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CONSECUTIVE SQUARE-FREE VALUES OF THE TYPE $x^2+y^2+z^2+k,\ x^2+y^2+z^2+k+1$

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Abstract. We show that for any given integer k there exist infinitely many consecutive square-free numbers of the type $x^2+y^2+z^2+k$, $x^2+y^2+z^2+k+1$. We also establish an asymptotic formula for $1\leqslant x,y,z\leqslant H$ such that $x^2+y^2+z^2+k$, $x^2+y^2+z^2+k+1$ are square-free. The method we used in this paper is due to Tolev.

Keywords: square-free number; Salié sum; Gauss sum

MSC 2020: 11N37, 11L05, 11L40

1. Notations

Let H be a sufficiently large positive number and k be any given integer. The letters m, n, l, t, q, q_1, q_2 stand for positive integers. Let ε be an arbitrarily small positive real number which may not be the same in all appearances. Put $e(y)=\mathrm{e}^{2\pi \mathrm{i} y}$ and $e_q(y)=\mathrm{e}^{2\pi \mathrm{i} y/q}$. For any odd q, the Jacobi symbol is denoted by $(\frac{\cdot}{q})$. The symbol $\bar{n}_q$ means the inverse of n modulo q for $(n,q)=1$. The greatest common divisor of $a_1,\ldots,a_n$ is denoted by $(a_1,\ldots,a_n)$. Let $\mu(n)$ be Möbius function. As usual, the functions $\tau(n)$ and $\omega(n)$ represent the number of positive divisors of n and the number of distinct prime divisors of n, respectively.

Define the Gauss sum as follows:

$$G(q;n,m)=\sum_{1\leqslant x\leqslant q} e_q(nx^2+mx),\quad G(q;n)=\sum_{1\leqslant x\leqslant q} e_q(nx^2).$$

The Salié sum is defined as

$$S(q;n,m)=\sum_{\substack{1\leqslant x\leqslant q\\ (x,q)=1}}\Big(\frac{x}{q}\Big)e_q(nx+m\bar{x}).$$

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Define

$$\lambda(q_1, q_2; n, m, l) = \sum_{x,y,z:\ (1.1)} e_{q_1 q_2}(nx + my + lz),$$

where the summation is taken over the integers x, y, z satisfying the conditions

$$\begin{cases} 1 \leqslant x, y, z \leqslant q_1 q_2, \\ x^2 + y^2 + z^2 + k \equiv 0 \pmod{q_1}, \\ x^2 + y^2 + z^2 + k + 1 \equiv 0 \pmod{q_2}. \end{cases} \tag{1.1}$$

We also define

$$\lambda(q_1, q_2) = \lambda(q_1, q_2; 0, 0, 0). \tag{1.2}$$

2. Introduction

The problem for the consecutive square-free numbers was first studied by Carlitz in 1932, see [1]. For any $\varepsilon > 0$, Carlitz proved that

$$\sum_{1 \leqslant x \leqslant H} \mu^2(x)\mu^2(x+1) = \prod_p \Bigl(1 - \frac{2}{p^2}\Bigr) H + O(H^{2/3+\varepsilon}).$$

The error term was improved by Heath-Brown (see [5]) to $H^{7/11} \log^7 x$ and Reuss (see [7]) to $H^{(26+\sqrt{433})/81}$. In 2012, Tolev in [8] studied the square-free values of the polynomial with two variables. For any $\varepsilon > 0$, he proved that

$$\Gamma(H) = \prod_p \Bigl(1 - \frac{\lambda_2(p^2)}{p^4}\Bigr) H^2 + O(H^{4/3+\varepsilon}),$$

where $\Gamma(H)$ is the number of the square-free values of $x^2 + y^2 + 1$ with $1 \leqslant x, y \leqslant H$ and $\lambda_2(q)$ is the number of the integer solutions to the congruence equation

$$1 \leqslant x, y \leqslant q, \quad x^2 + y^2 + 1 \equiv 0 \pmod{q}.$$

With the method developed by Tolev, Dimitrov in [2] studied consecutive square-free numbers of the form x^2+y^2+1, x^2+y^2+2. He also established an asymptotic formula

$$\sum_{1 \leqslant x, y \leqslant H} \mu^2(x^2 + y^2 + 1)\mu^2(x^2 + y^2 + 2) = cH^2 + O(H^{8/5+\varepsilon})$$

for an absolute constant c. It is worth mentioning that Dimitrov in [3] studied the pairs of square-free values of the type n^2+1, n^2+2 and gave the asymptotic formula

$$\sum_{1 \leqslant n \leqslant X} \mu^2(n^2 + 1)\mu^2(n^2 + 2) = \sigma X + O(X^{8/9+\varepsilon})$$

for an absolute constant σ.

Recently, with Tolev's method and an estimate for the Salié sum, Zhou and Ding in [9] studied the asymptotic formula of square-free values represented by the polynomial $x^2 + y^2 + z^2 + k$. They obtained the asymptotic formula

$$\sum_{1\leqslant x,y,z\leqslant H} \mu^2(x^2 + y^2 + z^2 + k) = c_1 H^3 + O(H^{7/3+\varepsilon}),$$

where c_1 is an absolute constant.

Inspired by the work of Tolev, Dimitrov, Zhou and Ding, combining their methods, we shall prove the following theorem.

Theorem 2.1. *Let*

$$S(H) = \sum_{1\leqslant x,y,z\leqslant H} \mu^2(x^2 + y^2 + z^2 + k)\mu^2(x^2 + y^2 + z^2 + k + 1).$$

For any given integer k and any $\varepsilon > 0$, we have

$$S(H) = \prod_p \Big(1 - \frac{\lambda(p^2, 1) + \lambda(1, p^2)}{p^6}\Big) H^3 + O(H^{7/3+\varepsilon}). \tag{2.1}$$

3. Auxiliary lemmas

To prove Theorem 2.1, we need the following lemmas. The first one gives some basic properties of the Gauss sum.

Lemma 3.1 ([4])**.** *For the Gauss sum we have:*

(1) *If $(q_1, q_2) = 1$, then*

$$G(q_1q_2; m_1q_2 + m_2q_1, n) = G(q_1; m_1q_2^2, n)G(q_2; m_2q_1^2, n).$$

(2) *If $(q, n) = d$, then*

$$G(q; n, m) = \begin{cases} dG(q/d; n/d, m/d) & \text{if } d \mid m, \\ 0 & \text{if } d \nmid m. \end{cases}$$

(3) *If $(q, 2n) = 1$, then*

$$G(q; n, m) = e_q(-\overline{(4n)}_q m^2)\Big(\frac{n}{q}\Big)G(q; 1).$$

The next lemma gives us an estimate for the Salié sum.

Lemma 3.2 ([6]). *We have*

$$|S(q;n,m)| \leqslant 2^{\omega(q)}\sqrt{q}.$$

The next lemma shows that $\lambda(q_1,q_2;n,m,l)$ has some multiplicative properties.

Lemma 3.3. *If* $(q_1'q_1'', q_2'q_2'') = (q_1', q_1'') = (q_2', q_2'') = 1$, *then*

$$\lambda(q_1'q_1'', q_2'q_2''; n,m,l) = \lambda(q_1', q_2'; \overline{q_1''q_2''}n, \overline{q_1''q_2''}m, \overline{q_1''q_2''}l) \cdot \lambda(q_1'', q_2''; \overline{q_1'q_2'}n, \overline{q_1'q_2'}m, \overline{q_1'q_2'}l),$$

where $\overline{q_1'q_2'}$ *is the inverse of* $q_1'q_2'$ *modulo* $q_1''q_2''$ *and* $\overline{q_1''q_2''}$ *is the inverse of* $q_1''q_2''$ *modulo* $q_1'q_2'$.

Proof. The proof is similar to Lemma 2.2 of [9]. We skip the details and leave it to the reader. □

We now use Lemmas 3.1, 3.2 and 3.3 to give the upper bound of $\lambda(q_1,q_2;n,m,l)$.

Lemma 3.4. *If* $8 \nmid q_1$, $8 \nmid q_2$ *and* $(q_1,q_2) = 1$, *then we have*

$$\lambda(q_1,q_2;n,m,l) \ll q_1q_2\tau(q_1q_2)2^{\omega(q_1q_2)}(q_1q_2,n,m,l).$$

In particular, we have

$$\lambda(q_1,q_2;n,m,l) \ll (q_1q_2)^{1+\varepsilon}(q_1q_2,n,m,l) \quad \text{and} \quad \lambda(q_1,q_2) \ll (q_1q_2)^{2+\varepsilon}.$$

Proof. We first consider the case $2 \nmid q_1q_2$. By Lemma 3.1 (1) and the orthogonality relations,

$$\sum_{1\leqslant h\leqslant q} e_q(ht) = \begin{cases} q & \text{if } t \equiv 0 \pmod q, \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$\begin{aligned}
&\lambda(q_1,q_2;n,m,l) \\
&\quad = (q_1q_2)^{-1} \sum_{1\leqslant x,y,z\leqslant q_1q_2} e_{q_1q_2}(nx+my+lz) \sum_{1\leqslant h_1\leqslant q_1} e_{q_1}(h_1(x^2+y^2+z^2+k)) \\
&\qquad \times \sum_{1\leqslant h_2\leqslant q_2} e_{q_2}(h_2(x^2+y^2+z^2+k+1)) \\
&\quad = (q_1q_2)^{-1} \sum_{1\leqslant h_1\leqslant q_1} e_{q_1}(h_1k) \sum_{1\leqslant h_2\leqslant q_2} e_{q_2}(h_2(k+1)) G(q_1q_2; h_1q_2+h_2q_1, n) \\
&\qquad \times G(q_1q_2; h_1q_2+h_2q_1, m) G(q_1q_2; h_1q_2+h_2q_1, l) \\
&\quad = (q_1q_2)^{-1} \sum_{1\leqslant h_1\leqslant q_1} e_{q_1}(h_1k) G(q_1; h_1q_2^2, n) G(q_1; h_1q_2^2, m) G(q_1; h_1q_2^2, l) \\
&\qquad \times \sum_{1\leqslant h_2\leqslant q_2} e_{q_2}(h_2(k+1)) G(q_2; h_2q_1^2, n) G(q_2; h_2q_1^2, m) G(q_2; h_2q_1^2, l).
\end{aligned}$$

Since $2 \nmid q_1q_2$ and $(q_1,q_2)=1$, by Lemma 3.1 (2) and (3), the quantity $\lambda(q_1,q_2;n,m,l)$ equals to

$$(3.1)\qquad (q_1q_2)^2 \sum_{\substack{d_1|q_1\\(q_1/d_1)|(n,m,l)}} d_1^{-3} \sum_{\substack{1\leqslant r_1\leqslant d_1\\(r_1,d_1)=1}} e_{d_1}(r_1k)G(d_1;r_1q_2^2,nd_1q_1^{-1})$$
$$\times G(d;r_1q_2^2,md_1q_1^{-1})G(d;r_1q_2^2,ld_1q_1^{-1})$$
$$\times \sum_{\substack{d_2|q_2\\(q_2/d_2)|(n,m,l)}} d_2^{-3} \sum_{\substack{1\leqslant r_2\leqslant d_2\\(r_2,d_2)=1}} e_{d_2}(r_2(k+1))G(d_2;r_2q_1^2,nd_2q_2^{-1})$$
$$\times G(d_2;r_2q_1^2,md_2q_2^{-1})G(d_2;r_2q_1^2,ld_2q_2^{-1}).$$

Note that if $2\nmid q$, $d|q$, $(q',q)=1$ and $(r,d)=1$, then we have $(d,2rq')=1$. By Lemma 3.1 (3), we obtain

$$(3.2)\qquad G(d;rq',ndq^{-1})G(d;rq',mdq^{-1})G(d;rq',ldq^{-1})$$
$$=\left(\frac{rq'}{d}\right)^3 G(d;1)^3 e_d(-\overline{(4rq')}_d(n^2+m^2+l^2)d^2q^{-2}).$$

Thus, by equations (3.1), (3.2) and the properties of Jacobi symbol, we get

$$\begin{aligned}
&\lambda(q_1,q_2;n,m,l)\\
&\quad=(q_1q_2)^2 \sum_{\substack{d_1|q_1\\(q_1/d_1)|(n,m,l)}} d_1^{-3}G(d_1;1)^3\\
&\qquad\times \sum_{\substack{1\leqslant r_1\leqslant d_1\\(r_1,d_1)=1}} \left(\frac{r_1}{d_1}\right)^3 e_{d_1}(r_1k-\overline{4}_{d_1}(n^2+m^2+l^2)d_1^2q_1^{-2}\overline{r_1q_2^2}_{d_1})\\
&\qquad\times \sum_{\substack{d_2|q_2\\(q_2/d_2)|(n,m,l)}} d_2^{-3}G(d_2;1)^3\\
&\qquad\times \sum_{\substack{1\leqslant r_2\leqslant d_2\\(r_2,d_2)=1}} \left(\frac{r_2}{d_2}\right)^3 e_{d_2}(r_2(k+1)-\overline{4}_{d_2}(n^2+m^2+l^2)d_2^2q_2^{-2}\overline{r_2q_1^2}_{d_2})\\
&\quad=(q_1q_2)^2 \sum_{\substack{d_1|q_1\\(q_1/d_1)|(n,m,l)}} d_1^{-3}G(d_1;1)^3S(d_1;k,-\overline{4}_{d_1}(n^2+m^2+l^2)d_1^2q_1^{-2}\overline{q_2^2}_{d_1})\\
&\qquad\times \sum_{\substack{d_2|q_2\\(q_2/d_2)|(n,m,l)}} d_2^{-3}G(d_2;1)^3S(d_2;k+1,-\overline{4}_{d_2}(n^2+m^2+l^2)d_2^2q_2^{-2}\overline{q_1^2}_{d_2}).
\end{aligned}$$

It is well known that $|G(d,1)| = \sqrt{d}$ for $2 \nmid d$, so we have by Lemma 3.2

$$\begin{aligned}\lambda(q_1,q_2;n,m,l) &\ll (q_1q_2)^2 \sum_{\substack{d_1|q_1\\(q_1/d_1)|(q_1,n,m,l)}} d_1^{-3} d_1^{3/2} d_1^{1/2} 2^{\omega(d_1)}\\ &\times \sum_{\substack{d_2|q_2\\(q_2/d_2)|(q_2,n,m,l)}} d_2^{-3} d_2^{3/2} d_2^{1/2} 2^{\omega(d_2)}\\ &\ll (q_1q_2)^2 \sum_{(d_1/q_1)|(q_1,n,m,l)} d_1^{-1}2^{\omega(d_1)} \sum_{(d_2/q_2)|(q_2,n,m,l)} d_2^{-1}2^{\omega(d_2)}\\ &\ll (q_1q_2)^2 2^{\omega(q_1)}2^{\omega(q_2)} \sum_{r_1|(q_1,n,m,l)} q_1^{-1}r_1 \sum_{r_2|(q_2,n,m,l)} q_2^{-1}r_2\\ &\ll q_1q_2\tau(q_1q_2)2^{\omega(q_1q_2)}(q_1q_2,n,m,l).\end{aligned}$$

Now, in the general case $8 \nmid q_1$, $8 \nmid q_2$ and $(q_1,q_2) = 1$, we can suppose that $q_1 = 2^h q_1'$ with $2 \nmid q_1'$ and $h \leqslant 2$. Our lemma follows from Lemma 3.3 and the trivial estimate $\lambda(2^h,1;n,m,l) \leqslant 2^{3h} \leqslant 64$. □

Now, we use the above lemma to give an estimate which shall be encountered in the proof of our theorem.

Lemma 3.5. *Let $H \geqslant 2$, $8 \nmid q_1$, $8 \nmid q_2$ and $(q_1,q_2) = 1$. Then for the sums*

$$U = \sum_{1\leqslant n\leqslant H} \frac{|\lambda(q_1,q_2;n,0,0)|}{n}, \quad V = \sum_{1\leqslant n,m\leqslant H} \frac{|\lambda(q_1,q_2;n,m,0)|}{nm},$$
$$W = \sum_{1\leqslant n,m,l\leqslant H} \frac{|\lambda(q_1,q_2;n,m,l)|}{nml},$$

we have

$$U \ll (q_1q_2)^{1+\varepsilon}H^{\varepsilon}, \quad V \ll (q_1q_2)^{1+\varepsilon}H^{\varepsilon}, \quad W \ll (q_1q_2)^{1+\varepsilon}H^{\varepsilon}.$$

P r o o f. By Lemma 3.4, we have

$$\begin{aligned}U &\ll (q_1q_2)^{1+\varepsilon} \sum_{1\leqslant n\leqslant H} \frac{(q_1q_2,n)}{n} \ll (q_1q_2)^{1+\varepsilon} \sum_{d|q_1q_2} d \sum_{\substack{1\leqslant n\leqslant H\\ d|n}} \frac{1}{n}\\ &= (q_1q_2)^{1+\varepsilon} \sum_{d|q_1q_2} d \sum_{1\leqslant l\leqslant H/d} \frac{1}{ld} \ll (q_1q_2)^{1+\varepsilon}\omega(q_1q_2)\log H\\ &\ll (q_1q_2)^{1+\varepsilon}H^{\varepsilon}.\end{aligned}$$

Similarly,

$$V \ll (q_1q_2)^{1+\varepsilon} \sum_{1\leqslant n,m\leqslant H} \frac{(q_1q_2,n,m)}{nm} \ll (q_1q_2)^{1+\varepsilon} \sum_{1\leqslant m\leqslant H} \frac{1}{m} \sum_{1\leqslant n\leqslant H} \frac{(q_1q_2,n)}{n}$$
$$\ll (q_1q_2)^{1+\varepsilon} H^{\varepsilon}$$

and

$$W \ll (q_1q_2)^{1+\varepsilon} \sum_{1\leqslant n,m,l\leqslant H} \frac{(q_1q_2,n,m,l)}{nm} \ll (q_1q_2)^{1+\varepsilon} \Big(\sum_{1\leqslant m\leqslant H} \frac{1}{m}\Big)^2 \sum_{1\leqslant n\leqslant H} \frac{(q_1q_2,n)}{n}$$
$$\ll (q_1q_2)^{1+\varepsilon} H^{\varepsilon}.$$

This completes the proof. □

The next one is a standard result in the Fourier analysis.

Lemma 3.6 ([8])**.** *Let $\varrho(t) = \frac{1}{2} - \{t\}$, where $\{t\}$ is the fractional part of t. For any $H \geqslant 2$ we have*

$$\varrho(t) = \sum_{1\leqslant |n|\leqslant H} \frac{e(nt)}{2\pi i n} + O(g(H,t)),$$

where $g(H,t)$ is a positive, infinitely many times differentiable and periodic with period one, function of t. It can be represented as a Fourier series

$$g(H,t) = \sum_{n\in\mathbb{Z}} c_H(n)e(nt),$$

with coefficients $c_H(n)$ satisfying

$$c_H(n) \ll \frac{\log H}{H} \quad \text{for all } n, \quad \text{and} \quad \sum_{|n|>H^{1+\varepsilon}} |c_H(n)| \ll H^{-A}.$$

Here $A > 0$ is arbitrarily large and the constant in the above equality depends on A and ε.

The finial three lemmas dealt with certain estimates involved with sums of fractions.

Lemma 3.7. *Let $H \geqslant 2$, $8 \nmid q_1$, $8 \nmid q_2$, $(q_1,q_2) = 1$ and $q_1q_2 < H^{\alpha}$ for a given $\alpha > 0$. Define*

$$N_1(H,q_1,q_2) = \sum_{x,y,z\colon (1.1)} \Big(\varrho\Big(\frac{H-x}{q_1q_2}\Big) - \varrho\Big(\frac{-x}{q_1q_2}\Big)\Big),$$

then

$$N_1(H,q_1,q_2) \ll H^{\varepsilon-1}(q_1q_2)^{2+\varepsilon} + H^{\varepsilon}(q_1q_2)^{1+\varepsilon},$$

where the implied constant may depend on ε and α.

P r o o f. By Lemma 3.6, we have

$$\sum_{x,y,z\colon (1.1)} \varrho\Big(\frac{H-x}{q_1q_2}\Big) = \Sigma_1 + O(\Sigma_2),$$

where

$$\Sigma_1 = \sum_{x,y,z\colon (1.1)} \sum_{1\leqslant |n|\leqslant H} \frac{e_{q_1q_2}(n(H-x))}{2\pi i n} = \sum_{1\leqslant |n|\leqslant H} \frac{e_{q_1q_2}(nH)\lambda(q_1,q_2;-n,0,0)}{2\pi i n}$$

and

$$\Sigma_2 = \sum_{x,y,z\colon (1.1)} g\Big(H, \frac{H-x}{q_1q_2}\Big).$$

From Lemma 3.5 it follows that

$$\Sigma_1 \ll (q_1q_2)^{1+\varepsilon} H^{\varepsilon}.$$

By Lemmas 3.4, 3.5 and 3.6, we get

$$\begin{aligned}
\Sigma_2 &= \sum_{x,y,z\colon (1.1)} \Big(c_H(0) + \sum_{1\leqslant |n|\leqslant H^{1+\varepsilon}} c_H(n) e_{q_1q_2}(n(H-x)) + H^{-A}\Big)\\
&= c_H(0)\lambda(q_1,q_2) + \sum_{1\leqslant |n|\leqslant H^{1+\varepsilon}} c_H(n) e_{q_1q_2}(nH)\lambda(q_1,q_2;-n,0,0) + O(1)\\
&\ll H^{\varepsilon-1}(q_1q_2)^{2+\varepsilon} + H^{\varepsilon-1} \sum_{1\leqslant |n|\leqslant H^{1+\varepsilon}} |\lambda(q_1,q_2;-n,0,0)| + 1\\
&\ll H^{\varepsilon-1}(q_1q_2)^{2+\varepsilon} + H^{\varepsilon} \sum_{1\leqslant n\leqslant H^{1+\varepsilon}} \frac{|\lambda(q_1,q_2;-n,0,0)|}{n} + 1\\
&\ll H^{\varepsilon-1}(q_1q_2)^{2+\varepsilon} + H^{\varepsilon}(q_1q_2)^{1+\varepsilon}.
\end{aligned}$$

Therefore,

$$\sum_{x,y,z\colon (1.1)} \varrho\Big(\frac{H-x}{q_1q_2}\Big) \ll H^{\varepsilon-1}(q_1q_2)^{2+\varepsilon} + H^{\varepsilon}(q_1q_2)^{1+\varepsilon}.$$

The same argument gives that

$$\sum_{x,y,z\colon (1.1)} \varrho\Big(\frac{-x}{q_1q_2}\Big) \ll H^{\varepsilon-1}(q_1q_2)^{2+\varepsilon} + H^{\varepsilon}(q_1q_2)^{1+\varepsilon}.$$

This completes the proof. □

Lemma 3.8. *Let* $H \geqslant 2$, $8 \nmid q_1$, $8 \nmid q_2$, $(q_1, q_2) = 1$ *and* $q_1 q_2 < H^\alpha$ *for a given* $\alpha > 0$. *Define*

$$N_2(H, q_1, q_2) = \sum_{x,y,z\colon (1.1)} \Big(\varrho\Big(\frac{H-x}{q_1q_2}\Big)\varrho\Big(\frac{H-y}{q_1q_2}\Big) + \varrho\Big(\frac{-x}{q_1q_2}\Big)\varrho\Big(\frac{-y}{q_1q_2}\Big)\Big) - 2\sum_{x,y,z\colon (1.1)} \varrho\Big(\frac{H-x}{q_1q_2}\Big)\varrho\Big(\frac{-y}{q_1q_2}\Big),$$

then

$$N_2(H, q_1, q_2) \ll H^{\varepsilon-1}(q_1q_2)^{2+\varepsilon} + H^\varepsilon(q_1q_2)^{1+\varepsilon},$$

where the implied constant may depend on ε *and* α.

P r o o f. By Lemma 3.6, we have

$$\varrho\Big(\frac{H-x}{q_1q_2}\Big)\varrho\Big(\frac{H-y}{q_1q_2}\Big) = \sum_{1\leqslant|n|,|m|\leqslant H} \frac{e_{q_1q_2}((n+m)H)e_{q_1q_2}(-nx-my)}{(2\pi\mathrm{i})^2 nm} + O\Big(\sum_{1\leqslant|n|\leqslant H} \frac{1}{n} g\Big(H, \frac{H-x}{q_1q_2}\Big) + g\Big(H, \frac{H-x}{q_1q_2}\Big) g\Big(H, \frac{H-y}{q_1q_2}\Big)\Big).$$

By the estimate of $\sum_2$ in Lemma 3.7, we have

$$\sum_{x,y,z\colon (1.1)} \sum_{1\leqslant|n|\leqslant H} \frac{1}{n} g\Big(H, \frac{H-x}{q_1q_2}\Big) \ll H^{\varepsilon-1}(q_1q_2)^{2+\varepsilon} + H^\varepsilon(q_1q_2)^{1+\varepsilon}.$$

It is easy to see

$$\sum_{x,y,z\colon (1.1)} g\Big(H, \frac{H-x}{q_1q_2}\Big) g\Big(H, \frac{H-y}{q_1q_2}\Big) \ll H^{\varepsilon-1}(q_1q_2)^{2+\varepsilon} + H^\varepsilon(q_1q_2)^{1+\varepsilon},$$

by some crude adjustments with the estimate of Σ_2. Combining these together, we have

$$\begin{aligned}\sum_{x,y,z\colon (1.1)} \varrho\Big(\frac{H-x}{q_1q_2}\Big)\varrho\Big(\frac{H-y}{q_1q_2}\Big) &= \sum_{1\leqslant|n|,|m|\leqslant H} \frac{e_{q_1q_2}((n+m)H)\lambda(q_1, q_2; -n, -m, 0)}{(2\pi\mathrm{i})^2 nm} \\ &\quad + O(H^{\varepsilon-1}(q_1q_2)^{2+\varepsilon} + H^\varepsilon(q_1q_2)^{1+\varepsilon}) \\ &\ll \sum_{1\leqslant|n|,|m|\leqslant H} \frac{|\lambda(q_1, q_2; -n, -m, 0)|}{nm} \\ &\quad + H^{\varepsilon-1}(q_1q_2)^{2+\varepsilon} + H^\varepsilon(q_1q_2)^{1+\varepsilon} \\ &\ll H^{\varepsilon-1}(q_1q_2)^{2+\varepsilon} + H^\varepsilon(q_1q_2)^{1+\varepsilon},\end{aligned}$$

where the last estimate of the double summation evidently follows from Lemma 3.6. The other terms of $N_2(H,q_1,q_2)$ can be bounded by similar arguments. Hence,

$$N_2(H,q_1,q_2) \ll H^{\varepsilon-1}(q_1q_2)^{2+\varepsilon} + H^{\varepsilon}(q_1q_2)^{1+\varepsilon}.$$

This completes the proof. □

Lemma 3.9. *Let $H \geqslant 2$, $8 \nmid q_1$, $8 \nmid q_2$, $(q_1,q_2)=1$ and $q_1q_2 < H^{\alpha}$ for a given $\alpha > 0$. Define*

$$\begin{aligned} N_3(H,q_1,q_2) = &\sum_{x,y,z\colon (1.1)} \Big(\varrho\Big(\frac{H-x}{q_1q_2}\Big)\varrho\Big(\frac{H-y}{q_1q_2}\Big)\varrho\Big(\frac{H-z}{q_1q_2}\Big)\Big) \\ &- 3\sum_{x,y,z\colon (1.1)} \Big(\varrho\Big(\frac{H-x}{q_1q_2}\Big)\varrho\Big(\frac{H-y}{q_1q_2}\Big)\varrho\Big(\frac{-z}{q_1q_2}\Big)\Big) \end{aligned}$$

and

$$\begin{aligned} N_3'(H,q_1,q_2) = 3 &\sum_{x,y,z\colon (1.1)} \varrho\Big(\frac{H-x}{q_1q_2}\Big)\varrho\Big(\frac{-y}{q_1q_2}\Big)\varrho\Big(\frac{-z}{q_1q_2}\Big) \\ &- \sum_{x,y,z\colon (1.1)} \varrho\Big(\frac{-x}{q_1q_2}\Big)\varrho\Big(\frac{-y}{q_1q_2}\Big)\varrho\Big(\frac{-z}{q_1q_2}\Big), \end{aligned}$$

then

$$N_3(H,q_1,q_2) + N_3'(H,q_1,q_2) \ll H^{\varepsilon-1}(q_1q_2)^{2+\varepsilon} + H^{\varepsilon}(q_1q_2)^{1+\varepsilon},$$

where the implied constant may depend on ε and α.

P r o o f. The proof of this lemma shares the same spirit as the former ones. Namely,

$$\begin{aligned} &\sum_{x,y,z\colon (1.1)} \varrho\Big(\frac{H-x}{q_1q_2}\Big)\varrho\Big(\frac{H-y}{q_1q_2}\Big)\varrho\Big(\frac{H-z}{q_1q_2}\Big) \\ &\quad = \sum_{1\leqslant |n|,|m|,|l|\leqslant H} \frac{e_{q_1q_2}((n+m)H)\lambda(q_1,q_2;-n,-m,-l)}{(2\pi \mathrm{i})^3 nml} \\ &\qquad + O(H^{\varepsilon-1}(q_1q_2)^{2+\varepsilon} + H^{\varepsilon}(q_1q_2)^{1+\varepsilon}) \\ &\quad \ll \sum_{1\leqslant |n|,|m|,|l|\leqslant H} \frac{|\lambda(q;-n,-m,-l)|}{nml} + H^{\varepsilon-1}(q_1q_2)^{2+\varepsilon} + H^{\varepsilon}(q_1q_2)^{1+\varepsilon} \\ &\quad \ll H^{\varepsilon-1}(q_1q_2)^{2+\varepsilon} + H^{\varepsilon}(q_1q_2)^{1+\varepsilon}. \end{aligned}$$

All terms of $N_3(H,q_1,q_2)$ and $N_3'(H,q_1,q_2)$ can be bounded by similar arguments, so we have

$$N_3(H,q_1,q_2) + N_3'(H,q_1,q_2) \ll H^{\varepsilon-1}(q_1q_2)^{2+\varepsilon} + H^{\varepsilon}(q_1q_2)^{1+\varepsilon}.$$

This completes the proof. □

4. Proof of Theorem 2.1

Let

$$D(H, q_1, q_2) = \sum_{x,y,z:\ (1.1)} 1.$$

By the well-known identity

$$\mu^2(n) = \sum_{d^2|n} \mu(d),$$

we obtain

$$\begin{aligned}
(4.1)\qquad S(H) &= \sum_{1\leqslant x,y,z\leqslant H} \sum_{d_1^2|x^2+y^2+z^2+k} \mu(d_1) \sum_{d_2^2|x^2+y^2+z^2+k+1} \mu(d_2) \\
&= \sum_{\substack{d_1,d_2\\(d_1,d_2)=1}} \mu(d_1)\mu(d_2) \sum_{x,y,z:\ (1.1)} 1 \\
&= \sum_{\substack{d_1,d_2\\(d_1,d_2)=1}} \mu(d_1)\mu(d_1)D(H, d_1^2, d_2^2).
\end{aligned}$$

We split the above sum into the following three parts:

$$(4.2)\qquad S(H) = S_1(H) + S_2(H) + S_3(H),$$

where

$$\begin{aligned}
S_1(H) &= \sum_{\substack{d_1d_2\leqslant\eta\\(d_1,d_2)=1}} \mu(d_1)\mu(d_2)D(H, d_1^2, d_2^2), \\
S_2(H) &= \sum_{\substack{\eta<d_1d_2\leqslant H^{4/3}\\(d_1,d_2)=1}} \mu(d_1)\mu(d_2)D(H, d_1^2, d_2^2), \\
S_3(H) &= \sum_{\substack{d_1d_2>H^{4/3}\\(d_1,d_2)=1}} \mu(d_1)\mu(d_2)D(H, d_1^2, d_2^2),
\end{aligned}$$

and $\eta < H^{4/3}$ is a parameter to be decided later.

Firstly, we bound $S_2(H)$. Let, according to the Chinese remainder theorem, θ be the solution of the system

$$\begin{cases}
1 \leqslant x, y, z \leqslant d_1^2d_2^2, \\
x^2 + y^2 + z^2 \equiv -k \pmod{d_1^2}, \\
x^2 + y^2 + z^2 \equiv -k - 1 \pmod{d_2^2}
\end{cases}$$

such that $1 \leqslant \theta \leqslant d_1^2 d_2^2$. Hence,

$$
\begin{aligned}
(4.3)\quad S_2(H) &= \sum_{\substack{\eta < d_1 d_2 \leqslant H^{4/3} \\ (d_1, d_2) = 1}} \mu(d_1)\mu(d_2) \sum_{\substack{1 < x, y, z \leqslant H \\ x^2+y^2+z^2 \equiv \theta \ (\mathrm{mod}\ d_1^2 d_2^2)}} 1 \\
&\ll \sum_{\eta < d_1 d_2 \leqslant H^{4/3}} \ \sum_{0 \leqslant l \leqslant (3H^2-\theta)(d_1 d_2)^{-2}} \ \sum_{1 \leqslant z \leqslant H} \ \sum_{\substack{1 < x, y \leqslant H \\ x^2+y^2 = l d_1^2 d_2^2 - z^2 + \theta}} 1 \\
&\ll \sum_{\eta < d_1 d_2 \leqslant H^{4/3}} \ \sum_{0 \leqslant l \leqslant (3H^2-\theta)(d_1 d_2)^{-2}} \ \sum_{1 \leqslant z \leqslant H} (l d_1^2 d_2^2 - z^2 + \theta)^{\varepsilon} \ll H^{3+\varepsilon}\eta^{-1}.
\end{aligned}
$$

Secondly, we bound $S_3(H)$. We can write

$$
|S_3(H)| \ll (\log H)^2 \sum_{D_1 \leqslant d_1 < 2D_1} \ \sum_{D_2 \leqslant d_2 < 2D_2} \ \sum_{\substack{t \leqslant (3H^2+k) d_1^{-2} \\ t d_1^2 + 1 \equiv 0 \ (\mathrm{mod}\ d_2^2)}} \ \sum_{1 \leqslant z \leqslant H} \ \sum_{\substack{1 \leqslant x, y \leqslant H \\ x^2+y^2 = t d_1^2 - z^2 - k}} 1,
$$

where

$$
\frac{1}{2} \leqslant D_1, D_2 \leqslant \sqrt{3H^2+k+1}, \quad D_1 D_2 \geqslant \frac{H^{1/3}}{4}.
$$

Thus, we have

$$
\begin{aligned}
|S_3(H)| &\ll H^{1+\varepsilon} \sum_{D_1 \leqslant d_1 < 2D_1} \ \sum_{t \leqslant (3H^2+k) D_1^{-2}} \ \sum_{D_2 \leqslant d_2 < 2D_2} \ \sum_{\substack{l \leqslant (3H^2+k+1) d_2^{-2} \\ t d_1^2 + 1 = l d_2^2}} 1 \\
&\ll H^{1+\varepsilon} \sum_{D_1 \leqslant d_1 < 2D_1} \ \sum_{t \leqslant (3H^2+k) D_1^{-2}} \tau(t d_1^2 + 1) \\
&\ll H^{1+\varepsilon} \sum_{D_1 \leqslant d_1 < 2D_1} \ \sum_{t \leqslant (3H^2+k) D_1^{-2}} 1 \ll H^{3+\varepsilon} D_1^{-1}.
\end{aligned}
$$

Similarly, we have $|S_3(H)| \ll H^{3+\varepsilon} D_2^{-1}$. It follows that

$$
(4.4)\qquad |S_3(H)| \ll H^{7/3+\varepsilon}.
$$

From now on, we suppose that $q_1 = d_1^2$, $q_2 = d_2^2$, where d_1 and d_2 are square-free, $(q_1, q_2) = 1$, and $d_1 d_2 \leqslant \eta$ by the definition of $D(H, q_1, q_2)$. We have

$$
(4.5)\qquad D(H, q_1, q_2) = \sum_{x, y, z : (1.1)} N(H, q_1, q_2, x) N(H, q_1, q_2, y) N(H, q_1, q_2, z),
$$

where $N(H, q_1, q_2, \xi)$ is the number of positive integers $h \leqslant H$ satisfying $h \equiv \xi$ $(\mathrm{mod}\ q_1 q_2)$. It is clear that

$$
N(H, q_1, q_2, \xi) = \Big[\frac{H-\xi}{q_1 q_2}\Big] - \Big[\frac{-\xi}{q_1 q_2}\Big] = \frac{H}{q_1 q_2} + \varrho\Big(\frac{H-\xi}{q_1 q_2}\Big) - \varrho\Big(\frac{-\xi}{q_1 q_2}\Big).
$$

This together with equations (4.5) give us that

$$D(H,d_1^2,d_2^2)=\frac{H^3\lambda(d_1^2,d_2^2)}{(d_1d_2)^6}+\frac{3H^2}{(d_1d_2)^4}N_1(H,d_1^2,d_2^2)+\frac{3H}{(d_1d_2)^2}N_2(H,d_1^2,d_2^2)$$
$$+N_3(H,d_1^2,d_2^2)+N_3'(H,d_1^2,d_2^2),$$

where the symbols $N_1(H,d_1^2,d_2^2)$, $N_2(H,d_1^2,d_2^2)$, $N_3(H,d_1^2,d_2^2)$ and $N_3'(H,d_1^2,d_2^2)$ are defined in Lemmas 3.7, 3.8 and 3.9, respectively. Therefore, by these lemmas we have

$$D(H,d_1^2,d_2^2)=\frac{H^3\lambda(d_1^2,d_2^2)}{(d_1d_2)^6}$$
$$+O\Big(\Big(\frac{H^2}{(d_1d_2)^4}+\frac{H}{(d_1d_2)^2}+1\Big)(H^{\varepsilon-1}(d_1d_2)^{4+\varepsilon}+H^{\varepsilon}(d_1d_2)^{2+\varepsilon})\Big).$$

Substituting this into equation (4.1) and with (4.2), (4.3), (4.4), we find that

$$S(H)=H^3\sum_{d=1}^{\infty}\frac{\mu(d_1)\mu(d_2)\lambda(d_1^2,d_2^2)}{(d_1d_2)^6}$$
$$+O(H^{1+\varepsilon}\eta+H^{2+\varepsilon}\eta^{-1}+H^{\varepsilon-1}\eta^5+H^{\varepsilon}\eta^3+H^{3+\varepsilon}\eta^{-1}+H^{7/3+\varepsilon}).$$

From definition (1.2), Lemma 3.3 and $(d_1,d_2)=1$ it follows that

$$\lambda(d_1^2,d_2^2)=\lambda(d_1^2,1)\lambda(1,d_2^2).$$

Then, by (55)–(59) of [2] and choosing the parameter $\eta=H^{2/3}$, we obtain the asymptotic formula (2.1).

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