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MULTISCALE HOMOGENIZATION OF NONLINEAR HYPERBOLIC-PARABOLIC EQUATIONS

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Abstract. The main purpose of the present paper is to study the asymptotic behavior (when $\varepsilon \to 0$) of the solution related to a nonlinear hyperbolic-parabolic problem given in a periodically heterogeneous domain with multiple spatial scales and one temporal scale. Under certain assumptions on the problem's coefficients and based on a priori estimates and compactness results, we establish homogenization results by using the multiscale convergence method.

Keywords: nonlinear hyperbolic-parabolic equation; homogenization; multiscale convergence method

MSC 2020: 35B27, 35B40, 34M10

1. Introduction

We study the asymptotic behavior, when $\varepsilon \to 0$, of the solution u_ε in relation to the nonlinear hyperbolic-parabolic problem

$$(1.1)\qquad \begin{cases} \alpha^\varepsilon u_\varepsilon'' + \beta^\varepsilon u_\varepsilon' - \operatorname{div}(A^\varepsilon(x,t)\nabla u_\varepsilon) + F^\varepsilon(x,u_\varepsilon) = f & \text{in } \Omega\times(0,T),\\ u_\varepsilon = 0 & \text{on } \partial\Omega\times(0,T),\\ u_\varepsilon(x,0) = g(x),\ \ \alpha^\varepsilon(x)u_\varepsilon'(x,0) = \sqrt{\alpha^\varepsilon(x)}\varrho(x) & \text{in } \Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ with a Lipschitz-continuous boundary, T is a real positive number and $u' = \partial u/\partial t$. Here div denotes the divergence operator in Ω and we assume that $(f,g,\varrho) \in L^2(\Omega_T)\times H_0^1(\Omega)\times L^2(\Omega)$, α^ε, β^ε, A^ε and F^ε are functions of the form

$$\alpha^\varepsilon(x) = \alpha\Big(\frac{x}{\varepsilon_1},\frac{x}{\varepsilon_2},\ldots,\frac{x}{\varepsilon_m}\Big),\quad \beta^\varepsilon(x) = \beta\Big(\frac{x}{\varepsilon_1},\frac{x}{\varepsilon_2},\ldots,\frac{x}{\varepsilon_m}\Big),\quad x\in\Omega,$$
$$A^\varepsilon(x,t) = A\Big(\frac{x}{\varepsilon_1},\frac{x}{\varepsilon_2},\ldots,\frac{x}{\varepsilon_m},\frac{t}{\varepsilon_1'}\Big),\quad (x,t)\in\Omega\times(0,T),$$

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and

$$F^\varepsilon(x,\lambda)=F\Big(\frac{x}{\varepsilon_1},\frac{x}{\varepsilon_2},\dots,\frac{x}{\varepsilon_m},\lambda\Big),\ \text{a.e. } x\in\Omega \text{ and } \forall\,\lambda\in\mathbb{R},$$

respectively, where α, β, A, and F are functions satisfying a certain hypothesis.

Here ε_i, $1\leqslant i\leqslant m$, and ε_1' are positive scales of $\varepsilon>0$, which converge to zero as ε tends to zero and satisfy

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon_1}\Big(\frac{\varepsilon_1'}{\varepsilon_1}\Big)=0,\quad \lim_{\varepsilon\to0}\frac{1}{\varepsilon_1'}\Big(\frac{\varepsilon_2}{\varepsilon_1'}\Big)=0\quad\text{and}\quad \lim_{\varepsilon\to0}\frac{1}{\varepsilon_k}\Big(\frac{\varepsilon_{k+1}}{\varepsilon_k}\Big)=0 \tag{1.2}$$

for $2\leqslant k\leqslant m-1$. Using the multiscale convergence method, it is shown that under the above assumptions, when $\varepsilon\to0$, the sequence $(u_\varepsilon)_{\varepsilon>0}$ of the solutions of the problem (1.1) converges to u_0 strongly in $L^2(0,T;L^2(\Omega))$ and u_0 is the solution of a problem of the same type as (1.1).

In literature, several problems similar to ours have been studied. For example, the linear case, where $F^\varepsilon\equiv0$, was studied by Bensoussan, Lions and Papanicolaou [2] in 1977. In 1996 Migorski [10] was the first to address the homogenization problem for hyperbolic-parabolic equations in perforated domains using the energy method, too. Yang, Zhao [13] in 2016 treated the same problem as Migorski using the periodic unfolding method. One year later, Douanla, Tetsadjio [6] considered the same problem as Bensoussan by means of reiterated homogenization in domains with tiny holes.

In this paper, we use multiscale convergence method to treat the homogenization of our problem. This method was developed in 1996 by Allaire and Briane [1] as a generalisation of the two-scale convergence [11]. Two-scale convergence was used in 2005 by Holmbom [9] to homogenize linear parabolic problems with both spatial and temporal microscale.

Here, we prove new results for homogenization of a hyperbolic-parabolic problem with multiple spatial scales and one temporal scale, when we take a nonlinear term F and consider the matrix A^ε to be depending only on space and time.

The paper is organized as follows: In Section 2, we specify the assumptions about our problem and we give results about the multiscale convergence method, and then we give a result about the existence and uniqueness of the homogenized problem. In Section 3, we state and prove the main results of this paper and finish by comments and perspectives.

In this paper, we use the following notations:

▷ $\Omega\times(0,T)$ is denoted by Ω_T,

▷ $(\cdot,\cdot)_2$ ($\|\cdot\|_2$, respectively) denotes the scalar product in $L^2(\Omega)$ (the norm of $L^2(\Omega)$, respectively),

▷ $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$,

- ▷ if X is a Banach space, then we denote by $L^2(\Omega_T, X)$ the Banach space of vector valued functions $v\colon \Omega_T \to X$ which are measurable and $\|v(x,t)\|_X \in L^2(\Omega_T)$ with the norm $\|v\|^2_{L^2(\Omega_T,X)} = \int_{\Omega_T} \|v(x,t)\|^2_X \,\mathrm{d}x\,\mathrm{d}t$,
- ▷ E is a fundamental sequence (it means any ordinary sequence of real numbers $0 < \varepsilon_n \leqslant 1$ such that $\varepsilon_n \to 0$ as $n \to \infty$),
- ▷ $Y = (0,1)^N$ is the unit cube in $\mathbb{R}^N$ and $\mathcal{T} = (0,1)$,
- ▷ $B(\mathbb{R}^N)$ is a given function space,
- ▷ $B_{\mathrm{per}}(\mathbb{R}^N)$ is the space of functions in $B_{\mathrm{loc}}(\mathbb{R}^N)$ which are Y-periodic,
- ▷ $B_{\#}(\mathbb{R}^N)$ is the space of functions $v \in B_{\mathrm{per}}(\mathbb{R}^N)$ such that $\int_Y v(y)\,\mathrm{d}y = 0$,
- ▷ $Y_k = Y$, $Y^m = Y_1 \times \ldots \times Y_m$, $y^m = (y_1, \ldots, y_m)$ and $\mathrm{d}y^m = \mathrm{d}y_1 \ldots \mathrm{d}y_m$,
- ▷ $\mathcal{M}_{Y^m}(\varphi) = \int_{Y^m} \varphi(y^m)\,\mathrm{d}y^m$ for $\varphi \in L^1(Y^m)$,
- ▷ $\otimes$ is the tensor product defined as follows: for the function spaces $B(\Omega_T)$, $B(Y_1), \ldots, B(Y_m)$ and $B(\mathcal{T})$, $B(\Omega_T) \otimes B(Y_1) \otimes \ldots \otimes B(Y_m) \otimes B(\mathcal{T})$ stands for the space of linear combinations of elements $\psi_0 \otimes \psi_1 \otimes \ldots \otimes \psi_{m+1}$ such that for every $(x,t,y^m,\tau) \in \Omega \times (0,T) \times Y^m \times \mathcal{T}$, we have

$$(\psi_0 \otimes \psi_1 \otimes \ldots \otimes \psi_{m+1})(x,t,y^m,\tau) = \psi_0(x,t)\psi_1(y_1)\ldots\psi_m(y_m)\psi_{m+1}(\tau),$$

- ▷ c and C denote generic constants which do not depend on ε.

2. Assumptions, multi-scale convergence and preliminary result

2.1. Assumptions. Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set with Lipschitz continuous boundary. We consider the nonlinear hyperbolic-parabolic problem (1.1) and we assume that:

(A.1) α^ε and β^ε are functions defined as $\alpha^\varepsilon(x) = \alpha(x/\varepsilon_1, \ldots, x/\varepsilon_m)$ and $\beta^\varepsilon(x) = \beta(x/\varepsilon_1, \ldots, x/\varepsilon_m)$ with $\alpha(y^m) \in L^\infty(\mathbb{R}^{mN})$, $\beta(y^m) \in L^\infty(\mathbb{R}^{mN})$ that satisfy

$$\begin{aligned} \alpha(y^m) &\geqslant 0 && \text{a.e. in } \mathbb{R}^{mN},\\ \beta(y^m) &\geqslant \beta_0 > 0 && \text{a.e. in } \mathbb{R}^{mN}. \end{aligned}$$

(A.2) The mapping $(y^m, \tau) \mapsto a_{ij}(y^m, \tau)$ satisfies the following properties:

(i) $A(y^m,\tau) = (a_{ij}(y^m,\tau))_{1 \leqslant i,j \leqslant N} \in (L^\infty(\mathbb{R}^{mN+1}))^{N\times N}$ is real and symmetric matrix.

(ii) there exist positive constants μ_1, μ_2 with $0 < \mu_1 \leqslant \mu_2$ such that

$$A\xi_i\xi_j \geqslant \mu_1|\xi|^2 \quad \text{for a.e. } (y^m, \tau) \in \mathbb{R}^{mN+1} \quad \text{and all} \quad \xi \in \mathbb{R}^N,$$
$$\|a_{ij}\|_{L^\infty(\mathbb{R}^{mN+1})} \leqslant \mu_2 \quad \forall \leqslant i, j \leqslant N.$$

(iii) for almost every $y^m \in \mathbb{R}^{mN}$ and for all $1 \leqslant i,\ j \leqslant N$, the function $\tau \to a_{ij}(y^m, \tau)$ admits a time derivative $\partial a_{ij}/\partial\tau$ which is a measurable function and there exist $\eta_0 > 0$ and a positive constant μ_3 such that

$$\eta\Big|\frac{\partial a_{ij}}{\partial\tau}(y^m, \tau)\Big| \leqslant \mu_3 \quad \forall\, \eta \geqslant \eta_0 \text{ and a.e. } (y^m, \tau) \in \mathbb{R}^{mN+1}.$$

(A.3) The periodicity of sum functions:

(i) For any $1 \leqslant i, j \leqslant N$, the function $(y^m, \tau) \mapsto a_{ij}(y^m, \tau)$ is $Y^m \times \mathcal{T}$-periodic.

(ii) The functions $y^m \mapsto \alpha(y^m), \beta(y^m)$ are Y^m-periodic and the function α satisfies

$$\mathcal{M}_{Y^m}(\alpha) > 0.$$

(iii) The function $y^m \mapsto F(y^m, \cdot)$ is Y^m-periodic.

(A.4) For all $\lambda \in \mathbb{R}$ and for all $y^m \in \mathbb{R}^{mN}$, the function $(y^m, \lambda) \mapsto F(y^m, \lambda)$ from $\mathbb{R}^{mN+1}$ to $\mathbb{R}$ is continuous, monotonously non-decreasing with respect to λ for any y^m and satisfies:

(i) $F(y^m, 0) = 0$ for any $y^m \in \mathbb{R}^{mN}$.

(ii) There exists a constant q satisfying $q > 0$ if $N = 1, 2$ and if $N \geqslant 3$, $0 < q < 2/(N-2)$ and there exists a positive constant c_0 such that

$$|F(y^m, \lambda)| \leqslant c_0(1 + |\lambda|^{q+1}) \quad \text{a.e. in } \mathbb{R}^{mN} \quad \forall\, \lambda \text{ in } \mathbb{R}.$$

(iii) For any $y^m \in \mathbb{R}^{mN}$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ there exists $c_1 > 0$ such that

$$|F(y^m, \lambda_1) - F(y^m, \lambda_2)| \leqslant c_1|\lambda_1 - \lambda_2|.$$

(iv) $(\cdot, \lambda) \mapsto G(\cdot, \lambda) = \int_0^\lambda F(\cdot, s)\,\mathrm{d}s$ is a non-negative function for any $\lambda \in \mathbb{R}$.

Remark 2.1. For each ε fixed, Theorems 1 and 2 in [4] prove the existence and uniqueness of (1.1). See also [5] and [14]. The assumptions on the data assure the existence and uniqueness of the solution u_ε related to the problem (1.1) such that:

(i) $u_\varepsilon \in L^\infty(0, T; H_0^1(\Omega))$,

(ii) $u'_\varepsilon \in L^2(0, T; L^2(\Omega))$.

2.2. Multi-scale convergence method. As mentioned above, the homogenization of the problem (1.1) is obtained by using the $(m+1,2)$-scale convergence method. In this section, we recall the definition and some results related to this method.

Note that according to (1.2), the lists $\{\varepsilon_1,\dots,\varepsilon_m\}$ and $\{\varepsilon'_1\}$ of spatial and temporal scales, respectively, are jointly well-separated in sense of Definition 2 in [7]. Then, the followings results are direct consequences of Definition 1 in [7], Proposition 2 in [8] and Theorem 2.10 in [12], respectively.

Definition 2.1. A sequence (u_ε) in $L^2(\Omega_T)$ is said to $(m+1,2)$-scale converge to a limit $v\in L^2(\Omega_T\times Y^m\times\mathcal{T})$ (denoted by $u_\varepsilon \overset{(m+1,2)}{\rightharpoonup} v$) if, as $\varepsilon\to 0$,

$$\lim_{\varepsilon\to 0}\int_{\Omega_T} u_\varepsilon(x,t)\varphi\Big(x,t,\frac{x}{\varepsilon_1},\dots,\frac{x}{\varepsilon_m},\frac{t}{\varepsilon'_1}\Big)\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T}\int_{Y^m}\int_{\mathcal{T}} v(x,t,y^m,\tau)\varphi(x,t,y^m,\tau)\,\mathrm{d}\tau\,\mathrm{d}y^m\,\mathrm{d}x\,\mathrm{d}t \tag{2.1}$$

for all $\varphi\in L^2(\Omega_T;\mathcal{C}_{\mathrm{per}}(Y^m\times\mathcal{T}))$.

Proposition 2.1. *Suppose that $\varphi\in L^2(\Omega_T;\mathcal{C}_{\mathrm{per}}(Y^m\times\mathcal{T}))$. Then*

$$\varphi\Big(x,t,\frac{x}{\varepsilon_1},\dots,\frac{x}{\varepsilon_m},\frac{t}{\varepsilon'_1}\Big)\overset{(m+1,2)}{\rightharpoonup}\varphi(x,t,y^m,\tau),$$

as ε goes to zero.

Theorem 2.1. *Let $(u_\varepsilon)_\varepsilon$ be a bounded sequence in $L^2(0,T;H^1_0(\Omega))$ with $(\partial_t u_\varepsilon)_\varepsilon$ bounded in $L^2(0,T;H^{-1}(\Omega))$. Then, up to a subsequence, as $\varepsilon\to 0$,*

$$\begin{aligned} u_\varepsilon &\to u_0 \quad \text{in } L^2(\Omega_T),\\ u_\varepsilon &\rightharpoonup u_0 \quad \text{in } L^2(0,T;H^1_0(\Omega)),\\ \nabla u_\varepsilon &\overset{(m+1,2)}{\rightharpoonup} \nabla u_0+\sum_{i=1}^m \nabla_{y_i}u_i, \end{aligned}$$

where $u_0\in L^2(0,T;H^1_0(\Omega)), \partial_t u_0\in L^2(0,T;L^2(\Omega))$, $u_1\in L^2(\Omega_T\times\mathcal{T};H^1_\#(Y_1))$ and $u_k\in L^2(\Omega_T\times Y^{k-1}\times\mathcal{T};H^1_\#(Y_k)))$, $2\leqslant k\leqslant m$.

Remark 2.2. If $(u_\varepsilon)_\varepsilon\subset L^2(\Omega_T)$ and $v\in L^2(\Omega_T\times Y^m\times\mathcal{T})$ are such that $u_\varepsilon\overset{(m+1,2)}{\rightharpoonup} v$, then (2.1) still holds for $\varphi\in\mathcal{C}(\overline{\Omega}_T;L^\infty_{\mathrm{per}}(Y^m\times\mathcal{T}))$.

2.3. Preliminary results. Let $V=\{v\in L^2(0,T;H_0^1(\Omega))\colon\ v'\in L^2(0,T;L^2(\Omega))\}$ be the Banach space with the norm

$$\|v\|_V=\|v\|_{L^2(0,T;H_0^1(\Omega))}+\|v'\|_{L^2(0,T;L^2(\Omega))}.$$

So, the space

$$V_0=\Big\{v\in V\colon\ v(0)=g\text{ and }v'(0)=\frac{\mathcal{M}_{Y^m}(\sqrt{\alpha})}{\mathcal{M}_{Y^m}(\alpha)}\varrho\Big\}$$

is a Banach space with the relative topology on V. Introduce the space

$$\mathcal{V}=V_0\times L^2(\Omega_T;H^1_{\#}(Y_1))\times\ldots\times L^2(\Omega_T;L^2(Y^{m-1}\times\mathcal{T};H^1_{\#}(Y_m)))$$

with the norm

$$\begin{aligned}\|u\|^2_{\mathcal{V}}=\|u_0\|^2_{V_0}&+\|\nabla_{y_1}u_1\|^2_{L^2(\Omega_T;L^2(Y_1))}+\|\nabla_{y_2}u_2\|^2_{L^2(\Omega_T;L^2(Y^2\times\mathcal{T}))}\\&+\ldots+\|\nabla_{y_m}u_m\|^2_{L^2(\Omega_T;L^2(Y^m\times\mathcal{T}))},\quad u=(u_0,u_1,\ldots,u_m)\in\mathcal{V},\end{aligned}$$

and set

$$\mathcal{E}=\mathcal{D}(\Omega)\otimes\mathcal{D}(]0,T[),\quad \mathcal{E}_1=\mathcal{E}\otimes\mathcal{C}^\infty_{\#}(Y_1)$$

and

$$\mathcal{E}_k=\mathcal{E}\otimes\mathcal{C}^\infty_{\rm per}(Y_1)\otimes\ldots\otimes\mathcal{C}^\infty_{\#}(Y_k)\otimes\mathcal{C}^\infty_{\rm per}(\mathcal{T}),\quad 2\leqslant k\leqslant m.$$

Then $\mathcal{V}$ is a Banach space and further $\mathcal{E}\times\mathcal{E}_1\times\ldots\times\mathcal{E}_m$ is dense in $\mathcal{V}$.

Consider the following variational problem:

$$\left\{\begin{aligned}&\text{Find }u=(u_0,\ldots,u_m)\in\mathcal{V}:\mathcal{M}_{Y^m}(\alpha)(u_0'',v_0)_{L^2(\Omega_T)}\\&\qquad+\mathcal{M}_{Y^m}(\beta)(u_0',v_0)_{L^2(\Omega_T)}\\&\qquad+\int_0^T a(u(t),v(t))\,\mathrm{d}t+(H(u_0),v_0)_{L^2(\Omega_T)}\\&\quad=(f,v_0)_{L^2(\Omega_T)}\quad\forall\,v=(v_0,\ldots,v_m)\in\mathcal{V},\end{aligned}\right.\tag{2.2}$$

where the bilinear form a and the function H are defined as

(2.3)
$$a(u,v)=\sum_{i,j=1}^N\int_{\Omega_T}\int_{Y^m}\int_{\mathcal{T}}a_{ij}\partial_j u\partial_i v\,\mathrm{d}x\,\mathrm{d}y^m\,\mathrm{d}\tau\text{ and }H(\lambda)=\int_{Y^m}F(y^m,\lambda)\,\mathrm{d}y^m$$

for $u,v\in\mathcal{V},\lambda\in\mathbb{R}$ and

$$\partial_j w=\frac{\partial w_0}{\partial x_j}+\sum_{k=1}^m\frac{\partial w_k}{(\partial y_k)_j}$$

for $w=(w_0,w_1,\ldots,w_m)\in\mathcal{V}$.

Lemma 2.1. *The variational problem* (2.2) *has at most one solution.*

P r o o f. The variational problem (2.2) is equivalent to the system consisting of the following equations:

$$(2.4)\quad \begin{cases} u_0 \in V_0\colon\ \mathcal{M}_{Y^m}(\alpha)\displaystyle\int_{\Omega_T} u_0'' v_0 \,\mathrm{d}x\,\mathrm{d}t + \mathcal{M}_{Y^m}(\beta)\int_{\Omega_T} u_0' v_0 \,\mathrm{d}x\,\mathrm{d}t \\ \qquad + \displaystyle\int_{\Omega_T} H(u_0) v_0 \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T}\int_{Y^m}\int_{\mathcal{T}} A(y^m,\tau)\nabla u \nabla v_0 \,\mathrm{d}\tau\,\mathrm{d}y^m\,\mathrm{d}x\,\mathrm{d}t \\ = (f, v_0)_{L^2(\Omega_T)} \quad \forall\, v_0 \in V_0, \end{cases}$$

$$(2.5)\quad \begin{cases} u_1 \in L^2(\Omega_T; H^1_{\#}(Y_1)) : \displaystyle\int_{\Omega_T}\int_{Y^m}\int_{\mathcal{T}} A(y^m,\tau)\nabla u \nabla_{y_1} v_1 \,\mathrm{d}\tau\,\mathrm{d}y^m\,\mathrm{d}x\,\mathrm{d}t = 0 \\ \qquad\qquad \forall\, v_1 \in L^2(\Omega_T; H^1_{\#}(Y_1)), \end{cases}$$

and

$$(2.6)\quad \begin{cases} u_k \in L^2(\Omega_T; L^2(Y^{k-1}\times\mathcal{T}; H^1_{\#}(Y_k))) : \\ \displaystyle\int_{\Omega_T}\int_{Y^m}\int_{\mathcal{T}} A(y^m,\tau)\nabla u \nabla_{y_k} v_k \,\mathrm{d}\tau\,\mathrm{d}y^m\,\mathrm{d}x\,\mathrm{d}t = 0 \\ \qquad \forall\, v_k \in L^2(\Omega_T; L^2(Y^{k-1}\times\mathcal{T}; H^1_{\#}(Y_k))), \quad 2 \leqslant k \leqslant m, \end{cases}$$

where $\nabla u = \nabla u_0 + \sum\limits_{k=1}^{m} \nabla_{y_k} u_k$. Setting $k = m$ in (2.6) and taking $v_m = \varphi_0 \otimes \varphi_1 \otimes \ldots \otimes \varphi_m \otimes \psi$ with $\varphi_0 \in \mathcal{D}(\Omega_T), \varphi_k \in \mathcal{C}^\infty_{\mathrm{per}}(Y_k)$ $(1 \leqslant k \leqslant m-1)$, $\varphi_m \in \mathcal{C}^\infty_{\#}(Y_m)$ and $\psi \in \mathcal{C}^\infty_{\mathrm{per}}(\mathcal{T})$, we obtain, for almost all $(x,t,y^{m-1},\tau) \in \Omega_T \times Y^{m-1} \times \mathcal{T}$, that the function $y_m \mapsto u_m(x,t,y^{m-1},y_m,\tau)$ solves the equation

$$(2.7)\qquad -\mathrm{div}_{y_m}(A\nabla_{y_m} u_m) = \Big(\nabla u_0 + \sum_{i=1}^{m-1} \nabla_{y_i} u_i\Big)\Big(\mathrm{div}_{y_m} \sum_j a_{ij}\cdot e_j\Big) \quad \text{in } Y_m,$$

where $e_j \in \mathbb{R}^N$ is the j-th canonical unit vector having 1 in its j-th component and zeros elsewhere. Moreover, if u_m^1 and u_m^2 are two solutions to (2.7), then $u_m^1 - u_m^2$ is the solution of

$$(2.8)\qquad \int_{Y_m} A\nabla_{y_m}(u_m^1 - u_m^2)\nabla\xi \,\mathrm{d}y_m = 0 \quad \forall\, \xi \in H^1_{\mathrm{per}}(Y_m).$$

Let $\xi = u_m^1 - u_m^2$ in (2.8). Then, the properties of the matrix A give

$$\mu_1 \|\nabla_{y_m}(u_m^1 - u_m^2)\|_{L^2(Y_m)} \leqslant 0.$$

So, $\nabla_{y_m}(u_m^1 - u_m^2) = 0$ almost everywhere in Y_m. Then $u_m^1 - u_m^2$ is constant in Y_m. But $\mathcal{M}_{Y_m}(u_m^1 - u_m^2) = 0$ which implies that $u_m^1 - u_m^2 = 0$ in Y_m. By the same methodology, we deduce that each problem in (2.6) and also the problem (2.5) have a unique solution. Note that if u_0^1 and u_0^2 are solutions to (2.4), then, by the uniqueness of u_k, $1 \leqslant k \leqslant m$, we can suppose that $u^1 = (u_0^1, u_1, \ldots, u_m)$ and $u^2 = (u_0^2, u_1, \ldots, u_m)$ are solutions to (2.2). Then $w = (u_0^1 - u_0^2, 0, \ldots, 0)$ is the solution to

$$
\begin{aligned}
(2.9) \qquad & (\mathcal{M}_{Y^m}(\alpha) w_0'' + \mathcal{M}_{Y^m}(\beta) w_0' + H(u_0^1) - H(u_0^2), \varphi)_{L^2(\Omega_T)} \\
& + \int_{\Omega_T} \int_{Y^m} \int_{\mathcal{T}} A(y^m, \tau) \nabla w_0 \nabla \varphi \, \mathrm{d}\tau \, \mathrm{d}y^m \, \mathrm{d}x \, \mathrm{d}t = 0 \quad \forall \varphi \in V_0,
\end{aligned}
$$

where $w_0 = u_0^1 - u_0^2$. Taking $\varphi \in \mathcal{E}$ in (2.9), we obtain the equation

$$
\mathcal{M}_{Y^m}(\alpha) w_0'' + \mathcal{M}_{Y^m}(\beta) w_0' - \operatorname{div}(A^* \nabla w_0) + (H(u_0^1) - H(u_0^2)) = 0 \quad \text{in } \mathcal{E}',
$$

where $A^* := \int_{Y^m} \int_{\mathcal{T}} A(y^m, \tau) \, \mathrm{d}y^m \, \mathrm{d}\tau$. Multiplying the above equation by w_0', integrating over Ω and then integrating by parts, we get

$$
\begin{aligned}
& \frac{1}{2} \frac{\partial}{\partial t} \left(\mathcal{M}_{Y^m}(\alpha) \|w_0'\|_2^2 + \int_\Omega A^* \nabla w_0 \nabla w_0 \, \mathrm{d}x \right) \\
& \qquad + \mathcal{M}_{Y^m}(\beta) \|w_0'\|_2^2 + (H(u_0^1) - H(u_0^2), w_0')_2 = 0.
\end{aligned}
$$

Integrating the above equation over $]0, t[$, $t > 0$, with the fact that $w_0(0) = 0$ and $w_0'(0) = 0$ gives

$$
\begin{aligned}
\mathcal{M}_{Y^m}(\alpha) \frac{1}{2} \|w_0'(t)\|_2^2 + \mathcal{M}_{Y^m}(\beta) \int_0^t \|w_0'(s)\|_2^2 \, \mathrm{d}s + \frac{1}{2} \int_\Omega A^* \nabla w_0 \nabla w_0 \, \mathrm{d}x \\
= - \int_0^t (H(u_0^1(s)) - H(u_0^2(s)), w_0'(s))_2 \, \mathrm{d}s.
\end{aligned}
$$

Assumption (A.4) (iii), the Hölder and Poincaré inequalities imply that

$$
\begin{aligned}
\left| \int_0^t (H(u_0^1(s)) - H(u_0^2(s)), w_0'(s))_2 \, \mathrm{d}s \right| & \leqslant \frac{c_1}{2} \int_0^t \|w_0(s)\|_2^2 \, \mathrm{d}s + \frac{c_1}{2} \int_0^t \|w_0'(s)\|_2^2 \, \mathrm{d}s \\
& \leqslant \frac{c_1 c_\Omega}{2} \int_0^t \|\nabla w_0(s)\|_2^2 \, \mathrm{d}s + \frac{c_1}{2} \int_0^t \|w_0'(s)\|_2^2 \, \mathrm{d}s
\end{aligned}
$$

and by assumption (A.2) we have

$$
\int_\Omega A^* \nabla w_0 \nabla w_0 \, \mathrm{d}x \geqslant \mu_1 \int_\Omega |\nabla w_0(t)|^2 \, \mathrm{d}x.
$$

We deduce the existence of a real number $C > 0$ depending only on Ω and c_1 such that

$$
\mathcal{M}_{Y^m}(\alpha) \frac{1}{2} \|w_0'(t)\|_2^2 + \mu_1 \int_\Omega |\nabla w_0(t)|^2 \, \mathrm{d}x \leqslant C \int_0^t (\|\nabla w_0(s)\|_2^2 + \|w_0'(s)\|_2^2) \, \mathrm{d}s.
$$

Applying Gronwalls lemma in the last inequality we obtain $w_0 = 0$, i.e.,

$$
u_0^1 = u_0^2 \quad \text{a.e. in } \Omega_T.
$$

□

3. Homogenization results

We can now state the main results of this paper.

Theorem 3.1. *Assume that the hypotheses* (A.1)–(A.4) *hold. Let* $(u_\varepsilon)_{\varepsilon>0}$ *be the sequence of solutions to* (1.1), E *being a fundamental sequence. Then, there exist* $m+1$ *functions* $(u_0, u_1, \ldots, u_m) \in \mathcal{V}$ *such that, as* $\varepsilon \to 0$, u_ε *converges weakly to* u_0 *in* $L^2(0,T;H_0^1(\Omega))$ *and* ∇u_ε $(m+1,2)$*-scale converges to* $\nabla u_0 + \sum_{k=1}^{m} \nabla_{y_k} u_k$, $(u_0, u_1, \ldots, u_m)$ *is the unique solution to* (2.2). *The functions* u_1, u_k $(2 \leqslant k \leqslant m-1)$ *and* u_m *are defined as the solutions of the elliptic equations*

$$
(3.1) \qquad \begin{cases}
-\mathrm{div}_{y_1}\Bigl(\displaystyle\int_{Y_2}\cdots\int_{Y_m}\int_{\mathcal{T}}\Bigl(A\Bigl(\nabla u_0 + \sum_{i=1}^{m}\nabla_{y_i}u_i\Bigr)\Bigr)\,\mathrm{d}y_2\ldots\mathrm{d}y_m\,\mathrm{d}\tau\Bigr) = 0,\\[2ex]
-\mathrm{div}_{y_k}\Bigl(\displaystyle\int_{Y_{k+1}}\cdots\int_{Y_m}\Bigl(A\Bigl(\nabla u_0 + \sum_{i=1}^{m}\nabla_{y_i}u_i\Bigr)\Bigr)\,\mathrm{d}y_{k+1}\ldots\mathrm{d}y_m\Bigr) = 0,\\
\hfill 2 \leqslant k \leqslant m-1,\\[1ex]
-\mathrm{div}_{y_m}\Bigl(A\Bigl(\nabla u_0 + \displaystyle\sum_{i=1}^{m}\nabla_{y_i}u_i\Bigr)\Bigr) = 0,
\end{cases}
$$

respectively.

Corollary 3.1. *The function* u_0 *is also the unique solution of the homogenized problem*

$$
(3.2) \qquad \begin{cases}
\mathcal{M}_{Y^m}(\alpha)u_0'' + \mathcal{M}_{Y^m}(\beta)u_0' - \mathrm{div}(A^*\nabla u_0) + H(u_0) = f & \text{in } \Omega_T,\\[1ex]
u_0(x,0) = g(x), \ u_0'(x,0) = \dfrac{\mathcal{M}_{Y^m}(\sqrt{\alpha})}{\mathcal{M}_{Y^m}(\alpha)}\varrho(x) & \text{in } \Omega,
\end{cases}
$$

where the matrix A^* *is defined by*

$$
A^*\xi = \int_{Y_1}\cdots\int_{Y_m}\int_{\mathcal{T}} A(y_1,\ldots,y_m,\tau)\Bigl(\xi + \sum_{j=1}^{m}\nabla_{y_j}u_j\Bigr)\,\mathrm{d}y^m\,\mathrm{d}\tau
$$

for all $\xi \in \mathbb{R}^N$ *with* u_k $(1 \leqslant k \leqslant m)$ *are functions defined in* (3.1). *The nonlinear term* H *is defined in* (2.3).

To prove the main results, we need to establish the limits of some terms in (1.1) in order to obtain the problem (2.2). To begin with we formulate the following preliminary results.

Proposition 3.1. *Under the hypotheses* (A.1)–(A.4), *the following estimates hold:*

$$\|u_\varepsilon\|_{L^\infty(0,T;H_0^1(\Omega))} \leqslant C, \tag{3.3}$$

$$\|u_\varepsilon'\|_{L^2(0,T;L^2(\Omega))} \leqslant C, \tag{3.4}$$

$$\big\|\sqrt{\alpha^\varepsilon}u_\varepsilon'\big\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C, \tag{3.5}$$

$$\|F^\varepsilon(\cdot,u_\varepsilon)\|_{L^2(0,T;L^2(\Omega))} \leqslant C, \tag{3.6}$$

where C is a positive constant which does not depend on ε.

P r o o f. The variational formulation of (1.1) is

$$\begin{cases} \text{Find } u_\varepsilon \in L^\infty(0,T;H_0^1(\Omega)) \text{ and } \sqrt{\alpha^\varepsilon}u_\varepsilon' \in L^\infty(0,T;L^2(\Omega)) \text{ such that:} \\ \langle \alpha^\varepsilon u_\varepsilon''(t),v\rangle + (\beta^\varepsilon u_\varepsilon'(t),v)_2 + (A^\varepsilon(t)\nabla u_\varepsilon,\nabla v)_2 \\ \qquad +(F^\varepsilon(x,u_\varepsilon(t)),v)_2 = (f(t),v)_2 \quad \text{in } \mathcal{D}'(]0,T[) \quad \forall\, v\in H_0^1(\Omega). \end{cases} \tag{3.7}$$

Let $t\in\,]0,T[$. Taking $v=u_\varepsilon'(t)$ in (3.7) and using (A.4) (iv), we obtain

$$\begin{aligned} &\frac{\partial}{\partial t}\Big[\frac12\big\|\sqrt{\alpha^\varepsilon}u_\varepsilon'\big\|_2^2 + \frac12\sum_{i,j}\int_\Omega a_{ij}^\varepsilon(t)\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial u_\varepsilon}{\partial x_j}\,\mathrm{d}x + \int_\Omega G^\varepsilon(x,u_\varepsilon)\,\mathrm{d}x\Big] + (\beta^\varepsilon u_\varepsilon',u_\varepsilon')_2 \\ &\qquad = \frac{1}{2\varepsilon_1'}\int_\Omega\Big(\frac{\partial A}{\partial\tau}\Big)^\varepsilon(t)\nabla u_\varepsilon\nabla u_\varepsilon\,\mathrm{d}x + (f,u_\varepsilon')_2, \end{aligned} \tag{3.8}$$

where $G^\varepsilon(x,v) = G(x/\varepsilon_1,\dots,x/\varepsilon_m,v)\geqslant 0$ $(v\in\mathbb{R})$. Using assumption (A.2) (iii), we get

$$\exists\,\varepsilon_0>0:\frac{1}{\varepsilon_1'}\Big|\Big(\frac{\partial A}{\partial\tau}\Big)^\varepsilon(x,t)\Big| \leqslant \mu_3 \quad \forall\, 0<\varepsilon_1'\leqslant\varepsilon_0 \text{ and a.e. } (x,t)\in\Omega_T. \tag{3.9}$$

Integrating the inequality (3.8) over the interval $]0,t[$ and using (3.9), (A.1) and the fact that $G^\varepsilon\geqslant 0$, we obtain

$$\begin{aligned} &\frac12\big\|\sqrt{\alpha^\varepsilon}u_\varepsilon'(t)\big\|_2^2 + \beta_0\int_0^t\|u_\varepsilon'(s)\|_2^2\,\mathrm{d}s + \frac12\sum_{i,j}\int_\Omega a_{ij}^\varepsilon(t)\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial u_\varepsilon}{\partial x_j}\,\mathrm{d}x \\ &\quad\leqslant \frac{\mu_3}{2}\int_0^t\|\nabla u_\varepsilon(s)\|_2^2\,\mathrm{d}s + \int_0^t(f(s),u_\varepsilon'(s))_2\,\mathrm{d}s + \frac12\|\varrho(x)\|_2^2 \\ &\qquad + \int_\Omega G^\varepsilon(x,g(x))\,\mathrm{d}x + \frac12\int_\Omega A^\varepsilon(x,0)\nabla g(x)\cdot\nabla g(x)\,\mathrm{d}x. \end{aligned}$$

Since $(f, g, \varrho) \in L^2(\Omega_T) \times H_0^1(\Omega) \times L^2(\Omega)$, by the continuous embedding theorem, the Hölder inequality and assumption (A.2), we have

$$\begin{aligned}\frac{1}{2}\big\|\sqrt{\alpha^\varepsilon} u_\varepsilon'(t)\big\|_2^2 + \beta_0 \int_0^t \|u_\varepsilon'(s)\|_2^2 \,\mathrm{d}s + \frac{\mu_1}{2}\|\nabla u_\varepsilon(t)\|_2^2 \\ \leqslant \frac{\mu_3}{2}\int_0^t \|\nabla u_\varepsilon(s)\|_2^2 \,\mathrm{d}s + \frac{\beta_0}{2}\int_0^t \|u_\varepsilon'(s)\|_2^2 \,\mathrm{d}s \\ + \mu_2\|\nabla g(x)\|_{L^2(\Omega)}^2 + c\|g\|_{H_0^1(\Omega)}^{q+2} + K\end{aligned}$$

with β_0 being the constant defined in (A.1). Since $g \in H_0^1(\Omega)$, then

$$\frac{1}{2}\big\|\sqrt{\alpha^\varepsilon} u_\varepsilon'(t)\big\|_2^2 + \frac{\beta_0}{2}\int_0^t \|u_\varepsilon'(s)\|_2^2 \,\mathrm{d}s + \frac{\mu_1}{2}\|\nabla u_\varepsilon(t)\|_2^2 \leqslant c\int_0^t \|\nabla u_\varepsilon(s)\|_2^2 \,\mathrm{d}s + C.$$

Gronwall's inequality implies that there exists a constant C which is independent of ε such that

$$\big\|\sqrt{\alpha^\varepsilon} u_\varepsilon'(t)\big\|_2^2 + \int_0^t \|u_\varepsilon'(s)\|_2^2 \,\mathrm{d}s + \|u_\varepsilon(t)\|_{H_0^1(\Omega)}^2 \leqslant C$$

for all $t \in]0, T[$, which implies that the estimates (3.3)–(3.5) hold. Moreover, in view of part (ii) of assumption (A.4) and the continuous embedding theorem, we have

$$\begin{aligned}\|F^\varepsilon(x, u_\varepsilon)\|_{L^2(0,T;L^2(\Omega))} &\leqslant \|c_0(1 + |u_\varepsilon|^{q+1})\|_{L^2(0,T;L^2(\Omega))} \\ &\leqslant c_{\Omega,T} + c_0\bigg(\int_0^T \|u_\varepsilon(t)\|_{(L^{2(q+1)}(\Omega))}^{q+1} \,\mathrm{d}t\bigg)^{1/2} \\ &\leqslant c_{\Omega,T} + c_0 c_\Omega\bigg(\int_0^T \|u_\varepsilon(t)\|_{H_0^1(\Omega)}^{q+1} \,\mathrm{d}t\bigg)^{1/2} \leqslant C,\end{aligned}$$

where C is a constant independent of ε. Then, we have the estimate (3.6). $\square$

In the following, we present a result about the convergence of the problem (1.1).

Proposition 3.2. *Let* $(u_\varepsilon)_{\varepsilon>0}$ *be the sequence of solutions to* (1.1), E *being a fundamental sequence and let* $F\colon \mathbb{R}^{mN} \times \mathbb{R} \to \mathbb{R}$ *be a function satisfying assumptions* (A.3) (iii) *and* (A.4) (ii)–(iii). *Then, up to a subsequence, as* $E \ni \varepsilon \to 0$,

$$u_\varepsilon \to u_0 \quad \text{in } L^2(\Omega_T), \tag{3.10}$$

$$u_\varepsilon' \rightharpoonup u_0' \quad \text{in } L^2(\Omega_T), \tag{3.11}$$

$$\nabla u_\varepsilon \overset{(m+1,2)}{\rightharpoonup} \nabla u_0 + \sum_{k=1}^m \nabla_{y_k} u_k, \tag{3.12}$$

$$F^\varepsilon(x, u_\varepsilon) \overset{(m+1,2)}{\rightharpoonup} F(y^m, u_0), \tag{3.13}$$

where $u_0 \in L^2(0, T; H_0^1(\Omega))$, $\partial_t u_0 \in L^2(0, T; L^2(\Omega))$, $u_1 \in L^2(\Omega_T \times \mathcal{T}; H_\#^1(Y_1))$ *and* $u_k \in L^2(\Omega_T \times Y^{k-1} \times \mathcal{T}; H_\#^1(Y_k)), 2 \leqslant k \leqslant m$.

P r o o f. The inequalities (3.3), (3.4) and Theorem 2.1 imply that there exists $(u_0, u_1, \ldots, u_m) \in L^2(0,T; H_0^1(\Omega)) \times L^2(\Omega_T \times \mathcal{T}; H^1_\#(Y_1)) \times \ldots \times L^2(\Omega_T \times Y^{m-1} \times \mathcal{T}; H^1_\#(Y_m))$ such that (3.10)–(3.12) hold as $E \ni \varepsilon \to 0$. By assumptions (A.3) (iii), (A.4) (ii) and the continuous embedding theorem, it follows that $F(y^m, u_0(x,t))$ lies in $L^2(\Omega_T; \mathcal{C}_{\rm per}(Y^m))$ which implies, by Proposition 2.1, that

$$F^\varepsilon(x, u_0(x,t)) \overset{(m+1,2)}{\rightharpoonup} F(y^m, u_0(x,t)),$$

and for all $\psi \in L^2(\Omega_T; \mathcal{C}_{\rm per}(Y^m))$ we can write

$$\begin{aligned}\int_{\Omega_T} F^\varepsilon(x,u_\varepsilon)\psi^\varepsilon \,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_T}\int_{Y^m} F(y^m, u_0(x,t))\psi \,\mathrm{d}y^m\,\mathrm{d}x\,\mathrm{d}t \\ = \int_{\Omega_T} (F^\varepsilon(x,u_\varepsilon) - F^\varepsilon(x,u_0))\psi^\varepsilon \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} F^\varepsilon(x,u_0)\psi^\varepsilon\,\mathrm{d}x\,\mathrm{d}t \\ - \int_{\Omega_T}\int_{Y^m} F(y^m, u_0(x,t))\psi\,\mathrm{d}y^m\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

The convergence (3.10) and Assumption (A.4) (iii) imply that the auxiliary result (3.13) holds. □

Lemma 3.1. *The function u_1 given by Proposition 3.2 lies in $L^2(\Omega_T; H^1_\#(Y_1))$ (i.e., the function u_1 is independent of τ).*

P r o o f. Let φ be the mapping of $\Omega_T \times \mathbb{R}^N \times \mathbb{R}$ into $\mathbb{R}$ given by

$$\varphi(x,t,y_1,\tau) = \varphi_1(x,t)\varphi_2(y_1)\varphi_3(\tau), \quad (x,t) \in \Omega_T,\ y_1 \in Y_1,\ \tau \in \mathcal{T},$$

with $\varphi_1 \in \mathcal{D}(\Omega)\otimes\mathcal{D}(]0,T[)$, $\varphi_2 \in \mathcal{C}^\infty_\#(Y_1)$ and $\varphi_3 \in \mathcal{C}^\infty_{\rm per}(\mathcal{T})$. We consider the function $\varphi^\varepsilon \in \mathcal{D}(\Omega_T)$ where

$$\varphi^\varepsilon(x,t) = \frac{(\varepsilon_1')^2}{\varepsilon_1}\varphi\Big(x,t,\frac{x}{\varepsilon_1},\frac{t}{\varepsilon_1'}\Big), \quad (x,t) \in \Omega_T,$$

and $\varphi \in \mathcal{D}(\Omega_T) \otimes \mathcal{C}^\infty_\#(Y_1) \otimes \mathcal{C}^\infty_{\rm per}(\mathcal{T})$. Multiplying the equation (1.1) by φ^ε and integrating over Ω_T, we get

$$\begin{aligned}\int_{\Omega_T} \alpha^\varepsilon u_\varepsilon''\varphi^\varepsilon\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T}(\beta^\varepsilon u_\varepsilon' + F^\varepsilon(x,u_\varepsilon) - f)\varphi^\varepsilon\,\mathrm{d}x\,\mathrm{d}t \\ + \int_{\Omega_T} A^\varepsilon\nabla u_\varepsilon\nabla\varphi^\varepsilon\,\mathrm{d}x\,\mathrm{d}t = 0.\end{aligned} \tag{3.14}$$

At first, by the Hölder inequality we have

$$\int_{\Omega_T} (\beta^\varepsilon u_\varepsilon' + F^\varepsilon(x,u_\varepsilon) - f)\varphi^\varepsilon \,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{(\varepsilon_1')^2}{\varepsilon_1} \|\beta^\varepsilon u_\varepsilon' + F^\varepsilon(x,u_\varepsilon) - f\|_{L^2(\Omega_T)} \|\varphi\|_{L^2(\Omega_T)}$$

and

$$\begin{aligned}\int_{\Omega_T} A^\varepsilon \nabla u_\varepsilon \nabla \varphi^\varepsilon \,\mathrm{d}x\,\mathrm{d}t &= \int_{\Omega_T} A^\varepsilon \nabla u_\varepsilon \Big(\frac{(\varepsilon_1')^2}{\varepsilon_1}\nabla\varphi + \Big(\frac{\varepsilon_1'}{\varepsilon_1}\Big)^2 \nabla_{y_1}\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant \|A^\varepsilon \nabla u_\varepsilon\|_{L^2(\Omega_T)} \Big(\frac{(\varepsilon_1')^2}{\varepsilon_1}\|\nabla\varphi\|_{L^2(\Omega_T)} + \Big(\frac{\varepsilon_1'}{\varepsilon_1}\Big)^2 \|\nabla_{y_1}\varphi\|_{L^2(\Omega_T)}\Big).\end{aligned}$$

As $E \ni \varepsilon \to 0$, the separation of the scales ε_1 and ε_1' in (1.2), assumptions (A.1) and (A.2) (i), and a priori estimates (3.3) and (3.6) give that the second and the third term in the left-hand side in (3.14) converge to zero. Concerning the first term in (3.14), we observe that

$$\int_{\Omega_T} \alpha^\varepsilon u_\varepsilon'' \varphi^\varepsilon \,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} \Big[\frac{(\varepsilon_1')^2}{\varepsilon_1}\alpha^\varepsilon u_\varepsilon \Big(\frac{\partial^2\varphi}{\partial t^2}\Big)^\varepsilon + 2\frac{\varepsilon_1'}{\varepsilon_1}\alpha^\varepsilon u_\varepsilon \Big(\frac{\partial^2\varphi}{\partial t\partial\tau}\Big)^\varepsilon + \alpha^\varepsilon \frac{u_\varepsilon}{\varepsilon_1}\Big(\frac{\partial^2\varphi}{\partial\tau^2}\Big)^\varepsilon\Big]\,\mathrm{d}x\,\mathrm{d}t.$$

Using Theorem 2.6 in [3], (1.2), (3.3) and Theorem 7 in [7] we obtain

$$\int_{\Omega_T} \alpha^\varepsilon u_\varepsilon'' \varphi^\varepsilon \,\mathrm{d}x\,\mathrm{d}t \to 0 + \mathcal{M}_{Y^m}(\alpha) \int_{\Omega_T} \iint_{Y_1 \mathcal{T}} u_1(x,t,y_1,\tau) \frac{\partial\varphi}{\partial\tau^2}\,\mathrm{d}\tau\,\mathrm{d}y_1\,\mathrm{d}x\,\mathrm{d}t.$$

Then,

$$\mathcal{M}_{Y^m}(\alpha)\int_{\Omega_T}\int_{Y_1} \varphi_1(x,t)\varphi_2(y_1)\Big(\int_{\mathcal{T}} u_1(x,t,y_1,\tau)\frac{\partial\varphi_3}{\partial\tau^2}(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}y_1\,\mathrm{d}x\,\mathrm{d}t = 0.$$

It results from the arbitrariness of φ_1, φ_2 and assumption (A.3) (ii) that

$$\int_{\mathcal{T}} u_1(x,t,y_1,\tau)\frac{\partial^2\varphi_3}{\partial\tau^2}(\tau)\,\mathrm{d}s = 0 \quad \forall\,\varphi_3 \in \mathcal{C}^\infty_{\mathrm{per}}(\mathcal{T}).$$

Choosing $\varphi_3(\tau) = \mathrm{e}^{-2\mathrm{i}\pi l\tau}(l \in \mathbb{Z}\setminus\{0\})$, we obtain that

$$\int_{\mathcal{T}} u_1(x,t,y_1,\tau)\mathrm{e}^{-2\mathrm{i}\pi l\tau}\,\mathrm{d}\tau = 0 \quad \forall\, l \in \mathbb{Z}\setminus\{0\}. \tag{3.15}$$

The properties of the Fourier series expansion and the periodicity of the function $u_1\colon \mathcal{T} \to \mathbb{R}$ give

$$u_1(x,t,y_1,\tau) = \sum_{l\in\mathbb{Z}} a_l \mathrm{e}^{2\mathrm{i}\pi l\tau} \text{ where } a_l = \int_{\mathcal{T}} u_1(x,t,y_1,\tau)\mathrm{e}^{-2\mathrm{i}\pi l\tau}\,\mathrm{d}\tau.$$

Thanks to (3.15) this gives $a_l = 0$ for all $l \in \mathbb{Z}\setminus\{0\}$, so $u_1 = a_0 = \int_0^1 u_1(x,t,y_1,\tau)\,\mathrm{d}\tau$. That means that the function u_1 does not depend on the variable τ. □

P r o o f of Theorem 3.1. According to Lemma 3.1 we can use the test functions of the form

$$v_\varepsilon(x,t) = v_0(x,t) + \varepsilon_1 v_1\Big(x,t,\frac{x}{\varepsilon_1}\Big) + \sum_{k=2}^{m} \varepsilon_k v_k\Big(x,t,\frac{x}{\varepsilon_1},\ldots,\frac{x}{\varepsilon_k},\frac{t}{\varepsilon_1'}\Big)$$

with $(v_0, v_1, \ldots, v_m) \in \mathcal{E} \times \mathcal{E}_1 \times \ldots \times \mathcal{E}_m$. Multiplying the equation (1.1) by v_ε, and integrating over Ω_T, we obtain

$$\tag{3.16} \int_{\Omega_T} \alpha^\varepsilon u_\varepsilon v_\varepsilon'' \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} \beta^\varepsilon u_\varepsilon' v_\varepsilon \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} A^\varepsilon \nabla u_\varepsilon \nabla v_\varepsilon \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} F^\varepsilon(x,u_\varepsilon) v_\varepsilon \,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} f v_\varepsilon \,\mathrm{d}x\,\mathrm{d}t.$$

The first term in the left-hand side of this equality is expanded as

$$\tag{3.17} \begin{aligned} \int_{\Omega_T} \alpha^\varepsilon u_\varepsilon v_\varepsilon'' \,\mathrm{d}x\,\mathrm{d}t &= \int_{\Omega_T} \alpha^\varepsilon u_\varepsilon v_0'' \,\mathrm{d}x\,\mathrm{d}t + \sum_{k=1}^{m} \varepsilon_k \int_{\Omega_T} \alpha^\varepsilon u_\varepsilon (v_k'')^\varepsilon \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + 2\sum_{k=2}^{m} \frac{\varepsilon_k}{\varepsilon_1'} \int_{\Omega_T} \alpha^\varepsilon u_\varepsilon \Big(\frac{\partial^2 v_k}{\partial t \partial \tau}\Big)^\varepsilon \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \sum_{k=2}^{m} \frac{1}{\varepsilon_1'}\Big(\frac{\varepsilon_k}{\varepsilon_1'}\Big) \int_{\Omega_T} \alpha^\varepsilon u_\varepsilon \Big(\frac{\partial^2 v_k}{\partial \tau^2}\Big)^\varepsilon \,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Taking into account the separatedness of the scales in (1.2) and using Theorem 2.6 in [3], (3.12) we get, by passing to the limit in (3.17) as $E \ni \varepsilon \to 0$,

$$\int_{\Omega_T} \alpha\Big(\frac{x}{\varepsilon_1},\ldots,\frac{x}{\varepsilon_m}\Big) u_\varepsilon v_\varepsilon'' \,\mathrm{d}x\,\mathrm{d}t \to \mathcal{M}_{Y^m}(\alpha) \int_{\Omega_T} u_0'' v_0 \,\mathrm{d}x\,\mathrm{d}t.$$

The third term in the left-hand side of (3.16) is equal to

$$\tag{3.18} \begin{aligned} &\int_{\Omega_T} A^\varepsilon \nabla u_\varepsilon \Big(\nabla v_0 + \sum_{k=1}^{m} \nabla_{y_k} v_k\Big) \,\mathrm{d}x\,\mathrm{d}t + \sum_{k=1}^{m} \varepsilon_k \int_{\Omega_T} A^\varepsilon \nabla u_\varepsilon \nabla v_k \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_{\Omega_T} A^\varepsilon \nabla u_\varepsilon \Big(\sum_{k=2}^{m}\sum_{j=1}^{k-1} \frac{\varepsilon_k}{\varepsilon_j} \nabla_{y_j} v_k\Big) \,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Using the separatedness of scales in (1.2) and using (3.12), Theorem 2.6 in [3] and Remark 2.2, we conclude that as $E \ni \varepsilon \to 0$, (3.18) converges to

$$\int_{\Omega_T} \int_{Y^m} \int_{\mathcal{T}} A(y^m,\tau)\Big(\nabla u_0 + \sum_{k=1}^{m} \nabla_{y_k} u_k\Big) \cdot \Big(\nabla v_0 + \sum_{k=1}^{m} \nabla_{y_k} v_k\Big) \,\mathrm{d}y^m\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}t.$$

For the second and the fourth term in the left-hand side of (3.16), as $E \ni \varepsilon \to 0$, (3.11), (3.13), Theorem 2.6 in [3] and Remark 2.2 imply that

$$\int_{\Omega_T} \beta^\varepsilon u'_\varepsilon v_\varepsilon \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} F^\varepsilon v_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \to \mathcal{M}_{Y^m}(\beta) \int_{\Omega_T} u'_0 v_0 \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} H(u_0) v_0 \,\mathrm{d}x\,\mathrm{d}t.$$

Finally, the right-hand term of (3.16) tends to

$$\int_{\Omega_T} f(x,t) v_0 \,\mathrm{d}x\,\mathrm{d}t.$$

Thus, by passing to the limit in (3.16) as $E \ni \varepsilon \to 0$, we find that $(u_0, u_1, \ldots, u_m) \in \mathcal{V}$ is the solution of problem

$$\begin{aligned}\mathcal{M}_{Y^m}(\alpha) \int_{\Omega_T} u''_0 v_0 \,\mathrm{d}x\,\mathrm{d}t &+ \mathcal{M}_{Y^m}(\beta) \int_{\Omega_T} u'_0 v_0 \,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^T a(u,v)\,\mathrm{d}t + \int_{\Omega_T} H(u_0) v_0 \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{\Omega_T} f v_0 \,\mathrm{d}x\,\mathrm{d}t \quad \forall\, v = (v_0, v_1, \ldots, v_m) \in \mathcal{E} \times \mathcal{E}_1 \times \ldots \times \mathcal{E}_m,\end{aligned} \tag{3.19}$$

where $a(u,v)$ and H are defined in (2.3). The uniqueness of the solution to (3.19) and the arbitrariness of the fundamental sequence E in the limit passage prove that we have $u_\varepsilon \to u_0$ in $L^2(\Omega_T)$ for the whole generalised sequence ε. □

Proof of Corollary 3.1. The equation in (3.2) is a direct consequence of (2.4). Hence, it remains to determine the initial condition. So, multiplying the equation (1.1) by $\psi(x)\xi(t)$, where $\psi \in \mathcal{D}(\Omega)$ and $\xi \in \mathcal{D}([0,T])$ with $\xi(T) = 0$, and integrating over Ω_T, we obtain

$$\begin{aligned}-\int_{\Omega_T} \alpha^\varepsilon u'_\varepsilon \xi'(t)\psi(x)\,\mathrm{d}x\,\mathrm{d}t &- \int_\Omega \sqrt{\alpha^\varepsilon}\varrho(x)\psi(x)\xi(0)\,\mathrm{d}x \\ &+ \int_{\Omega_T} \beta^\varepsilon u'_\varepsilon \psi(x)\xi(t)\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} A^\varepsilon \nabla u_\varepsilon \nabla\psi(x)\xi(t)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_{\Omega_T} F^\varepsilon(x, u_\varepsilon)\psi(x)\xi(t)\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} f(x,t)\psi(x)\xi(t)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Passing to the limit as $\varepsilon \to 0$ and using the same arguments as above we get

$$\begin{aligned}-\mathcal{M}_{Y^m}(\alpha) \int_{\Omega_T} u'_0(x,t)\xi'(t)\psi(x)\,\mathrm{d}x\,\mathrm{d}t &- \mathcal{M}_{Y^m}(\sqrt{\alpha}) \int_\Omega \varrho(x)\psi(x)\xi(0)\,\mathrm{d}x \\ &+ \mathcal{M}_{Y^m}(\beta) \int_{\Omega_T} u'_0 \psi(x)\xi(t)\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} A^* \nabla u_0 . \nabla\psi(x)\xi(t)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_{\Omega_T} H(u_0)\psi(x)\xi(t)\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} f(x,t)\psi(x)\xi(t)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{3.20}$$

Integrating by parts with respect to the variable t in the first term of (3.20) and using the equation in (3.2), we obtain

$$\int_\Omega \mathcal{M}_{Y^m}(\alpha)u_0'(x,0)\psi(x)\xi(0)\,\mathrm{d}x - \mathcal{M}_{Y^m}(\sqrt{\alpha})\int_\Omega \varrho(x)\psi(x)\xi(0)\,\mathrm{d}x = 0.$$

By the arbitrariness of ψ and ξ, we get

$$\mathcal{M}_{Y^m}(\alpha)u_0'(x,0) = \mathcal{M}_{Y^m}(\sqrt{\alpha})\varrho(x) \quad \text{in } \Omega.$$

In a similar way, choosing $\psi \in \mathcal{D}(\Omega)$, $\xi \in \mathcal{D}([0,T])$ with $\xi(T) = \xi(0) = \xi'(T) = 0$, integrating by parts twice and passing to the limit again, we get

$$u_0(0) = g \quad \text{in } \Omega.$$

$\square$

Remark 3.1. The example, when the scales $\varepsilon_k(\varepsilon)$, $1 \leqslant k \leqslant m$, $\varepsilon_1'(\varepsilon)$ are defined as

$$\varepsilon_k = \varepsilon^{r_k} \quad \text{and} \quad \varepsilon_1' = \varepsilon^{r_1'},$$

where $0 < 2r_1 < r_1' < 0.5r_2$ and $r_k < r_{k+1}$, $2 \leqslant k \leqslant m$, can be taken as an application of our results.

Comments and perspectives

The main result of this paper is not achieved in the absence of imposing the separation of the scales in (1.2) due to the difficulty of passing to the limit in the relations (3.17) and (3.18). Even in the linear case, there is a very limited study of these equations, for example, Douanla, Tetsadjio [6] considered homogenization of hyperbolic-parabolic equations in porous media with tiny holes and with a special case of spatial and temporal scales. In case of studying:

- ▷ multiscale homogenization of nonlinear hyperbolic-parabolic equations with potentially arbitrary finite number of both spatial and temporal scales

$$(*) \qquad \alpha^\varepsilon u_\varepsilon'' + \beta^\varepsilon u_\varepsilon' - \nabla\cdot(a(x/\varepsilon_1,\ldots,x/\varepsilon_n,t/\varepsilon_1',\ldots,t/\varepsilon_m')) + F^\varepsilon(u_\varepsilon) = f,$$

- ▷ multiscale homogenization of $(*)$ in a periodically perforated domain,
- ▷ multiscale stochastic homogenization of quasilinear hyperbolic-parabolic problems,

they are not addressed or resolved in approximation theory. We hope to study these equations in the future when possible.

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