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ON THE REGULARITY OF BILINEAR MAXIMAL OPERATOR

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Abstract. We study the regularity properties of bilinear maximal operator. Some new bounds and continuity for the above operators are established on the Sobolev spaces, Triebel-Lizorkin spaces and Besov spaces. In addition, the quasicontinuity and approximate differentiability of the bilinear maximal function are also obtained.

Keywords: bilinear maximal operator; Triebel-Lizorkin space; Besov space; Lipschitz space; p-quaiscontinuous; approximate differentiability

MSC 2020: 42B25, 46E35

1. Introduction

In recent years there has been considerable interest in investigating the regularity properties of various maximal operators. The first work was due to Kinnunen (see [9]) who proved that the usual centered Hardy–Littlewood maximal operator M is bounded in the first order Sobolev space $W^{1,p}(\mathbb{R}^n)$ for $1 < p \leqslant \infty$, where $n \geqslant 1$ and $W^{1,p}(\mathbb{R}^n)$ is defined as

$$W^{1,p}(\mathbb{R}^n) := \{f\colon \mathbb{R}^n \to \mathbb{R}\colon \|f\|_{W^{1,p}(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)} + \|\nabla f\|_{L^p(\mathbb{R}^n)} < \infty\},$$

where $\nabla f = (D_1 f, \dots, D_n f)$ is the weak gradient of f. Kinnunen's result was later extended to various variants. For example, see [10] for the local case, [11] for the fractional case, [1] for the bilinear case, [16] for the multisublinear case. Since the maximal operator lacks the sublinearity at the derivative level, the continuity of $M\colon W^{1,p}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$ for $1 < p < \infty$ was posed by Hajłasz and Onninen (see [7]) and addressed by Luiro, see [18]. On the other hand, the regularity properties

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of maximal operators on other smooth function spaces have been studied by many authors. Korry in [12], [13] established the boundedness of M on the fractional Sobolev spaces $W^{s,p}(\mathbb{R}^n)$, the inhomogeneous Triebel-Lizorkin spaces $F_s^{p,q}(\mathbb{R}^n)$ and inhomogeneous Besov spaces $B_s^{p,q}(\mathbb{R}^n)$ for $0 < s < 1$ and $1 < p, q < \infty$. The continuity of the Hardy-Littlewood maximal operator on the inhomogeneous Triebel-Lizorkin (or Besov) spaces was proved by Luiro (see [19]) or Liu and Wu, see [17]. It should be pointed out that in the bilinear setting, the regularity properties for the maximal operators are more complex and refined. Particularly, Carneiro and Moreira in [1] firstly considered the Sobolev regularity of the bilinear maximal operator

$$\mathfrak{M}(f,g)(x) = \sup_{r>0} \frac{1}{|B(O,r)|} \int_{B(O,r)} |f(x+y)g(x-y)|\,\mathrm{d}y,$$

where $x \in \mathbb{R}^n$ and $O = (0,0,\ldots,0) \in \mathbb{R}^n$. This type of maximal operator $\mathfrak{M}$ was originally introduced by Calderón and was studied by Lacey, see [14]. The main result of [1] can be listed as follows.

Theorem A ([1]). *Let $1 < p_1, p_2 < \infty$, $1 \leqslant p < \infty$ and $1/p = 1/p_1 + 1/p_2$. The map $\mathfrak{M}\colon W^{1,p_1}(\mathbb{R}^n) \times W^{1,p_2}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$ is bounded and continuous. In particular, if $f \in W^{1,p_1}(\mathbb{R}^n)$ and $g \in W^{1,p_2}(\mathbb{R}^n)$, then the following pointwise estimate holds:*

$$|\nabla \mathfrak{M}(f,g)(x)| \leqslant \mathfrak{M}(|\nabla f|, g)(x) + \mathfrak{M}(f, |\nabla g|)(x)$$

for almost every $x \in \mathbb{R}^n$.

Recently, Liu, Liu and Zhang in [15] established the following results.

Theorem B ([15]). *Let $0 < s < 1$, $1 < p_1, p_2, p, q < \infty$ and $1/p = 1/p_1 + 1/p_2$. Then both the maps $\mathfrak{M}\colon F_s^{p_1,q}(\mathbb{R}^n) \times F_s^{p_2,q}(\mathbb{R}^n) \to F_s^{p,q}(\mathbb{R}^n)$ and $\mathfrak{M}\colon B_s^{p_1,q}(\mathbb{R}^n) \times B_s^{p_2,q}(\mathbb{R}^n) \to B_s^{p,q}(\mathbb{R}^n)$ are bounded and continuous.*

From Theorems A and B we see that $\mathfrak{M}(f,g)$ enjoys better regularity properties if both f and g belong to the same kind of smooth function spaces. It is natural and interesting to ask whether the bilinear maximal operator enjoys better regularity properties when the above operator acts on two distinct functions from smooth function spaces. This is our main motivation for this work.

In order to formulate our main results, let us introduce Lipschitz space.

Definition 1.1. Let $0 < \gamma \leqslant 1$. The *homogeneous* Lipschitz space $Lip_\gamma(\mathbb{R}^n)$ is defined as

$$Lip_\gamma(\mathbb{R}^n) := \{f\colon \mathbb{R}^n \to \mathbb{C} \text{ continuous } \|f\|_{Lip_\gamma(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{Lip_\gamma(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n} \sup_{h\in\mathbb{R}^n\setminus\{0\}} \frac{|f(x+h)-f(x)|}{|h|^\gamma}.$$

The *inhomogeneous* Lipschitz space $\mathrm{Lip}_\gamma(\mathbb{R}^n)$ is given by

$$\mathrm{Lip}_\gamma(\mathbb{R}^n) := \{f\colon \mathbb{R}^n \to \mathbb{C} \text{ continuous}\colon \|f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)} := \|f\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{Lip_\gamma(\mathbb{R}^n)} < \infty\}.$$

Remark 1.2. By Rademacher's theorem, if $f \in Lip_1(\mathbb{R}^n)$, then the weak partial derivatives $D_i b$, $i = 1, \ldots, n$, exist almost everywhere. Moreover, we have

$$D_i b(x) = \lim_{h\to 0} \frac{b(x+he_i)-b(x)}{h} \quad \text{and} \quad |D_i b(x)| \leqslant \|b\|_{Lip_1(\mathbb{R}^n)}$$

for almost every $x \in \mathbb{R}^n$. Here $e_i = (0, \ldots, 0, i, 0, \ldots, 0)$ is the canonical ith base vector in $\mathbb{R}^n$ for $i = 1, \ldots, n$.

We now list the main results as follows.

Theorem 1.3. *Let $0 < \gamma \leqslant 1$, $0 < s < \gamma$ and $1 < p, q < \infty$. Then $\mathfrak{M}$ is bounded and continuous from $\mathrm{Lip}_\gamma(\mathbb{R}^n) \times F_s^{p,q}(\mathbb{R}^n)$ to $F_s^{p,q}(\mathbb{R}^n)$. Moreover, there exists a constant $C > 0$ such that*

$$\|\mathfrak{M}(f,g)\|_{F_s^{p,q}(\mathbb{R}^n)} \leqslant C\|f\|_{Lip_\gamma(\mathbb{R}^n)}\|g\|_{F_s^{p,q}(\mathbb{R}^n)} \tag{1.1}$$

for all $f \in Lip_\gamma(\mathbb{R}^n)$ and $g \in F_s^{p,q}(\mathbb{R}^n)$.

Theorem 1.4. *Let $0 < \gamma \leqslant 1$, $0 < s < \gamma$ and $1 < p, q < \infty$. Then $\mathfrak{M}$ is bounded and continuous from $\mathfrak{M}\colon \mathrm{Lip}_\gamma(\mathbb{R}^n) \times B_s^{p,q}(\mathbb{R}^n)$ to $B_s^{p,q}(\mathbb{R}^n)$. Moreover, there exists a constant $C > 0$ such that*

$$\|\mathfrak{M}(f,g)\|_{B_s^{p,q}(\mathbb{R}^n)} \leqslant C\|f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\|g\|_{B_s^{p,q}(\mathbb{R}^n)} \tag{1.2}$$

for all $f \in \mathrm{Lip}_\gamma(\mathbb{R}^n)$ and $g \in B_s^{p,q}(\mathbb{R}^n)$.

It was pointed out in [5] that $F_s^{p,2}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n)$ for any $s > 0$ and $1 < p < \infty$. Applying Theorem 1.3, we can get the following result.

Corollary 1.5. *Let $0 < \gamma \leqslant 1$, $0 < s < \gamma$ and $1 < p < \infty$. Then $\mathfrak{M}$ is bounded and continuous from $\mathrm{Lip}_\gamma(\mathbb{R}^n) \times W^{s,p}(\mathbb{R}^n)$ to $W^{s,p}(\mathbb{R}^n)$. Moreover, there exists a constant $C > 0$ such that*

$$\|\mathfrak{M}(f,g)\|_{W^{s,p}(\mathbb{R}^n)} \leqslant C\|f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\|g\|_{W^{s,p}(\mathbb{R}^n)}$$

for all $f \in \mathrm{Lip}_\gamma(\mathbb{R}^n)$ and $g \in W^{s,p}(\mathbb{R}^n)$.

The third result focuses on the quasicontinuity of bilinear maximal function, which is based on the Sobolev regularity of the bilinear maximal function. Let us recall some definitions.

Definition 1.6 ([8], Sobolev p-capacity)**.** Let $1 < p < \infty$ and set

$$\mathcal{A}(E) = \{f \in W^{1,p}(\mathbb{R}^n)\colon\ f \geqslant 1 \text{ on a neighbourhood of } E\}.$$

The Sobolev p-capacity of the set $E \subset \mathbb{R}^n$ is defined by

$$C_p(E) := \inf_{f\in\mathcal{A}(E)} \int_{\mathbb{R}^n} (|f(y)|^p + |\nabla f(y)|^p)\,\mathrm{d}y.$$

We set $C_p(E) = \infty$ if $\mathcal{A}(E) = \emptyset$.

Definition 1.7 ([8], p-quasicontinuous and p-quasieverywhere)**.** A function f is said to be p-quasicontinuous in $\mathbb{R}^n$ if for every $\varepsilon > 0$ there exists a set $F \subset \mathbb{R}^n$ such that $C_p(F) < \varepsilon$, the set $\mathbb{R}^n \setminus F$ is closed and the restriction of f to $\mathbb{R}^n \setminus F$ is continuous. A property holds p-quasieverywhere if it holds outside a set of the Sobolev p-capacity zero.

It was shown in [2] that the Sobolev p-capacity is a monotone and a countably subadditive set function. Also, it is an outer measure over $\mathbb{R}^n$. It is well known that each Sobolev function has a quasicontinuous representative, that is, for each $u \in W^{1,p}(\mathbb{R}^n)$ there is a p-quasicontinuous function $v \in W^{1,p}(\mathbb{R}^n)$ such that $u = v$ a.e. in $\mathbb{R}^n$. This representative is unique in the sense that if v and w are p-quasicontinuous and $v = w$ a.e. in $\mathbb{R}^n$, then $w = v$ p-quasieverywhere in $\mathbb{R}^n$, see [2] for more details.

In 1997, Kinnunen in [9] proved that Mf is p-quasicontinuous if $f \in W^{1,p}(\mathbb{R}^n)$ for any $1 < p < \infty$. Motivated by Kinnunen's work (see [9]), Liu et al. (see [15]) established the p-quasicontinuity of $\mathfrak{M}(f,g)$, provided that $f \in W^{1,p_1}(\mathbb{R}^n)$, $g \in W^{1,p_2}(\mathbb{R}^n)$, $1 < p_1, p_2, p < \infty$ and $1/p = 1/p_1 + 1/p_2$.

In this paper we shall establish the following result.

Theorem 1.8. *Let $1 < p < \infty$. If $f \in \mathrm{Lip}_1(\mathbb{R}^n)$ and $g \in W^{1,p}(\mathbb{R}^n)$, then we have*

$$\|\mathfrak{M}(f,g)\|_{W^{1,p}(\mathbb{R}^n)} \leqslant C\|f\|_{\mathrm{Lip}_1(\mathbb{R}^n)}\|g\|_{W^{1,p}(\mathbb{R}^n)}. \tag{1.3}$$

Moreover, the function $\mathfrak{M}(f,g)$ is p-quasicontinuous.

Finally, we study the approximate differentiability for the bilinear maximal function.

Definition 1.9 ([6], Approximate differentiability)**.** Let f be a real-valued function defined on a set $E \subset \mathbb{R}^n$. We say that f is approximately differentiable at $x_0 \in E$ if there is a vector $L = (L_1, L_2, \dots, L_n) \in \mathbb{R}^n$ such that for any $\varepsilon > 0$ the set

$$A_\varepsilon = \Big\{x \in \mathbb{R}^n\colon\ \frac{|f(x) - f(x_0) - L(x - x_0)|}{|x - x_0|} < \varepsilon\Big\}$$

has x_0 as a density point. If this is the case, then x_0 is a density point of E and L is uniquely determined. The vector L is called the *approximate differential* of f at x_0 and is denoted by $\nabla f(x_0)$.

In 2010, Hajłasz and Malý in [6] proved that Mf is approximately differentiable a.e. if $f \in L^1(\mathbb{R}^n)$ is approximately differentiable a.e. Motivated by Hajłasz and Malý's work, we shall establish the following result.

Theorem 1.10. *Let $f \in \mathrm{Lip}_1(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$ be approximately differentiable a.e. Then $\mathfrak{M}(f, g)$ is approximately differentiable a.e.*

Remark 1.11. Note that u is approximately differentiable a.e. if $u \in W^{1,1}(\mathbb{R}^n)$. Moreover, Theorem 1.8 yields that $\mathfrak{M}(f, g)$ is approximately differentiable a.e. if $f \in Lip_1(\mathbb{R}^n)$ and $g \in W^{1,1}(\mathbb{R}^n)$.

Remark 1.12. Let $1 < p < \infty$. From Theorem 1.8 we see that $\mathfrak{M}$ is bounded from $\mathrm{Lip}_1(\mathbb{R}^n) \times W^{1,p}(\mathbb{R}^n)$ to $W^{1,p}(\mathbb{R}^n)$. It is unknown whether $\mathfrak{M}$ is continuous from $\mathrm{Lip}_1(\mathbb{R}^n) \times W^{1,p}(\mathbb{R}^n)$ to $W^{1,p}(\mathbb{R}^n)$. Another interesting question is whether the map $\mathfrak{M}\colon \mathrm{Lip}_\gamma(\mathbb{R}^n) \times W^{1,p}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$ is bounded or continuous when $\gamma \neq 1$.

This paper will be organized as follows. Section 2 is devoted to presenting the proofs of Theorems 1.3 and 1.4. In Section 3 we prove Theorem 1.8. The proof of Theorem 1.10 will be given in Section 4.

We would like to remark that the proofs of Theorems 1.3 and 1.4 are based on [15]. The proof of Theorem 1.8 (or Theorem 1.10) is motivated by the idea in [9] (or [6]). However, some new techniques are needed. Throughout this paper, the letter C will stand for positive constants not necessarily the same one at each occurrence but is independent of the essential variables. In what follows, let $\mathfrak{R}_n = \{\zeta \in \mathbb{R}^n\colon \frac{1}{2} < |\zeta| \leqslant 1\}$. For any $p \in (1, \infty)$, we let p' denote the dual exponent to p defined as $1/p + 1/p' = 1$. For any $h \in \mathbb{R}^n$ and arbitrary function $u\colon \mathbb{R}^n \to \mathbb{R}$, we define the first order difference of u by $\Delta_h u(x) := u_h(x) - u(x)$, where $u_h(x) = u(x + h)$.

2. Proofs of Theorems 1.3 and 1.4

In this section we will present the proofs of Theorems 1.3 and 1.4.

2.1. Proof of Theorem 1.3. In order to prove Theorem 1.3, let us introduce some notation. For a measurable function $g\colon \mathbb{R}^n \times \mathbb{Z} \times \mathfrak{R}_n \to \mathbb{R}$ we set

$$\|g\|_{p,q,r,s} := \left\| \left(\sum_{k\in\mathbb{Z}} 2^{ksq} \left(\int_{\mathfrak{R}_n} |g(x,k,\zeta)|^r \,\mathrm{d}\zeta \right)^{q/r} \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)}.$$

Denote by $\dot{F}_s^{p,q}(\mathbb{R}^n)$ the homogeneous Triebel-Lizorkin spaces. In [22] Yabuta showed that

$$\|f\|_{\dot{F}_s^{p,q}(\mathbb{R}^n)} \sim \|\Delta_{2^{-k}\zeta} f\|_{p,q,r,s} \quad \text{for } 0 < s < 1,\ 1 < p < \infty,\ 1 < q \leqslant \infty,\ 1 \leqslant r < \min\{p,q\}. \tag{2.1}$$

The following properties for the Triebel-Lizorkin spaces are known, see [3], [5], [20]:

$$\|f\|_{F_s^{p,q}(\mathbb{R}^n)} \sim \|f\|_{\dot{F}_s^{p,q}(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)} \quad \text{for } s > 0,\ 1 < p, q < \infty, \tag{2.2}$$

$$\|f\|_{F_{s_1}^{p,q}(\mathbb{R}^n)} \leqslant \|f\|_{F_{s_2}^{p,q}(\mathbb{R}^n)} \quad \text{for } s_1 \leqslant s_2,\ 0 < p,\ q < \infty, \tag{2.3}$$

$$\|f\|_{F_s^{p,q_2}(\mathbb{R}^n)} \leqslant \|f\|_{F_s^{p,q_1}(\mathbb{R}^n)} \quad \text{for } s \in \mathbb{R},\ 0 < p < \infty,\ 0 < q_1 \leqslant q_2 < \infty. \tag{2.4}$$

In order to prove Theorem 1.3, we need the following lemmas.

Lemma 2.1 ([22]). *For any $1 < p,\ q,\ r < \infty$ we have*

$$\left\| \left(\sum_{k\in\mathbb{Z}} \|Mf_{k,\zeta}\|_{L^r(\mathfrak{R}_n)}^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \leqslant C_{p,q,r} \left\| \left(\sum_{k\in\mathbb{Z}} \|f_{k,\zeta}\|_{L^r(\mathfrak{R}_n)}^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)}.$$

Lemma 2.2. *Let $0 < q < \infty$, $0 < r \leqslant \infty$, $0 < \gamma < \infty$ and $0 < s < \gamma$. If $f \in \mathrm{Lip}_\gamma(\mathbb{R}^n)$, then*

$$\left(\sum_{k\in\mathbb{Z}} 2^{ksq} \| \|\Delta_{2^{-k}\zeta} f\|_{L^\infty(\mathbb{R}^n)} \|_{L^r(\mathfrak{R}_n)}^q \right)^{1/q} \leqslant C \|f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}.$$

Proof. Note that for any $k \in \mathbb{Z}$ and $\zeta \in \mathfrak{R}_n$,

$$\|\Delta_{2^{-k}\zeta} f\|_{L^\infty(\mathbb{R}^n)} \leqslant \min\{2\|f\|_{L^\infty(\mathbb{R}^n)}, 2^{-k\gamma}\|f\|_{Lip_\gamma(\mathbb{R}^n)}\}.$$

It follows that

$$\| \|\Delta_{2^{-k}\zeta} f\|_{L^\infty(\mathbb{R}^n)} \|_{L^r(\mathfrak{R}_n)} \leqslant |\mathfrak{R}_n|^{1/r} \min\{2\|f\|_{L^\infty(\mathbb{R}^n)}, 2^{-k\gamma}\|f\|_{Lip_\gamma(\mathbb{R}^n)}\}.$$

Therefore, we have

$$\begin{aligned}\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\|\|\Delta_{2^{-k}\zeta}f\|_{L^\infty(\mathbb{R}^n)}\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}&\\ \leqslant 2\|f\|_{L^\infty(\mathbb{R}^n)}\Big(\sum_{k=-\infty}^{0} 2^{ksq}\Big)^{1/q} &+ \|f\|_{Lip_\gamma(\mathbb{R}^n)}\Big(\sum_{k=1}^{\infty}2^{k(s-\gamma)q}\Big)^{1/q}\\ \leqslant C\|f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}.&\end{aligned}$$

□

We now turn to prove Theorem 1.3.

Proof of Theorem 1.3. We divide the proof of Theorem 1.3 into two parts:

Step 1: Proof of the boundedness part. Fix $x, h \in \mathbb{R}^n$, we have

$$\begin{aligned}|\Delta_h\mathfrak{M}(f,g)(x)| &= |(\mathfrak{M}(f,g))_h(x) - \mathfrak{M}(f,g)(x)|\\ &= |\mathfrak{M}(f_h,g_h)(x) - \mathfrak{M}(f,g)(x)|\\ &\leqslant \mathfrak{M}(\Delta_h f, \Delta_h g)(x) + \mathfrak{M}(f,\Delta_h g)(x) + \mathfrak{M}(\Delta_h f, g)(x).\end{aligned}\tag{2.5}$$

In view of (2.5), for any $k \in \mathbb{Z}$, $\zeta \in \mathfrak{R}_n$ and $x \in \mathbb{R}^n$, we have

$$\begin{aligned}|\Delta_{2^{-k}\zeta}\mathfrak{M}(f,g)(x)| \leqslant \mathfrak{M}(\Delta_{2^{-k}\zeta}f, \Delta_{2^{-k}\zeta}g)(x) &+ \mathfrak{M}(\Delta_{2^{-k}\zeta}f, g)(x)\\ &+ \mathfrak{M}(f, \Delta_{2^{-k}\zeta}g)(x).\end{aligned}\tag{2.6}$$

By (2.1), (2.6) and Minkowski's inequality, we write

$$\begin{aligned}\|\mathfrak{M}(f,g)\|_{\dot F^{p,q}_s(\mathbb{R}^n)} &\leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}|\Delta_{2^{-k}\zeta}(\mathfrak{M}(f,g))|\,\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\ &\leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\mathfrak{M}(\Delta_{2^{-k}\zeta}f,\Delta_{2^{-k}\zeta}g)\,\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\ &\quad + C\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\mathfrak{M}(\Delta_{2^{-k}\zeta}f,g)\,\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\ &\quad + C\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\mathfrak{M}(f,\Delta_{2^{-k}\zeta}g)\,\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\ &=: \sum_{i=1}^{3} I_i.\end{aligned}\tag{2.7}$$

It is clear that for any arbitrary functions u, v defined on $\mathbb{R}^n$,

$$\mathfrak{M}(u,v)(x) \leqslant \|u\|_{L^\infty(\mathbb{R}^n)}Mv(x).\tag{2.8}$$

Next we estimate I_1, I_2, I_3, respectively.

Estimate for I_1. Let $q_1,\, q_2 \in (1,\infty)$ be such that $1/q = 1/q_1 + 1/q_2$ and $r \in (1, \min\{q_1, p\})$. By (2.1), (2.2), (2.4) and (2.8), Lemmas 2.1 and 2.2 and Hölder's inequality, we get

$$
\begin{aligned}
(2.9)\quad I_1 &\leqslant C\bigg\|\bigg(\sum_{k\in\mathbb{Z}} 2^{ksq}\bigg(\int_{\mathfrak{R}_n} \|\Delta_{2^{-k}\zeta} f\|_{L^\infty(\mathbb{R}^n)} M(\Delta_{2^{-k}\zeta} g)\,\mathrm{d}\zeta\bigg)^q\bigg)^{1/q}\bigg\|_{L^p(\mathbb{R}^n)} \\
&\leqslant C\bigg\|\bigg(\sum_{k\in\mathbb{Z}} 2^{ksq} \|\|\Delta_{2^{-k}\zeta} f\|_{L^\infty(\mathbb{R}^n)}\|^q_{L^{r'}(\mathfrak{R}_n)} \|M(\Delta_{2^{-k}\zeta} g)\|^q_{L^r(\mathfrak{R}_n)}\bigg)^{1/q}\bigg\|_{L^p(\mathbb{R}^n)} \\
&\leqslant C\bigg\|\bigg(\sum_{k\in\mathbb{Z}} 2^{ksq_2} \|\|\Delta_{2^{-k}\zeta} f\|_{L^\infty(\mathbb{R}^n)}\|^{q_2}_{L^{r'}(\mathfrak{R}_n)}\bigg)^{1/q_2} \\
&\quad\times \bigg(\sum_{k\in\mathbb{Z}} 2^{ksq_1} \|M(\Delta_{2^{-k}\zeta} g)\|^{q_1}_{L^r(\mathfrak{R}_n)}\bigg)^{1/q_1}\bigg\|_{L^p(\mathbb{R}^n)} \\
&\leqslant C\|f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\bigg\|\bigg(\sum_{k\in\mathbb{Z}} 2^{ksq_1} \|\Delta_{2^{-k}\zeta} g\|^{q_1}_{L^r(\mathfrak{R}_n)}\bigg)^{1/q_1}\bigg\|_{L^p(\mathbb{R}^n)} \\
&\leqslant C\|f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\|g\|_{\dot F^{p,q_1}_s(\mathbb{R}^n)} \leqslant C\|f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\|g\|_{F^{p,q_1}_s(\mathbb{R}^n)} \\
&\leqslant C\|f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\|g\|_{F^{p,q}_s(\mathbb{R}^n)}.
\end{aligned}
$$

Estimate for I_2. In view of (2.1), (2.2), Lemma 2.2 and Hölder's inequality, we have

$$
\begin{aligned}
(2.10)\qquad I_2 &\leqslant C\bigg\|\bigg(\sum_{k\in\mathbb{Z}} 2^{ksq}\bigg(\int_{\mathfrak{R}_n} \|\Delta_{2^{-k}\zeta} f\|_{L^\infty(\mathbb{R}^n)} Mg\,\mathrm{d}\zeta\bigg)^q\bigg)^{1/q}\bigg\|_{L^p(\mathbb{R}^n)} \\
&\leqslant C\bigg\|Mg\bigg(\sum_{k\in\mathbb{Z}} 2^{ksq} \|\|\Delta_{2^{-k}\zeta} f\|_{L^\infty(\mathbb{R}^n)}\|^q_{L^1(\mathfrak{R}_n)}\bigg)^{1/q}\bigg\|_{L^p(\mathbb{R}^n)} \\
&\leqslant C\|f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\|Mg\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\|g\|_{L^p(\mathbb{R}^n)} \\
&\leqslant C\|f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\|g\|_{F^{p,q}_s(\mathbb{R}^n)}.
\end{aligned}
$$

Estimate for I_3. Fix $\alpha \in (1, \min\{q, p\})$. Applying (2.1), (2.2), Lemma 2.1 and Hölder's inequality, one obtains

$$
\begin{aligned}
(2.11)\qquad I_3 &\leqslant C\bigg\|\bigg(\sum_{k\in\mathbb{Z}} 2^{ksq}\bigg(\int_{\mathfrak{R}_n} \|f\|_{L^\infty(\mathbb{R}^n)} M(\Delta_{2^{-k}\zeta} g)\,\mathrm{d}\zeta\bigg)^q\bigg)^{1/q}\bigg\|_{L^p(\mathbb{R}^n)} \\
&\leqslant C\|f\|_{L^\infty(\mathbb{R}^n)}\bigg\|\bigg(\sum_{k\in\mathbb{Z}} 2^{ksp} \|M(\Delta_{2^{-k}\zeta} g)\|^q_{L^1(\mathfrak{R}_n)}\bigg)^{1/q}\bigg\|_{L^p(\mathbb{R}^n)} \\
&\leqslant C\|f\|_{L^\infty(\mathbb{R}^n)}\bigg\|\bigg(\sum_{k\in\mathbb{Z}} 2^{ksp} \|M(\Delta_{2^{-k}\zeta} g)\|^q_{L^\alpha(\mathfrak{R}_n)}\bigg)^{1/q}\bigg\|_{L^p(\mathbb{R}^n)}
\end{aligned}
$$

$$\leqslant C\|f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\left\|\left(\sum_{k\in\mathbb{Z}} 2^{ksp}\|\Delta_{2^{-k}\zeta}g\|^q_{L^\alpha(\mathbb{R}^n)}\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)}$$
$$\leqslant C\|f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\|g\|_{\dot F^{p,q}_s(\mathbb{R}^n)} \leqslant C\|f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\|g\|_{F^{p,q}_s(\mathbb{R}^n)}.$$

It follows from (2.7) and (2.9)–(2.11) that

$$\|\mathfrak{M}(f,g)\|_{\dot F^{p,q}_s(\mathbb{R}^n)} \leqslant C\|f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\|g\|_{F^{p,q}_s(\mathbb{R}^n)}. \tag{2.12}$$

On the other hand, it is easy to see that

$$\|\mathfrak{M}(f,g)\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{L^\infty(\mathbb{R}^n)}\|g\|_{L^p(\mathbb{R}^n)}. \tag{2.13}$$

Then (1.1) follows from (2.2), (2.12) and (2.13).

Step 2: Proof of the continuity part. Let $f_j \to f$ in $\mathrm{Lip}_\gamma(\mathbb{R}^n)$ and $g_j \to g$ in $F^{p,q}_s(\mathbb{R}^n)$ as $j\to\infty$. It suffices to show that

$$\|\mathfrak{M}(f_j,g_j) - \mathfrak{M}(f,g)\|_{F^{p,q}_s(\mathbb{R}^n)} \to 0 \quad \text{as } j\to\infty. \tag{2.14}$$

From (2.2) we see that $g_j \to g$ in $\dot F^{p,q}_s(\mathbb{R}^n)$ and in $L^p(\mathbb{R}^n)$ as $j\to\infty$. By the sublinearity of $\mathfrak{M}$, (2.13) and Minkowski's inequality,

$$\begin{aligned}
(2.15)\qquad & \|\mathfrak{M}(f_j,g_j) - \mathfrak{M}(f,g)\|_{L^p(\mathbb{R}^n)} \\
&\leqslant \|\mathfrak{M}(f_j-f,g_j-g)\|_{L^p(\mathbb{R}^n)} + \|\mathfrak{M}(f_j-f,g)\|_{L^p(\mathbb{R}^n)} \\
&\quad + \|\mathfrak{M}(f,g_j-g)\|_{L^p(\mathbb{R}^n)} \\
&\leqslant C(\|f_j-f\|_{L^\infty(\mathbb{R}^n)}\|g_j-g\|_{L^p(\mathbb{R}^n)} + \|f_j-f\|_{L^\infty(\mathbb{R}^n)}\|g\|_{L^p(\mathbb{R}^n)} \\
&\quad + \|f\|_{L^\infty(\mathbb{R}^n)}\|g_j-g\|_{L^p(\mathbb{R}^n)}) \to 0 \quad \text{as } j\to\infty.
\end{aligned}$$

In view of (2.2) and (2.15), for (2.14) it is enough to prove that

$$\|\mathfrak{M}(f_j,g_j) - \mathfrak{M}(f,g)\|_{\dot F^{p,q}_s(\mathbb{R}^n)} \to 0 \quad \text{as } j\to\infty. \tag{2.16}$$

We shall prove (2.16) by contradiction. If (2.16) is false, then we may assume without loss of generality that there exists a constant $c>0$ such that

$$\|\mathfrak{M}(f_j,g_j) - \mathfrak{M}(f,g)\|_{\dot F^{p,q}_s(\mathbb{R}^n)} > c \quad \forall\, j\geqslant 1. \tag{2.17}$$

Since $\mathfrak{M}(f_j,g_j)\to\mathfrak{M}(f,g)$ in $L^p(\mathbb{R}^n)$ as $j\to\infty$, by extracting a subsequence we may assume without loss of generality that $|\mathfrak{M}(f_j,g_j)(x) - \mathfrak{M}(f,g)(x)|\to 0$ as $j\to\infty$ for almost every $x\in\mathbb{R}^n$. It follows that

$$\Delta_{2^{-k}\zeta}(\mathfrak{M}(f_j,g_j) - \mathfrak{M}(f,g))(x) \to 0 \quad \text{as } j\to\infty \tag{2.18}$$

for every $(k,\zeta)\in\mathbb{Z}\times\mathfrak{R}_n$ and almost every $x\in\mathbb{R}^n$. For convenience, we set

$$\begin{aligned}\Psi_j(x,k,\zeta) &:= \mathfrak{M}(\Delta_{2^{-k}\zeta}(f_j-f),\Delta_{2^{-k}\zeta}(g_j-g))(x)+\mathfrak{M}(\Delta_{2^{-k}\zeta}(f_j-f),\Delta_{2^{-k}\zeta}g)(x)\\ &\quad+\mathfrak{M}(\Delta_{2^{-k}\zeta}f,\Delta_{2^{-k}\zeta}(g_j-g))(x)+\mathfrak{M}(\Delta_{2^{-k}\zeta}(f_j-f),g_j-g)(x)\\ &\quad+\mathfrak{M}(\Delta_{2^{-k}\zeta}(f_j-f),g)(x)+\mathfrak{M}(\Delta_{2^{-k}\zeta}f,g_j-g)(x)\\ &\quad+\mathfrak{M}(f_j-f,\Delta_{2^{-k}\zeta}(g_j-g))(x)+\mathfrak{M}(f_j-f,\Delta_{2^{-k}\zeta}g)(x)\\ &\quad+\mathfrak{M}(f,\Delta_{2^{-k}\zeta}(g_j-g))(x),\\ \Psi(x,k,\zeta) &:= 2\mathfrak{M}(\Delta_{2^{-k}\zeta}f,\Delta_{2^{-k}\zeta}g)(x)+2\mathfrak{M}(\Delta_{2^{-k}\zeta}f,g)(x)+2\mathfrak{M}(f,\Delta_{2^{-k}\zeta}g)(x).\end{aligned}$$

By (2.6), it is not difficult to check that

$$(2.19)\qquad |\Delta_{2^{-k}\zeta}(\mathfrak{M}(f_j,g_j)-\mathfrak{M}(f,g))(x)|\leqslant\Psi_j(x,k,\zeta)+\Psi(x,k,\zeta)$$

for all $(x,k,\zeta)\in\mathbb{R}^n\times\mathbb{Z}\times\mathfrak{R}_n$. Some computations similar to (2.9)–(2.11) imply that

$$\begin{aligned}(2.20)\quad \|\Psi_j\|_{p,q,1,s} &\leqslant C(\|f_j-f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\|g_j-g\|_{F_s^{p,q}(\mathbb{R}^n)}\\ &\quad+\|f_j-f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\|g\|_{F_s^{p,q}(\mathbb{R}^n)}+\|f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\|g_j-g\|_{F_s^{p,q}(\mathbb{R}^n)}),\\ (2.21)\quad \|\Psi\|_{p,q,1,s} &\leqslant C\|f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\|g\|_{F_s^{p,q}(\mathbb{R}^n)}.\end{aligned}$$

By (2.18)–(2.21) and the arguments similar to those used to derive the proof of Theorem 1.1 in [15], one can get a contradiction for (2.16). The details are omitted. □

2.2. Proof of Theorem 1.4. Let $\dot B_s^{p,q}(\mathbb{R}^n)$ denote the homogeneous Besov spaces. The following properties are known, see [3], [5], [20], [22]:

$$(2.22)\qquad \|f\|_{B_s^{p,q}(\mathbb{R}^n)}\sim\|f\|_{\dot B_s^{p,q}(\mathbb{R}^n)}+\|f\|_{L^p(\mathbb{R}^n)}\quad\text{for } s>0,\ 1<p,q<\infty,$$

$$(2.23)\qquad \|f\|_{B_s^{p,q_1}(\mathbb{R}^n)}\leqslant\|f\|_{B_s^{p,q_2}(\mathbb{R}^n)}\quad\text{for } s\in\mathbb{R},\ 0<p<\infty,\ 0<q_2<q_1,$$

$$(2.24)\qquad \|f\|_{\dot B_s^{p,q}(\mathbb{R}^n)}\sim\bigg(\sum_{k\in\mathbb{Z}}2^{ksq}\bigg\|\bigg(\int_{\mathfrak{R}_n}|\Delta_{2^{-k}\zeta}f|^r\,\mathrm{d}\zeta\bigg)^{1/r}\bigg\|_{L^p(\mathbb{R}^n)}^q\bigg)^{1/q}$$

if $0<s<1$, $1\leqslant p<\infty$, $1\leqslant q\leqslant\infty$ and $1\leqslant r\leqslant p$.

For a measurable function $g\colon\ \mathbb{R}^n\times\mathbb{Z}\times\mathfrak{R}_n\to\mathbb{R}$ we set

$$\|g\|_{p,q,s}:=\bigg(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}|g(x,k,\zeta)|^p\,\mathrm{d}x\,\mathrm{d}\zeta\Big)^{q/p}\bigg)^{1/q}.$$

By (2.24) and Fubini's theorem, the following is valid:

$$(2.25)\qquad \|f\|_{\dot B_s^{p,q}(\mathbb{R}^n)}\sim\|\Delta_{2^{-k}\zeta}f\|_{p,q,s}\quad\text{for } 0<s<1,\ 1\leqslant p<\infty,\ 1\leqslant q\leqslant\infty.$$

We now present the proof of Theorem 1.4.

P r o o f of Theorem 1.4. We divide the proof into two parts.

Step 1: Proof of the boundedness part. Let $0 < s < 1$ and $1 < p, q < \infty$. In view of (2.6) and (2.25), one has

(2.26)
$$
\begin{aligned}
\|\mathfrak{M}(f,g)\|_{\dot{B}^{p,q}_s(\mathbb{R}^n)} &\leqslant C\biggl(\sum_{k\in\mathbb{Z}} 2^{ksq}\biggl(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n} |\Delta_{2^{-k}\zeta}\mathfrak{M}(f,g)(x)|^p \,\mathrm{d}x\,\mathrm{d}\zeta\biggr)^{q/p}\biggr)^{1/q}\\
&\leqslant C\biggl(\sum_{k\in\mathbb{Z}} 2^{ksq}\biggl(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n} (\mathfrak{M}(\Delta_{2^{-k}\zeta}f,\Delta_{2^{-k}\zeta}g)(x))^p \,\mathrm{d}x\,\mathrm{d}\zeta\biggr)^{q/p}\biggr)^{1/q}\\
&\quad + C\biggl(\sum_{k\in\mathbb{Z}} 2^{ksq}\biggl(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n} (\mathfrak{M}(\Delta_{2^{-k}\zeta}f,g)(x))^p \,\mathrm{d}x\,\mathrm{d}\zeta\biggr)^{q/p}\biggr)^{1/q}\\
&\quad + C\biggl(\sum_{k\in\mathbb{Z}} 2^{ksq}\biggl(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n} (\mathfrak{M}(f,\Delta_{2^{-k}\zeta}g)(x))^p \,\mathrm{d}x\,\mathrm{d}\zeta\biggr)^{q/p}\biggr)^{1/q}\\
&=: \sum_{i=1}^{3} J_i.
\end{aligned}
$$

Estimate for J_1. Let $q_1, q_2 \in (1,\infty)$ be such that $1/q = 1/q_1 + 1/q_2$. By (2.13), Hölder's inequality, Lemma 2.2, (2.22), (2.23) and (2.25) one has

(2.27)
$$
\begin{aligned}
J_1 &\leqslant C\biggl(\sum_{k\in\mathbb{Z}} 2^{ksq}\biggl(\int_{\mathfrak{R}_n} \|\Delta_{2^{-k}\zeta}f\|^p_{L^\infty(\mathbb{R}^n)}\|\Delta_{2^{-k}\zeta}g\|^p_{L^p(\mathbb{R}^n)}\,\mathrm{d}\zeta\biggr)^{q/p}\biggr)^{1/q}\\
&\leqslant C\biggl(\sum_{k\in\mathbb{Z}} 2^{ksq}\|\Delta_{2^{-k}\zeta}f\|^q_{L^\infty(\mathbb{R}^n\times\mathfrak{R}_n)}\|\Delta_{2^{-k}\zeta}g\|^q_{L^p(\mathbb{R}^n\times\mathfrak{R}_n)}\biggr)^{1/q}\\
&\leqslant C\biggl(\sum_{k\in\mathbb{Z}} 2^{ksq_2}\|\Delta_{2^{-k}\zeta}f\|^{q_2}_{L^\infty(\mathbb{R}^n\times\mathfrak{R}_n)}\biggr)^{1/q_2}\biggl(\sum_{k\in\mathbb{Z}} 2^{ksq_1}\|\Delta_{2^{-k}\zeta}g\|^{q_1}_{L^p(\mathbb{R}^n\times\mathfrak{R}_n)}\biggr)^{1/q_1}\\
&\leqslant C\|f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\biggl(\sum_{k\in\mathbb{Z}} 2^{ksq_1}\|\Delta_{2^{-k}\zeta}g\|^{q_1}_{L^p(\mathbb{R}^n\times\mathfrak{R}_n)}\biggr)^{1/q_1}\\
&\leqslant C\|f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\|g\|_{\dot{B}^{p,q_1}_s(\mathbb{R}^n)} \leqslant C\|f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\|g\|_{B^{p,q_1}_s(\mathbb{R}^n)}\\
&\leqslant C\|f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\|g\|_{B^{p,q}_s(\mathbb{R}^n)}.
\end{aligned}
$$

Estimate for J_2. By (2.13), (2.22), (2.23) and Lemma 2.2, we have

$$
\begin{aligned}
J_2 &\leqslant C\biggl(\sum_{k\in\mathbb{Z}} 2^{ksq}\biggl(\int_{\mathfrak{R}_n} \|\Delta_{2^{-k}\zeta}f\|^p_{L^\infty(\mathbb{R}^n)}\|g\|^p_{L^p(\mathbb{R}^n)}\,\mathrm{d}\zeta\biggr)^{q/p}\biggr)^{1/q}\\
&\leqslant C\|g\|_{L^p(\mathbb{R}^n)}\biggl(\sum_{k\in\mathbb{Z}} 2^{ksq}\|\|\Delta_{2^{-k}\zeta}f\|_{L^\infty(\mathbb{R}^n)}\|^q_{L^p(\mathfrak{R}_n)}\biggr)^{1/q}\\
&\leqslant C\|f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\|g\|_{B^{p,q}_s(\mathbb{R}^n)}.
\end{aligned}
\tag{2.28}
$$

Estimate for J_3. By (2.13), (2.22) and (2.25), we have

$$J_3 \leqslant C\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} \|f\|_{L^\infty(\mathbb{R}^n)}\|\Delta_{2^{-k}\zeta}g\|^p_{L^p(\mathbb{R}^n)}\,\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q} \tag{2.29}$$
$$\leqslant C\|f\|_{L^\infty(\mathbb{R}^n)}\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} \|\Delta_{2^{-k}\zeta}g\|^p_{L^p(\mathbb{R}^n)}\,\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}$$
$$\leqslant C\|f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\|g\|_{\dot{B}^{p,q}_s(\mathbb{R}^n)} \leqslant C\|f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\|g\|_{B^{p,q}_s(\mathbb{R}^n)}.$$

Finally, it follows from (2.26)–(2.29) that

$$\|\mathfrak{M}(f,g)\|_{\dot{B}^{p,q}_s(\mathbb{R}^n)} \leqslant C\|f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\|g\|_{B^{p,q}_s(\mathbb{R}^n)}.$$

This together with (2.13) and (2.22) implies (1.2).

Step 2: Proof of the continuity part. Assume that $f_j \to f$ in $\mathrm{Lip}_\gamma(\mathbb{R}^n)$ and $g_j \to g$ in $B^{p,q}_s(\mathbb{R}^n)$ as $j\to\infty$. By (2.22) we see that $g_j \to g$ in $\dot{B}^{p,q}_s(\mathbb{R}^n)$ and in $L^p(\mathbb{R}^n)$ as $j\to\infty$. From (2.15) we have that $\mathfrak{M}(f_j,g_j)\to\mathfrak{M}(f,g)$ in $L^p(\mathbb{R}^n)$ as $j\to\infty$. Hence, it is enough to show that

$$\|\mathfrak{M}(f_j,g_j)\to\mathfrak{M}(f,g)\|_{\dot{B}^{p,q}_s(\mathbb{R}^n)} \to 0 \quad \text{as } j\to\infty. \tag{2.30}$$

We now prove (2.30) by contradiction. Assume that (2.30) is not true. Without loss of generality we may assume that there exists a constant $c>0$ such that

$$\|\mathfrak{M}(f_j,g_j)-\mathfrak{M}(f,g)\|_{\dot{B}^{p,q}_s(\mathbb{R}^n)} > c \quad \forall\, j\geqslant 1. \tag{2.31}$$

Since $\mathfrak{M}(f_j,g_j)\to\mathfrak{M}(f,g)$ in $L^p(\mathbb{R}^n)$ as $j\to\infty$, then by extracting a subsequence we may assume without loss of generality that $|\mathfrak{M}(f_j,g_j)(x)-\mathfrak{M}(f,g)(x)|\to 0$ as $j\to\infty$ for almost every $x\in\mathbb{R}^n$. Hence, we have

$$\Delta_{2^{-k}\zeta}(\mathfrak{M}(f_j,g_j)-\mathfrak{M}(f,g))(x)\to 0 \quad \text{as } j\to\infty$$

for every $(k,\zeta)\in\mathbb{Z}\times\mathfrak{R}_n$ and almost every $x\in\mathbb{R}^n$. Let Ψ_j and Ψ be given as in (2.19). Some arguments similar to (2.27)–(2.29) show that

$$\begin{aligned}
\|\Psi\|_{p,q,s} &\leqslant C\|f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\|g\|_{B^{p,q}_s(\mathbb{R}^n)},\\
\|\Psi_j\|_{p,q,s} &\leqslant C(\|f_j-f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\|g_j-g\|_{B^{p,q}_s(\mathbb{R}^n)} + \|f_j-f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\|g\|_{B^{p,q}_s(\mathbb{R}^n)}\\
&\qquad + \|f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\|g_j-g\|_{B^{p,q}_s(\mathbb{R}^n)}).
\end{aligned}$$

The rest of the proof follows from the arguments similar to the proof of Theorem 1.3. We omit the details. $\square$

3. Proof of Theorem 1.8

In this section we shall present the proof of Theorem 1.8. Before presenting the proof, let us introduce some notation. Let $u \in L^p(\mathbb{R}^n)$ with $p \geqslant 1$. For all $h \in \mathbb{R}$, $|h| > 0$, $y \in \mathbb{R}^n$ and $i = 1, \ldots, n$, we define the function $u_{h,i}$ by setting

$$u_{h,i}(x) = \frac{u(x + he_i) - u(x)}{h}, \quad x \in \mathbb{R}^n.$$

Here $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the canonical ith base vector in $\mathbb{R}^n$. Set

$$G(u; p) := \limsup_{h \to 0} \frac{\|\Delta_h u\|_{L^p(\mathbb{R}^n)}}{|h|}.$$

According to [4], Section 7.11, we have

$$u \in W^{1,q}(\mathbb{R}^n), \quad 1 < q < \infty \Leftrightarrow u \in L^q(\mathbb{R}^n) \quad \text{and} \quad G(u; q) < \infty. \tag{3.1}$$

It is well known that for $p \geqslant 1$, $u_{h,i} \to D_i u$ in $L^p(\mathbb{R}^n)$ when $h \to 0$ if $u \in W^{1,p}(\mathbb{R}^n)$. Moreover, if $u \in L^p(\mathbb{R}^n)$ for $p \geqslant 1$, then $u_h \to u$ in $L^p(\mathbb{R}^n)$ when $|h| \to 0$.

We now prove Theorem 1.8.

P r o o f of Theorem 1.8. We divide the proof of Theorem 1.8 into three steps:

Step 1: Proof of $\mathfrak{M}(f, g) \in W^{1,p}(\mathbb{R}^n)$. Applying Minkowski's inequality and (2.5), one has that when $|h| < 1$,

$$\begin{aligned} &\|\Delta_h \mathfrak{M}(f, g)\|_{L^p(\mathbb{R}^n)} \\ &\quad \leqslant \|\mathfrak{M}(\Delta_h f, \Delta_h g)\|_{L^p(\mathbb{R}^n)} + \|\mathfrak{M}(f, \Delta_h g)\|_{L^p(\mathbb{R}^n)} + \|\mathfrak{M}(\Delta_h f, g)\|_{L^p(\mathbb{R}^n)} \\ &\quad \leqslant C(\|\Delta_h f\|_{L^\infty(\mathbb{R}^n)} \|\Delta_h g\|_{L^p(\mathbb{R}^n)} + \|f\|_{L^\infty(\mathbb{R}^n)} \|\Delta_h g\|_{L^p(\mathbb{R}^n)} \\ &\qquad + \|\Delta_h f\|_{L^\infty(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)}) \\ &\quad \leqslant C(\|f\|_{Lip_1(\mathbb{R}^n)} \|\Delta_h g\|_{L^p(\mathbb{R}^n)} |h| + \|f\|_{L^\infty(\mathbb{R}^n)} \|\Delta_h g\|_{L^p(\mathbb{R}^n)} \\ &\qquad + \|f\|_{Lip_1(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)} |h|) \\ &\quad \leqslant C(\|f\|_{\mathrm{Lip}_1(\mathbb{R}^n)} \|\Delta_h g\|_{L^p(\mathbb{R}^n)} + \|f\|_{Lip_1(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)} |h|). \end{aligned} \tag{3.2}$$

This together with the fact that $G(g; p) < \infty$ leads to

$$\begin{aligned} G(\mathfrak{M}(f, g); p) &= \limsup_{h \to 0} \frac{\|\Delta_h (\mathfrak{M}(f, g))\|_{L^p(\mathbb{R}^n)}}{|h|} \\ &\leqslant C \limsup_{h \to 0} \frac{1}{|h|} (\|f\|_{\mathrm{Lip}_1(\mathbb{R}^n)} \|\Delta_h g\|_{L^p(\mathbb{R}^n)} + \|f\|_{Lip_1(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)} |h|) \\ &\leqslant C(\|f\|_{\mathrm{Lip}_1(\mathbb{R}^n)} G(g; p) + \|f\|_{Lip_1(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)}) < \infty. \end{aligned}$$

Combining this with (2.13) and (3.1) implies that $\mathfrak{M}(f, g) \in W^{1,p}(\mathbb{R}^n)$.

Step 2: Proof of (1.3). Let us fix $l \in \{1, 2, \dots, n\}$. By Step 1 we see that $\mathfrak{M}(f, g) \in W^{1,p}(\mathbb{R}^n)$. It follows that $(\mathfrak{M}(f, g))_{h,l} \to D_l\mathfrak{M}(f, g)$ in $L^p(\mathbb{R}^n)$ as $h \to 0$. Moreover, $g_{h,l} \to D_l g$ in $L^p(\mathbb{R}^n)$ as $h \to 0$. By Riesz theorem, there exists a sequence of numbers $\{h_k\}$ with $\lim\limits_{k\to\infty} h_k = 0$ and a measurable set E satisfying $|\mathbb{R}^n \setminus E| = 0$ such that $(\mathfrak{M}(f, g))_{h_k,l}(x) \to D_l\mathfrak{M}(f, g)(x)$ and $g_{h_k,l}(x) \to D_l g(x)$ as $k \to \infty$ for all $x \in E$. Applying Fatou's Lemma and (3.2), we obtain

$$\begin{aligned}
\|D_l\mathfrak{M}(f, g)\|_{L^p(\mathbb{R}^n)} &= \Big\|\liminf_{k\to\infty}(\mathfrak{M}(f, g))_{h_k,l}\Big\|_{L^p(\mathbb{R}^n)} \leqslant \liminf_{k\to\infty} \|(\mathfrak{M}(f, g))_{h_k,l}\|_{L^p(\mathbb{R}^n)} \\
&\leqslant C \liminf_{k\to\infty}(\|f\|_{\mathrm{Lip}_1(\mathbb{R}^n)}\|g_{h_k,l}\|_{L^p(\mathbb{R}^n)} + \|f\|_{Lip_1(\mathbb{R}^n)}\|g\|_{L^p(\mathbb{R}^n)}) \\
&\leqslant C\|f\|_{Lip_1(\mathbb{R}^n)}\|g\|_{L^p(\mathbb{R}^n)} + C \limsup_{k\to\infty} \|f\|_{\mathrm{Lip}_1(\mathbb{R}^n)}\|g_{h_k,l}\|_{L^p(\mathbb{R}^n)} \\
&\leqslant C\|f\|_{Lip_1(\mathbb{R}^n)}\|g\|_{L^p(\mathbb{R}^n)} + C\|f\|_{\mathrm{Lip}_1(\mathbb{R}^n)}\Big\|\limsup_{k\to\infty} g_{h_k,l}\Big\|_{L^p(\mathbb{R}^n)} \\
&\leqslant C\|f\|_{\mathrm{Lip}_1(\mathbb{R}^n)}\|g\|_{W^{1,p}(\mathbb{R}^n)}.
\end{aligned}$$

This together with (2.13) leads to (1.3).

Step 3: Proof of the p-quasicontinuity. We first prove that $\mathfrak{M}(f, g)$ is continuous on $\mathbb{R}^n$ if $f \in \mathrm{Lip}_1(\mathbb{R}^n)$ and $g \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. Fix $x, h \in \mathbb{R}^n$, by the mean value theorem for differentials one has that $|\Delta_h g(x)| \leqslant C|h|$ for a constant $C > 0$. Then we get from (2.5) that

$$\begin{aligned}
|\Delta_h\mathfrak{M}(f, g)(x)| \leqslant{}& \|\Delta_h f\|_{L^\infty(\mathbb{R}^n)}\|\Delta_h g\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(\mathbb{R}^n)}\|\Delta_h g\|_{L^\infty(\mathbb{R}^n)} \\
&+ \|\Delta_h f\|_{L^\infty(\mathbb{R}^n)}\|g\|_{L^\infty(\mathbb{R}^n)} \\
\leqslant{}& (3\|f\|_{Lip_1(\mathbb{R}^n)}\|g\|_{L^\infty(\mathbb{R}^n)} + C\|f\|_{L^\infty(\mathbb{R}^n)})|h|,
\end{aligned}$$

which leads to the continuity of $\mathfrak{M}(f, g)$ at x. Thus, $\mathfrak{M}(f, g)$ is continuous on $\mathbb{R}^n$.

Next we assume that $f \in \mathrm{Lip}_1(\mathbb{R}^n)$ and $g \in W^{1,p}(\mathbb{R}^n)$ for $1 < p < \infty$. Without loss of generality we may assume that $g \geqslant 0$. There exists a sequence of functions $\{g_k\}_{k\geqslant 1} \subset \mathcal{C}_0^\infty(\mathbb{R}^n)$ such that $0 \leqslant g_k(x) \leqslant g(x)$ for any $k \geqslant 1$ and $x \in \mathbb{R}^n$. Moreover, $g_k \to g$ in $W^{1,p}(\mathbb{R}^n)$ as $k \to \infty$. We also assume that

$$\|g_k - g\|_{W^{1,p}(\mathbb{R}^n)} \leqslant 2^{-2k} \quad \text{for } k \geqslant 1. \tag{3.3}$$

For $\lambda > 0$, we set $O_\lambda = \{x \in \mathbb{R}^n \colon \mathfrak{M}(f, g)(x) > \lambda\}$. In view of (1.3), we can get the following weak type inequality for the Sobolev capacity:

$$\begin{aligned}
(C_p(O_\lambda))^{1/p} &\leqslant \frac{1}{\lambda}\bigg(\int_{\mathbb{R}^n} ((\mathfrak{M}(f, g)(x))^p + |\nabla\mathfrak{M}(f, g)(x)|^p)\,\mathrm{d}x\bigg)^{1/p} \\
&\leqslant \frac{1}{\lambda}\|\mathfrak{M}(f, g)\|_{W^{1,p}(\mathbb{R}^n)} \leqslant C\frac{1}{\lambda}\|f\|_{\mathrm{Lip}_1(\mathbb{R}^n)}\|g\|_{W^{1,p}(\mathbb{R}^n)}.
\end{aligned} \tag{3.4}$$

For any $k \geqslant 1$, let $A_k := \{x \in \mathbb{R}^n \colon \mathfrak{M}(f,g)(x) - \mathfrak{M}(f,g_k)(x) > 2^{-k}\}$. Since $\mathfrak{M}(f,g)$ is lower-semicontinuous and $\mathfrak{M}(f,g_k)$ is continuous, then $\mathfrak{M}(f,g)-\mathfrak{M}(f,g_k)$ is lower-semicontinuous. Thus, for any $k \geqslant 1$, the set A_k is open. By the sublinearity of $\mathfrak{M}$, we have

$$\mathfrak{M}(f,g)(x) - \mathfrak{M}(f,g_k)(x) \leqslant \mathfrak{M}(f,g-g_k)(x).$$

This together with (3.4) yield that

$$(C_p(A_k))^{1/p} \leqslant C2^k \|f\|_{\mathrm{Lip}_1(\mathbb{R}^n)} \|g-g_k\|_{W^{1,p}(\mathbb{R}^n)} \leqslant C\|f\|_{\mathrm{Lip}_1(\mathbb{R}^n)} 2^{-k}. \tag{3.5}$$

Let $B_k = \bigcup_{i=k}^{\infty} A_i$ with $k \geqslant 1$. By the subadditivity and (3.5),

$$C_p(B_k) \leqslant \sum_{i=k}^{\infty} C_p(A_i) \leqslant C\|f\|^p_{\mathrm{Lip}_1(\mathbb{R}^n)} \sum_{i=k}^{\infty} 2^{-ip} \leqslant C2^{-(k-1)p} \quad \forall\, k \geqslant 1.$$

This gives that $\lim_{k\to\infty} C_p(B_k) = 0$. It follows that for a fixed $\varepsilon > 0$, there exists a positive integer K_0 such that $C_p(B_{K_0}) < \varepsilon$. On the other hand, we see that for $x \in \mathbb{R}^n \setminus B_{K_0}$,

$$\mathfrak{M}(f,g)(x) - \mathfrak{M}(f,g_k)(x) \leqslant 2^{-k} \quad \forall\, k \geqslant K_0.$$

Thus, $\{\mathfrak{M}(f,g_k)\}_{k\geqslant 1}$ converges to $\mathfrak{M}(f,g)$ uniformly in $\mathbb{R}^n \setminus B_{K_0}$. Note that $\mathfrak{M}(f,g_k) \in \mathcal{C}(\mathbb{R}^n)$. Therefore, $\mathfrak{M}(f,g)$ is continuous in $\mathbb{R}^n \setminus B_{K_0}$. It is easy to see that $\mathbb{R}^n \setminus B_{K_0}$ is closed. The above facts imply that $\mathfrak{M}(f,g)$ is p-quasicontinuous. □

4. Proof of Theorem 1.10

This section is devoted to presenting the proof of Theorem 1.10. Let us introduce a characterization of a.e. approximate differentiable function.

Lemma 4.1 ([21], Theorem 1)**.** *Let $f\colon E \to \mathbb{R}$ be measurable, $E \subset \mathbb{R}^n$. Then the following conditions are equivalent:*

(i) *f is approximately differentiable a.e.*

(ii) *For any $\varepsilon > 0$, there is a closed set $F \subset E$ and a locally Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}$ such that $f = g$ on $x \in F$ and $|E \setminus F| < \varepsilon$.*

In order to prove Theorem 1.10, we need the following result.

Lemma 4.2. *Let $f \in \mathrm{Lip}_1(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$. For any $\varepsilon > 0$, we define the truncated bilinear maximal operator $\mathfrak{M}_\varepsilon$ by*

$$\mathfrak{M}_\varepsilon(f,g)(x) = \sup_{r\geqslant\varepsilon} \frac{1}{|B(O,r)|} \int_{\mathbb{R}^n} |f(x+y)g(x-y)|\,\mathrm{d}y, \quad x \in \mathbb{R}^n.$$

Then for every $\varepsilon > 0$, the function $\mathfrak{M}_\varepsilon(f,g)$ is Lipschitz continuous.

P r o o f. Let $x, y \in \mathbb{R}$ and $\varepsilon > 0$. It is enough to show that

$$|\mathfrak{M}_\varepsilon(f,g)(x) - \mathfrak{M}_\varepsilon(f,g)(y)| \leqslant C|x-y|, \tag{4.1}$$

where $C > 0$ is independent of x, y. Without loss of generality we may assume that $\mathfrak{M}_\varepsilon(f,g)(x) > \mathfrak{M}_\varepsilon(f,g)(y)$. When $|x-y| \geqslant \varepsilon/n$, inequality (4.1) follows from the following trivial estimate:

$$|\mathfrak{M}_\varepsilon(f,g)(x)| \leqslant C_n \varepsilon^{-n} \|f\|_{L^\infty(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)}. \tag{4.2}$$

Next we consider the case $|x-y| < \varepsilon/n$. For convenience, we can redefine $\mathfrak{M}_\varepsilon$ by

$$\mathfrak{M}_\varepsilon(f,g)(z) = \sup_{r \geqslant \varepsilon} \frac{1}{|B(O,r)|} \int_{B(z,r)} |f(u)g(2z-u)| \, \mathrm{d}u, \quad z \in \mathbb{R}^n.$$

Let $r \geqslant \varepsilon$. Note that $B(x,r) \subset B(y, r+|x-y|)$. Then we have

$$\begin{aligned}
\frac{1}{|B(y,r+|x-y|)|} & \int_{B(y,r+|x-y|)} |g(u)f(2y-u)| \, \mathrm{d}u \\
& \geqslant \frac{1}{|B(y,r+|x-y|)|} \int_{B(x,r)} |g(u)f(2y-u)| \, \mathrm{d}u \\
& \geqslant \frac{1}{|B(y,r+|x-y|)|} \int_{B(x,r)} |g(u)f(2x-u)| \, \mathrm{d}u \\
& \quad + \frac{1}{|B(y,r+|x-y|)|} \int_{B(x,r)} |g(u)|(|f(2y-u)| - |f(2x-u)|) \, \mathrm{d}u.
\end{aligned}$$

Noting that for any $r \geqslant a > 0$, $b \geqslant 0$ and $\delta \geqslant 0$,

$$\Big(\frac{r}{r+b}\Big)^\delta - 1 = \Big(\frac{1}{1+b/r}\Big)^\delta - 1^\delta = \int_0^{b/r} \frac{-\delta}{(1+x)^{\delta+1}} \, \mathrm{d}x \geqslant -\frac{b}{r}\delta \geqslant -\frac{b}{a}\delta.$$

It follows that

$$\begin{aligned}
\frac{1}{|B(y,r+|x-y|)|} & \int_{B(x,r)} |g(u)f(2x-u)| \, \mathrm{d}u \\
& = \frac{|B(x,r)|}{|B(y,r+|x-y|)|} \frac{1}{|B(x,r)|} \int_{B(x,r)} |g(u)f(2x-u)| \, \mathrm{d}u \\
& \geqslant (1 - n\varepsilon^{-1}|x-y|) \frac{1}{|B(x,r)|} \int_{B(x,r)} |g(u)f(2x-u)| \, \mathrm{d}u.
\end{aligned}$$

On the other hand, one gets

$$\begin{aligned}
\Big| \frac{1}{|B(y,r+|x-y|)|} & \int_{B(x,r)} |g(u)|(|f(2y-u)| - |f(2x-u)|) \, \mathrm{d}u \Big| \\
& \leqslant \frac{1}{|B(y,r+|x-y|)|} \int_{B(x,r)} |g(u)||f(2y-u) - f(2x-u)| \, \mathrm{d}u \\
& \leqslant C_n \|f\|_{Lip_1(\mathbb{R}^n)} \varepsilon^{-n} \|g\|_{L^1(\mathbb{R}^n)} |x-y|.
\end{aligned}$$

Thus, we have

$$\begin{aligned}\frac{1}{|B(y,r+|x-y|)|}&\int_{B(y,r+|x-y|)}|g(u)f(2y-u)|\,\mathrm{d}u\\&\geqslant(1-n\varepsilon^{-1}|x-y|)\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(u)g(2x-u)|\,\mathrm{d}u\\&\quad-C_n\|f\|_{Lip_1(\mathbb{R}^n)}\varepsilon^{-n}\|g\|_{L^1(\mathbb{R}^n)}|x-y|.\end{aligned}$$

Since $(1-n\varepsilon^{-1}|x-y|)>0$, we have

$$\mathfrak{M}_\varepsilon(f,g)(y)\geqslant(1-n\varepsilon^{-1}|x-y|)\mathfrak{M}_\varepsilon(f,g)(x)-C_n\|f\|_{Lip_1(\mathbb{R}^n)}\varepsilon^{-n}\|g\|_{L^1(\mathbb{R}^n)}|x-y|.$$

This together with (4.2) implies that

$$\begin{aligned}&\mathfrak{M}_\varepsilon(f,g)(x)-\mathfrak{M}_\varepsilon(f,g)(y)\\&\quad\leqslant n\varepsilon^{-1}|x-y|\mathfrak{M}_\varepsilon(f,g)(x)+C_n\|f\|_{Lip_1(\mathbb{R}^n)}\varepsilon^{-n}\|g\|_{L^1(\mathbb{R}^n)}|x-y|\\&\quad\leqslant C_n\varepsilon^{-n}\|f\|_{L^\infty(\mathbb{R}^n)}\|g\|_{L^1(\mathbb{R}^n)}n\varepsilon^{-1}|x-y|+C_n\|f\|_{Lip_1(\mathbb{R}^n)}\varepsilon^{-n}\|g\|_{L^1(\mathbb{R}^n)}|x-y|\\&\quad\leqslant C|x-y|.\end{aligned}$$

This proves (4.1) and completes the proof of Lemma 4.1. □

We now prove Theorem 1.10.

Proof of Theorem 1.10. We set

$$E=\Big\{x\in\mathbb{R}^n\colon\ \lim_{r\to0}\frac{1}{|B(O,r)|}\int_{B(O,r)}|f(x+y)g(x-y)|\,\mathrm{d}y=|f(x)g(x)|\Big\}.$$

By an argument similar to the one that proves that almost every point is a Lebesgue point, one can conclude that $|\mathbb{R}^n\setminus E|=0$. Let $x\in E$ be such that $\mathfrak{M}(f,g)(x)>|f(x)g(x)|$. We first prove that there exists $r>0$ such that

$$\mathfrak{M}(f,g)(x)=\frac{1}{|B(O,r)|}\int_{B(O,r)}|f(x+y)g(x-y)|\,\mathrm{d}y.$$

If not, we have

$$\mathfrak{M}(f,g)(x)=\sup_{r\geqslant k}\frac{1}{|B(O,r)|}\int_{B(O,r)}|f(x+y)g(x-y)|\,\mathrm{d}y\quad\forall k\geqslant1.$$

It is clear that

$$\mathfrak{M}(f,g)(x)\leqslant C_nk^{-n}\|f\|_{L^\infty(\mathbb{R}^n)}\|g\|_{L^1(\mathbb{R}^n)}\quad\forall k\geqslant1.$$

This yields that $\mathfrak{M}(f,g)(x) = 0$ by letting $k \to \infty$. This is a contradiction. So we can write

$$\mathbb{R}^n = (\mathbb{R}^n \setminus E) \cup \{x \in \mathbb{R}^n \colon \mathfrak{M}(f,g)(x) = |f(x)g(x)|\} \cup F,$$

where $F = \bigcup_{k=1}^{\infty} F_k$ and

$$F_k = \{x \in \mathbb{R}^n \colon \mathfrak{M}(f,g)(x) = \mathfrak{M}_{1/k}(f,g)(x)\}.$$

Obviously, $F_k \subset F_{k+1}$. We get by Lemma 4.2 that $\mathfrak{M}_{1/k}(f,g)$ is Lipschitz continuous for any $k \geqslant 1$. Hence, for all $k \geqslant 1$ the function $\mathfrak{M}(f,g)\chi_{F_{k+1}\setminus F_k}$ is approximately differentiable almost everywhere. Write

$$\mathfrak{M}(f,g)\chi_F = \mathfrak{M}(f,g)\chi_{F_1} + \sum_{k=1}^{\infty} \mathfrak{M}(f,g)\chi_{F_{k+1}\setminus F_k}.$$

Invoking Lemma 4.1, we have that $\mathfrak{M}(f,g)\chi_F$ is approximately differentiable a.e. On the other hand, from Lemma 4.1 we see that $|fg|$ is approximately differentiable a.e. So $\mathfrak{M}(f,g)$ is approximately differentiable a.e. in the set $\{x \in \mathbb{R}^n \colon \mathfrak{M}(f,g)(x) = |f(x)g(x)|\}$. Note that $|\mathbb{R}^n \setminus E| = 0$. Therefore, $\mathfrak{M}(f,g)$ is approximately differentiable a.e. $\square$

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