, a contradiction. Hence, $(N :_R M) \subseteq \sqrt{\mathrm{Ann}_R(N)}$.

(2) By (1), $(N :_R M) \subseteq \sqrt{\mathrm{Ann}_R(N)}$. Let $r \in (N :_R M)$. Then, $rM \subseteq N$. Since N is a $G.n$-submodule, $r \in \sqrt{\mathrm{Ann}_R(N)}$. Therefore, there exists $k \in \mathbb{N}$ such that $r^k \in \mathrm{Ann}_R(N)$. Hence, $r^{(k+1)}M = 0$. This implies that $r \in \sqrt{\mathrm{Ann}_R(M)}$.

(3) Let $rm \in \bigcap_{i \in I} N_i$ for $r \in R$ and $m \in M - \bigcap_{i \in I} N_i$. Then $m \notin N_j$ for some $j \in I$. Since N_j is a $G.n$-submodule of M, we obtain $r \in \sqrt{\mathrm{Ann}_R N_j} \subseteq \sqrt{\mathrm{Ann}_R(\bigcap_{i \in I} N_i)}$. So, $\bigcap_{i \in I} N_i$ is a $G.n$-submodule.

(4) Let $rm \in \bigcup_{i\in I} N_i$ for $r \in R$ and $m \in M - \bigcup_{i\in I} N_i$. Then, $m \notin N_i$ for any $i \in I$. Since N_i is a $G.n$-submodule for any $i \in I$, we conclude that $r \in \sqrt{\mathrm{Ann}_R(N_i)}$ and thus, the fact that I is a finite set implies $r \in \sqrt{\mathrm{Ann}_R\Big(\bigcup_{i\in I} N_i\Big)}$. Therefore, $\bigcup_{i\in I} N_i$ is a $G.n$-submodule. □

Proposition 3.3. *Let K and L be submodules of M, and I be an ideal of R such that $I \nsubseteq \sqrt{\mathrm{Ann}_R(K)} \cup \sqrt{\mathrm{Ann}_R(L)}$. Then, the following statements are true.*

(1) *If K and L are $G.n$-submodules of M with $IK = IL$, then $K = L$.*

(2) *If IK is a $G.n$-submodule of M, then $IK = K$.*

Proof. (1) Since K is a $G.n$-submodule and $IL \subseteq K$, Theorem 3.1 shows that $L \subseteq K$. Likewise, $K \subseteq L$.

(2) Since IK is a $G.n$-submodule and $IK \subseteq IK$, we conclude that $K \subseteq IK$. This completes the proof. □

By Lemma 3.1, the next lemmas provide a useful characterization of modules that have $G.n$-submodules.

Lemma 3.3. *Let M be a torsion-free R-module. Then, the zero submodule is a $G.n$-submodule of M.*

Lemma 3.4. *Let M be a multiplication R-module.*

(1) *If M is torsion-free, then the zero submodule is the only $G.n$-submodule of M.*

(2) *If $\mathrm{Ann}_R(M)$ is a radical ideal, then the zero submodule is the only $G.n$-submodule of M.*

Proposition 3.4. *Let N be a proper submodule of M. Then, N is a $G.n$-submodule if and only if for every $m \in M$, $(N :_R m) = R$ or $(N :_R m) \subseteq \sqrt{\mathrm{Ann}_R(N)}$.*

Proof. Assume that N is a $G.n$-submodule. If $(N :_R m) \nsubseteq \sqrt{\mathrm{Ann}_R(N)}$, then there exists $r \in (N :_R m) - \sqrt{\mathrm{Ann}_R(N)}$. So, $rm \in N$, where $r \notin \sqrt{\mathrm{Ann}_R(N)}$. Since N is a $G.n$-submodule, then $m \in N$. Hence, $(N :_R m) = R$. Conversely, let $rm \in N$ for $r \in R$ and $m \in M$, where $r \notin \sqrt{\mathrm{Ann}_R(N)}$. Then, $r \in (N :_R m) - \sqrt{\mathrm{Ann}_R(N)}$. By the assumption, $(N :_R m)$ and therefore, $m \in N$. □

Corollary 3.1. *Let N be a proper submodule of M. Then, N is a $G.n$-submodule if and only if for every $m \in M - N$, $(N :_R m) \subseteq \sqrt{\mathrm{Ann}_R(N)}$.*

Recall that $r \in R$ is said to be a *zero divisor* of an R-module M if there exists a nonzero element $m \in M$ such that $rm = 0$.

Theorem 3.2. *Let M be an R-module and N be a submodule of M. Then, N is a $G.n$-submodule if and only if every zero divisor of the R-module $\frac{M}{N}$ is in $\sqrt{\mathrm{Ann}_R(N)}$.*

Proof. Let N be a $G.n$-submodule and r be a zero divisor of $\frac{M}{N}$. Then, there exists $m \in M - N$ such that $rm \in N$. Since N is a $G.n$-submodule, $r \in \sqrt{\mathrm{Ann}_R(N)}$. For the converse, assume that $rm \in N$ for $r \in R$ and $m \in M$, where $m \notin N$. Then, r is a zero divisor of $\frac{M}{N}$ and thus $r \in \sqrt{\mathrm{Ann}_R(N)}$. □

Proposition 3.5. *A prime submodule N of M is a $G.n$-submodule if and only if $(N :_R M) \subseteq \sqrt{\mathrm{Ann}_R(N)}$.*

Proof. Suppose that N is a prime submodule of M. If N is a $G.n$-submodule, then by Proposition 3.2 (1), $(N :_R M) \subseteq \sqrt{\mathrm{Ann}_R(N)}$. For the converse, assume that $(N :_R M) \subseteq \sqrt{\mathrm{Ann}_R(N)}$. Now, we show that N is a $G.n$-submodule. Let $am \in N$ and $a \notin \sqrt{\mathrm{Ann}_R(N)}$ for $a \in R$ and $m \in M$. Since N is a prime submodule and $a \notin (N :_R M)$, we find that $m \in N$ and thus N is a $G.n$-submodule. □

Lemma 3.5. *Let N be a $G.n$-submodule of an R-module M such that $(N :_R M) = \sqrt{\mathrm{Ann}_R(N)}$. Then, N is a prime submodule.*

Proof. The proof is straightforward. □

Proposition 3.6. *Every $G.n$-submodule is an r-submodule.*

Proof. Let N be a $G.n$-submodule of M, and $am \in N$ for some $a \in R$ and $m \in M$, with $\mathrm{ann}_M(a) = 0$. Assume that $a \in \sqrt{\mathrm{Ann}_R(N)}$. Then, there exists $n \in \mathbb{N}$ such that $a^n N = 0$. Choose the smallest positive integer n such that $a^n N = 0$. Then, $a^{n-1} N \neq 0$. Since $a(a^{n-1}N) = a^n N = 0$, $a^{n-1}N \subseteq \mathrm{ann}_M(a) = 0$ and so, $a^{n-1}N = 0$, which is a contradiction. Thus, $a \notin \sqrt{\mathrm{Ann}_R(N)}$. Since N is a $G.n$-submodule and $am \in N$, we conclude that $m \in N$. Hence, N is an r-submodule of M. □

Lemma 3.6. *Let M and N be R-modules such that $N \subseteq M$. If L is a $G.n$-submodule of N and N is a $G.n$-submodule of M, then L is a $G.n$-submodule of M.*

Proof. Let $am \in L$ for $a \in R - \sqrt{\mathrm{Ann}_R(L)}$ and $m \in M$. Since N is a $G.n$-submodule of M and $\sqrt{\mathrm{Ann}_R(N)} \subseteq \sqrt{\mathrm{Ann}_R(L)}$, then $m \in N$. Since L is a $G.n$-submodule of N and $m \in N$, then $m \in L$. □

Proposition 3.7. *Suppose that $R = R_1 \times R_2 \times \ldots \times R_n$ and $M = M_1 \times M_2 \times \ldots \times M_n$, where M_i is a nonzero R_i-module for $1 \leqslant i \leqslant n$. If N is a $G.n$-submodule of M, then there exists j, $1 \leqslant j \leqslant n$, such that $N = N_1 \times N_2 \times \ldots \times N_n$, N_j is a $G.n$-submodule and for any i with $i \neq j$, $N_i = 0$.*

P r o o f. Assume that N is a $G.n$-submodule. Since $N \neq M$, there exists j, $1 \leqslant j \leqslant n$, such that $(0, \ldots, 0, \underbrace{m}_{j\text{th}}, 0, \ldots, 0) \in M - N$. Then,

$$(1, \ldots, 1, \underbrace{0}_{j\text{th}}, 1, \ldots, 1)(0, \ldots, 0, \underbrace{m}_{j\text{th}}, 0, \ldots, 0) \in N.$$

So, $(1, \ldots, 1, \underbrace{0}_{j\text{th}}, 1, \ldots, 1) \in \sqrt{\mathrm{Ann}_R(N)}$. This implies that $N = 0 \times \ldots \times 0 \times \underbrace{K}_{j\text{th}} \times 0 \times \ldots \times 0$, where K is a submodule of M_j. We have $\mathrm{Ann}_R(N) = R_1 \times \ldots \times R_{j-1} \times \mathrm{Ann}_{R_j}(K) \times R_{j+1} \times \ldots \times R_n$. Assume that $am \in K$ for $a \in R_j - \sqrt{\mathrm{Ann}_{R_j}(K)}$ and $m \in M_j$. Since $(0, \ldots, 0, \underbrace{a}_{j\text{th}}, 0, \ldots, 0)(0, \ldots, 0, \underbrace{m}_{j\text{th}}, 0, \ldots, 0) \in N$, $(0, \ldots, 0, \underbrace{m}_{j\text{th}}, 0, \ldots, 0) \in N$. It follows that $m \in K$. Hence, K is a $G.n$-submodule of M_j. □

Proposition 3.8. *Let M be an R-module. If N is a $G.n$-submodule of M such that $(N :_R M) \neq \mathrm{Ann}_R(N)$, then $(N :_M \mathrm{Ann}_R(N)^k)$ is a $G.n$-submodule of M for $k \in \mathbb{N}$.*

P r o o f. We have $(\mathrm{Ann}_R(N))^{k+1} \subseteq \mathrm{Ann}_R((N :_M (\mathrm{Ann}_R(N))^k)) \subseteq \mathrm{Ann}_R(N)$. Let $am \in (N :_M (\mathrm{Ann}_R(N))^k)$ for $a \in R$ and $m \in M$, with

$$a \notin \sqrt{\mathrm{Ann}_R((N :_M (\mathrm{Ann}_R(N))^k))}.$$

Then, $a(\mathrm{Ann}_R(N))^k m \subseteq N$ and since N is a $G.n$-submodule, $(\mathrm{Ann}_R(N))^k m \subseteq N$. Hence, $m \in (N :_M (\mathrm{Ann}_R(N))^k)$. □

Proposition 3.9. *Let M be an R-module and R be an Artinian ring. If every proper submodule of M is a $G.n$-submodule, then every ascending chain of its cyclic submodules stops.*

P r o o f. Let $Rm_1 \subset Rm_2 \subset Rm_3 \subset \ldots \subset Rm_k \subset \ldots$ be a chain of cyclic submodules of M. Then

$$m_1 = r_2 m_2 = r_2 r_3 m_3 = \ldots = r_2 \ldots r_k m_k = \ldots,$$

for $r_1, r_2, \ldots \in R$. Since Rm_i is a $G.n$-submodule, $r_i \in \sqrt{\mathrm{Ann}_R(m_{i-1})}$. On the other hand, since $\sqrt{\mathrm{Ann}_R(m_i)} \subseteq \sqrt{\mathrm{Ann}_R(m_{i-1})}$, we conclude the existence of $n \in \mathbb{N}$ such that $\sqrt{\mathrm{Ann}_R(m_i)} = \sqrt{\mathrm{Ann}_R(m_n)}$ for all $n \leqslant i$. So, $m_1 = r_2 \ldots r_n r_{n+1} m_{n+1} = 0$. It follows that $m_i = 0$ for all $i \in \mathbb{N}$, which is a contradiction. Therefore, every ascending chain of cyclic submodules of M stops. □

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