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CONTINUOUS DEPENDENCE AND GENERAL DECAY OF SOLUTIONS FOR A WAVE EQUATION WITH A NONLINEAR MEMORY TERM

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Abstract. We study existence, uniqueness, continuous dependence, general decay of solutions of an initial boundary value problem for a viscoelastic wave equation with strong damping and nonlinear memory term. At first, we state and prove a theorem involving local existence and uniqueness of a weak solution. Next, we establish a sufficient condition to get an estimate of the continuous dependence of the solution with respect to the kernel function and the nonlinear terms. Finally, under suitable conditions to obtain the global solution, we prove the general decay property with positive initial energy for this global solution.

Keywords: viscoelastic equations; strong damping; nonlinear memory; general decay

MSC 2020: 35L20, 35L70

1. Introduction

In this paper, we study the following Dirichlet problem for a wave equation with strong damping and nonlinear memory:

$$
(1.1)\qquad \begin{cases} u_{tt} - \lambda u_{txx} - \dfrac{\partial^2}{\partial x^2}(\mu(x,t,u(x,t))) + \displaystyle\int_0^t g(t-s)\dfrac{\partial^2}{\partial x^2}(\bar{\mu}(x,s,u(x,s)))\,\mathrm{d}s \\ \qquad\qquad = f(x,t,u,u_t,u_x,u_{tx}), \quad 0<x<1,\ 0<t<T, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x), \quad u_t(x,0) = \tilde{u}_1(x), \end{cases}
$$

where $\lambda > 0$ is a given constant and f, g, μ, $\bar{\mu}$, $\tilde{u}_0$, $\tilde{u}_1$ are given functions.

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Problem (1.1) is a type of viscoelastic problems, the Volterra integral in the first equation of (1.1) is a memory term, so-called viscoelastic term, responsible for viscoelastic damping. The wave equations with memory terms are arised in studies about viscoelastic materials, which possess a capacity of storage and dissipation of mechanical energy. The dynamic properties of viscoelastic materials are of great importance and interest as they appear in many applications to natural sciences, for literatures on this topic, see [10]–[20] and references therein.

The viscoelastic problem of the form (1.1) has been studied by many authors, for example, we refer to [4], [15], [16], [22]–[24], [26]–[32], [34]. By using different methods together with various techniques in functional analysis, several results concerning the existence/global existence and the properties of solutions of viscoelastic problems such as blow-up, decay, stability have been established.

For more details, there have been a lot of investigations dedicated to the following viscoelastic wave equation:

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)\,\mathrm{d}s - \lambda\Delta u_t + \gamma h(u_t) = \mathcal{F}(x,t,u), \tag{1.2}$$

where the kernel g and the source f are C^1 functions satisfying some appropriate hypotheses, and h is a linear or nonlinear function of u_t.

In general, the most common forms of the nonlinear damping h and the source $\mathcal{F}$ in (1.2) are exponential types, especially $h = |u_t|^{m-2}u_t$ and $\mathcal{F} = |u|^{p-2}u$. In [4], Cavalcanti et al. proved that, as $\lambda = 0$, $\gamma = 0$, $\mathcal{F} = 0$ and together with nonlinear boundary damping, the energy of solutions of the corresponding problem goes uniformly to zero at infinity. In [29], Messaoudi considered (1.2) with $\lambda = 0$, $\gamma = 0$, $\mathcal{F} = |u|^{p-2}u$, and showed that, for certain class of relaxation functions and certain initial data, the solution energy decayed at a similar rate of decay of the relaxation function, which was not necessarily decaying in a polynomial or exponential fashion. In [28], Messaoudi studied (1.2) in the case of $\lambda = 0$, $h = a|u_t|^{m-2}u_t$, $\mathcal{F} = b|u|^{p-2}u$, and proved a blow-up result for solutions with negative initial energy if $p > m$ and a global existence result for $p \leqslant m$. Latterly, Kafini and Messaoudi [22] also obtained a blow-up result of a Cauchy problem for a nonlinear viscoelastic equation in the form (1.2) with $m = 2$. In [27], Mesloub and Boulaaras studied a viscoelastic equation for more general decaying kernels and established some general decay results, from which the usual exponential and polynomial rates are only special cases. In the presence of the strong damping $-\Delta u_t$ and the linear damping u_t $(m = 2)$, Li and He [24] proved the global existence of solutions and established a general decay rate estimate for the corresponding problem given by

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)\,\mathrm{d}s - \Delta u_t + u_t = u|u|^{p-2}, \tag{1.3}$$

where the relaxation g is a C^1 function satisfying some suitable hypotheses.

On the other hand, the finite-time blow-up results of solutions with both negative initial energy and positive initial energy were also obtained. In [23], with addition of the dispersion $-\Delta u_{tt}$, Kafini and Mustafa also investigated (1.3) on the whole space $\mathbb{R}^n$, and the authors proved a blow-up result by imposing conditions on the kernel g. For more results related to (1.2) and (1.3), such as general decay or blow up in finite time, one can refer to [15], [16], [31], [34].

In [26], Long et al. studied a specific form of (1.2) with $\lambda = 0$, $\gamma = 1$, $h = |u_t|^{m-2}u_t$, i.e., the authors considered the viscoelastic equation

$$u_{tt} - u_{xx} + \int_0^t g(t-s)u_{xx}(x,s)\,\mathrm{d}s + \alpha|u_t|^{p-2}u_t = \mathcal{F}(x,t,u), \tag{1.4}$$

associated with mixed nonhomogeneous conditions. Under a certain local Lipschitzian condition on the source $\mathcal{F}$ and certain class of relaxation functions and suitable initial datum, a global existence was proved and an asymptotic behavior of solutions as $t \to \infty$ was studied. Recently, Quynh et al. [34] has considered (1.4), in which an N-order recurrent sequence has been established and its convergence to the unique solution of (1.4) satisfying an estimation of convergent rate in N-order has been proved. Furthermore, by using finite-difference approximation, the authors constructed an algorithm to find numerical solutions via the 2-order iterative scheme.

However, to the best of our knowledge, there are relatively few works devoted to the study of partial differential equations with nonlinear memory, for example [7], [8], [18], [21], [32], [35]. In the paper published in 1985 [18], Hrusa considered a one-dimensional nonlinear viscoelastic equation of the form

$$u_{tt} - cu_{xx} + \int_0^t g(t-s)(\Psi(u_x(x,s)))_x\,\mathrm{d}s = f(x,t), \tag{1.5}$$

the author established several global existence results for large data and proved an exponential decay result for strong solutions when $g(s) = \mathrm{e}^{-s}$ and Ψ satisfies some conditions. In [35], Shang and Guo proved the existence, uniqueness, and regularity of the global strong solution and gave some conditions of the nonexistence of global solution to the one-dimension pseudoparabolic equation with the nonlinear memory term $\int_0^t g(t-s)(\sigma(u(x,s),u_x(x,s)))_x\,\mathrm{d}s$. In [32], Ngoc et al. proved the local existence of the wave equation with strong damping and nonlinear viscoelastic term as follows:

$$(1.6)\quad \begin{aligned} &u_{tt} - \lambda u_{xxt} - \frac{\partial}{\partial x}[\mu_1(x,t,u(x,t),\|u(t)\|^2,\|u_x(t)\|^2)u_x] \\ &\quad + \int_0^t g(t-s)\frac{\partial}{\partial x}[\mu_2(x,s,u(x,s),\|u(s)\|^2,\|u_x(s)\|^2)u_x(x,s)]\,\mathrm{d}s \\ &= F(x,t,u,u_x,u_t,\|u(t)\|^2,\|u_x(t)\|^2), \end{aligned}$$

$0 < x < 1$, $0 < t < T$, associated with Robin-Dirichlet boundary conditions and initial conditions, where $\lambda > 0$ is a constant, μ_1, μ_2, g, f are given functions which satisfy certain conditions and the norm $\|\cdot\|$ is defined by $\|u(t)\|^2 = \int_0^1 u^2(x,t)\,\mathrm{d}x$. Moreover, the authors established an asymptotic expansion of solutions, i.e., the solutions of (1.6) can be approximated by an N-order polynomial in small parameter. Recently, Kaddour and Reissig [21] have proved the global (in time) well-posedness results for Sobolev solutions to the following Cauchy problem for a damped wave equation with nonlinear memory on the right-hand side:

$$(1.7)\quad \begin{cases} u_{tt} - \Delta u + (1+t)^r u_t = \int_0^t (t-\tau)^{-\gamma}|u(\tau,x)|^p\,\mathrm{d}\tau, & (t,x)\in(0,\infty)\times\mathbb{R}^n, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), & x\in\mathbb{R}^n, \end{cases}$$

where $r \in (-1,1)$ and $\gamma \in (0,1)$. Moreover, for another investigation of (1.7) given in [21], they have also proved a blow-up result for local (in time) Sobolev solutions.

On the other hand, it seems that there are no results relating to continuous dependence and general decay of solutions of initial boundary value problems with nonlinear memory term. The topic of continuous dependence on datum has received important attention since 1960 with the earlier works of Douglis [9] and John [20]. After that, Benilan and Crandall [2] discussed the continuous dependence on the nonlinearities of solutions of the Cauchy problem for the equation

$$(1.8)\quad \begin{cases} u_t - \Delta\varphi(u) = 0 & \text{in } \mathbb{R}^n\times\mathbb{R}^+, \\ u(x,0) = u_0(x), & x\in\mathbb{R}^n. \end{cases}$$

The authors defined the continuous dependence of solutions in the sense (see [2], page 162)

$$\|u_m(t) - u_\infty(t)\|_{L^1(\mathbb{R}^n)} \to 0 \text{ as } \varphi_m \to \varphi_\infty \text{ with } \varphi_m \text{ instead of } \varphi,$$

where $\varphi_m\colon \mathbb{R}\to\mathbb{R}$ are continuous and nondecreasing functions, $\varphi_m(0) = 0$, and u_m are solutions of the Cauchy problem (1.8). In [33], Pan proved the following estimation which showed the continuous dependence of solutions for the parabolic equation with exponential nonlinearity

$$\int_0^\infty \int_0^1 |u(x,t,m) - u(x,t,m_0)| \,\mathrm{d}t \,\mathrm{d}x \leqslant C^*|m - m_0|,$$

where u is a solution of the proposed problem, $0 < m$, $m_0 \leqslant 1$ and C^* is an explicit constant. Recently, Bayraktar and Gür [1] have studied the continuous dependence of solutions on dispersive δ and r and dissipative b coefficients of the damped improved Boussinesq equation

$$u_{tt} - b\Delta u - \delta\Delta u_{tt} - r\Delta u_t = \Delta(-u|u|^{p-2}),$$

in which the effects of small perturbations of parameters on solutions have been obtained. For similar results, we refer to [5], [14].

Motivated by the above-mentioned inspiring works, in this paper, we consider Problem (1.1) and we first prove the existence and uniqueness of solutions for this problem (Theorem 3.6) by applying the linearization method together with Faedo-Galerkin method and the weak compact method. Next, we consider the continuous dependence of solutions on the nonlinearities of Problem (1.1). Precisely, if $u = u(\mu, \bar{\mu}, f, g)$ and $u_j = u(\mu_j, \bar{\mu}_j, f_j, g_j)$ are the solutions of Problem (1.1) respectively depending on the datum $(\mu, \bar{\mu}, f, g)$ and $(\mu_j, \bar{\mu}_j, f_j, g_j)$, such that

$$\begin{cases} \sup\limits_{M>0} \max\limits_{|\beta|\leqslant 3} \|D^\beta \mu_j - D^\beta \mu\|_{C^0(A_M)} \to 0 & \text{as } j \to \infty, \\ \sup\limits_{M>0} \max\limits_{|\beta|\leqslant 3} \|D^\beta \bar{\mu}_j - D^\beta \bar{\mu}\|_{C^0(A_M)} \to 0 & \text{as } j \to \infty, \\ \sup\limits_{M>0} \max\limits_{|\alpha|\leqslant 1} \|D^\alpha f_j - D^\alpha f\|_{C^0(\tilde{A}_M)} \to 0 & \text{as } j \to \infty, \\ \|g_j - g\|_{H^1(0,T^*)} \to 0 & \text{as } j \to \infty, \end{cases} \tag{1.9}$$

where T^* is fixed positive constant, A_M, $\tilde{A}_M$ are compact sets depending on a positive constant M, $D^\alpha f$ are partial derivatives with order less than or equal $|\alpha|$, then u_j converges to u in $W_1(T)$ as $j \to \infty$ (Theorem 4.1).

Finally, we consider a specific case of Problem (1.1) with $\mu = \mu(t,u)$, $\bar{\mu} = u$, $f = -\lambda_1 u_t + f(u) - \frac{1}{2}D_2^2\mu(t,u)u_x^2 + F(x,t)$, and we prove the general decay of solutions to Problem (1.1) in this case (Theorem 5.6). It is well known that in order to assure the general decay of solutions, the essential assumption for the relaxation function g usually satisfies a relation of the form

$$g'(t) \leqslant -\xi(t)g(t), \tag{1.10}$$

where ξ is a differentiable nonincreasing positive function, see [11], [15], [30]. Recently, condition (1.10) has been relaxed by Mesloub and Boulaaras [27], Boumaza and Boulaaras [3], Conti and Pata [6]—the kernel g does not have to be necessarily decreasing. In the present paper, the relaxation function g also satisfies (1.10), however, it is necessary to set some assumptions for the nonlinear quantity μ; we shall give an example in which μ satisfies a relatively wide class of C^3-functions.

We also note that the decay property is a form of asymptotic behavior/stability in which the energy of solutions tends to zero at infinity. For topic on asymptotic behavior of solutions, there have been many interesting results for models related to (1.1) with memory term, for example, we refer to [19], [17], [24] and the references therein.

The paper consits of five sections. In Section 2, we present some preliminaries. In Section 3, we state and prove the theorem of the existence and uniqueness of Problem (1.1). Sections 4 and 5 are devoted to the continuous dependence and the general decay of solutions of Probblem (1.1). The results obtained here may be considered as relative generalizations of those in [28]–[30], [34].

2. Preliminaries

In this section, we present some notations and materials in order to present the main results. Let $\Omega = (0,1)$, $Q_T = (0,1) \times (0,T)$ and we define the scalar product in L^2 by

$$\langle u, v\rangle = \int_0^1 u(x)v(x)\,\mathrm{d}x,$$

and the corresponding norm $\|\cdot\|$, i.e., $\|u\|^2 = \langle u, u\rangle$. Let us denote the standard function spaces by $C^m(\overline{\Omega})$, $L^p = L^p(\Omega)$ and $H^m = H^m(\Omega)$ for $1 \leqslant p \leqslant \infty$ and $m \in \mathbb{N}$. Also, we denote that $\|\cdot\|_X$ is a norm in a Banach space X, and $L^p(0,T;X)$, $1 \leqslant p \leqslant \infty$, is the Banach space of real functions $u\colon (0,T) \to X$ measurable with the corresponding norm $\|\cdot\|_{L^p(0,T;X)}$ defined by

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p\,\mathrm{d}t\right)^{1/p} < \infty \quad \text{for } 1 \leqslant p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{0<t<T} \|u(t)\|_X \quad \text{for } p = \infty.$$

On H^1, we use the norm

$$\|v\|_{H^1} = (\|v\|^2 + \|v_x\|^2)^{1/2}. \tag{2.1}$$

The following lemma is known.

Lemma 2.1 (see [25]). *The imbeddings* $H^1 \hookrightarrow C^0(\overline{\Omega})$ *and* $H_0^1 \hookrightarrow C^0(\overline{\Omega})$ *are compact and*

$$\begin{aligned} \|v\|_{C^0(\overline{\Omega})} &\leqslant \sqrt{2}\|v\|_{H^1} \quad \forall\, v \in H^1, \\ \|v\|_{C^0(\overline{\Omega})} &\leqslant \|v_x\| \qquad\quad \forall\, v \in H_0^1, \end{aligned} \tag{2.2}$$

where $H_0^1 = \{v \in H^1\colon\ v(0) = v(1) = 0\}$.

Remark 2.2. By (2.1) and (2.2), it is easy to prove that on H_0^1, the two norms $v \mapsto \|v\|_{H^1}$ and $v \mapsto \|v_x\|$ are equivalent.

Throughout this paper, we write $u(t)$, $u'(t) = u_t(t) = \dot{u}(t)$, $u''(t) = u_{tt}(t) = \ddot{u}(t)$, $u_x(t) = \triangledown u(t)$, $u_{xx}(t) = \Delta u(t)$, to denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively. With $f \in C^k([0,1] \times [0,T^*] \times \mathbb{R}^4)$, $f = f(x,t,y_1,\dots,y_4)$, we define $D_1 f = \frac{\partial f}{\partial x}$, $D_2 f = \frac{\partial f}{\partial t}$, $D_{2+i} f = \frac{\partial f}{\partial y_i}$, $i = 1,\dots,4$ and $D^\alpha f = D_1^{\alpha_1} \dots D_6^{\alpha_6} f$; $\alpha = (\alpha_1, \dots, \alpha_6) \in \mathbb{Z}_+^6$, $|\alpha| = \alpha_1 + \dots + \alpha_6 \leqslant k$; $D^{(0,\dots,0)} f = f$. Similarly, with $\mu \in C^k([0,1] \times [0,T^*] \times \mathbb{R})$, $\mu = \mu(x,t,y)$, we define $D_1 \mu = \frac{\partial \mu}{\partial x}$, $D_2 \mu = \frac{\partial \mu}{\partial t}$, $D_3 \mu = \frac{\partial \mu}{\partial y}$ and $D^\beta \mu = D_1^{\beta_1} \dots D_3^{\beta_3} \mu$, $\beta = (\beta_1, \dots, \beta_3) \in \mathbb{Z}_+^3$, $|\beta| = \beta_1 + \dots + \beta_3 \leqslant k$; $D^{(0,\dots,0)} \mu = \mu$.

3. Local existence and uniqueness

In this section, we consider the local existence and uniqueness of Problem (1.1). By using the linearization method together with Faedo-Galerkin method, we prove that there exists a recurrent sequence which converges to the weak solution of (1.1). Let $T^* > 0$. We make the following assumptions:

(H_1) $\tilde{u}_0, \tilde{u}_1 \in H^2 \cap H_0^1$;

(H_2) $\mu, \bar{\mu} \in C^3([0,1] \times [0,T^*] \times \mathbb{R})$ and $D_3 \mu(x,t,y) \geqslant \mu_* > 0$ for all $(x,t,y) \in [0,1] \times [0,T^*] \times \mathbb{R}$;

(H_3) $g \in H^1(0,T^*)$;

(H_4) $f \in C^1([0,1] \times [0,T^*] \times \mathbb{R}^4)$, such that

- (i) $f(0,t,0,0,y_3,y_4) = f(1,t,0,0,y_3,y_4) = 0$ for all $(t,y_3,y_4) \in [0,T^*] \times \mathbb{R}^2$,
- (ii) there exists a positive constant σ such that $\sigma < \sqrt{\bar{\mu}_*}/(3\sqrt{2})$, with $\bar{\mu}_* = \min\{1, \mu_*, 2\lambda\}$ and

$$\|D_6 f\|_{C^0(\tilde{A}_M)} \leqslant \sigma \quad \forall\, M > 0,$$

where $\tilde{A}_M = [0,1] \times [0,T^*] \times [-M,M]^2 \times [-\sqrt{2}M, \sqrt{2}M]^2$.

A typical example for the function f satisfying assumption (H_4) is

$$f(x,t,y_1,\dots,y_4) = f_1(x,t,y_1,\dots,y_3) + \frac{\sigma y_1^2}{1+y_1^2} \sin y_4,$$

$(x,t,y_1,\dots,y_4)\in[0,1]\times[0,T^*]\times\mathbb{R}^4$, where $f_1\in C^1([0,1]\times[0,T^*]\times\mathbb{R}^3)$, such that

$$f_1(0,t,0,0,y_3)=f_1(1,t,0,0,y_3)=0\quad\forall\,(t,y,y_3)\in[0,T^*]\times\mathbb{R},$$

and $0<\sigma<\sqrt{\bar\mu_*}/(3\sqrt{2})$, with $\bar\mu_*=\min\{1,\mu_*,\lambda\}$.

One can easily verify that $f\in C^1([0,1]\times[0,T^*]\times\mathbb{R}^4)$ and (H_4) (i) is fulfilled. By

$$|D_6f(x,t,y_1,\dots,y_4)|=\frac{\sigma y_1^2}{1+y_1^2}|\cos y_4|\leqslant\sigma$$

for all $(x,t,y_1,\dots,y_4)\in\tilde A_M$ for all $M>0$, it follows that $\|D_6f\|_{C^0(\tilde A_M)}\leqslant\sigma$ for all $M>0$. Then condition (H_4) (ii) also holds.

Definition 3.1. A function u is called a weak solution of the initial-boundary value problem (1.1) if

$$\begin{aligned}u\in W_T=\{u\in L^\infty(0,T;H^2\cap H_0^1)\colon\ &u'\in L^\infty(0,T;H^2\cap H_0^1),\\ &u''\in L^2(0,T;H_0^1)\cap L^\infty(0,T;L^2)\},\end{aligned}$$

and u satisfies the variational equation

$$\langle u''(t),v\rangle+\lambda\langle u_x'(t),v_x\rangle+a(t;u(t),v)=\int_0^t g(t-s)\bar a(s;u(s),v)\,\mathrm{d}s+\langle f[u](t),v\rangle \tag{3.1}$$

for all $v\in H_0^1$, a.e. $t\in(0,T)$, together with the initial conditions

$$u(0)=\tilde u_0,\quad u'(0)=\tilde u_1, \tag{3.2}$$

where

$$\begin{aligned}f[u](x,t)&=f(x,t,u(x,t),u'(x,t),u_x(x,t),u_x'(x,t)),\\ a(t;u(t),v)&=\Big\langle\frac{\partial}{\partial x}(\mu(t,u(t))),v_x\Big\rangle=\langle D_1\mu(t,u(t))+D_3\mu(t,u(t))u_x(t),v_x\rangle,\\ \bar a(t;u(t),v)&=\Big\langle\frac{\partial}{\partial x}(\bar\mu(t,u(t))),v_x\Big\rangle=\langle D_1\bar\mu(t,u(t))+D_3\bar\mu(t,u(t))u_x(t),v_x\rangle.\end{aligned}$$

Let $T^*>0$ be fixed. For $M>0$ we put

$$\begin{cases}K_M(\mu)=\|\mu\|_{C^3(A_M)}=\max\limits_{|\beta|\leqslant3}\|D^\beta\mu\|_{C^0(A_M)},\\ K_M(\bar\mu)=\|\bar\mu\|_{C^3(A_M)}=\max\limits_{|\beta|\leqslant3}\|D^\beta\bar\mu\|_{C^0(A_M)},\\ \widetilde K_M(f)=\|f\|_{C^1(\tilde A_M)}=\max\limits_{|\alpha|\leqslant1}\|D^\alpha f\|_{C^0(\tilde A_M)},\\ \|\mu\|_{C^0(A_M)}=\sup\limits_{(x,t,y)\in A_M}|\mu(x,t,y)|,\\ \|f\|_{C^0(\tilde A_M)}=\sup\limits_{(x,t,y_1,\dots,y_4)\in\tilde A_M}|f(x,t,y_1,\dots,y_4)|,\end{cases}$$

where $A_M = [0,1] \times [0,T^*] \times [-M,M]$ and $\tilde{A}_M = [0,1] \times [0,T^*] \times [-M,M]^2 \times [-\sqrt{2}M, \sqrt{2}M]^2$. For any $T \in (0,T^*]$ we consider the set

$$V_T = \{v \in L^\infty(0,T;H^2 \cap H_0^1)\colon\ v' \in L^\infty(0,T;H^2 \cap H_0^1),\ v'' \in L^2(0,T;H_0^1)\}.$$

Then V_T is a Banach space with respect to the norm (see Lions [25])

$$\|v\|_{V_T} = \max\{\|v\|_{L^\infty(0,T;H^2\cap H_0^1)},\ \|v'\|_{L^\infty(0,T;H^2\cap H_0^1)},\ \|v''\|_{L^2(0,T;H_0^1)}\}.$$

Also, we define the sets

$$\begin{cases} W(M,T) = \{v \in V_T\colon\ \|v\|_{V_T} \leqslant M\}, \\ W_1(M,T) = \{v \in W(M,T)\colon\ v'' \in L^\infty(0,T;L^2)\}. \end{cases} \tag{3.3}$$

In the following, we shall establish a linear recurrent sequence $\{u_m\}$ by choosing the first iteration $u_0 \equiv \tilde{u}_0$, and suppose that

$$u_{m-1} \in W_1(M,T). \tag{3.4}$$

Then we shall find u_m in $W_1(M,T)$ satisfying the problem

$$\begin{cases} \langle u_m''(t), v\rangle + \lambda\langle u_{mx}'(t), v_x\rangle + a_m(t; u_m(t), v) \\ \qquad = \displaystyle\int_0^t g(t-s)\bar{a}_m(s; u_m(s), v)\,\mathrm{d}s + \langle F_m(t), v\rangle \quad \forall\, v \in H_0^1, \\ u_m(0) = \tilde{u}_0,\ u_m'(0) = \tilde{u}_1, \end{cases} \tag{3.5}$$

in which

$$\begin{aligned} F_m(x,t) &= f[u_{m-1}](x,t) \\ &= f(x,t,u_{m-1}(x,t), u_{m-1}'(x,t), \nabla u_{m-1}(x,t), \nabla u_{m-1}'(x,t)), \\ a_m(t;u,v) &= \langle D_1\mu(t,u_{m-1}(t)) + D_3\mu(t,u_{m-1}(t))u_x, v_x\rangle, \\ \bar{a}_m(t;u,v) &= \langle D_1\bar{\mu}(t,u_{m-1}(t)) + D_3\bar{\mu}(t,u_{m-1}(t))u_x, v_x\rangle,\ u,v \in H_0^1. \end{aligned}$$

Note that $a_m(t;u,v)$, $\bar{a}_m(t;u,v)$ can be rewritten in the form

$$\begin{aligned} a_m(t;u,v) &= A_m(t;u,v) + \langle \mu_{1m}(t), v_x\rangle, \\ \bar{a}_m(t;u,v) &= \bar{A}_m(t;u,v) + \langle \bar{\mu}_{1m}(t), v_x\rangle, \quad u,v \in H_0^1, \end{aligned}$$

where

$$\begin{aligned} A_m(t;u,v) &= \langle \mu_{3m}(t)u_x, v_x\rangle,\ \bar{A}_m(t;u,v) = \langle \bar{\mu}_{3m}(t)u_x, v_x\rangle, \quad u,v \in H_0^1, \\ \mu_{3m}(x,t) &= D_3\mu(x,t,u_{m-1}(x,t)), \quad \mu_{1m}(x,t) = D_1\mu(x,t,u_{m-1}(x,t)), \\ \bar{\mu}_{3m}(x,t) &= D_3\bar{\mu}(x,t,u_{m-1}(x,t)), \quad \bar{\mu}_{1m}(x,t) = D_1\bar{\mu}(x,t,u_{m-1}(x,t)). \end{aligned}$$

Then Problem (3.5) is equivalent to

$$(3.6)\qquad \begin{cases} \langle u_m''(t), v\rangle + \lambda\langle u_{mx}'(t), v_x\rangle + A_m(t; u_m(t), v) \\ \qquad = \displaystyle\int_0^t g(t-s)\bar{A}_m(s; u_m(s), v)\,\mathrm{d}s + \langle \widehat{F}_m(t), v\rangle \quad \forall\, v \in H_0^1, \\ u_m(0) = \tilde{u}_0,\ u_m'(0) = \tilde{u}_1, \end{cases}$$

where $\widehat{F}_m(t)\colon\ H_0^1 \to \mathbb{R}$ is a linear continuous functional on H_0^1, which is defined by

$$(3.7)\quad \langle \widehat{F}_m(t), v\rangle = \langle F_m(t), v\rangle - \langle \mu_{1m}(t), v_x\rangle + \int_0^t g(t-s)\langle \bar{\mu}_{1m}(s), v_x\rangle\,\mathrm{d}s, \quad v \in H_0^1.$$

The existence of u_m is assured by the following theorem.

Theorem 3.2. *Under assumptions* (H_1)–(H_4), *there exist positive constants* M, T *such that for* $u_0 \equiv \tilde{u}_0$ *there exists a recurrent sequence* $\{u_m\} \subset W_1(M,T)$ *defined by* (3.4), (3.6), *and* (3.7).

Proof of Theorem 3.2. The proof consists of several steps.

Step 1. The Galerkin approximation. Consider a special orthonormal basis $\{w_j\}$ on H_0^1: $w_j(x) = \sqrt{2}\sin(j\pi x)$, $j \in \mathbb{N}$, formed by the eigenfunctions of the Laplacian $-\Delta = -\frac{\partial^2}{\partial x^2}$. Put

$$u_m^{(k)}(t) = \sum_{j=1}^k c_{mj}^{(k)}(t) w_j,$$

where the coefficients $c_{mj}^{(k)}$ satisfy the system of linear integrodifferential equations

$$(3.8)\quad \begin{cases} \langle \ddot{u}_m^{(k)}(t), w_j\rangle + \lambda\langle \dot{u}_{mx}^{(k)}(t), w_{jx}\rangle + A_m(t; u_m^{(k)}(t), w_j) \\ \qquad = \displaystyle\int_0^t g(t-s)\bar{A}_m(s; u_m^{(k)}(s), w_j)\,\mathrm{d}s + \langle \widehat{F}_m(t), w_j\rangle, \quad 1 \leqslant j \leqslant k, \\ u_m^{(k)}(0) = \tilde{u}_{0k},\ \dot{u}_m^{(k)}(0) = \tilde{u}_{1k}, \end{cases}$$

in which

$$(3.9)\qquad \begin{cases} \tilde{u}_{0k} = \displaystyle\sum_{j=1}^k \alpha_j^{(k)} w_j \to \tilde{u}_0 \quad \text{strongly in } \ H^2 \cap H_0^1, \\ \tilde{u}_{1k} = \displaystyle\sum_{j=1}^k \beta_j^{(k)} w_j \to \tilde{u}_1 \quad \text{strongly in } H^2 \cap H_0^1. \end{cases}$$

System (3.8) is equivalent to the system of linear intergal equations which can be rewritten in the form

$$c_m^{(k)} = U[c_m^{(k)}],$$

where

$$
\begin{aligned}
c_m^{(k)} &= (c_{m1}^{(k)},\dots,c_{mk}^{(k)}),\\
U[c_m^{(k)}] &= (U_1[c_m^{(k)}],\dots,U_k[c_m^{(k)}]),\\
U_j[c_m^{(k)}](t) &= \mathcal{F}_j^{(k)}[c_m^{(k)}](t) + G_j^{(k)}(t),\\
\mathcal{F}_j^{(k)}[c_m^{(k)}](t) &= -\int_0^t\int_0^r \mathrm{e}^{-\lambda\lambda_j(r-s)}\sum_{i=1}^k A_{mij}(s)c_{mi}^{(k)}(s)\,\mathrm{d}s\,\mathrm{d}r\\
&\quad+\int_0^t\int_0^r\int_0^s \mathrm{e}^{-\lambda\lambda_j(r-s)}g(s-\tau)\sum_{i=1}^k \bar{A}_{mij}(\tau)c_{mi}^{(k)}(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\,\mathrm{d}r,\\
G_j^{(k)}(t) &= \alpha_j^{(k)} + \frac{\beta_j^{(k)}}{\lambda\lambda_j}(1-\mathrm{e}^{-\lambda\lambda_j t}) + \int_0^t\int_0^r \mathrm{e}^{-\lambda\lambda_j(r-s)}\langle \widehat{F}_m(s),w_j\rangle\,\mathrm{d}s\,\mathrm{d}r,\\
\lambda_j &= (j\pi)^2,\\
A_{mij}(t) &= A_m(t;w_i,w_j),\\
\bar{A}_{mij}(t) &= \bar{A}_m(t;w_i,w_j),\quad i,j=1,\dots,k,\ 0\leqslant t\leqslant T.
\end{aligned}
$$

Using Banach's contraction principle, which is similar to the one used in [32], it is not difficult to prove that the above fixed-point equation admits a unique solution $c_m^{(k)}\in C([0,T];\mathbb{R}^k)$, so let us omit the details.

Step 2. A priori estimate. Put

$$
\begin{aligned}
S_m^{(k)}(t) &= \|\dot{u}_m^{(k)}(t)\|^2 + \|\dot{u}_{mx}^{(k)}(t)\|^2 + \|\sqrt{\mu_{3m}(t)}u_{mx}^{(k)}(t)\|^2\\
&\quad+\|\sqrt{\mu_{3m}(t)}\Delta u_m^{(k)}(t)\|^2 + \lambda\|\Delta\dot{u}_m^{(k)}(t)\|^2\\
&\quad+2\lambda\int_0^t(\|\dot{u}_{mx}^{(k)}(s)\|^2+\|\Delta\dot{u}_m^{(k)}(s)\|^2)\,\mathrm{d}s + 2\int_0^t\|\ddot{u}_{mx}^{(k)}(s)\|^2\,\mathrm{d}s,
\end{aligned}
$$

then it follows from (3.8) that

$$
\begin{aligned}
S_m^{(k)}(t) &= S_m^{(k)}(0) + 2\langle\mu_{3mx}(0)\tilde{u}_{0kx},\Delta\tilde{u}_{0k}\rangle + 2\Big\langle\frac{\partial}{\partial x}(\mu_{3m}(0)\tilde{u}_{0kx}),\Delta\tilde{u}_{1k}\Big\rangle\\
&\quad+\int_0^t\mathrm{d}s\int_0^1\mu_{3m}'(x,s)(|u_{mx}^{(k)}(x,s)|^2+|\Delta u_m^{(k)}(x,s)|^2)\,\mathrm{d}x\\
&\quad+2\int_0^t g(t-s)\bar{A}_m(s;u_m^{(k)}(s),u_m^{(k)}(t))\,\mathrm{d}s\\
&\quad+2\int_0^t g(t-s)\Big\langle\frac{\partial}{\partial x}(\bar{\mu}_{3m}(s)u_{mx}^{(k)}(s)),\Delta u_m^{(k)}(t)+\Delta\dot{u}_m^{(k)}(t)\Big\rangle\,\mathrm{d}s\\
&\quad-2g(0)\int_0^t\Big\langle\frac{\partial}{\partial x}(\bar{\mu}_{3m}(s)u_{mx}^{(k)}(s)),\Delta u_m^{(k)}(s)+\Delta\dot{u}_m^{(k)}(s)\Big\rangle\,\mathrm{d}s\\
&\quad-2g(0)\int_0^t\bar{A}_m(s;u_m^{(k)}(s),u_m^{(k)}(s))\,\mathrm{d}s
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
&- 2\int_0^t \mathrm{d}\tau \int_0^\tau g'(\tau - s)\bar{A}_m(s; u_m^{(k)}(s), u_m^{(k)}(\tau))\,\mathrm{d}s \\
&- 2\int_0^t \mathrm{d}\tau \int_0^\tau g'(\tau - s) \\
&\times \Big\langle \frac{\partial}{\partial x}(\bar{\mu}_{3m}(s)u_{mx}^{(k)}(s)), \Delta u_m^{(k)}(\tau) + \Delta \dot{u}_m^{(k)}(\tau)\Big\rangle\,\mathrm{d}s \\
&+ 2\int_0^t \Big\langle \frac{\partial}{\partial s}(\mu_{3mx}(s)u_{mx}^{(k)}(s)), \Delta u_m^{(k)}(s)\Big\rangle\,\mathrm{d}s \\
&+ 2\int_0^t \Big\langle \frac{\partial^2}{\partial x\partial s}(\mu_{3m}(s)u_{mx}^{(k)}(s)), \Delta \dot{u}_m^{(k)}(s)\Big\rangle\,\mathrm{d}s \\
&- 2\langle \mu_{3mx}(t)u_{mx}^{(k)}(t), \Delta u_m^{(k)}(t)\rangle - 2\Big\langle \frac{\partial}{\partial x}(\mu_{3m}(t)u_{mx}^{(k)}(t)), \Delta \dot{u}_m^{(k)}(t)\Big\rangle \\
&+ 2\int_0^t \langle \widehat{F}_m(s), \dot{u}_m^{(k)}(s)\rangle\,\mathrm{d}s + 2\int_0^t \langle \widehat{F}_m(s), -\Delta \dot{u}_m^{(k)}(s)\rangle\,\mathrm{d}s \\
&+ 2\int_0^t \langle \widehat{F}_m(s), -\Delta \ddot{u}_m^{(k)}(s)\rangle\,\mathrm{d}s \\
= {}& S_m^{(k)}(0) + 2\langle \mu_{3mx}(0)\tilde{u}_{0kx}, \Delta \tilde{u}_{0k}\rangle \\
&+ 2\Big\langle \frac{\partial}{\partial x}(\mu_{3m}(0)\tilde{u}_{0kx}), \Delta \tilde{u}_{1k}\Big\rangle + \sum_{i=1}^{14} J_i.
\end{aligned}
$$

We shall estimate the terms J_i on the right-hand side of (3.10) as follows. First, we need the following lemma, whose proof is proved in Section 6.

Lemma 3.3. *Put*

$$
\begin{aligned}
\overline{S}_m^{(k)}(t) &= \|\dot{u}_m^{(k)}(t)\|^2 + \|\dot{u}_{mx}^{(k)}(t)\|^2 + \|\Delta \dot{u}_m^{(k)}(t)\|^2 + \|u_{mx}^{(k)}(t)\|^2 + \|\Delta u_m^{(k)}(t)\|^2 \\
&+ \int_0^t (\|\dot{u}_{mx}^{(k)}(s)\|^2 + \|\Delta \dot{u}_m^{(k)}(s)\|^2)\,\mathrm{d}s + \int_0^t \|\ddot{u}_{mx}^{(k)}(s)\|^2\,\mathrm{d}s.
\end{aligned} \tag{3.11}
$$

Then the following estimations are admitted:

(i) $|\mu'_{im}(x,t)| \leqslant (1+M)K_M(\mu)$, $i = 1,3$,
(ii) $\|\mu'_{im}(t)\| \leqslant (1+M)K_M(\mu)$, $i = 1,3$,
(iii) $|\mu_{imx}(x,t)| \leqslant (1+2M)K_M(\mu)$, $i = 1,3$,
(iv) $\|\mu_{imx}(t)\| \leqslant (1+M)K_M(\mu)$, $i = 1,3$,
(v) $|\mu'_{imx}(x,t)| \leqslant (1+5M+2M^2)K_M(\mu)$, $i = 1,3$,
(vi) $\|\mu'_{imx}(t)\| \leqslant (1+3M+M^2)K_M(\mu)$, $i = 1,3$,
(vii) $|\bar{A}_m(s; u_m^{(k)}(s), u_m^{(k)}(t))| \leqslant K_M(\bar{\mu})\|u_{mx}^{(k)}(s)\|\|u_{mx}^{(k)}(t)\|$,
(viii) $\|\Delta u_m^{(k)}(t) + \Delta \dot{u}_m^{(k)}(t)\| \leqslant \sqrt{2}\sqrt{\overline{S}_m^{(k)}(t)}$,
(ix) $\|u_{mx}^{(k)}(t)\|^2 \leqslant 2\|\tilde{u}_{0kx}\|^2 + 2T^* \int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s$,
(x) $\|\frac{\partial}{\partial x}(\mu_{3m}(t)u_{mx}^{(k)}(t))\| \leqslant 2(1+M)K_M(\mu)\sqrt{\overline{S}_m^{(k)}(t)}$,

(xi) $\|\frac{\partial}{\partial t}(\mu_{3mx}(t)u_{mx}^{(k)}(t))\| \leqslant (2+7M+2M^2)K_M(\mu)\sqrt{\overline{S}_m^{(k)}(t)}$,

(xii) $\|\frac{\partial^2}{\partial x\partial t}(\mu_{3m}(t)u_{mx}^{(k)}(t))\| \leqslant 2(2+4M+M^2)K_M(\mu)\sqrt{\overline{S}_m^{(k)}(t)}$.

Moreover, inequalities (i)–(xii) *are also valid with replacing* μ *by* $\bar{\mu}$.

By Lemma 3.3, the terms J_1–J_9 on the right-hand side of (3.10) are estimated as follows:

Using the inequality $S_m^{(k)}(t) \geqslant \bar{\mu}_*\overline{S}_m^{(k)}(t)$, where $\bar{\mu}_* = \min\{1,\mu_*,2\lambda\}$ and $2ab \leqslant \beta a^2 + \beta^{-1}b^2$ for all $a,b \in \mathbb{R}$ with $\beta = \beta_* = \frac{1}{10}\bar{\mu}_*$, the terms J_1–J_9 are, respectively, estimated by

$$
\begin{aligned}
(3.12)\quad J_1 &= \int_0^t \mathrm{d}s \int_0^1 \mu'_{3m}(x,s)(|u_{mx}^{(k)}(x,s)|^2 + |\Delta u_m^{(k)}(x,s)|^2)\,\mathrm{d}x \\
&\leqslant (1+M)K_M(\mu)\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s = C_1(M)\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s, \\
J_2 &= 2\int_0^t g(t-s)\bar{A}_m(s;u_m^{(k)}(s),u_m^{(k)}(t))\,\mathrm{d}s \\
&\leqslant \|\tilde{u}_{0kx}\|^2 + (T^* + 2K_M^2(\bar{\mu})\|g\|^2_{L^2(0,T^*)})\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s \\
&= \|\tilde{u}_{0kx}\|^2 + C_2(M)\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s, \\
J_3 &= 2\int_0^t g(t-s)\Big\langle \frac{\partial}{\partial x}(\bar{\mu}_{3m}(s)u_{mx}^{(k)}(s)), \Delta u_m^{(k)}(t) + \Delta\dot{u}_m^{(k)}(t)\Big\rangle\,\mathrm{d}s \\
&\leqslant \beta_*\overline{S}_m^{(k)}(t) + \frac{8}{\beta_*}(1+M)^2K_M^2(\bar{\mu})\|g\|^2_{L^2(0,T^*)}\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s \\
&= \beta_*\overline{S}_m^{(k)}(t) + C_3(M)\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s, \\
J_4 &= -2g(0)\int_0^t \Big\langle \frac{\partial}{\partial x}(\bar{\mu}_{3m}(s)u_{mx}^{(k)}(s)), \Delta u_m^{(k)}(s) + \Delta\dot{u}_m^{(k)}(s)\Big\rangle\,\mathrm{d}s \\
&\leqslant 4\sqrt{2}(1+M)K_M(\bar{\mu})|g(0)|\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s = C_4(M)\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s, \\
J_5 &= -2g(0)\int_0^t \bar{A}_m(s;u_m^{(k)}(s),u_m^{(k)}(s))\,\mathrm{d}s \\
&\leqslant 2|g(0)|K_M(\bar{\mu})\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s = C_5(M)\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s, \\
J_6 &= -2\int_0^t \mathrm{d}\tau\int_0^\tau g'(\tau-s)\bar{A}_m(s;u_m^{(k)}(s),u_m^{(k)}(\tau))\,\mathrm{d}s \\
&\leqslant 2K_M(\bar{\mu})\sqrt{T^*}\|g'\|_{L^2(0,T^*)}\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s = C_6(M)\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s,
\end{aligned}
$$

$$J_7 = -2\int_0^t \mathrm{d}\tau \int_0^\tau g'(\tau - s)\Big\langle \frac{\partial}{\partial x}(\bar\mu_{3m}(s)u_{mx}^{(k)}(s)), \Delta u_m^{(k)}(\tau) + \Delta\dot u_m^{(k)}(\tau)\Big\rangle\,\mathrm{d}s$$
$$\leqslant 4\sqrt{2}(1+M)K_M(\bar\mu)\sqrt{T^*}\|g'\|_{L^2(0,T^*)}\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s$$
$$= C_7(M)\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s,$$
$$J_8 = 2\int_0^t \Big\langle \frac{\partial}{\partial s}(\mu_{3mx}(s)u_{mx}^{(k)}(s)), \Delta u_m^{(k)}(s)\Big\rangle\,\mathrm{d}s$$
$$\leqslant 2(2+7M+2M^2)K_M(\mu)\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s = C_8(M)\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s,$$
$$J_9 = 2\int_0^t \Big\langle \frac{\partial^2}{\partial x\partial s}(\mu_{3m}(s)u_{mx}^{(k)}(s)), \Delta\dot u_m^{(k)}(s)\Big\rangle\,\mathrm{d}s$$
$$\leqslant 4(2+4M+M^2)K_M(\mu)\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s = C_9(M)\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s.$$

In order to estimate the terms J_{10} and J_{11}, we use the following lemma, whose proof can be found in Section 6.

Lemma 3.4. *The following estimations are valid:*

(i) $$\|\Delta u_m^{(k)}(t)\| \leqslant \|\Delta\tilde u_{0k}\| + \int_0^t \sqrt{\overline{S}_m^{(k)}(s)}\,\mathrm{d}s,$$

(ii) $$\|\mu_{3mx}(t)u_{mx}^{(k)}(t)\| \leqslant \|\mu_{3mx}(0)\tilde u_{0kx}\| + (2+7M+2M^2)K_M(\mu)\int_0^t \sqrt{\overline{S}_m^{(k)}(s)}\,\mathrm{d}s,$$

(iii) $$\Big\|\frac{\partial}{\partial x}(\mu_{3m}(t)u_{mx}^{(k)}(t))\Big\| \leqslant \Big\|\frac{\partial}{\partial x}(\mu_{3m}(0)\tilde u_{0kx})\Big\| + 2(2+4M+M^2)K_M(\mu)\int_0^t \sqrt{\overline{S}_m^{(k)}(s)}\,\mathrm{d}s.$$

Using Lemma 3.4 and the inequality $(a+b)^2 \leqslant 2a^2 + 2b^2$ for all $a,b\in\mathbb{R}$, the terms J_{10}, J_{11} are estimated as follows:

(3.13) $$J_{10} = -2\langle \mu_{3mx}(t)u_{mx}^{(k)}(t), \Delta u_m^{(k)}(t)\rangle$$
$$\leqslant 2(\|\mu_{3mx}(0)\tilde u_{0kx}\|^2 + \|\Delta\tilde u_{0k}\|^2) + 2T^*[1 + (2+7M+2M^2)^2K_M^2(\mu)]\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s$$
$$= 2(\|\mu_{3mx}(0)\tilde u_{0kx}\|^2 + \|\Delta\tilde u_{0k}\|^2) + C_{10}(M)\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s;$$

$$\begin{aligned}J_{11} = &-2\Big\langle \frac{\partial}{\partial x}(\mu_{3m}(t)u_{mx}^{(k)}(t)), \Delta \dot{u}_m^{(k)}(t)\Big\rangle \\ &\leqslant \beta_* \overline{S}_m^{(k)}(t) + \frac{2}{\beta_*}\Big\|\frac{\partial}{\partial x}(\mu_{3m}(0)\tilde{u}_{0kx})\Big\|^2 \\ &\quad + \frac{8}{\beta_*}(2+4M+M^2)^2 K_M^2(\mu) T^* \int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s \\ &= \beta_* \overline{S}_m^{(k)}(t) + \frac{2}{\beta_*}\Big\|\frac{\partial}{\partial x}(\mu_{3m}(0)\tilde{u}_{0kx})\Big\|^2 + C_{11}(M)\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s.\end{aligned}$$

The terms J_{12}–J_{14} are also estimated as above. By the fact that

$$\langle \widehat{F}_m(t), \dot{u}_m^{(k)}(t)\rangle = \langle F_m(t), \dot{u}_m^{(k)}(t)\rangle - \langle \mu_{1m}(t), \dot{u}_{mx}^{(k)}(t)\rangle + \int_0^t g(t-s)\langle \bar{\mu}_{1m}(s), \dot{u}_{mx}^{(k)}(t)\rangle\,\mathrm{d}s,$$

we have

$$\begin{aligned}|\langle \widehat{F}_m(t), \dot{u}_m^{(k)}(t)\rangle| &\leqslant (\widetilde{K}_M(f) + K_M(\mu) + K_M(\bar{\mu})\|g\|_{L^1(0,T^*)})\sqrt{\overline{S}_m^{(k)}(t)} \\ &\equiv \sqrt{C_{12}(M)}\sqrt{\overline{S}_m^{(k)}(t)}.\end{aligned}$$

Then

$$J_{12} = 2\int_0^t \langle \widehat{F}_m(s), \dot{u}_m^{(k)}(s)\rangle\,\mathrm{d}s \leqslant TC_{12}(M) + \int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s. \tag{3.14}$$

We also have

$$\begin{aligned}\langle \widehat{F}_m(t), -\Delta\dot{u}_m^{(k)}(t)\rangle = &\langle F_m(t), -\Delta\dot{u}_m^{(k)}(t)\rangle - \langle \mu_{1mx}(t), \Delta\dot{u}_m^{(k)}(t)\rangle \\ &+ \int_0^t g(t-s)\langle \bar{\mu}_{1mx}(s), \Delta\dot{u}_m^{(k)}(t)\rangle\,\mathrm{d}s,\end{aligned}$$

so

$$\begin{aligned}|\langle \widehat{F}_m(t), -\Delta\dot{u}_m^{(k)}(t)\rangle| &\leqslant \Big[\|F_m(t)\| + \|\mu_{1mx}(t)\| + \int_0^t |g(t-s)|\|\bar{\mu}_{1mx}(s)\|\,\mathrm{d}s\Big]\|\Delta\dot{u}_m^{(k)}(t)\| \\ &\leqslant [\widetilde{K}_M(f) + (1+M)(K_M(\mu) + K_M(\bar{\mu})\|g\|_{L^1(0,T^*)})]\sqrt{\overline{S}_m^{(k)}(t)} \\ &\equiv \sqrt{C_{13}(M)}\sqrt{\overline{S}_m^{(k)}(t)}.\end{aligned}$$

Then

$$J_{13} = 2\int_0^t \langle \widehat{F}_m(s), -\Delta\dot{u}_m^{(k)}(s)\rangle\,\mathrm{d}s \leqslant TC_{13}(M) + \int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s. \tag{3.15}$$

Similarly,

$$\langle \widehat{F}_m(t), -\Delta \ddot{u}_m^{(k)}(t)\rangle = \langle F_{mx}(t), \ddot{u}_{mx}^{(k)}(t)\rangle - \frac{\mathrm{d}}{\mathrm{d}t}\langle \mu_{1mx}(t), \Delta \dot{u}_m^{(k)}(t)\rangle + \langle \mu_{1mx}'(t), \Delta \dot{u}_m^{(k)}(t)\rangle$$
$$+ \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t g(t-s)\langle \bar{\mu}_{1mx}(s), \Delta \dot{u}_m^{(k)}(t)\rangle \,\mathrm{d}s - g(0)\langle \bar{\mu}_{1mx}(t), \Delta \dot{u}_m^{(k)}(t)\rangle$$
$$- \int_0^t g'(t-s)\langle \bar{\mu}_{1mx}(s), \Delta \dot{u}_m^{(k)}(t)\rangle \,\mathrm{d}s,$$

thus,

$$\begin{aligned} J_{14} &= 2\int_0^t \langle \widehat{F}_m(s), -\Delta \ddot{u}_m^{(k)}(s)\rangle \,\mathrm{d}s \\ &= 2\langle \mu_{1mx}(0), \Delta \tilde{u}_{1k}\rangle + 2\int_0^t \langle F_{mx}(s), \ddot{u}_{mx}^{(k)}(s)\rangle \,\mathrm{d}s - 2\langle \mu_{1mx}(t), \Delta \dot{u}_m^{(k)}(t)\rangle \\ &\quad + 2\int_0^t \langle \mu_{1mx}'(s), \Delta \dot{u}_m^{(k)}(s)\rangle \,\mathrm{d}s + 2\int_0^t g(t-s)\langle \bar{\mu}_{1mx}(s), \Delta \dot{u}_m^{(k)}(t)\rangle \,\mathrm{d}s \\ &\quad - 2g(0)\int_0^t \langle \bar{\mu}_{1mx}(s), \Delta \dot{u}_m^{(k)}(s)\rangle \,\mathrm{d}s \\ &\quad - 2\int_0^t \mathrm{d}\tau \int_0^\tau g'(\tau-s)\langle \bar{\mu}_{1mx}(s), \Delta \dot{u}_m^{(k)}(\tau)\rangle \,\mathrm{d}s \\ &= 2\langle \mu_{1mx}(0), \Delta \tilde{u}_{1k}\rangle + q_1 + \ldots + q_6. \end{aligned}$$

In order to estimate $q_1, \ldots, q_6$, we use the following inequalities:

$$\|F_{mx}(s)\| \leqslant (1+4M)\widetilde{K}_M(f),$$
$$\|\mu_{1mx}(t)\| \leqslant \|\mu_{1mx}(0)\| + \int_0^t \|\mu_{1mx}'(s)\| \,\mathrm{d}s \leqslant \|\mu_{1mx}(0)\| + T(1+3M+M^2)K_M(\mu).$$

Then

$$\begin{aligned} q_1 &= 2\int_0^t \langle F_{mx}(s), \ddot{u}_{mx}^{(k)}(s)\rangle \,\mathrm{d}s \\ &\leqslant \frac{1}{\beta_*}T(1+4M)^2\widetilde{K}_M^2(f) + \beta_*\overline{S}_m^{(k)}(t) = T\bar{q}_1(M) + \beta_*\overline{S}_m^{(k)}(t), \\ q_2 &= -2\langle \mu_{1mx}(t), \Delta \dot{u}_m^{(k)}(t)\rangle \\ &\leqslant \beta_*\overline{S}_m^{(k)}(t) + \frac{2}{\beta_*}\|\mu_{1mx}(0)\|^2 + \frac{2T}{\beta_*}T(1+3M+M^2)^2K_M^2(\mu) \\ &= \beta_*\overline{S}_m^{(k)}(t) + \frac{2}{\beta_*}\|\mu_{1mx}(0)\|^2 + T\bar{q}_2(M), \\ q_3 &= 2\int_0^t \langle \mu_{1mx}'(s), \Delta \dot{u}_m^{(k)}(s)\rangle \,\mathrm{d}s \\ &\leqslant T(1+3M+M^2)^2K_M^2(\mu) + \int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s = T\bar{q}_3(M) + \int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s, \end{aligned}$$

$$
\begin{aligned}
q_4 &= 2\int_0^t g(t-s)\langle \bar\mu_{1mx}(s), \Delta\dot u_m^{(k)}(t)\rangle \,\mathrm{d}s\\
&\leqslant \beta_* \overline{S}_m^{(k)}(t) + \frac{1}{\beta_*}T(1+M)^2K_M^2(\bar\mu)\|g\|^2_{L^2(0,T^*)} = \beta_*\overline{S}_m^{(k)}(t) + T\bar q_4(M),\\
q_5 &= -2g(0)\int_0^t \langle \bar\mu_{1mx}(s), \Delta\dot u_m^{(k)}(s)\rangle \,\mathrm{d}s\\
&\leqslant Tg^2(0)(1+M)^2K_M^2(\bar\mu) + \int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s = T\bar q_5(M) + \int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s,\\
q_6 &= -2\int_0^t \mathrm{d}\tau \int_0^\tau g'(\tau-s)\langle \bar\mu_{1mx}(s), \Delta\dot u_m^{(k)}(\tau)\rangle \,\mathrm{d}s\\
&\leqslant \int_0^t \overline{S}_m^{(k)}(\tau)\,\mathrm{d}\tau + TT^*(1+M)^2K_M^2(\bar\mu)\|g'\|^2_{L^2(0,T^*)}\\
&= \int_0^t \overline{S}_m^{(k)}(\tau)\,\mathrm{d}\tau + T\bar q_6(M).
\end{aligned}
$$

Thus, J_{14} is estimated by

(3.16)

$$
J_{14} \leqslant 2\langle \mu_{1mx}(0), \Delta\tilde u_{1k}\rangle + \frac{2}{\beta_*}\|\mu_{1mx}(0)\|^2 + TC_{14}(M) + 3\beta_*\overline{S}_m^{(k)}(t) + 3\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s,
$$

where $C_{14}(M) = \sum\limits_{j=1}^{6} \bar q_j(M)$.

Combining (3.12)–(3.16), it follows from (3.10) and (3.11) that

$$
\overline{S}_m^{(k)}(t) \leqslant \overline{S}_{0m}^{(k)} + TD_1(M) + D_2(M)\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s, \tag{3.17}
$$

where

$$
\begin{aligned}
\overline{S}_{0m}^{(k)} &= \frac{2}{\bar\mu_*}S_m^{(k)}(0) + \frac{4}{\bar\mu_*}[\langle \mu_{3mx}(0)\tilde u_{0kx}, \Delta\tilde u_{0k}\rangle + \langle \mu_{1mx}(0), \Delta\tilde u_{1k}\rangle] + \frac{2}{\bar\mu_*}\|\tilde u_{0kx}\|^2\\
&\quad + \frac{4}{\bar\mu_*}\Big\langle \frac{\partial}{\partial x}(\mu_{3m}(0)\tilde u_{0kx}), \Delta\tilde u_{1k}\Big\rangle + \frac{4}{\bar\mu_*}(\|\mu_{3mx}(0)\tilde u_{0kx}\|^2 + \|\Delta\tilde u_{0k}\|^2)\\
&\quad + \frac{40}{\bar\mu_*^2}(\|\frac{\partial}{\partial x}(\mu_{3m}(0)\tilde u_{0kx})\|^2 + \|\mu_{1mx}(0)\|^2),
\end{aligned} \tag{3.18}
$$

$$
\begin{aligned}
S_m^{(k)}(0) &= \|\tilde u_{1k}\|^2 + \|\tilde u_{1kx}\|^2 + \|\sqrt{\mu_{3m}(0)}\tilde u_{0kx}\|^2\\
&\quad + \|\sqrt{\mu_{3m}(0)}\Delta\tilde u_{0k}\|^2 + \lambda\|\Delta\tilde u_{1k}\|^2,\\
D_1(M) &= \frac{2}{\bar\mu_*}[C_{12}(M) + C_{13}(M) + C_{14}(M)],\\
D_2(M) &= \frac{2}{\bar\mu_*}\Big(5 + \sum_{j=1}^{11} C_j(M)\Big).
\end{aligned}
$$

The convergences given by (3.18) show that there exists a positive constant M independent of k and m such that

$$\overline{S}_{0m}^{(k)} \leqslant \frac{M^2}{2} \quad \forall\, m, \quad k \in \mathbb{N}. \tag{3.19}$$

The local existence is obtained by choosing T small enough as in the following lemma.

Lemma 3.5. *Suppose that there exists a positive constant M satisfying* (3.19). *For any $T \in (0, T^*]$, put*

$$k_T = 3\sqrt{D_1^*(M,T)}\exp(TD_2^*(M)), \tag{3.20}$$

where

$$\begin{aligned} D_1^*(M,T) &= \frac{2}{\bar{\mu}_*}(\sigma + 2\sqrt{T}\widetilde{K}_M(f))^2 + \frac{12T}{\bar{\mu}_*^2}(1+M)^2(K_M^2(\mu) + K_M^2(\bar{\mu})\|g\|_{L^2(0,T^*)}^2) \\ &\quad + \frac{T}{\bar{\mu}_*}(1+M)^2(|g(0)| + \sqrt{T^*}\|g'\|_{L^2(0,T^*)})K_M(\bar{\mu}), \\ D_2^*(M) &= \frac{1}{\bar{\mu}_*}[1 + (1+M)K_M(\mu) + 4(|g(0)| + \sqrt{T^*}\|g'\|_{L^2(0,T^*)})K_M(\bar{\mu})] \\ &\quad + \frac{6}{\bar{\mu}_*^2}K_M^2(\bar{\mu})\|g\|_{L^2(0,T^*)}. \end{aligned}$$

Then T can be chosen small enough such that

$$\Big(\frac{M^2}{2} + TD_1(M)\Big)\mathrm{e}^{TD_2(M)} \leqslant M^2 \quad \text{and} \quad k_T < 1. \tag{3.21}$$

P r o o f. By the assumption $0 < \sigma < \sqrt{\bar{\mu}_*}/(3\sqrt{2})$, it is easy to get that

$$\lim_{T\to 0_+} k_T = \lim_{T\to 0_+} 3\sqrt{D_1^*(M,T)}\exp(TD_2^*(M)) = 3\sqrt{\frac{2}{\bar{\mu}_*}}\sigma < 1$$

and

$$\lim_{T\to 0_+}\Big(\frac{M^2}{2} + TD_1(M)\Big)\mathrm{e}^{TD_2(M)} = \frac{M^2}{2} < M^2.$$

□

It follows from (3.17) and (3.21) that

$$\overline{S}_m^{(k)}(t) \leqslant M^2 \mathrm{e}^{-TD_2(M)} + D_2(M) \int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s.$$

By using Gronwall's Lemma, we deduce from the above inequality that

$$\overline{S}_m^{(k)}(t) \leqslant M^2 \mathrm{e}^{-TD_2(M)} \mathrm{e}^{tD_2(M)} \leqslant M^2$$

for all $t \in [0,T]$ for all $m,\ k \in \mathbb{N}$. Therefore we have

$$u_m^{(k)} \in W_1(M,T) \quad \forall\, m \ \text{ and } \ k \in \mathbb{N}. \tag{3.22}$$

Step 3. Limiting process. By (3.22), there exists a subsequence of $\{u_m^{(k)}\}$ with the same symbol, such that

$$\begin{cases} u_m^{(k)} \to u_m & \text{in } L^\infty(0,T;H^2\cap H_0^1) \text{ weak}*, \\ \dot{u}_m^{(k)} \to u_m' & \text{in } L^\infty(0,T;H^2\cap H_0^1) \text{ weak}*, \\ \ddot{u}_m^{(k)} \to u_m'' & \text{in } L^2(0,T;H_0^1) \text{ weak}, \\ u_m \in W(M,T). \end{cases} \tag{3.23}$$

Passing to the limit in (3.8) and (3.9), we have u_m satisfying (3.6) and (3.7) in $L^2(0,T)$.

On the other hand, we deduce from $(3.6)_1$ and $(3.23)_4$ that

$$\begin{aligned} u_m'' &= \lambda u_{mxx}' + \frac{\partial}{\partial x}(\mu_{1m}(t) + \mu_{3m}(t) u_{mx}(t)) \\ &\quad - \int_0^t g(t-s) \frac{\partial}{\partial x}(\bar{\mu}_{1m}(s) + \bar{\mu}_{3m}(s) u_{mx}(s))\,\mathrm{d}s + F_m \\ &\equiv \widetilde{F}_m \in L^\infty(0,T;L^2). \end{aligned}$$

Thus, $u_m \in W_1(M,T)$. Theorem 3.2 is proved. □

By using Theorem 3.2 and the compact imbedding theorems, we shall prove the existence and uniqueness of weak local solutions to Problem (1.1). We first introduce the Banach space (see Lions [25]) as

$$W_1(T) = \{u \in C^0([0,T];H_0^1) \cap C^1([0,T];L^2)\colon\ u' \in L^2(0,T;H_0^1)\},$$

with respect to the norm $\|u\|_{W_1(T)} = \|u\|_{C^0([0,T];H_0^1)} + \|u'\|_{C^0(0,T;L^2)} + \|u'\|_{L^2(0,T;H_0^1)}$.

Then we have the following theorem.

Theorem 3.6. *Suppose that assumptions* (H_1)–(H_4) *hold. Then the recurrent sequence* $\{u_m\}$ *defined by* (3.8)–(3.9) *strongly converges to* u *in* $W_1(T)$*. Furthermore,* u *is a unique weak solution of Problem* (1.1) *and* $u \in W_1(M,T)$*. On the other hand, the following estimation is valid:*

$$\|u_m - u\|_{W_1(T)} \leqslant C_T k_T^m \quad \forall\, m \in \mathbb{N},$$

where $k_T \in [0,1)$ *is defined as in* (3.20) *and* C_T *is a constant depending only on* T, f, g, μ, $\bar{\mu}$, $\tilde{u}_0$, $\tilde{u}_1$.

P r o o f of Theorem 3.6. First, we prove the local existence of Problem (1.1). We begin by proving that $\{u_m\}$ (in Theorem 3.2) is a Cauchy sequence in $W_1(T)$. Let $w_m = u_{m+1} - u_m$. Then w_m satisfies the variational problem

$$\text{(3.24)}\qquad \begin{cases} \langle w_m''(t), v\rangle + \lambda\langle w_{mx}'(t), v_x\rangle + B_m(t,v) \\ \qquad = \displaystyle\int_0^t g(t-s)\overline{B}_m(s,v)\,\mathrm{d}s + \langle F_{m+1}(t) - F_m(t), v\rangle \quad \forall\, v \in H_0^1, \\ w_m(0) = w_m'(0) = 0, \end{cases}$$

where

$$\begin{aligned} B_m(t,v) &= a_{m+1}(t; u_{m+1}(t), v) - a_m(t; u_m(t), v), \\ \overline{B}_m(t,v) &= \bar{a}_{m+1}(t; u_{m+1}(t), v) - \bar{a}_m(t; u_m(t), v), \quad v \in H_0^1. \end{aligned}$$

Taking $v = w_m'(t)$ in $(3.24)_1$ and then integrating in t, we get
(3.25)

$$\begin{aligned} \bar{\mu}_*\overline{S}_m(t) \leqslant{}& 2\int_0^t \langle F_{m+1}(s) - F_m(s), w_m'(s)\rangle\,\mathrm{d}s \\ &+ \int_0^t \mathrm{d}s \int_0^1 \mu_{3m+1}'(x,s) w_{mx}^2(x,s)\,\mathrm{d}x \\ &- 2\int_0^t \langle [\mu_{3m+1}(s) - \mu_{3m}(s)]u_{mx}(s) + \mu_{1m+1}(s) - \mu_{1m}(s), w_{mx}'(s)\rangle\,\mathrm{d}s \\ &+ 2\int_0^t g(t-s)\overline{B}_m(s, w_m(t))\,\mathrm{d}s - 2g(0)\int_0^t \overline{B}_m(s, w_m(s))\,\mathrm{d}s \\ &- 2\int_0^t \mathrm{d}\tau \int_0^\tau g'(\tau - s)\overline{B}_m(s, w_m(\tau))\,\mathrm{d}s \\ ={}& \sum_{j=1}^6 \overline{I}_j, \end{aligned}$$

where $\bar{\mu}_* = \min\{1, \mu_*, \lambda\}$ and

$$\text{(3.26)}\qquad \overline{S}_m(t) = \|w_m'(t)\|^2 + \|w_{mx}(t)\|^2 + \int_0^t \|w_{mx}'(s)\|^2\,\mathrm{d}s.$$

Next, the integrals on the right-hand side of (3.25) are estimated as follows. By the inequalities

$$\begin{aligned}
\|F_{m+1}(t)-F_m(t)\| &\leqslant 2\widetilde{K}_M(f)\|w_{m-1}\|_{W_1(T)}+\sigma\|\nabla w'_{m-1}(t)\|,\\
\Big(\int_0^t \|F_{m+1}(s)-F_m(s)\|^2\,\mathrm{d}s\Big)^{1/2} &\leqslant (2\sqrt{T}\widetilde{K}_M(f)+\sigma)\|w_{m-1}\|_{W_1(T)},\\
|\mu_{im+1}(x,t)-\mu_{im}(x,t)| &\leqslant K_M(\mu)|w_{m-1}(x,t)|\\
&\leqslant K_M(\mu)\|w_{m-1}\|_{W_1(T)}, \quad i=1,3,
\end{aligned}$$

the terms $\overline{I}_1$, $\overline{I}_2$, $\overline{I}_3$ are estimated by

$$\begin{aligned}
\overline{I}_1 &= 2\int_0^t \langle F_{m+1}(s)-F_m(s), w'_m(s)\rangle\,\mathrm{d}s \\
&\leqslant (2\sqrt{T}\widetilde{K}_M(f)+\sigma)^2\|w_{m-1}\|^2_{W_1(T)}+\int_0^t \overline{S}_m(s)\,\mathrm{d}s,\\
\overline{I}_2 &= \int_0^t \mathrm{d}s\int_0^1 \mu'_{3m+1}(x,s)w^2_{mx}(x,s)\,\mathrm{d}x \leqslant (1+M)K_M(\mu)\int_0^t \overline{S}_m(s)\,\mathrm{d}s,\\
\overline{I}_3 &= -2\int_0^t \langle[\mu_{3m+1}(s)-\mu_{3m}(s)]u_{mx}(s)+\mu_{1m+1}(s)-\mu_{1m}(s), w'_{mx}(s)\rangle\,\mathrm{d}s\\
&\leqslant \frac{\bar{\mu}_*}{6}\overline{S}_m(t)+\frac{6}{\bar{\mu}_*}T(1+M)^2K^2_M(\mu)\|w_{m-1}\|^2_{W_1(T)}.
\end{aligned} \tag{3.27}$$

For the integral $\overline{I}_4$, $\overline{I}_5$, $\overline{I}_6$, we note that

$$\begin{aligned}
\overline{B}_m(s,w_m(t)) &= \langle \bar{\mu}_{3m+1}(s)w_{mx}(s), w_{mx}(t)\rangle\\
&\quad + \langle[\bar{\mu}_{3m+1}(s)-\bar{\mu}_{3m}(s)]u_{mx}(s)+\bar{\mu}_{1m+1}(s)-\bar{\mu}_{1m}(s), w_{mx}(t)\rangle,
\end{aligned}$$

hence,

$$|\overline{B}_m(s,w_m(t))| \leqslant K_M(\bar{\mu})\Big[\sqrt{\overline{S}_m(s)}+(1+M)\|w_{m-1}\|_{W_1(T)}\Big]\sqrt{\overline{S}_m(t)}.$$

Then

$$\begin{aligned}
\overline{I}_4 &= 2\int_0^t g(t-s)\overline{B}_m(s,w_m(t))\,\mathrm{d}s\\
&\leqslant \frac{\bar{\mu}_*}{3}\overline{S}_m(t)+\frac{6}{\bar{\mu}_*}T(1+M)^2K^2_M(\bar{\mu})\|g\|^2_{L^2(0,T^*)}\|w_{m-1}\|^2_{W_1(T)}\\
&\quad +\frac{6}{\bar{\mu}_*}K^2_M(\bar{\mu})\|g\|^2_{L^2(0,T^*)}\int_0^t \overline{S}_m(s)\,\mathrm{d}s,\\
\overline{I}_5 &= -2g(0)\int_0^t \overline{B}_m(s,w_m(s))\,\mathrm{d}s \leqslant 4|g(0)|K_M(\bar{\mu})\int_0^t \overline{S}_m(s)\,\mathrm{d}s\\
&\quad +\frac{1}{2}T|g(0)|K_M(\bar{\mu})(1+M)^2\|w_{m-1}\|^2_{W_1(T)},
\end{aligned} \tag{3.28}$$

$$
\begin{aligned}
\overline{I}_6 = &-2\int_0^t \mathrm{d}\tau \int_0^\tau g'(\tau - s)\overline{B}_m(s, w_m(\tau))\,\mathrm{d}s \\
&\leqslant 4K_M(\bar{\mu})\sqrt{T^*}\|g'\|_{L^2(0,T^*)}\int_0^t \overline{S}_m(s)\,\mathrm{d}s \\
&\quad + \frac{1}{2}T(1+M)^2K_M(\bar{\mu})\sqrt{T^*}\|g'\|_{L^2(0,T^*)}\|w_{m-1}\|^2_{W_1(T)}.
\end{aligned}
$$

Combining estimations (3.27) and (3.28), we deduce from (3.26) that

$$
\overline{S}_m(t) \leqslant D_1^*(M,T)\|w_{m-1}\|^2_{W_1(T)} + 2D_2^*(M)\int_0^t \overline{S}_m(s)\,\mathrm{d}s,
$$

where $D_1^*(M,T)$, $D_2^*(M)$ are defined as in Lemma 3.5.

Using Gronwall's lemma, we get from (3.26) that

$$
\overline{S}_m(t) \leqslant D_1^*(M,T)\|w_{m-1}\|^2_{W_1(T)}\exp(2TD_2^*(M)). \tag{3.29}
$$

Hence, it leads to

$$
\|w_m\|_{W_1(T)} \leqslant k_T\|w_{m-1}\|_{W_1(T)} \quad \forall\, m \in \mathbb{N},
$$

where the constant $k_T \in [0,1)$ is defined as in (3.20), which implies that

$$
\|u_{m+p} - u_m\|_{W_1(T)} \leqslant \frac{2M}{1-k_T}k_T^m \quad \forall\, m,\ p \in \mathbb{N}.
$$

It follows that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Then there exists $u \in W_1(T)$ such that

$$
u_m \to u \ \text{ strongly in } \ W_1(T). \tag{3.30}
$$

Note that $u_m \in W(M,T)$, then there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$
\begin{cases}
u_{m_j} \to u & \text{in } L^\infty(0,T;H^2\cap H_0^1) \text{ weak}*,\\
u'_{m_j} \to u' & \text{in } L^\infty(0,T;H^2\cap H_0^1) \text{ weak}*,\\
u''_{m_j} \to u'' & \text{in } L^2(0,T;H_0^1) \text{ weak},\\
u \in W(M,T).
\end{cases} \tag{3.31}
$$

Since

$$
\|F_m - f[u]\|_{L^2(Q_T)} \leqslant (\sigma + 2\sqrt{T}\widetilde{K}_M(f))\|u_{m-1} - u\|_{W_1(T)}, \tag{3.32}
$$

by (3.30) and (3.32), we have

$$
F_m \to f[u] \ \text{ strongly in } \ L^2(Q_T). \tag{3.33}
$$

On the other hand, using the equality

$$\begin{aligned} a_m(t;u_m(t),v) - a(t;u(t),v) &= \langle \mu_{3m}(t)u_{mx}(t) - \mu_3[u](t)u_x(t) + \mu_{1m}(t) - \mu_1[u](t), v_x\rangle \\ &= \langle \mu_{3m}(t)[u_{mx}(t) - u_x(t)] + [\mu_{3m}(t) - \mu_3[u](t)]u_x(t), v_x\rangle \\ &\quad + \langle \mu_{1m}(t) - \mu_1[u](t), v_x\rangle \end{aligned}$$

and the inequality

$$\begin{aligned} |\mu_{im+1}(x,t) - \mu_i[u](x,t)| &\leqslant K_M(\mu)|u_{m-1}(x,t) - u(x,t)| \\ &\leqslant K_M(\mu)\|u_{m-1} - u\|_{W_1(T)}, \quad i = 1,3, \end{aligned}$$

we get

$$\begin{aligned} &|a_m(t;u_m(t),v) - a(t;u(t),v)| \\ &\qquad \leqslant K_M(\mu)[\|u_m - u\|_{W_1(T)} + (1+M)\|u_{m-1} - u\|_{W_1(T)}]\|v_x\|. \end{aligned}$$

Hence,

$$a_m(t;u_m(t),v) \to a(t;u(t),v) \quad \text{in } L^\infty(0,T) \text{ weak}{*}\ \forall\, v \in H_0^1. \tag{3.34}$$

Similarly,

$$\int_0^t g(t-s)\bar{a}_m(s;u_m(s),v)\,\mathrm{d}s \to \int_0^t g(t-s)\bar{a}(s;u(s),v)\,\mathrm{d}s \tag{3.35}$$

in $L^\infty(0,T)$ weak$*$ for all $v \in H_0^1$.

Passing to the limit in (3.8) and (3.9) as $m = m_j \to \infty$, it follows from (3.33), (3.34) and (3.35) that there exists $u \in W(M,T)$ satisfying (3.1), (3.2).

On the other hand, we derive from (3.1) and $(3.31)_4$ that

$$u'' = \lambda u'_{xx} + \frac{\partial^2}{\partial x^2}(\mu(t,u(t))) - \int_0^t g(t-s)\frac{\partial^2}{\partial x^2}(\bar{\mu}(s,u(s)))\,\mathrm{d}s + f[u] \equiv \widetilde{F} \in L^\infty(0,T;L^2).$$

Thus, $u \in W_1(M,T)$. The proof of the existence is completed.

Finally, we need to prove the uniqueness of solutions. Let u_1, $u_2 \in W_1(M,T)$ be two weak solutions of Problem (1.1). Then $u = u_1 - u_2$ satisfies the variational problem

$$\begin{cases} \langle u''(t),v\rangle + \lambda\langle u'_x(t),v_x\rangle + B(t,v) \\ \qquad = \displaystyle\int_0^t g(t-s)\overline{B}(s,v)\,\mathrm{d}s + \langle \overline{F}_1(t) - \overline{F}_2(t), v\rangle \quad \forall\, v \in H_0^1, \\ u(0) = u'(0) = 0, \end{cases} \tag{3.36}$$

where

$$
\begin{aligned}
B(t,v) &= \langle \mu_3[u_1](t)u_x(t) + [\mu_3[u_1](t) - \mu_3[u_2](t)]u_{2x}(t), v_x\rangle \\
&\quad + \langle \mu_1[u_1](t) - \mu_1[u_2](t), v_x\rangle, \\
\overline{B}(t,v) &= \langle \bar\mu_3[u_1](t)u_x(t) + [\bar\mu_3[u_1](t) - \bar\mu_3[u_2](t)]u_{2x}(t), v_x\rangle \\
&\quad + \langle \bar\mu_1[u_1](t) - \bar\mu_1[u_2](t), v_x\rangle, \quad v \in H_0^1, \\
\mu_i[u](x,t) &= D_i\mu(x,t,u(x,t)), \\
\bar\mu_i[u](x,t) &= D_i\bar\mu(x,t,u(x,t)), \quad i = 1,3, \\
\overline{F}_j(t) &= f[u_j](t), \quad j = 1,2.
\end{aligned}
$$

Taking $v = u'(t)$ in $(3.36)_1$ and integrating in time from 0 to t, we get

(3.37)

$$
\begin{aligned}
\bar\mu_*\overline{Z}(t) \leqslant\ & \int_0^t \mathrm{d}s \int_0^1 \mu_3'[u_1](x,s)u_x^2(x,s)\,\mathrm{d}x \\
& - 2\int_0^t \langle [\mu_3[u_1](s) - \mu_3[u_2](s)]u_{2x}(s), u_x'(s)\rangle\,\mathrm{d}s \\
& - 2\int_0^t \langle \mu_1[u_1](s) - \mu_1[u_2](s), u_x'(s)\rangle\,\mathrm{d}s + 2\int_0^t g(t-s)\overline{B}(s,u(t))\,\mathrm{d}s \\
& - 2g(0)\int_0^t \overline{B}(s,u(s))\,\mathrm{d}s - 2\int_0^t \mathrm{d}\tau \int_0^\tau g'(\tau-s)\overline{B}(s,u(\tau))\,\mathrm{d}s \\
& + 2\int_0^t \langle \overline{F}_1(s) - \overline{F}_2(s), u'(s)\rangle\,\mathrm{d}s,
\end{aligned}
$$

where $\overline{Z}(t) = \|u'(t)\|^2 + \|u_x(t)\|^2 + \int_0^t \|u_x'(s)\|^2\,\mathrm{d}s$.

Through similar calculations as in the proof of Theorem 3.2, we obtain from (3.37) that

$$
(\bar\mu_* - 2\sigma^2 - 2\gamma)\overline{Z}(t) \leqslant \eta(M,\gamma)\int_0^t \overline{Z}(s)\,\mathrm{d}s \tag{3.38}
$$

for all $\gamma > 0$, where

$$
\begin{aligned}
\eta(M,\gamma) = 1 &+ 16\widetilde{K}_M^2(f) + (1+M)K_M(\mu) \\
&+ 2(2+M)K_M(\bar\mu)(|g(0)| + \sqrt{T^*}\|g'\|_{L^2(0,T^*)}) \\
&+ \frac{1}{\gamma}[(1+M)^2K_M^2(\mu) + (2+M)^2K_M^2(\bar\mu)\|g\|^2_{L^2(0,T^*)}].
\end{aligned}
$$

Since $0 < \sigma < \sqrt{\bar\mu_*}/(3\sqrt{2})$, it follows that $\bar\mu_* - 2\sigma^2 > 0$. Then, by choosing $\gamma > 0$ such that $\bar\mu_* - 2\sigma^2 - 2\gamma > 0$ and using Gronwall's lemma, we deduce from (3.38) that $\overline{Z}(t) \equiv 0$, i.e., $u = u_1 - u_2 = 0$.

Therefore, the uniqueness is proved. The proof of Theorem 3.6 is done. □

4. Continuous dependence

In this section, we assume that $\lambda > 0$ and $\tilde{u}_0,\ \tilde{u}_1 \in H^2 \cap H_0^1$. By Theorem 3.6, Problem (1.1) admits a unique solution u depending on the datum μ, $\bar{\mu}$, f, g

$$u = u(\mu, \bar{\mu}, f, g),$$

where μ, $\bar{\mu}$, f, g satisfy assumptions (H_2)–(H_4).

First, we note that if the data $(\mu, \bar{\mu}, f, g)$, $(\mu_j, \bar{\mu}_j, f_j, g_j)$ satisfy (H_2)–(H_4) and in addition, the condition

$$\begin{aligned} d_1(\mu_j, \mu) &\equiv \sup_{M>0} \max_{|\beta|\leqslant 3} \|D^\beta \mu_j - D^\beta \mu\|_{C^0(A_M)} \to 0, \\ d_1(\bar{\mu}_j, \bar{\mu}) &\equiv \sup_{M>0} \max_{|\beta|\leqslant 3} \|D^\beta \bar{\mu}_j - D^\beta \bar{\mu}\|_{C^0(A_M)} \to 0, \\ \tilde{d}(f_j, f) &\equiv \sup_{M>0} \max_{|\alpha|\leqslant 1} \|D^\alpha f_j - D^\alpha f\|_{C^0(\tilde{A}_M)} \to 0, \quad \|g_j - g\|_{H^1(0,T^*)} \to 0 \end{aligned} \tag{4.1}$$

as $j \to \infty$ is fulfilled, then there exists $j_0 \in \mathbb{N}$ (independent of M) such that

$$\begin{cases} \|D^\beta \mu_j\|_{C^0(A_M)} \leqslant 1 + \|D^\beta \mu\|_{C^0(A_M)} & \forall\, \beta \in \mathbb{Z}_+^3,\ |\beta| \leqslant 3,\ \forall\, M > 0, \forall\, j \geqslant j_0, \\ \|D^\beta \bar{\mu}_j\|_{C^0(A_M)} \leqslant 1 + \|D^\beta \bar{\mu}\|_{C^0(A_M)} & \forall\, \beta \in \mathbb{Z}_+^3,\ |\beta| \leqslant 3,\ \forall\, M > 0, \forall\, j \geqslant j_0, \\ \|D^\alpha f_j\|_{C^0(\tilde{A}_M)} \leqslant 1 + \|D^\alpha f\|_{C^0(\tilde{A}_M)} & \forall\, \alpha \in \mathbb{Z}_+^6,\ |\alpha| \leqslant 1,\ \forall\, M > 0, \forall\, j \geqslant j_0, \\ \|g_j\|_{H^1(0,T^*)} \leqslant 1 + \|g\|_{H^1(0,T^*)} & \forall\, j \geqslant j_0. \end{cases}$$

By setting the constants $K_M(\mu)$, $K_M(\bar{\mu})$, $\widetilde{K}_M(f)$ and (H_3), we deduce from the above estimation that

$$\begin{cases} K_M(\mu_j) \leqslant 1 + K_M(\mu) & \forall\, M > 0, \forall\, j \geqslant j_0, \\ K_M(\bar{\mu}_j) \leqslant 1 + K_M(\bar{\mu}) & \forall\, M > 0, \forall\, j \geqslant j_0, \\ \widetilde{K}_M(f_j) \leqslant 1 + \widetilde{K}_M(f) & \forall\, M > 0, \forall\, j \geqslant j_0, \\ \|g_j\|_{H^1(0,T^*)} \leqslant 1 + \|g\|_{H^1(0,T^*)} & \forall\, j \geqslant j_0. \end{cases}$$

Thus, the Galerkin approximation sequence $\{u_m^{(k)}\}$ corresponding to $(\mu, \bar{\mu}, f, g) = (\mu_j, \bar{\mu}_j, f_j, g_j)$, $j \geqslant j_0$ also satisfies the a priori estimates as in Theorem 3.2 and

$$u_m^{(k)} \in W_1(M, T) \quad \forall\, m \ \text{ and } \ k \in \mathbb{N},$$

where M, T are constants independent of j. Indeed, in the process, we can choose the positive constants M and T as in (3.19) and (3.21) with replacing $K_M(\mu)$, $K_M(\bar{\mu})$,

$\widetilde{K}_M(f)$, $|g(0)|$, $|\mu_{1mx}(0)|$, $|\mu_{3mx}(0)|$ by $1+K_M(\mu)$, $1+K_M(\bar{\mu})$, $1+\widetilde{K}_M(f)$, $1+|g(0)|$, $1+|\mu_{1mx}(0)|$, $1+|\mu_{3mx}(0)|$, respectively.

Hence, the limitation u_j of $\{u_m^{(k)}\}$, as $k\to\infty$ and $m\to\infty$ later, is the unique weak solution of Problem (1.1) corresponding to $(\mu,\bar{\mu},f)=(\mu_j,\bar{\mu}_j,f_j)$, $j\geqslant j_0$ satisfying

$$u_j\in W_1(M,T)\quad \forall\, j\geqslant j_0.$$

Moreover, by the same argument as used in Theorem 3.6, we can prove that the limitation u of $\{u_j\}$ as $j\to\infty$ is the unique weak solution of Problem (1.1) and $u\in W_1(M,T)$.

Consequently, we have the following theorem.

Theorem 4.1. *For any $\lambda>0$, $\tilde{u}_0$, $\tilde{u}_1\in H^2\cap H_0^1$, suppose that* (H$_2$)–(H$_4$) *and condition* (4.1) *hold. Then there exists a positive constant T such that the solution of Problem* (1.1) *is continuous dependency on the data μ, $\bar{\mu}$, f, g, i.e., if $(\mu,\bar{\mu},f,g)$ and $(\mu_j,\bar{\mu}_j,f_j,g_j)$ satisfy* (H$_2$)–(H$_4$) *and* (4.1), *then*

$$u_j=u(\mu_j,\bar{\mu}_j,f_j,g_j)\to u \ \text{ strongly in } W_1(T) \text{ as } j\to\infty.$$

Moreover, we have the estimation

$$\|u_j-u\|_{W_1(T)}\leqslant C_T(d_1(\mu_j,\mu)+d_1(\bar{\mu}_j,\bar{\mu})+\tilde{d}(f_j,f)+\|g_j-g\|_{H^1(0,T^*)})\quad \forall\, j\geqslant j_0,$$

where C_T is a constant depending only on T, f, g, μ, $\bar{\mu}$, $\tilde{u}_0$ and $\tilde{u}_1$.

Proof of Theorem 4.1. Setting

$$\begin{aligned}\tilde{g}_j&=g_j-g,\\ \widetilde{F}_j(x,t)&=f_j[u_j](x,t)-f[u](x,t),\\ f_j[u_j](x,t)&=f_j(x,t,u_j(x,t),u_j'(x,t),u_{jx}(x,t),u_{jx}'(x,t)),\\ f[u](x,t)&=f(x,t,u(x,t),u'(x,t),u_x(x,t),u_x'(x,t)),\end{aligned}$$

$w_j=u_j-u$ satisfies the variational problem

$$\text{(4.2)}\qquad \begin{cases}\langle w_j''(t),v\rangle+\lambda\langle w_{jx}'(t),v_x\rangle+a_j(t;u_j(t),v)-a(t;u(t),v)\\ \qquad =\displaystyle\int_0^t[g_j(t-s)\bar{a}_j(s;u_j(s),v)-g(t-s)\bar{a}(s;u(s),v)]\,\mathrm{d}s\\ \qquad\quad +\langle\widetilde{F}_j(t),v\rangle\quad \forall\, v\in H_0^1,\\ w_j(0)=w_j'(0)=0,\end{cases}$$

where

$$\begin{aligned}
a_j(t;u_j(t),v) &= \langle D_3\mu_j(t,u_j(t))u_{jx}(t),v_x\rangle + \langle D_1\mu_j(t,u_j(t)),v_x\rangle,\\
a(t;u(t),v) &= \langle D_3\mu(t,u(t))u_x(t),v_x\rangle + \langle D_1\mu(t,u(t)),v_x\rangle,\\
\bar{a}_j(t;u_j(t),v) &= \langle D_3\bar{\mu}_j(t,u_j(t))u_{jx}(t),v_x\rangle + \langle D_1\bar{\mu}_j(t,u_j(t)),v_x\rangle,\\
\bar{a}(t;u(t),v) &= \langle D_3\bar{\mu}(t,u(t))u_x(t),v_x\rangle + \langle D_1\bar{\mu}(t,u(t)),v_x\rangle.
\end{aligned}$$

On the other hand, by the equalities

$$\begin{aligned}
&a_j(t;u_j(t),v) - a(t;u(t),v)\\
&\quad= \langle D_3\mu_j(t,u_j(t))w_{jx}(t),v_x\rangle + \langle[D_3\mu_j(t,u_j(t)) - D_3\mu(t,u(t))]u_x(t),v_x\rangle\\
&\qquad+ \langle D_1\mu_j(t,u_j(t)) - D_1\mu(t,u(t)),v_x\rangle,\\
&\bar{a}_j(s;u_j(s),v) - \bar{a}(s;u(s),v)\\
&\quad= \langle D_3\bar{\mu}_j(s,u_j(s))w_{jx}(s),v_x\rangle + \langle[D_3\bar{\mu}_j(s,u_j(s)) - D_3\bar{\mu}(s,u(s))]u_x(s),v_x\rangle\\
&\qquad+ \langle D_1\bar{\mu}_j(s,u_j(s)) - D_1\bar{\mu}(s,u(s)),v_x\rangle,\\
&g_j(t-s)\bar{a}_j(s;u_j(s),v) - g(t-s)\bar{a}(s;u(s),v)\\
&\quad= [g_j(t-s) - g(t-s)][\langle D_3\bar{\mu}_j(s,u_j(s))u_{jx}(s),v_x\rangle + \langle D_1\bar{\mu}_j(s,u_j(s)),v_x\rangle]\\
&\qquad+ g(t-s)\langle D_3\bar{\mu}_j(s,u_j(s))w_{jx}(s),v_x\rangle\\
&\qquad+ g(t-s)\langle[D_3\bar{\mu}_j(s,u_j(s)) - D_3\bar{\mu}(s,u(s))]u_x(s),v_x\rangle\\
&\qquad+ g(t-s)\langle D_1\bar{\mu}_j(s,u_j(s)) - D_1\bar{\mu}(s,u(s)),v_x\rangle,
\end{aligned}$$

we rewrite (4.2) as

$$\tag{4.3}
\begin{cases}
\langle w_j''(t),v\rangle + \lambda\langle w_{jx}'(t),v_x\rangle + \langle D_3\mu_j(t,u_j(t))w_{jx}(t),v_x\rangle\\
\qquad = \displaystyle\int_0^t g(t-s)\langle D_3\bar{\mu}_j(s,u_j(s))w_{jx}(s),v_x\rangle\,\mathrm{d}s\\
\qquad\quad + \displaystyle\int_0^t [g_j(t-s) - g(t-s)][\langle D_3\bar{\mu}_j(s,u_j(s))u_{jx}(s),v_x\rangle\\
\qquad\quad + \langle D_1\bar{\mu}_j(s,u_j(s)),v_x\rangle]\,\mathrm{d}s\\
\qquad\quad + \displaystyle\int_0^t g(t-s)\langle[D_3\bar{\mu}_j(s,u_j(s)) - D_3\bar{\mu}(s,u(s))]u_x(s),v_x\rangle\,\mathrm{d}s\\
\qquad\quad + \displaystyle\int_0^t g(t-s)\langle D_1\bar{\mu}_j(s,u_j(s)) - D_1\bar{\mu}(s,u(s)),v_x\rangle\,\mathrm{d}s\\
\qquad\quad - \langle[D_3\mu_j(t,u_j(t)) - D_3\mu(t,u(t))]u_x(t),v_x\rangle\\
\qquad\quad - \langle D_1\mu_j(t,u_j(t)) - D_1\mu(t,u(t)),v_x\rangle + \langle\widetilde{F}_j(t),v\rangle \quad \forall\, v\in H_0^1,\\
w_j(0) = w_j'(0) = 0.
\end{cases}$$

Taking $v = w_j'(t)$ in $(4.4)_1$ and then integrating in t, we get

$$
\begin{aligned}
(4.4)\qquad \bar{\mu}_* \overline{S}_j(t) \leqslant{} & \int_0^t \mathrm{d}s \int_0^1 \frac{\partial}{\partial s}[D_3\mu_j(x,s,u_j(x,s))] w_{jx}^2(x,s)\,\mathrm{d}x \\
& + 2\int_0^t \mathrm{d}\tau \int_0^\tau g(\tau - s)\langle D_3\bar{\mu}_j(s,u_j(s)) w_{jx}(s), w_{jx}'(\tau)\rangle\,\mathrm{d}s \\
& + 2\int_0^t \mathrm{d}\tau \int_0^\tau [g_j(\tau - s) - g(\tau - s)] \\
& \times \langle D_1\bar{\mu}_j(s,u_j(s)) + D_3\bar{\mu}_j(s,u_j(s)) u_{jx}(s), w_{jx}'(\tau)\rangle\,\mathrm{d}s \\
& + 2\int_0^t \mathrm{d}\tau \int_0^\tau g(\tau - s) \\
& \times \langle D_1\bar{\mu}_j(s,u_j(s)) - D_1\bar{\mu}(s,u(s)), w_{jx}'(\tau)\rangle\,\mathrm{d}s \\
& + 2\int_0^t \mathrm{d}\tau \int_0^\tau g(\tau - s) \\
& \times \langle [D_3\bar{\mu}_j(s,u_j(s)) - D_3\bar{\mu}(s,u(s))] u_x(s), w_{jx}'(\tau)\rangle\,\mathrm{d}s \\
& - 2\int_0^t \langle [D_3\mu_j(s,u_j(s)) - D_3\mu(s,u(s))] u_x(s), w_{jx}'(s)\rangle\,\mathrm{d}s \\
& - 2\int_0^t \langle D_1\mu_j(s,u_j(s)) - D_1\mu(s,u(s)), w_{jx}'(s)\rangle\,\mathrm{d}s \\
& + 2\int_0^t \langle \widetilde{F}_j(s), w_j'(s)\rangle\,\mathrm{d}s \\
={} & \sum_{j=1}^8 I_j,
\end{aligned}
$$

where $\bar{\mu}_* = \min\{1, \lambda, \mu_*\}$ and $\overline{S}_j(t) = \|w_j'(t)\|^2 + \|w_{jx}(t)\|^2 + \int_0^t \|w_{jx}'(s)\|^2\,\mathrm{d}s$.

We estimate the terms I_j on the right-hand side of (4.4) as follows.

Estimate of I_1. By the estimation

$$
\Bigl|\frac{\partial}{\partial s}[D_3\mu_j(x,s,u_j(x,s))]\Bigr| \leqslant K_M(\mu_j)(1 + |u_j'(x,s)|) \leqslant (1 + K_M(\mu))(1 + M),
$$

we have

$$
\begin{aligned}
(4.5)\qquad I_1 &= \int_0^t \mathrm{d}s \int_0^1 \frac{\partial}{\partial s}[D_3\mu_j(x,s,u_j(x,s))] w_{jx}^2(x,s)\,\mathrm{d}x \\
&\leqslant (1 + K_M(\mu))(1 + M) \int_0^t \|w_{jx}(s)\|^2\,\mathrm{d}s \\
&\leqslant (1 + K_M(\mu))(1 + M) \int_0^t \overline{S}_j(s)\,\mathrm{d}s.
\end{aligned}
$$

Estimate of I_2. By the estimation

$$|D_3\bar{\mu}_j(x,s,u_j(x,s))| \leqslant K_M(\bar{\mu}_j) \leqslant 1 + K_M(\bar{\mu}),$$

we obtain

$$\begin{aligned}
(4.6)\qquad I_2 &= 2\int_0^t \mathrm{d}\tau \int_0^\tau g(\tau - s)\langle D_3\bar{\mu}_j(s,u_j(s))w_{jx}(s), w'_{jx}(\tau)\rangle \,\mathrm{d}s \\
&\leqslant 2(1+K_M(\bar{\mu}))\int_0^t \|w'_{jx}(\tau)\| \,\mathrm{d}\tau \int_0^\tau |g(\tau-s)|\|w_{jx}(s)\| \,\mathrm{d}s \\
&\leqslant \beta\overline{S}_j(t) + \frac{1}{\beta}(1+K_M(\bar{\mu}))^2 T^*\|g\|^2_{L^2(0,T^*)}\int_0^t \overline{S}_j(s)\,\mathrm{d}s.
\end{aligned}$$

Estimate of I_3. Note that

$$\begin{aligned}
&\|D_1\bar{\mu}_j(s,u_j(s)) + D_3\bar{\mu}_j(s,u_j(s))u_{jx}(s)\| \\
&\qquad \leqslant K_M(\bar{\mu}_j)(1+\|u_{jx}(s)\|) \leqslant (1+K_M(\bar{\mu}))(1+M),
\end{aligned}$$

hence,

$$\begin{aligned}
(4.7)\qquad I_3 &= 2\int_0^t \mathrm{d}\tau \int_0^\tau [g_j(\tau-s) - g(\tau-s)] \\
&\quad \times \langle D_1\bar{\mu}_j(s,u_j(s)) + D_3\bar{\mu}_j(s,u_j(s))u_{jx}(s), w'_{jx}(\tau)\rangle \,\mathrm{d}s \\
&\leqslant 2(1+K_M(\bar{\mu}))(1+M)\int_0^t \|w'_{jx}(\tau)\| \,\mathrm{d}\tau \int_0^\tau |g_j(\tau-s) - g(\tau-s)| \,\mathrm{d}s \\
&\leqslant \beta\overline{S}_j(t) + \frac{1}{\beta}(1+K_M(\bar{\mu}))^2(1+M)^2 T^*\|g_j - g\|^2_{L^2(0,T^*)}.
\end{aligned}$$

Estimate of I_4. Using the estimation

$$\begin{aligned}
&|D_1\bar{\mu}_j(x,s,u_j(x,s)) - D_1\bar{\mu}(x,s,u(x,s))| \\
&\qquad \leqslant \sup_{(x,t,y)\in A_M} |D_1\bar{\mu}_j(x,s,y) - D_1\bar{\mu}(x,s,y)| + K_M(\bar{\mu})|u_j(x,s) - u(x,s)| \\
&\qquad \leqslant d_1(\bar{\mu}_j,\bar{\mu}) + K_M(\bar{\mu})\sqrt{\overline{S}_j(s)},
\end{aligned}$$

we get

$$\begin{aligned}
(4.8)\qquad I_4 &= 2\int_0^t \mathrm{d}\tau \int_0^\tau g(\tau-s)\langle D_1\bar{\mu}_j(s,u_j(s)) - D_1\bar{\mu}(s,u(s)), w'_{jx}(\tau)\rangle \,\mathrm{d}s \\
&\leqslant \beta\overline{S}_j(t) + \frac{2}{\beta}T^*\|g\|^2_{L^2(0,T^*)}\int_0^t [d_1^2(\bar{\mu}_j,\bar{\mu}) + K_M^2(\bar{\mu})\overline{S}_j(s)]\,\mathrm{d}s \\
&\leqslant \beta\overline{S}_j(t) + \frac{2}{\beta}(T^*\|g\|_{L^2(0,T^*)})^2 d_1^2(\bar{\mu}_j,\bar{\mu}) \\
&\quad + \frac{2}{\beta}T^*(\|g\|_{L^2(0,T^*)}K_M(\bar{\mu}))^2\int_0^t \overline{S}_j(s)\,\mathrm{d}s.
\end{aligned}$$

Estimate of I_5. By the above inequality, we obtain

$$\|[D_3\bar{\mu}_j(s,u_j(s)) - D_3\bar{\mu}(s,u(s))]u_x(s)\| \leqslant M\Big(d_1(\bar{\mu}_j,\bar{\mu}) + K_M(\bar{\mu})\sqrt{\overline{S}_j(s)}\Big).$$

Hence,

$$\begin{aligned}(4.9)\qquad I_5 &= 2\int_0^t \mathrm{d}\tau \int_0^\tau g(\tau - s)\langle[D_3\bar{\mu}_j(s,u_j(s)) - D_3\bar{\mu}(s,u(s))]u_x(s), w'_{jx}(\tau)\rangle\,\mathrm{d}s\\ &\leqslant \beta\overline{S}_j(t) + \frac{2}{\beta}(T^*M\|g\|_{L^2(0,T^*)})^2 d_1^2(\bar{\mu}_j,\bar{\mu})\\ &\quad+\frac{2}{\beta}T^*(M\|g\|_{L^2(0,T^*)}K_M(\bar{\mu}))^2\int_0^t \overline{S}_j(s)\,\mathrm{d}s.\end{aligned}$$

Estimate of I_6. Similarly, we verify that

$$\|[D_3\mu_j(s,u_j(s)) - D_3\mu(s,u(s))]u_x(s)\| \leqslant M\Big(d_1(\mu_j,\mu) + K_M(\mu)\sqrt{\overline{S}_j(s)}\Big),$$

so

$$\begin{aligned}(4.10)\qquad I_6 &= -2\int_0^t \langle[D_3\mu_j(s,u_j(s)) - D_3\mu(s,u(s))]u_x(s), w'_{jx}(s)\rangle\,\mathrm{d}s\\ &\leqslant \beta\overline{S}_j(t) + \frac{2}{\beta}T^*M^2d_1^2(\mu_j,\mu) + \frac{2}{\beta}M^2K_M^2(\mu)\int_0^t \overline{S}_j(s)\,\mathrm{d}s.\end{aligned}$$

Estimate of I_7. Repeating the estimation similarly to I_6, we obtain

$$|D_1\mu_j(x,s,u_j(x,s)) - D_1\mu(x,s,u(x,s))| \leqslant d_1(\mu_j,\mu) + K_M(\mu)\sqrt{\overline{S}_j(s)},$$

so it follows that

$$\begin{aligned}(4.11)\qquad I_7 &= -2\int_0^t \langle D_1\mu_j(s,u_j(s)) - D_1\mu(s,u(s)), w'_{jx}(s)\rangle\,\mathrm{d}s\\ &\leqslant \beta\overline{S}_j(t) + \frac{2}{\beta}T^*d_1^2(\mu_j,\mu) + \frac{2}{\beta}K_M^2(\mu)\int_0^t \overline{S}_j(s)\,\mathrm{d}s.\end{aligned}$$

Estimate of I_8. We note that

$$\widetilde{F}_j(t) = \widetilde{F}_j(x,t) = F_j(x,t) - F(x,t) = f_j[u_j](x,t) - f[u_j](x,t) + f[u_j](x,t) - f[u](x,t).$$

Since

$$\begin{aligned}|f_j[u_j](x,t) - f[u_j](x,t)| &= |f_j(x,t,u_j(x,t),u'_j(x,t),\nabla u_j(x,t),\nabla u'_j(x,t))\\ &\quad- f(x,t,u_j(x,t),u'_j(x,t),\nabla u_j(x,t),\nabla u'_j(x,t))|\\ &\leqslant \|f_j - f\|_{C^0(\tilde{A}_M)} \leqslant \tilde{d}(f_j,f),\end{aligned}$$

it follows that

$$\|f[u_j](t) - f[u](t)\| \leqslant 2\sqrt{2}\widetilde{K}_M(f)\sqrt{\overline{S}_j(t)} + \widetilde{K}_M(f)\|w'_{jx}(t)\|.$$

Then

$$\begin{aligned}\|\widetilde{F}_j(t)\| &\leqslant \|f_j[u_j](t) - f[u_j](t)\| + \|f[u_j](t) - f[u](t)\| \\ &\leqslant \tilde{d}(f_j, f) + 2\sqrt{2}\widetilde{K}_M(f)\sqrt{\overline{S}_j(t)} + \widetilde{K}_M(f)\|w'_{jx}(t)\|.\end{aligned}$$

Hence,

$$\begin{aligned}(4.12)\qquad I_8 &= 2\int_0^t \langle \widetilde{F}_j(s), w'_j(s)\rangle\,\mathrm{d}s \\ &\leqslant T^*\tilde{d}^2(f_j, f) + \beta\overline{S}_j(t) + \Big[1 + 4\sqrt{2}\widetilde{K}_M(f) + \frac{1}{\beta}\widetilde{K}_M^2(f)\Big]\int_0^t \overline{S}_j(s)\,\mathrm{d}s.\end{aligned}$$

Finally, by choosing $\beta = \bar{\mu}_*/14$, we get from (4.5)–(4.10) that

$$\overline{S}_j(t) \leqslant R_j(M) + D_M\int_0^t \overline{S}_j(s)\,\mathrm{d}s,$$

where

$$\begin{aligned}R_j(M) &= \frac{2}{\bar{\mu}_*}T^*\tilde{d}^2(f_j, f) + \frac{28}{\bar{\mu}_*^2}(1 + K_M(\bar{\mu}))^2(1 + M)^2T^*\|g_j - g\|^2_{L^2(0,T^*)} \\ &\quad + \frac{56}{\bar{\mu}_*^2}T^*(1 + M^2)[d_1^2(\mu_j, \mu) + (T^*\|g\|_{L^2(0,T^*)})^2d_1^2(\bar{\mu}_j, \bar{\mu})], \\ D_M &= \frac{2}{\bar{\mu}_*}\Big(1 + 4\sqrt{2}\widetilde{K}_M(f) + (1 + K_M(\mu))(1 + M) + \frac{14}{\bar{\mu}_*}\widetilde{K}_M^2(f)\Big) \\ &\quad + 2\frac{28}{\bar{\mu}_*^2}(1 + K_M(\bar{\mu}))^2T^*\|g\|^2_{L^2(0,T^*)} \\ &\quad + \frac{56}{\bar{\mu}_*^2}(1 + M^2)(K_M^2(\mu) + T^*\|g\|^2_{L^2(0,T^*)}K_M^2(\bar{\mu})).\end{aligned}$$

Using Gronwall's lemma, we have

$$\overline{S}_j(t) \leqslant R_j(M)\exp(TD_M).$$

This yields that

$$\begin{aligned}\|u_j - u\|_{W_1(T)} &\leqslant 3\sqrt{\exp(TD_M)R_j(M)} \\ &\leqslant C_T(d_1(\mu_j, \mu) + d_1(\bar{\mu}_j, \bar{\mu}) + \tilde{d}(f_j, f) + \|g_j - g\|_{H^1(0,T^*)}) \quad \forall\, j \geqslant j_0.\end{aligned}$$

Theorem 4.1 is proved. □

R e m a r k 4.2. We give here an example, in which condition (4.1) is satisfied.

(i) Consider $\{f_j\}$ defined by

$$f_j(x,t,y_1,\dots,y_4) = f(x,t,y_1,\dots,y_4) + \frac{x^2t^2y_1^2}{j(1+y_1^2)},$$

$(x,t,y_1,\dots,y_4) \in [0,1]\times[0,T^*]\times\mathbb{R}^4$, where $f \in C^1([0,1]\times[0,T^*]\times\mathbb{R}^4)$ satisfies (H_4).

It is easy to check that $f_j \in C^1([0,1]\times[0,T^*]\times\mathbb{R}^4)$ also satisfies (H_4) and

$$\tilde{d}(f_j,f) \equiv \sup_{M>0}\max_{|\alpha|\leqslant 1}\|D^\alpha f_j - D^\alpha f\|_{C^0(\tilde{A}_M)} \leqslant \frac{2}{j}\max\{(T^*)^2,T^*\} \to 0 \text{ as } j\to\infty.$$

(ii) Consider $\{\mu_j\}$ defined by

$$\mu_j(x,t,y) = \mu(x,t,y) + \frac{xty^2}{j(1+y^2)},$$

$(x,t,y) \in [0,1]\times[0,T^*]\times\mathbb{R}$, where $\mu \in C^3([0,1]\times[0,T^*]\times\mathbb{R})$ satisfies (H_2).

It is easy to check that $\mu_j \in C^3([0,1]\times[0,T^*]\times\mathbb{R})$ also satisfies (H_2) and

$$\begin{aligned} d_1(\mu_j,\mu) &\equiv \sup_{M>0}\sup\Bigl(\max_{|\beta|\leqslant 3}\|D^\beta\mu_j - D^\beta\mu\|_{C^0(A_M)}\Bigr) \\ &\leqslant \frac{1}{j}\max\{5,18T^*\} \to 0 \text{ as } j\to\infty. \end{aligned}$$

(iii) It is similar to give $\{\bar{\mu}_j\}$and $\{g_j\}$, so we omit it here.

5. Global existence and general decay

In this section, we investigate the general decay of solutions to Problem (1.1) in the specific case $\mu = \mu(t,u)$, $\bar{\mu}(u) = u$, $f = -\lambda_1 u_t + f(u) - \frac{1}{2}D_2^2\mu(t,u)u_x^2 + F(x,t)$. Precisely, we shall consider the problem

$$(5.1)\qquad \begin{cases} u_{tt} + \lambda_1 u_t - \lambda u_{txx} - \dfrac{\partial^2}{\partial x^2}\mu(t,u(x,t)) + \displaystyle\int_0^t g(t-s)u_{xx}(x,s)\,\mathrm{d}s \\ \qquad = f(u) - \dfrac{1}{2}D_2^2\mu(t,u)u_x^2 + F(x,t),\ 0<x<1,\ 0<t<T, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x), \quad u_t(x,0) = \tilde{u}_1(x), \end{cases}$$

where $\lambda > 0$, $\lambda_1 > 0$ are given constants and μ, g, f, F, $\tilde{u}_0$, $\tilde{u}_1$ are given functions satisfying the following assumptions.

We first note that by Theorem 3.6, under the assumptions corresponding to this special case, Problem (5.1) has a unique local weak solution u such that

$$u \in C([0,T]; H^2 \cap H_0^1) \cap C^1([0,T]; H_0^1) \cap L^\infty(0,T; H^2 \cap H_0^1),$$
$$u' \in C([0,T]; H_0^1) \cap L^\infty(0,T; H^2 \cap H_0^1), \quad u'' \in L^2(0,T; H_0^1) \cap L^\infty(0,T; L^2),$$

for T small enough. Furthermore, using the standard arguments of density, we can propose the assumptions to get the local existence and uniqueness of a weak solution for Problem (5.1) with less smoothness as follows.

($\overline{\mathrm{H}}_1$) $(\tilde{u}_0, \tilde{u}_1) \in H_0^1 \times L^2$;
($\overline{\mathrm{H}}_2$) $\mu \in C^3(\mathbb{R}_+ \times \mathbb{R})$ and there exists a positive constant μ_* such that
   (i) $D_2\mu(t,z) \geqslant \mu_* > 0$ for all $(t,z) \in \mathbb{R}_+ \times \mathbb{R}$,
   (ii) $D_1D_2\mu(t,z) \leqslant 0$ for all $(t,z) \in \mathbb{R}_+ \times \mathbb{R}$;
($\overline{\mathrm{H}}_3$) $g \in C^1(\mathbb{R}_+)$;
($\overline{\mathrm{H}}_4$) $f \in C^1(\mathbb{R})$ such that $f(0) = 0$ and $yf(y) > 0$ for all $y \in \mathbb{R}$;
($\widehat{\mathrm{H}}_5$) $F \in L^2((0,1) \times \mathbb{R}_+)$.

We then obtain the following theorem.

Theorem 5.1. *Let* ($\overline{\mathrm{H}}_1$), ($\widehat{\mathrm{H}}_2$), ($\overline{\mathrm{H}}_3$), ($\overline{\mathrm{H}}_4$), ($\widehat{\mathrm{H}}_5$) *hold. Then there exist* $T > 0$ *and a unique solution of Problem* (5.1) *such that*

$$u \in C^0([0,T]; H_0^1) \cap C^1([0,T]; L^2), \quad u' \in L^2(0,T; H_0^1). \tag{5.2}$$

We now prove the existence of global solution and the energy of the solution decays as $t \to \infty$. For this purpose, we strengthen the following assumptions.

($\overline{\mathrm{H}}_1$) $(\tilde{u}_0, \tilde{u}_1) \in H_0^1 \times L^2$;
(H_2^d) $\mu \in C^3(\mathbb{R}_+ \times \mathbb{R})$ and there exist positive constants μ_*, μ_{1*}, μ_{2*} such that
   (i) $D_2\mu(t,z) \geqslant \mu_* > 0$ for all $(t,z) \in \mathbb{R}_+ \times \mathbb{R}$,
   (ii) $D_1D_2\mu(t,z) \leqslant 0$ for all $(t,z) \in \mathbb{R}_+ \times \mathbb{R}$,
   (iii) $\frac{1}{2}zD_2^2\mu(t,z) + D_2\mu(t,z) \geqslant \mu_{1*} > 0$ for all $(t,z) \in \mathbb{R}_+ \times \mathbb{R}$,
   (iv) $zD_2^2\mu(t,z) \geqslant -\mu_{2*}$ for all $(t,z) \in \mathbb{R}_+ \times \mathbb{R}$;
(H_3^d) $g \in C^1(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$ such that
   (i) $L_* = \mu_* - \overline{g}(\infty) > 0$,
   (ii) there exists a function $\xi \in C^1(\mathbb{R}_+)$ such that
      (j) $\xi'(t) \leqslant 0 < \xi(t)$ for all $t \geqslant 0$, $\int_0^\infty \xi(s)\,\mathrm{d}s = \infty$,
      (jj) $g'(t) \leqslant -\xi(t)g(t) < 0$ for all $t \geqslant 0$, where $\overline{g}(t) = \int_0^t g(s)\,\mathrm{d}s$, $\overline{g}(\infty) = \int_0^\infty g(s)\,\mathrm{d}s$;

(H_4^d) $f \in C^1(\mathbb{R})$, $f(0) = 0$, $yf(y) > 0$ for all $y \in \mathbb{R}$ and there exist constants α, β, d_2, $\bar{d}_2 > 0$ with $\alpha > 2$, $\beta > 2$ such that
(i) $yf(y) \leqslant d_2 \int_0^y f(z)\,\mathrm{d}z$ for all $y \in \mathbb{R}$,
(ii) $\int_0^y f(z)\,\mathrm{d}z \leqslant \bar{d}_2(|y|^\alpha + |y|^\beta)$ for all $y \in \mathbb{R}$;

(H_5^d) $F \in L^\infty(\mathbb{R}_+; L^2) \cap L^1(\mathbb{R}_+; L^2)$ and there exist two positive constants C_0, γ_0 such that

$$\|F(t)\|^2 \leqslant C_0 \exp(-\gamma_0 t) \quad \forall\, t \geqslant 0;$$

(H_6^d) $p > \max\{2, d_2\}$, $\mu_* > \frac{1}{2}(p/d_2)\mu_{2*} + (1 + p/d_2)\overline{g}(\infty)$.

We next prove that if

$$\int_0^1 D_2\mu(0, \tilde{u}_0(x))\tilde{u}_{0x}^2(x)\,\mathrm{d}x - p\int_0^1 \mathrm{d}x \int_0^{\tilde{u}_0(x)} f(z)\,\mathrm{d}z > 0$$

and if the initial energy and $\|F(t)\|$ are small enough, then the solution is globally extended in time and its energy decays to zero as t tends to infinity. To achieve this goal, we first construct the Lyapunov functional in the form

$$\mathcal{L}(t) = E(t) + \delta\psi(t), \tag{5.3}$$

where δ is a positive constant suitably chosen and

$$E(t) = \frac{1}{2}\|u'(t)\|^2 + \Big(\frac{1}{2} - \frac{1}{p}\Big)[(g * u)(t) + N(u)] + \frac{1}{p}I(t), \tag{5.4}$$

$$\psi(t) = \langle u'(t), u(t)\rangle + \frac{\lambda_1}{2}\|u(t)\|^2 + \frac{\lambda}{2}\|u_x(t)\|^2, \tag{5.5}$$

with $(g * u)(t) = \int_0^t g(t-s)\|u_x(t) - u_x(s)\|^2\,\mathrm{d}s$ and

$$\begin{aligned} I(t) &= (g * u)(t) + N(u) - p\int_0^1 \mathcal{F}(u(x,t))\,\mathrm{d}x, \\ N(u) &= \int_0^1 D_2\mu(t, u(x,t))u_x^2(x,t)\,\mathrm{d}x - \overline{g}(t)\|u_x(t)\|^2, \ \mathcal{F}(y) = \int_0^y f(z)\,\mathrm{d}z. \end{aligned} \tag{5.6}$$

Lemma 5.2. *If* $(\overline{\mathrm{H}}_1)$, (H_2^d)–(H_6^d) *hold and* u *is the solution of* (5.1), *then the energy functional* $E(t)$ *satisfies*

$$\begin{aligned} E'(t) &\leqslant \frac{1}{2}\|F(t)\| + \frac{1}{2}\|F(t)\|\|u'(t)\|^2, \\ E'(t) &\leqslant -\Big(\lambda_1 - \frac{\varepsilon_1}{2}\Big)\|u'(t)\|^2 - \lambda\|u_x'(t)\|^2 - \frac{1}{2}\xi(t)(g * u)(t) + \frac{1}{2\varepsilon_1}\|F(t)\|^2 \end{aligned} \tag{5.7}$$

for all $\varepsilon_1 > 0$.

P r o o f. Multiplying (5.1) by $u'(x,t)$ and integrating over $[0,1]$, we get

$$E'(t) = -\lambda_1\|u'(t)\|^2 - \lambda\|u'_x(t)\|^2 + \frac{1}{2}(g' * u)(t) - \frac{1}{2}g(t)\|u_x(t)\|^2 + \frac{1}{2}\int_0^1 D_1D_2\mu(t,u(x,t))u_x^2(x,t)\,\mathrm{d}x + \langle F(t), u'(t)\rangle. \tag{5.8}$$

Using assumptions (H_2^d), (H_3^d), (H_5^d), we obtain

$$\begin{aligned} &\frac{1}{2}\int_0^1 D_1D_2\mu(t,u(x,t))u_x^2(x,t)\,\mathrm{d}x \leqslant 0, \\ &\frac{1}{2}(g' * u)(t) \leqslant -\frac{1}{2}\xi(t)(g * u)(t), \end{aligned} \tag{5.9}$$

so

$$E'(t) \leqslant \langle F(t), u'(t)\rangle \leqslant \frac{1}{2}\|F(t)\| + \frac{1}{2}\|F(t)\|\|u'(t)\|^2.$$

This assures (5.7_1).

By applying Cauchy-Schwartz inequality, we have

$$\langle F(t), u'(t)\rangle \leqslant \frac{1}{2\varepsilon_1}\|F_1(t)\|^2 + \frac{\varepsilon_1}{2}\|u'(t)\|^2 \quad \forall\,\varepsilon_1 > 0. \tag{5.10}$$

Then by using (5.8), (5.9) and (5.10), it is easy to see that (5.7_2) holds. Lemma 5.2 is proved. □

Lemma 5.3. *If* (H_2^d)–(H_6^d) *hold and* $(\tilde u_0, \tilde u_1) \in H_0^1 \times L^2$ *such that* $I(0) > 0$ *and*

$$\eta^* \equiv \mu_* - \overline{g}(\infty) - p\bar d_2(R_*^{\alpha-2} + R_*^{\beta-2}) > \frac{p}{2d_2}\mu_{2*} + \frac{p}{d_2}\overline{g}(\infty), \tag{5.11}$$

where

$$\begin{aligned} &R_* = \Big(\frac{2pE_*}{(p-2)L_*}\Big)^{1/2}, \quad E_* = \Big(E(0) + \frac{1}{2}\varrho_1\Big)\exp(\varrho_1), \\ &\varrho_1 = \int_0^\infty \|F(t)\|\,\mathrm{d}t, \quad L_* = \mu_* - \overline{g}(\infty) > 0, \end{aligned}$$

then $I(t) \geqslant 0$ *for all* $t \geqslant 0$.

We also note that condition (5.11) holds if $\overline{g}(\infty)$, E_* is chosen small enough and $\mu_* > 0$ is suitably large.

P r o o f. By the continuity of $I(t)$ and $I(0)>0$, there exists $\widetilde{T}>0$ such that

$$I(t)=I(u(t))>0 \quad \forall\, t\in[0,\widetilde{T}].$$

From (5.4) and (5.6), we get

$$\begin{aligned} E(t) &\geqslant \frac{1}{2}\|u'(t)\|^2+\Big(\frac{1}{2}-\frac{1}{p}\Big)[(g*u)(t)+N(u)] \\ &\geqslant \frac{1}{2}\|u'(t)\|^2+\Big(\frac{1}{2}-\frac{1}{p}\Big)[(g*u)(t)+L_*\|u_x(t)\|^2] \\ &\geqslant \frac{1}{2}\|u'(t)\|^2+\frac{p-2}{2p}(g*u)(t)+\frac{(p-2)L_*}{2p}\|u_x(t)\|^2 \quad \forall\, t\in[0,\widetilde{T}]. \end{aligned} \tag{5.12}$$

Combining (5.7_1) and (5.12) and using Gronwall's inequality, we obtain

$$\begin{aligned} \|u_x(t)\|^2 &+ \frac{p}{(p-2)L_*}\|u'(t)\|^2+\frac{1}{L_*}(g*u)(t) \\ &\leqslant \frac{2pE(t)}{(p-2)L_*}\leqslant \frac{2pE_*}{(p-2)L_*}\equiv R_*^2 \quad \forall\, t\in[0,\widetilde{T}]. \end{aligned} \tag{5.13}$$

Then it follows from (H_4^d) (ii) and (5.13) that

$$\begin{aligned} p\int_0^1 \mathcal{F}(u(x,t))\,\mathrm{d}x &\leqslant p\bar{d}_2(\|u(t)\|_{L^\alpha}^\alpha+\|u(t)\|_{L^\beta}^\beta) \\ &\leqslant p\bar{d}_2(\|u_x(t)\|^\alpha+\|v_x(t)\|^\beta) \\ &\leqslant p\bar{d}_2(R_*^{\alpha-2}+R_*^{\beta-2})\|u_x(t)\|^2. \end{aligned}$$

Thus,

$$I(t)\geqslant (g*u)(t)+\eta^*\|u_x(t)\|^2\geqslant 0 \quad \forall\, t\in[0,\widetilde{T}], \tag{5.14}$$

where the positive constant η^* is defined as in (5.11).

Next, we prove that $I(t)>0$ for all $t\geqslant 0$. Put $T_\infty=\sup\{\widetilde{T}>0\colon\ I(t)>0$ for all $t\in[0,\widetilde{T}]\}$, we have to show that $T_\infty=\infty$. Indeed, if $T_\infty<\infty$, then by the continuity of $I(t)$, we have $I(T_\infty)\geqslant 0$.

In the case of $I(T_\infty)>0$, by the same arguments as above, we can reduce that there exists $\widetilde{T}_\infty>T_\infty$ such that $I(t)>0$ for all $t\in[0,\widetilde{T}_\infty]$. This is contrary to the definition of T_∞.

In the case of $I(T_\infty)=0$, it follows from (5.14) that

$$0=I(T_\infty)\geqslant (g*u)(T_\infty)+\eta^*\|u_x(T_\infty)\|^2\geqslant 0.$$

Therefore,

$$\|u(T_\infty)\|=(g*u)(T_\infty)=0.$$

Due to the function $s \mapsto g(T_\infty - s)\|u_x(T_\infty) - u_x(s)\|^2$ being continuous on $[0, T_\infty]$ and $g(T_\infty - s) > 0$ for all $s \in [0, T_\infty]$, we have

$$(g * u)(T_\infty) = \int_0^{T_\infty} g(T_\infty - s)\|u_x(s)\|^2 \, \mathrm{d}s = 0.$$

It follows that $\|u_x(s)\|^2 = 0$ for all $s \in [0, T_\infty]$. Thus, $u(0) = 0$. This is contrary to $I(0) > 0$.

Consequently, $T_\infty = \infty$, i.e. $I(t) > 0$ for all $t \geqslant 0$. Lemma 5.3 is proved. □

It is clear to see that Lemmas 5.2, 5.3 assures the global existence of the solution for Problem (5.1).

Next, we put

$$E_1(t) = \|u'(t)\|^2 + \|u_x(t)\|^2 + N(u) + (g * u)(t) + I(t). \tag{5.15}$$

In order to discuss general decay, we need the following lemmas.

Lemma 5.4. *If the assumptions of Lemma* 5.3 *hold, there exist positive constants* β_1, $\overline{\beta}_1$, β_2, $\overline{\beta}_2$ *such that*

$$\begin{aligned} \beta_1 E_1(t) \leqslant \mathcal{L}(t) \leqslant \beta_2 E_1(t) \quad \forall\, t \geqslant 0, \\ \overline{\beta}_1 E_1(t) \leqslant E(t) \leqslant \overline{\beta}_2 E_1(t) \quad \forall\, t \geqslant 0 \end{aligned} \tag{5.16}$$

for δ *small enough.*

P r o o f. Lemma 5.4 is proved by using some simple estimates, hence we omit the details. □

Lemma 5.5. *If the assumptions of Lemma* 5.3 *hold, then the functional* $\psi(t)$ *defined by* (5.5) *satisfies the estimation*

(5.17)

$$\begin{aligned} \psi'(t) \leqslant \|u'(t)\|^2 &+ \frac{1}{2\varepsilon_2}\|F(t)\|^2 + \Big(\frac{d_2}{p} + \frac{1}{2\varepsilon_2}\Big)(g * u)(t) \\ &- \frac{\delta_1 d_2}{p} I(t) - \Big(1 - \frac{d_2}{p} - \delta_*\Big) N(u) \\ &- \Big[\frac{d_2}{p}(1-\delta_1)\eta^* + \delta_* \mu_{1*} - \frac{1}{2}(1-\delta_*)\mu_{2*} - \frac{\varepsilon_2}{2} - \Big(1 + \frac{\varepsilon_2}{2}\Big)\overline{g}(\infty)\Big] \|u_x(t)\|^2 \end{aligned}$$

for all $\varepsilon_2 > 0$, δ_*, $\delta_1 \in (0, 1)$.

P r o o f. Multiplying $(5.1)_1$ by $u(x,t)$ and integrating over $[0,1]$, we obtain

$$\begin{aligned}\psi'(t) &= \|u'(t)\|^2 - \frac{1}{2}\langle D_2^2\mu(t,u(t))u_x^2(t), u(t)\rangle - \langle D_2\mu(t,u(t))u_x(t), u_x(t)\rangle \\ &\quad + \langle F(t), u(t)\rangle + \int_0^t g(t-s)\langle u_x(s), u_x(t)\rangle \,\mathrm{d}s + \langle f(u(t)), u(t)\rangle \\ &= \|u'(t)\|^2 + \langle F(t), u(t)\rangle + \int_0^t g(t-s)\langle u_x(s), u_x(t)\rangle \,\mathrm{d}s + \langle f(u(t)), u(t)\rangle \\ &\quad - \delta_* \int_0^1 \Big[\frac{1}{2}u(x,t)D_2^2\mu(t,u(x,t)) + D_2\mu(t,u(x,t))\Big] u_x^2(x,t)\,\mathrm{d}x \\ &\quad - (1-\delta_*) \int_0^1 \Big[\frac{1}{2}u(x,t)D_2^2\mu(t,u(x,t)) + D_2\mu(t,u(x,t))\Big] u_x^2(x,t)\,\mathrm{d}x.\end{aligned}$$

Using Cauchy-Schwartz inequality, we have

$$\begin{gathered}\langle F(t), u(t)\rangle \leqslant \frac{\varepsilon_2}{2}\|u_x(t)\|^2 + \frac{1}{2\varepsilon_2}\|F(t)\|^2, \\ \int_0^t g(t-s)\langle u_x(s), u_x(t)\rangle\,\mathrm{d}s \leqslant \Big(1+\frac{\varepsilon_2}{2}\Big)\overline{g}(t)\|u_x(t)\|^2 + \frac{1}{2\varepsilon_2}(g*u)(t), \\ I(t) \geqslant \eta^*\|u_x(t)\|^2\end{gathered} \tag{5.18}$$

for all $\varepsilon_2 > 0$. By asumption (H_2^d) (iii) and (H_2^d) (iv), we get

$$\begin{aligned}&- \delta_* \int_0^1 \Big[\frac{1}{2}u(x,t)D_2^2\mu(t,u(x,t)) + D_2\mu(t,u(x,t))\Big] u_x^2(x,t)\,\mathrm{d}x \leqslant -\delta_*\mu_{1*}\|u_x(t)\|^2, \\ &- (1-\delta_*) \int_0^1 \Big[\frac{1}{2}u(x,t)D_2^2\mu(t,u(x,t)) + D_2\mu(t,u(x,t))\Big] u_x^2(x,t)\,\mathrm{d}x \\ &\qquad = -\frac{1}{2}(1-\delta_*)\int_0^1 u(x,t)D_2^2\mu(t,u(x,t))u_x^2(x,t) - (1-\delta_*) \\ &\qquad\quad \times [N(u) + \overline{g}(t)\|u_x(t)\|^2] \\ &\qquad \leqslant \frac{1}{2}(1-\delta_*)\mu_{2*}\|u_x(t)\|^2 - (1-\delta_*)N(u).\end{aligned} \tag{5.19}$$

On the other hand, by asumption (H_4^d) (i) and definition of $I(t)$ given by (5.6), we obtain

$$\begin{aligned}\langle f(u(t)), u(t)\rangle \leqslant d_2\int_0^1 \mathcal{F}(u(x,t))\,\mathrm{d}x &= \frac{d_2}{p}[(g*u)(t) + N(u) - \delta_1 I(t) - (1-\delta_1)I(t)] \\ &\leqslant \frac{d_2}{p}[(g*u)(t) + N(u) - \delta_1 I(t) - (1-\delta_1)\eta^*\|u_x(t)\|^2].\end{aligned} \tag{5.20}$$

Then it follows from (5.18)–(5.20) that inequality (5.17) is valid.

Lemma 5.5 is proved completely. □

Using Lemmas 5.2–5.5, we state and prove our main result in this section as follows.

Theorem 5.6. *If* (H_2^d)–(H_6^d) *hold and* $(\tilde{u}_0, \tilde{u}_1) \in H_0^1 \times L^2$ *satisfy* $I(0) > 0$ *and* (5.11), *then there exist positive constants* $\overline{C}$, $\overline{\gamma}$ *such that*

$$\|u'(t)\|^2 + \|u_x(t)\|^2 \leqslant \overline{C} \exp\Big(-\overline{\gamma} \int_0^t \xi(s)\,\mathrm{d}s\Big) \quad \forall\, t \geqslant 0. \tag{5.21}$$

Remark 5.7. The general decay obtained in Theorem 5.6 contains the decay results of exponential or polynomial or logarithmic type. Then the following examples describe the different decay depending on the form of $\zeta(t)$, and so lead to the corresponding decay of the solution u.

(i) Let $\zeta(t) \equiv a = \mathrm{const} > 0$. Then the assumption (H_3^d) is satisfied with $\zeta(t)$. So $\exp(-\overline{\gamma} \int_0^t \xi(s)\,\mathrm{d}s) = \exp(-a\overline{\gamma}t)$, and (5.21) becomes

$$\|u'(t)\|^2 + \|u_x(t)\|^2 \leqslant \overline{C} \exp(-a\gamma_* t) \quad \forall\, t \geqslant 0 \quad \text{(exponential decay)}.$$

(ii) Let $\zeta(t) = a/(1+t)$, $a = \mathrm{const} > 0$. Then (H_3^d) is satisfied with $\zeta(t)$. So $\exp(-\overline{\gamma} \int_0^t \xi(s)\,\mathrm{d}s) = 1/(1+t)^{a\overline{\gamma}}$, and (5.21) becomes

$$\|u'(t)\|^2 + \|u_x(t)\|^2 \leqslant \frac{\overline{C}}{(1+t)^{a\overline{\gamma}}} \quad \forall\, t \geqslant 0 \quad \text{(polynominal decay)}.$$

(iii) Let $\zeta(t) = a/((1+t)(1+\ln(1+t)))$, $a = \mathrm{const} > 0$. Then (H_3^d) is satisfied with $\zeta(t)$. So $\exp(-\overline{\gamma} \int_0^t \xi(s)\,\mathrm{d}s) = 1/(1+\ln(1+t))^{a\overline{\gamma}}$, and (5.21) becomes

$$\|u'(t)\|^2 + \|u_x(t)\|^2 \leqslant \frac{\overline{C}}{(1+\ln(1+t))^{a\gamma_*}} \quad \forall\, t \geqslant 0 \text{ (logarit decay)}.$$

Proof of Theorem 5.6. First, due to the definition of $\mathcal{L}(t)$ and the inequalities (5.7_2), (5.17), we deduce that

$$\begin{aligned} \mathcal{L}'(t) \leqslant & -\Big(\lambda_1 - \frac{\varepsilon_1}{2} - \delta\Big)\|u'(t)\|^2 - \frac{1}{2}\xi(t)(g * u)(t) + \frac{1}{2}\Big(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\Big)\|F(t)\|^2 \\ & + \delta\Big(\frac{d_2}{p} + \frac{1}{2\varepsilon_2}\Big)(g * u)(t) - \frac{\delta\delta_1 d_2}{p} I(t) - \delta\theta_1 N(u) - \delta\theta_2\|u_x(t)\|^2, \end{aligned} \tag{5.22}$$

where

$$\theta_1 = \theta_1(\delta_*) = 1 - \frac{d_2}{p} - \delta_*,$$

$$\theta_2 = \theta_2(\delta_*, \delta_1, \varepsilon_2) = \frac{d_2}{p}(1-\delta_1)\eta^* + \delta_*\mu_{1*} - \frac{1}{2}(1-\delta_*)\mu_{2*} - \frac{\varepsilon_2}{2} - \Big(1 + \frac{\varepsilon_2}{2}\Big)\overline{g}(\infty).$$

Clearly

$$\lim_{\delta_*\to 0_+} \theta_1(\delta_*) = 1 - \frac{d_2}{p} > 0,$$
$$\lim_{\substack{\delta_*\to 0_+,\ \delta_1\to 0_+\\ \varepsilon_2\to 0_+}} \theta_2(\delta_*, \delta_1, \varepsilon_2) = \frac{d_2}{p}\eta^* - \frac{1}{2}\mu_{2*} - \overline{g}(\infty) > 0.$$

Then we can choose δ_*, $\delta_1 \in (0,1)$ and $\varepsilon_2 > 0$ small enough such that $\theta_1 = \theta_1(\delta_*) > 0$, $\theta_2 = \theta_2(\delta_*, \delta_1, \varepsilon_2) > 0$. Moreover, we also choose $\varepsilon_1 > 0$, $\delta > 0$ small enough and satisfying

$$\bar\theta_1 = \lambda_1 - \frac{\varepsilon_1}{2} - \delta > 0, \quad 0 < \delta < \min\Big\{1;\ \frac{(p-2)(1-\varepsilon_*)L_*}{p}\Big\}.$$

Putting

$$\bar\theta_* = \min\Big\{\bar\theta_1, \delta\theta_1, \delta\theta_2, \frac{\delta\delta_1 d_2}{p}\Big\}, \quad \bar\theta_3 = \delta\Big(\frac{d_2}{p} + \frac{1}{2\varepsilon_2}\Big), \tag{5.23}$$

we get from (5.22) and (5.23) that

$$\mathcal{L}'(t) \leqslant -\bar\theta_* E_1(t) + (\bar\theta_* + \bar\theta_3)(g*u)(t) + \frac{1}{2}\Big(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\Big)\|F(t)\|^2. \tag{5.24}$$

Combining (5.7_2) and (5.24), we obtain

(5.25)
$$\begin{aligned}
\xi(t)\mathcal{L}'(t) \leqslant\ & -\bar\theta_*\xi(t)E_1(t) + (\bar\theta_* + \bar\theta_3)\xi(t)(g*u)(t) + \frac{1}{2}\Big(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\Big)\xi(0)\|F(t)\|^2\\
\leqslant\ & -\bar\theta_*\xi(t)E_1(t) + 2(\bar\theta_* + \bar\theta_3)\Big[-E'(t) + \frac{1}{2\varepsilon_1}\|F(t)\|^2\Big]\\
& + \frac{1}{2}\Big(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\Big)\xi(0)\|F(t)\|^2\\
\leqslant\ & -\bar\theta_*\xi(t)E_1(t) - 2(\bar\theta_* + \bar\theta_3)E'(t) + \overline{C}_0 e^{-\gamma_0 t},
\end{aligned}$$

where

$$\overline{C}_0 = \Big[\frac{\bar\theta_* + \bar\theta_3}{\varepsilon_1} + \frac{1}{2}\Big(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\Big)\Big]\xi(0)C_0.$$

Setting the functional $L(t) = \xi(t)\mathcal{L}(t) + 2(\bar\theta_* + \bar\theta_3)E(t)$, we have

$$L(t) \leqslant [\xi(0)\beta_2 + 2(\bar\theta_* + \bar\theta_3)\overline{\beta}_2]E_1(t) \equiv \hat\beta_2 E_1(t)$$

and

$$\begin{aligned}
L'(t) &= \xi'(t)\mathcal{L}(t) + \xi(t)\mathcal{L}'(t) + 2(\bar\theta_* + \bar\theta_3)E'(t)\\
&\leqslant -\bar\theta_*\xi(t)E_1(t) + \overline{C}_0 e^{-\gamma_0 t} \leqslant -\frac{\bar\theta_*}{\hat\beta_2}\xi(t)L(t) + \overline{C}_0 e^{-\gamma_0 t}.
\end{aligned}$$

By choosing $0 < \overline{\gamma} < \min\{\bar{\theta}_*/\hat{\beta}_2, \gamma_0/\xi(0)\}$, we get

$$L'(t) + \overline{\gamma}\xi(t)L(t) \leqslant \overline{C}_0 e^{-\gamma_0 t}.$$

Integrating the above inequality, we deduce

$$L(t) \leqslant \Big(L(0) + \frac{\overline{C}_0}{\gamma_0 - \overline{\gamma}\xi(0)}\Big)\exp\Big(-\overline{\gamma}\int_0^t \xi(\tau)\,\mathrm{d}\tau\Big). \tag{5.26}$$

On the other hand,

$$\begin{aligned} L(t) &= \xi(t)\mathcal{L}(t) + 2(\bar{\theta}_* + \bar{\theta}_3)E(t) \geqslant 2(\bar{\theta}_* + \bar{\theta}_3)\overline{\beta}_1 E_1(t) \\ &\geqslant 2(\bar{\theta}_* + \bar{\theta}_3)\overline{\beta}_1(\|u'(t)\|^2 + \|u_x(t)\|^2). \end{aligned} \tag{5.27}$$

Then by (5.26) and (5.27), we get (5.21). Theorem 5.6 is proved completely. □

R e m a r k 5.8. We also give here an example in which μ satisfies assumption (H_2^d). We shall consider the function $\mu(t,z) = \mu_* z + \bar{\mu}_* e^{-t}|z|^{k-1}z$, where $\mu_* > 0$, $\bar{\mu}_* > 0$, $k > 3$ are constants. By direct computations, we have

$$\begin{aligned} D_2\mu(t,z) &= \mu_* + k\bar{\mu}_* e^{-t}|z|^{k-1} \geqslant \mu_* > 0, \\ D_1D_2\mu(t,z) &= -k\bar{\mu}_* e^{-t}|z|^{k-1} \leqslant 0, \\ zD_2^2\mu(t,z) &= k(k-1)\bar{\mu}_* e^{-t}|z|^{k-1} \geqslant 0 > -\mu_{2*}, \\ \frac{1}{2}zD_2^2\mu(t,z) + D_2\mu(t,z) &= \frac{1}{2}(k-1)[D_2\mu(t,z) - \mu_*] + D_2\mu(t,z) \\ &\geqslant D_2\mu(t,z) \geqslant \mu_* = \mu_{1*} > 0. \end{aligned}$$

This claims that (H_2^d) holds.

6. Appendix

P r o o f of Lemma 3.3.

Cases (i), (ii): By $\mu_{im}(x,t) = D_i\mu(x,t,u_{m-1}(x,t))$, $i = 1,3$, we have

$$\mu'_{im}(x,t) = D_2D_i\mu(x,t,u_{m-1}(x,t)) + D_3D_i\mu(x,t,u_{m-1}(x,t))u'_{m-1}(x,t), \quad i = 1,3,$$

hence

$$\begin{aligned} |\mu'_{im}(x,t)| &\leqslant K_M(\mu)(1 + |u'_{m-1}(x,t)|) \\ &\leqslant K_M(\mu)(1 + \|\nabla u'_{m-1}(t)\|) \leqslant K_M(\mu)(1+M), \quad i = 1,3, \end{aligned}$$

and

$$\|\mu'_{im}(t)\| \leqslant K_M(\mu)(1 + \|\nabla u'_{m-1}(t)\|) \leqslant K_M(\mu)(1+M), \quad i = 1,3.$$

Thus (i), (ii) hold.

Cases (iii), (iv): By

$$\mu_{imx}(x,t) = D_1 D_i \mu(x,t,u_{m-1}(x,t)) + D_3 D_i \mu(x,t,u_{m-1}(x,t)) \nabla u_{m-1}(x,t),$$

$i = 1,3$, we have

$$\begin{aligned} |\mu_{imx}(x,t)| &\leqslant K_M(\mu)(1 + |\nabla u_{m-1}(x,t)|) \\ &\leqslant K_M(\mu)(1 + \sqrt{2}\|\nabla u_{m-1}(t)\|_{H^1}) \leqslant K_M(\mu)(1+2M), \quad i = 1,3, \end{aligned}$$

and

$$\|\mu_{imx}(t)\| \leqslant K_M(\mu)(1 + \|\nabla u_{m-1}(t)\|) \leqslant K_M(\mu)(1+M), \quad i = 1,3.$$

Thus (iii) and (iv) hold.

Cases (v), (vi): By

$$\begin{aligned} \mu'_{imx}(x,t) = {} & D_2 D_1 D_i \mu(x,t,u_{m-1}(x,t)) + D_3 D_1 D_i \mu(x,t,u_{m-1}(x,t)) u'_{m-1}(x,t) \\ & + [D_2 D_3 D_i \mu(x,t,u_{m-1}(x,t)) \\ & + D_3^2 D_i \mu(x,t,u_{m-1}(x,t)) u'_{m-1}(x,t)] \nabla u_{m-1}(x,t) \\ & + D_3 D_i \mu(x,t,u_{m-1}(x,t)) \nabla u'_{m-1}(x,t), \quad i = 1,3, \end{aligned}$$

we have

$$\begin{aligned} |\mu'_{imx}(x,t)| &\leqslant K_M(\mu)(1 + |u'_{m-1}(x,t)|) + K_M(\mu)(1 + |u'_{m-1}(x,t)|)|\nabla u_{m-1}(x,t)| \\ &\quad + K_M(\mu)|\nabla u'_{m-1}(x,t)| \\ &\leqslant K_M(\mu)(1+M) + K_M(\mu)(1+M)\sqrt{2}\|\nabla u_{m-1}(t)\|_{H^1} \\ &\quad + K_M(\mu)\sqrt{2}\|\nabla u'_{m-1}(t)\|_{H^1} \\ &\leqslant K_M(\mu)(1+M) + K_M(\mu)(1+M)2M + K_M(\mu)2M \\ &= (1 + 5M + 2M^2) K_M(\mu) \end{aligned}$$

and

$$\begin{aligned} \|\mu'_{imx}(t)\| &\leqslant K_M(\mu)(1 + \|u'_{m-1}(t)\|) + K_M(\mu)(1 + \|\nabla u'_{m-1}(t)\|)\|\nabla u_{m-1}(t)\| \\ &\quad + K_M(\mu)\|\nabla u'_{m-1}(t)\| \\ &\leqslant K_M(\mu)(1+M) + K_M(\mu)(1+M)M + K_M(\mu)M \\ &= (1 + 3M + M^2) K_M(\mu). \end{aligned}$$

Thus (v), (vi) hold.

Case (vii): By $\bar{A}_m(t;u,v) = \langle \bar{\mu}_{3m}(t) u_x, v_x \rangle$, $u, v \in H_0^1$ we get $|\bar{A}_m(t;u,v)| \leqslant K_M(\mu)\|u_x\|\|v_x\|$. Thus (vii) holds.

Case (viii): We have

$$\begin{aligned}\|\Delta u_m^{(k)}(t) + \Delta \dot{u}_m^{(k)}(t)\| &\leqslant \|\Delta u_m^{(k)}(t)\| + \|\Delta \dot{u}_m^{(k)}(t)\| \\ &\leqslant \sqrt{2}\sqrt{\|\Delta u_m^{(k)}(t)\|^2 + \|\Delta \dot{u}_m^{(k)}(t)\|^2} \leqslant \sqrt{2}\sqrt{\overline{S}_m^{(k)}(t)}.\end{aligned}$$

Thus (viii) holds.

Case (ix): We have

$$\|u_{mx}^{(k)}(t)\| \leqslant \|u_{mx}^{(k)}(0)\| + \int_0^t \|\dot{u}_{mx}^{(k)}(s)\| \, \mathrm{d}s \leqslant \|\tilde{u}_{0kx}\| + \int_0^t \sqrt{\overline{S}_m^{(k)}(s)} \, \mathrm{d}s,$$

which implies that (ix) holds.

Case (x): By (iii), we obtain

$$\begin{aligned}\Big\|\frac{\partial}{\partial x}(\mu_{3m}(t)u_{mx}^{(k)}(t))\Big\| &= \|\mu_{3mx}(t)u_{mx}^{(k)}(t) + \mu_{3m}(t)\Delta u_m^{(k)}(t)\| \\ &\leqslant \|\mu_{3mx}(t)u_{mx}^{(k)}(t)\| + \|\mu_{3m}(t)\Delta u_m^{(k)}(t)\| \\ &\leqslant K_M(\mu)(1+2M)\|u_{mx}^{(k)}(t)\| + K_M(\mu)\|\Delta u_m^{(k)}(t)\| \\ &\leqslant 2(1+M)K_M(\mu)\sqrt{\overline{S}_m^{(k)}(t)},\end{aligned}$$

which implies that (x) holds.

Case (xi): We deduce from (iii), (v) that

$$\begin{aligned}\Big\|\frac{\partial}{\partial t}(\mu_{3mx}(t)u_{mx}^{(k)}(t))\Big\| &= \|\mu_{3mx}'(t)u_{mx}^{(k)}(t) + \mu_{3mx}(t)\dot{u}_{mx}^{(k)}(t)\| \\ &\leqslant \|\mu_{3mx}'(t)\|_{L^\infty}\|u_{mx}^{(k)}(t)\| + \|\mu_{3mx}(t)\|_{L^\infty}\|\dot{u}_{mx}^{(k)}(t)\| \\ &\leqslant (1+5M+2M^2)K_M(\mu)\sqrt{\overline{S}_m^{(k)}(t)} + K_M(\mu)(1+2M)\sqrt{\overline{S}_m^{(k)}(t)} \\ &= (2+7M+2M^2)K_M(\mu)\sqrt{\overline{S}_m^{(k)}(t)}.\end{aligned}$$

Thus (xi) holds.

Case (xii): We deduce from (i), (iii), (v) that

$$\begin{aligned}&\Big\|\frac{\partial^2}{\partial x \partial t}(\mu_{3m}(t)u_{mx}^{(k)}(t))\Big\| \\ &\quad = \Big\|\frac{\partial}{\partial t}[\mu_{3mx}(t)u_{mx}^{(k)}(t) + \mu_{3m}(t)\Delta u_m^{(k)}(t)]\Big\| \\ &\quad = \|\mu_{3mx}'(t)u_{mx}^{(k)}(t) + \mu_{3mx}(t)\dot{u}_{mx}^{(k)}(t) + \mu_{3m}'(t)\Delta u_m^{(k)}(t) + \mu_{3m}(t)\Delta \dot{u}_m^{(k)}(t)\| \\ &\quad \leqslant [(1+5M+2M^2) + (1+2M) + (1+M) + 1]K_M(\mu)\sqrt{\overline{S}_m^{(k)}(t)} \\ &\quad = 2(2+4M+M^2)K_M(\mu)\sqrt{\overline{S}_m^{(k)}(t)}.\end{aligned}$$

Thus (xii) holds and proof of Lemma 3.3 is complete. □

Proof of Lemma 3.4.

Case (i): We have

$$\|\Delta u_m^{(k)}(t)\| \leqslant \|\Delta u_m^{(k)}(t)\| + \int_0^t \|\Delta \dot{u}_m^{(k)}(s)\|\,\mathrm{d}s \leqslant \|\Delta \tilde{u}_{0k}\| + \int_0^t \sqrt{\overline{S}_m^{(k)}(s)}\,\mathrm{d}s.$$

Thus (i) holds.

Case (ii): We deduce from Lemma 3.3 (xi) that

$$\begin{aligned}\|\mu_{3mx}(t)u_{mx}^{(k)}(t)\| &\leqslant \|\mu_{3mx}(0)u_{mx}^{(k)}(0)\| + \int_0^t \Big\|\frac{\partial}{\partial s}(\mu_{3mx}(s)u_{mx}^{(k)}(s))\Big\|\,\mathrm{d}s \\ &\leqslant \|\mu_{3mx}(0)\tilde{u}_{0kx}\| + (2+7M+2M^2)K_M(\mu)\int_0^t \sqrt{\overline{S}_m^{(k)}(s)}\,\mathrm{d}s.\end{aligned}$$

Thus (ii) holds.

Case (iii): We deduce from Lemma 3.3 (xi) that

$$\begin{aligned}\Big\|\frac{\partial}{\partial x}(\mu_{3m}(t)u_{mx}^{(k)}(t))\Big\| &\leqslant \Big\|\frac{\partial}{\partial x}(\mu_{3m}(0)u_{mx}^{(k)}(0))\Big\| + \int_0^t \Big\|\frac{\partial^2}{\partial x\partial s}(\mu_{3m}(s)u_{mx}^{(k)}(s))\Big\|\,\mathrm{d}s \\ &\leqslant \Big\|\frac{\partial}{\partial x}(\mu_{3m}(0)\tilde{u}_{0kx})\Big\| + 2(2+4M+M^2)K_M(\mu)\int_0^t \sqrt{\overline{S}_m^{(k)}(s)}\,\mathrm{d}s.\end{aligned}$$

Thus (iii) holds and proof of Lemma 3.4 is complete. □

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References

[1] *S. Bayraktar, Ş. Gür*: Continuous dependence of solutions for damped improved Boussinesq equation. Turk. J. Math. *44* (2020), 334–341. zbl MR doi

[2] *P. Benilan, M. G. Crandall*: The continuous dependence on φ of solutions of $u_t - \Delta\varphi(u) = 0$. Indiana Univ. Math. J. *30* (1981), 161–177. zbl MR doi

[3] *N. Boumaza, S. Boulaaras*: General decay for Kirchhoff type in viscoelasticity with not necessarily decreasing kernel. Math. Methods Appl. Sci. *41* (2018), 6050–6069. zbl MR doi

[4] *M. M. Cavalcanti, V. N. Domingos Cavalcanti, P. Martinez*: General decay rate estimates for viscoelastic dissipative systems. Nonlinear Anal., Theory Methods Appl., Ser. A *68* (2008), 177–193. zbl MR doi

[5] *B. Cockburn, G. Gripenber*: Continuous dependence on the nonlinearities of solutions of degenerate parabolic equations. J. Differ. Equations *151* (1999), 231–251. zbl MR doi

[6] *M. Conti, V. Pata*: General decay properties of abstract linear viscoelasticity. Z. Angew. Math. Phys. *71* (2020), Article ID 6, 21 pages. zbl MR doi

[7] *M. D'Abbicco*: The influence of a nonlinear memory on the damped wave equation. Nonlinear Anal., Theory Methods Appl., Ser. A *95* (2014), 130–145. zbl MR doi

[8] *M. D'Abbicco, S. Lucente*: The beam equation with nonlinear memory. Z. Angew. Math. Phys. *67* (2016), Article ID 60, 18 pages. zbl MR doi
[9] *A. Douglis*: The continuous dependence of generalized solutions of non-linear partial differential equations upon initial data. Commun. Pure Appl. Math. *14* (1961), 267–284. zbl MR doi
[10] *G. Duvaut, J. L. Lions*: Inequalities in Mechanics and Physics. Grundlehren der mathematischen Wissenschaften 219. Springer, Berlin, 1976. zbl MR doi
[11] *F. Ekinci, E. Pişkin, S. M. Boulaaras, I. Mekawy*: Global existence and general decay of solutions for a quasilinear system with degenerate damping terms. J. Funct. Spaces *2021* (2021), Article ID 4316238, 10 pages. zbl MR doi
[12] *A. Z. Fino*: Critical exponent for damped wave equations with nonlinear memory. Nonlinear Anal., Theory Methods Appl., Ser. A *74* (2011), 5495–5505. zbl MR doi
[13] *G. Gripenberg*: Global existence of solutions of Volterra integrodifferential equations of parabolic type. J. Differ. Equations *102* (1993), 382–390. zbl MR doi
[14] *Ş. Gür, M. E. Uysal*: Continuous dependence of solutions to the strongly damped nonlinear Klein-Gordon equation. Turk. J. Math. *42* (2018), 904–910. zbl MR doi
[15] *X. Han, M. Wang*: General decay of energy for a viscoelastic equation with nonlinear damping. J. Franklin Inst. *347* (2010), 806–817. zbl MR doi
[16] *J. Hao, H. Wei*: Blow-up and global existence for solution of quasilinear viscoelastic wave equation with strong damping and source term. Bound. Value Probl. *2017* (2017), Article ID 65, 12 pages. zbl MR doi
[17] *J. H. Hassan, S. A. Messaoudi*: General decay results for a viscoelastic wave equation with a variable exponent nonlinearity. Asymptotic Anal. *125* (2021), 365–388. doi
[18] *W. J. Hrusa*: Global existence and asymptotic stability for a semilinear hyperbolic Volterra equation with large initial data. SIAM J. Math. Anal. *16* (1985), 110–134. zbl MR doi
[19] *M. Jleli, B. Samet, C. Vetro*: Large time behavior for inhomogeneous damped wave equations with nonlinear memory. Symmetry *12* (2020), Article ID 1609, 12 pages. doi
[20] *F. John*: Continuous dependence on data for solutions of partial differential equations with a prescribed bound. Commun. Pure Appl. Math. *13* (1960), 551–586. zbl MR doi
[21] *T. H. Kaddour, M. Reissig*: Global well-posedness for effectively damped wave models with nonlinear memory. Commun. Pure Appl. Anal. *20* (2021), 2039–2064. zbl MR doi
[22] *M. Kafini, S. A. Messaoudi*: A blow-up result in a Cauchy viscoelastic problem. Appl. Math. Lett. *21* (2008), 549–553. zbl MR doi
[23] *M. Kafini, M. I. Mustafa*: Blow-up result in a Cauchy viscoelastic problem with strong damping and dispersive. Nonlinear Anal., Real World Appl. *20* (2014), 14–20. zbl MR doi
[24] *Q. Li, L. He*: General decay and blow-up of solutions for a nonlinear viscoelastic wave equation with strong damping. Bound. Value Probl. *2018* (2018), Article ID 153, 22 pages. MR doi
[25] *J. L. Lions*: Quelques méthodes de résolution des problèmes aux limites non linéaires. Etudes mathematiques. Dunod, Paris, 1969. (In French.) zbl MR
[26] *N. T. Long, A. P. N. Dinh, L. X. Truong*: Existence and decay of solutions of a nonlinear viscoelastic problem with a mixed nonhomogeneous condition. Numer. Funct. Anal. Optim. *29* (2008), 1363–1393. zbl MR doi
[27] *F. Mesloub, S. Boulaaras*: General decay for a viscoelastic problem with not necessarily decreasing kernel. J. Appl. Math. Comput. *58* (2018), 647–665. zbl MR doi
[28] *S. A. Messaoudi*: Blow up and global existence in a nonlinear viscoelastic wave equation. Math. Nachr. *260* (2003), 58–66. zbl MR doi
[29] *S. A. Messaoudi*: General decay of the solution energy in a viscoelastic equation with a nonlinear source. Nonlinear Anal., Theory Methods Appl., Ser. A *69* (2008), 2589–2598. zbl MR doi
[30] *M. I. Mustafa*: General decay result for nonlinear viscoelastic equations. J. Math. Anal. Appl. *457* (2018), 134–152. zbl MR doi

[31] *M. I. Mustafa*: Optimal decay rates for the viscoelastic wave equation. Math. Methods Appl. Sci. *41* (2018), 192–204. zbl MR doi

[32] *L. T. P. Ngoc, D. T. N. Quynh, N. A. Triet, N. T. Long*: Linear approximation and asymptotic expansion associated to the Robin-Dirichlet problem for a Kirchhoff-Carrier equation with a viscoelastic term. Kyungpook Math. J. *59* (2019), 735–769. MR

[33] *J. Q. Pan*: The continuous dependence on nonlinearities of solutions of the Neumann problem of a singular parabolic equation. Nonlinear Anal., Theory Methods Appl., Ser. A *67* (2007), 2081–2090. zbl MR doi

[34] *D. T. N. Quynh, B. D. Nam, L. T. M. Thanh, T. T. M. Dung, N. H. Nhan*: High-order iterative scheme for a viscoelastic wave equation and numerical results. Math. Probl. Eng. *2021* (2021), Article ID 9917271, 27 pages. MR doi

[35] *Y. Shang, B. Guo*: On the problem of the existence of global solutions for a class of nonlinear convolutional intergro-differential equations of pseudoparabolic type. Acta Math. Appl. Sin. *26* (2003), 511–524. (In Chinese.) zbl MR

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