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CONDITIONAL FOURIER-FEYNMAN TRANSFORM GIVEN INFINITE DIMENSIONAL CONDITIONING FUNCTION ON ABSTRACT WIENER SPACE

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Abstract. We study a conditional Fourier-Feynman transform (CFFT) of functionals on an abstract Wiener space (H, B, ν). An infinite dimensional conditioning function is used to define the CFFT. To do this, we first present a short survey of the conditional Wiener integral concerning the topic of this paper. We then establish evaluation formulas for the conditional Wiener integral on the abstract Wiener space B. Using the evaluation formula, we next provide explicit formulas for CFFTs of functionals in the Kallianpur and Bromley Fresnel class $\mathcal{F}(B)$ and we finally investigate some Fubini theorems involving CFFT.

Keywords: abstract Wiener space; conditional Wiener integral; conditional Fourier-Feynman transform; Fubini theorem

MSC 2020: 28C20, 42B10, 46G12, 46B09

1. Prologue: A short survey of conditional Fourier-Feynman transform and motivation

We start this paper with historical backgrounds and a motivation of the topic of this paper. To do this, we first provide a brief illustration of abstract Wiener spaces.

Let H be a real infinite dimensional Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and associated norm $|\cdot|$, and let B be a real separable Banach space with the norm $\|\cdot\|$. It is assumed that H is continuously, linearly, and densely embedded in B. The natural injection (i.e., embedding) is denoted by $\iota\colon H \hookrightarrow B$. Let ν be a centered Gaussian probability measure on $(B, \mathcal{B}(B))$, where $\mathcal{B}(B)$ is the Borel σ-field of B.

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The triple (H, B, ν) is called an *abstract Wiener space* if

$$\int_B \exp\{\mathrm{i}(\theta, x)\}\,\mathrm{d}\nu(x) = \exp\Big\{-\frac{1}{2}|\iota^*(\theta)|^2\Big\} = \exp\Big\{-\frac{1}{2}|\theta|^2\Big\}$$

for any $\theta \in B^*$, where $(\cdot,\cdot)$ denotes the natural dual pairing (B^*-B pairing) and $\iota^*\colon B^* \to H^*$ is the dual map to the natural injection $\iota\colon H \hookrightarrow B$, and where B^* and H^* are the topological duals of B and H, respectively. The space B^* is identified as a dense subspace of $H^* \approx H$ in the sense that for all $y \in B^*$ and $x \in H$,

$$\langle y, x\rangle = (y, x). \tag{1.1}$$

Thus, we have the triple $B^* \subset H^* \approx H \subset B$. The Hilbert space H is called the *Cameron-Martin space in the abstract Wiener space* B. For more details, see [13], [22], [23], [34], [37].

We consider the case $(H, B, \nu) = (C_0'[0,T], C_0[0,T], m_w)$, where $C_0[0,T]$ is the one parameter Wiener space, i.e., the space of all continuous functions x on the time interval $[0,T]$ with $x(0) = 0$, m_w is the Wiener measure characterized by

$$m_w(\{x\colon\ x(t) \leqslant a\}) = (2\pi t)^{-1/2}\int_{-\infty}^{a}\exp\Big\{-\frac{u^2}{2t}\Big\}\,\mathrm{d}u$$

for every $t \in \mathbb{R}$, and $C_0'[0,T]$ is the Cameron-Martin space in $C_0[0,T]$ defined by

$$C_0'[0,T] = \Big\{h \in C_0[0,T]\colon\ h(t) = \int_0^t v(s)\,\mathrm{d}s,\ \ v \in L_2[0,T]\Big\}$$

which is a real Hilbert space with the inner product $\langle h_1, h_2\rangle = \int_0^T Dh_1(t)Dh_2(t)\,\mathrm{d}t$, where $Dh = \mathrm{d}h/\mathrm{d}t$ (see [13], [37]).

1.1. Why is the conditional Wiener integral important. By the Feynman-Kac formula, solutions of heat equations are given by certain Wiener integral. Furthermore, the conditional Wiener integral of the Feynman-Kac functionals (see equation (1.2) below) is important in the study of Feynman integration theory, see [43]. In fact, the conditional Wiener integrals (or conditional Feynman integrals) provide solutions of the integral equations which are formally equivalent to the heat (or Schrödinger) equation, respectively.

The Feynman-Kac functionals on $C_0[0,T]$ are given by

$$K(x) = \exp\Big\{-\int_0^t \theta(s, x(s))\,\mathrm{d}t\Big\} \tag{1.2}$$

where θ is a complex-valued potential on $[0,T] \times \mathbb{R}$. Many physical problems concerning the heat equation can be formulated in terms of the conditional Wiener integral $E(K \mid X_t)$ of the functional K, where $X_t(x) = x(t)$. It is indeed known from a result of Kac (see [33]) that the function U defined on $[0,T] \times \mathbb{R}$ by

$$U(t,\xi) = (2\pi t)^{-1/2} \exp\Big\{-\frac{(\xi-\xi_0)}{2t}\Big\} E(K(x(t)+\xi_0) \mid X_t(x) = \xi - \xi_0)$$

is the solution of the partial differential equation

$$\frac{\partial U}{\partial t} = \frac{1}{2}\frac{\partial^2 U}{\partial^2 \xi} - \theta U$$

under the initial condition $U(\xi,0) = \delta(\xi - \xi_0)$, where δ denotes the Dirac delta function. Functionals like equation (1.2) and the integral equation have appeared in many papers, including [4]–[7], [14], [19], [29], [30], [32], [42], [43], [53].

1.2. Conditional Wiener integral. The importance of the study of conditional Wiener integral is provided in Subsection 1.1 above.

In [52], Yeh introduced an inversion formula for conditional expectations. The Fourier inversion formula tells us that a conditional expectation of a random variable Y can be found by a specific Lebesgue integral of Fourier transform of the random variable Y and can change the conditional expectation into nonconditional expectation. Since then, using the inversion formula for the conditional Wiener integral given one-dimensional conditioning function, Yeh in [53] obtains very useful results including the Kac-Feynman integral equation. See [9], [14] for further work involving Yeh's inversion formula. However, as commented in [42], Yeh's inversion formula is very difficult and complicated in its applications when the conditioning function in the conditional Wiener integral is vector-valued. In [42], Park and Skoug derived a simple formula for conditional Wiener integrals with the conditioning function given by $X_{\vec{t}}(x) = (x(t_1),\dots,x(t_n))$ for $x \in C_0[0,T]$ and finite time moments $\{t_1,\dots,t_n\}$ with $0 = t_0 < t_1 < \dots < t_n = T$. In their simple formula, they expressed the conditional Wiener integral in terms of ordinary Wiener integral, which generalizes Yeh's inversion formula. In [15]–[17], Chung and Kang extended the results of [42] to abstract Wiener spaces. Finally, in [44], Park and Skoug derived a simple formula for the conditional Wiener integral with much more general conditioning functions on the Wiener space $C_0[0,T]$. The simple formula studied in [44] need not depend upon the values of $x \in C_0[0,T]$ at only finitely many time moments $\{t_1,\dots,t_n\}$ in $(0,T]$.

1.3. Fourier-Feynman transform. The theory of the analytic Fourier-Feynman transform (FFT) suggested by Brue (see [3]) now is playing a central role in the analytic Feynman integration theory and applications in mathematical physics. The analytic FFT and several analogies for functionals on the classical Wiener space $C_0[0,T]$

have been improved in various research articles. For instance, [8], [24]–[28], [31], [46], [47]. This transform and its properties are similar in many respects to the ordinary Fourier transform of functions on Euclidean space. On the other hand, the studies of the analytic FFT for functionals on abstract Wiener spaces can be found in [10], [11], [36]. Also see the references in [49] for further information and results of the analytic FFT on classical and abstract Wiener spaces.

1.4. Conditional Fourier-Feynman transform. In [24]–[26], [47], Huffman, Park, Skoug, and Storvick established basic relationships between the FFT and the corresponding convolution product (CP) for various functionals on $C_0[0,T]$:

$$T_q^{(1)}((F*G)_q)(y) = T_q^{(1)}(F)\Big(\frac{y}{\sqrt{2}}\Big)T_q^{(1)}(G)\Big(\frac{y}{\sqrt{2}}\Big) \tag{1.3}$$

for scale-almost every $y \in C_0[0,T]$, where $(F*G)_q$ indicates the CP of functionals F and G on $C_0[0,T]$. In view of equation (1.3), we can say that the FFT $T_q^{(1)}$ satisfies a homomorphism structure with convolution $*$.

The concept of CFFT was introduced by Park and Skoug in [45]. In order to define the CFFT and the conditional CP (CCP), Park and Skoug used ideas from [18], [27], [42], [43] with the conditioning function

$$X_h(x) = \mathcal{Z}_h(x,T) = \int_0^T Dh(s)\,\widetilde{\mathrm{d}}x(s), \quad h \in C_0'[0,T], \tag{1.4}$$

where $\mathcal{Z}_h\colon\ C_0[0,T]\times[0,T]\to\mathbb{R}$ given by

$$\mathcal{Z}_h(x,t) = \int_0^t Dh(s)\,\widetilde{\mathrm{d}}x(s) = \int_0^T Dh(s)\chi_{[0,t]}(s)\,\widetilde{\mathrm{d}}x(s)$$

is a Gaussian process [12], [18], [27], [43], [45] and where $\int_0^T u(s)\,\widetilde{\mathrm{d}}x(s)$ denotes the Paley-Wiener-Zygmund (PWZ) stochastic integral of functions u in $L_2[0,T]$. For a more detailed study of the PWZ stochastic integral, see [39]–[41]. Then, under appropriate conditions for functionals F and G on $C_0[0,T]$, they showed that

$$\begin{aligned} &T_q^{(1)}((F*G)_q \mid X_h(\cdot,\xi_1) \mid X)(y,\xi_2) \\ &\quad = T_q^{(1)}(F \mid X_h)\Big(\frac{y}{\sqrt{2}},\frac{\xi_2+\xi_1}{\sqrt{2}}\Big)T_q^{(1)}(G \mid X_h)\Big(\frac{y}{\sqrt{2}},\frac{\xi_2-\xi_1}{\sqrt{2}}\Big), \end{aligned} \tag{1.5}$$

where $T_q^{(1)}(F \mid X_h)$ and $((F*G)_q \mid X_h)$ denote the CFFT and CCP, respectively, given X_h for functionals F and G on $C_0[0,T]$.

On the other hand, in [12], Chang, Park and Skoug using the conditioning function (1.4) obtained the Cameron-Martin translation theorem for generalized CFFTs

$$
\begin{aligned}
(1.6)\quad T_q^{(1)}(F \mid X_h)(y+x_0,\xi) &= T_q^{(1)}(F^* \mid X_h)(y,\xi+x_0(T)) \\
&\quad\times \exp\biggl\{\mathrm{i}q\int_0^T \frac{Dg(t)}{(Dh(t))^2}\,\tilde{\mathrm{d}}x(t) + \frac{\mathrm{i}q}{2}\int_0^T \frac{Dg(t)}{Dh(t)}\,\mathrm{d}t \\
&\quad + \frac{\mathrm{i}q x_0(T)}{2\langle h,h\rangle}\int_0^T \Bigl[\frac{Dg(t)}{Dh(t)}\Bigr]^2\,\mathrm{d}t + \Bigl(\xi + \frac{x_0(T)}{2}\Bigr)\biggr\}
\end{aligned}
$$

for appropriate h and g in $C_0'[0,T]$, where $x_0(t) = \int_0^t Dg(s)\,\mathrm{d}s$ and

$$F^*(\mathcal{Z}_h(x,\cdot)) = \exp\biggl\{-\mathrm{i}q\int_0^T \frac{Dg(s)}{(Dh(s))^2}\,\tilde{\mathrm{d}}\mathcal{Z}_h(x,s)\biggr\}F(\mathcal{Z}_h(x,\cdot)).$$

The structure of CFFT is based on the conditional Feynman integral (see [18], [19]) and, in particular, the conditional Wiener integral, see [42], [53]. In [45], Park and Skoug, using the conditioning function X_h given by (1.4) on the one-parameter Wiener space $C_0[0,T]$ and using ideas from [18], [19], [27], [42], defined the concept of a CFFT, $T_q^{(1)}(F \mid X_h)$, and the concept of a CCP, $((F*G)_q \mid X_h)$, for functionals on $C_0[0,T]$. Also, in [45], the authors established a relationship between the CFFT and CCP (see equation (1.5) above) as the relation between the Fourier transform and the convolution of functions on Euclidean spaces. In [12], [14]–[19], [42], [45], [53], the studies of the conditional Wiener and the conditional Feynman integrals given finite dimensional conditioning functions were performed with related topics. In [44], Park and Skoug provided an evaluation formula for the conditional Wiener integral given an infinite dimensional conditioning function. However, the examples and the applications presented in [44] are concerned only with finite dimensional conditioning functions.

In this paper, we extend the ideas of [44], [45] from the conditional Wiener integral for functionals on $C_0[0,T]$ to the conditional abstract Wiener integral for functionals on the abstract Wiener space B. In particular, we also provide an extended definition of CFFT for functionals on B. Precisely speaking, we have found the concept of CFFT given infinite dimensional conditioning functions on B.

This paper is organized as follows. In Section 2, we state the definition of the analytic Feynman integral and the analytic FFT for functionals on B. In Section 3, we analyze the structure of the conditional Wiener integral associated with infinite dimensional conditioning functions. In Section 4, we first establish evaluation formulas for conditional Wiener integrals on the abstract Wiener space B. We then, in Section 5, define the CFFT given infinite dimensional conditioning function (see equation (3.3) below) for functionals on B and provide explicit formulas for CFFT of functionals in the Kallianpur and Bromley Fresnel class $\mathcal{F}(B)$. In Section 6, we proceed to establish some Fubini theorems involving CFFT.

2. Background

In order to establish our evaluation formula for the conditional Wiener integral on B, we follow the exposition of [13], [34], [35], [37]. Let (H, B, ν) be an abstract Wiener space and let $\{e_n\}_{n=1}^{\infty}$ be a complete orthonormal set in H such that e_j's are in B^*. For every $h \in H$ and $x \in B$, a stochastic inner product $(h, x)^\sim$ is defined by

$$(2.1) \qquad (h, x)^\sim = \begin{cases} \lim\limits_{n\to\infty} \sum\limits_{j=1}^{n} \langle h, e_j\rangle (e_j, x) & \text{if the limit exists,} \\ 0 & \text{otherwise.} \end{cases}$$

By definition of the stochastic inner product $(\cdot,\cdot)^\sim$ and (1.1), it is clear that $(\theta, x)^\sim = (\theta, x)$ for all $\theta \in B^*$ and $x \in B$. It is well known [13], [34], [37] that for every nonzero h in H, $(h, x)^\sim$ is a Gaussian random variable on B with mean 0 and variance $|h|^2$. The stochastic inner product $(h, x)^\sim$ given by (2.1) is essentially independent of the choice of the complete orthonormal set used in its definition. Also, if both h and x are in H, then Parseval's identity gives $(h, x)^\sim = \langle h, x\rangle$. Furthermore, $(h, \lambda x)^\sim = (\lambda h, x)^\sim = \lambda (h, x)^\sim$ for any $\lambda \in \mathbb{R}$, $h \in H$ and $x \in B$. We also see that if $\{h_1, \ldots, h_n\}$ is an orthogonal set in H, then the Gaussian random variables $(h_j, x)^\sim$ are independent. It is well known that for any $h_1, h_2 \in H$,

$$(2.2) \qquad \int_B (h_1, x)^\sim (h_2, x)^\sim \, \mathrm{d}\nu(x) = \langle h_1, h_2\rangle.$$

Remark 2.1.

(i) By the Kallianpur and Bromley's results [34], pages 222–223, 225–227, the limit in (2.1) exists for ν-a.e. $x \in B$ and for every $h \in H (\approx H^*)$, $(h, \cdot)^\sim$ is in $L_2(B)$. For a more detailed study, see [37], Section 1.4.

(ii) We note [13], [37] that if $B = C_0[0, T]$ and $H = C_0'[0, T]$, then for $h \in H$ and $x \in B$, $(h, x)^\sim = \int_0^T Dh(s) \, \widetilde{\mathrm{d}}x(t)$ is the PWZ stochastic integral of Dh [39]–[41].

Let $\mathcal{W}(B)$ be the class of ν-Carathéodory measurable subsets of B. In order to study CFFTs as an application of the evaluation formula for conditional Wiener integral, we consider the complete probability space $(B, \mathcal{W}(B), \nu)$ and we denote the Wiener integral of a Wiener integrable functional F by

$$E[F] \equiv E_x[F(x)] = \int_B F(x) \, \mathrm{d}\nu(x).$$

A subset W of B is said to be scale-invariant measurable (see [13]) provided ϱW is $\mathcal{W}(B)$-measurable for every $\varrho > 0$ and a scale-invariant measurable subset N of B is said to be scale-invariant null provided $\nu(\varrho N) = 0$ for every $\varrho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). A functional F on B is said to be scale-invariant measurable provided F is defined on a scale-invariant measurable set and $F(\varrho\cdot)$ is $\mathcal{W}(B)$-measurable for every $\varrho > 0$.

We now introduce the Kallianpur and Bromley Fresnel class $\mathcal{F}(B)$ of abstract Wiener spaces. Let $\mathcal{M}(H)$ denote the space of complex-valued countably additive (and hence finite) Borel measures on H. Under total variation norm $\|\cdot\|$ and with convolution as multiplication, $\mathcal{M}(H)$ is a commutative Banach algebra with identity, see [20], [48]. The class $\mathcal{F}(B)$ is defined as the space of all s-equivalence classes of stochastic Fourier transforms of complex measures σ in $\mathcal{M}(H)$, that is,

$$\mathcal{F}(B) = \left\{[F]\colon\ F(x) = \int_H \exp\{\mathrm{i}(h,x)^{\sim}\}\,\mathrm{d}\sigma(h),\ x \in B,\ \sigma \in \mathcal{M}(H)\right\}.$$

Remark 2.2.

(i) It is known from [34], [35] that $\mathcal{F}(B)$ is a Banach algebra with the norm $\|F\| = \|\sigma\|$.

(ii) The mapping $\sigma \mapsto [F]$ is a Banach algebra isomorphism where $\sigma \in \mathcal{M}(H)$ is related to F by

$$F(x) = \int_H \exp\{\mathrm{i}(h,x)^{\sim}\}\,\mathrm{d}\sigma(h) \tag{2.3}$$

for s-a.e. $x \in B$. In [34], Kallianpur and Bromley carried out these arguments in detail. For further work with functionals in the class $\mathcal{F}(B)$, see [1], [10], [11], [36].

Throughout the rest of this paper, let $\mathbb{C}_+$ and $\widetilde{\mathbb{C}}_+$ denote complex numbers with a positive real part, and nonzero complex numbers with a nonnegative real part, respectively.

Given a $\mathcal{W}(B)$-measurable functional $F\colon\ B \to \mathbb{C}$ such that

$$J_F(\lambda) = E[F(\lambda^{-1/2}\cdot)] \equiv E_x[F(\lambda^{-1/2}x)] = \int_B F(\lambda^{-1/2}x)\,\mathrm{d}\nu(x)$$

exists as a finite number for all $\lambda > 0$, if there exists a function $J_F^*(\cdot)$ analytic on $\mathbb{C}_+$ such that $J_F^*(\lambda) = J_F(\lambda)$ for all $\lambda > 0$, $J_F^*(\lambda)$ is called the *analytic Wiener integral* of F over B with parameter λ. For $\lambda \in \mathbb{C}_+$ we write

$$E^{\mathrm{an}\,w_\lambda}[F] \equiv E_x^{\mathrm{an}\,w_\lambda}[F(x)] \equiv \int_B^{\mathrm{an}\,w_\lambda} F(x)\,\mathrm{d}\nu(x) = J_F^*(\lambda).$$

Let $q \neq 0$ be a real number and let F be a scale-invariant measurable functional whose analytic Wiener integral $E^{\text{an}\, w_\lambda}[F]$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the analytic Feynman integral of F with parameter q and we write

$$E^{\text{an}\, f_q}[F] \equiv E_x^{\text{an}\, f_q}[F(x)] \equiv \int_B^{\text{an}\, f_q} F(x)\, \mathrm{d}\nu(x) = \lim_{\lambda \to -\mathrm{i}q} E_x^{\text{an}\, w_\lambda}[F(x)]$$

where λ approaches $-\mathrm{i}q$ through values in $\mathbb{C}_+$.

Next, we state the definition of the L_1 analytic FFT. For $\lambda \in \mathbb{C}_+$ and $y \in B$, let

$$T_\lambda(F)(y) = E_x^{\text{an}\, w_\lambda}[F(y + x)].$$

We define the L_1 analytic FFT, $T_q^{(1)}(F)$ of F, by the formula

$$T_q^{(1)}(F)(y) = \lim_{\substack{\lambda \to -\mathrm{i}q \\ \lambda \in \mathbb{C}_+}} T_\lambda(F)(y)$$

for s-a.e. $y \in B$, whenever this limit exists.

Let $F \in \mathcal{F}(B)$ be given by (2.3). Then it has been shown that for all $q \in \mathbb{R} \setminus \{0\}$,

$$E^{\text{an}\, f_q}[F] = \int_H \exp\Big\{-\frac{\mathrm{i}}{2q}|h|^2\Big\}\, \mathrm{d}\sigma(h)$$

and

$$T_q^{(1)}(F)(y) = \int_H \exp\Big\{\mathrm{i}(h, y)^\sim - \frac{\mathrm{i}}{2q}|h|^2\Big\}\, \mathrm{d}\sigma(h)$$

for s-a.e. $y \in B$. For more details, see [1], [3], [10]–[12], [24]–[28], [31], [36], [46], [47].

3. Conditional Wiener integral

Let X be an $\mathbb{R}^n$-valued measurable function and Y a complex-valued integrable function on $(B, \mathcal{W}(B), \nu)$. Let $\mathcal{F}(X)$ denote the σ-field generated by X. Then by the definition, the conditional expectation of Y given $\mathcal{F}(X)$, written as $E(Y \mid X)$, is any real valued $\mathcal{F}(X)$-measurable function on B such that

$$\int_A Y(x)\, \mathrm{d}\nu = \int_A E(Y \mid X)(x)\, \mathrm{d}\nu(x) \quad \text{for } A \in \mathcal{F}(X).$$

It is well known that there exists a Borel measurable and P_X-integrable function ψ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), P_X)$ such that $E(Y \mid X) = \psi \circ X$, where $\mathcal{B}(\mathbb{R}^n)$ denotes the Borel σ-field of Borel subsets in $\mathbb{R}^n$ and P_X is the probability distribution of X defined by $P_X(U) = \nu(X^{-1}(U))$ for $U \in \mathcal{B}(\mathbb{R}^n)$. The function $\psi(\vec{\xi})$, $\vec{\xi} \in \mathbb{R}^n$, is unique up to Borel null sets in $\mathbb{R}^n$. Following Yeh (see [53]) the function $\psi(\vec{\xi})$, written as $E(Y \mid X = \vec{\xi})$, is called the conditional abstract Wiener integral of Y given X.

Let $\mathcal{H}$ be an infinite dimensional subspace of H with a (complete) orthonormal basis $\{g_1, g_2, \ldots\}$. Then the corresponding stochastic inner products

$$\gamma_j(x) = (g_j, x)^{\sim}, \quad j = 1, 2, \ldots, \tag{3.1}$$

form a set of independent Gaussian random variables on B.

For every $n \in \mathbb{N}$, let $\mathcal{H}_n$ be the subspace of H spanned by $\{g_1, \ldots, g_n\}$, and let $X_n \colon B \to \mathbb{R}^n$ and $X_\infty \colon B \to \mathbb{R}^{\mathbb{N}}$ be defined by

$$X_n(x) = ((g_1, x)^{\sim}, \ldots, (g_n, x)^{\sim}) = (\gamma_1(x), \ldots, \gamma_n(x)), \tag{3.2}$$

and

$$X_\infty(x) = ((g_1, x)^{\sim}, (g_2, x)^{\sim}, \ldots) = (\gamma_1(x), \gamma_2(x), \ldots). \tag{3.3}$$

A set of the type

$$I = \{x \in B \colon X_n(x) \in U\} \equiv X_n^{-1}(U), \quad U \in \mathcal{B}(\mathbb{R}^n),$$

is called a *Borel cylinder* (or a *quasi-Wiener interval*). It is well known that

$$\nu(I) = \int_U K_n(\vec{\xi}) \,\mathrm{d}\vec{\xi},$$

where

$$K_n(\vec{\xi}) = \prod_{j=1}^{n} \Big[(2\pi)^{-1/2} \Big\{ -\frac{1}{2}\xi_j^2 \Big\} \Big].$$

Let $\mathcal{F}_n$ be the σ-field formed by the sets $\{X_n^{-1}(U) \colon U \in \mathcal{B}(\mathbb{R}^n)\}$ and let $\mathcal{F}$ be the σ-field generated by $\bigcup\limits_{n=1}^{\infty} \mathcal{F}_n$. Then, by the definition of conditional expectation (see Doob [21], Tucker [50] and Yeh [51], [53]) for every $F \in L_1(B)$,

$$\begin{aligned} \int_{X_n^{-1}(U)} F(x) \,\mathrm{d}\nu(x) &= \int_{X_n^{-1}(U)} E(F \mid \mathcal{F}_n)(x) \,\mathrm{d}\nu(x) = \int_U E(F \mid X_n = \vec{\xi}) \,\mathrm{d}P_{X_n}(\vec{\xi}) \\ &= \int_U E(F \mid (g_j, \cdot)^{\sim} = \xi_j, j = 1, \ldots, n) \,\mathrm{d}P_{X_n}(\vec{\xi}), \quad U \in \mathcal{B}(\mathbb{R}^n) \end{aligned} \tag{3.4}$$

where $P_{X_n}(U) = \nu(X_n^{-1}(U))$ and $E(F \mid X_n = \vec{\xi})$ is a Lebesgue measurable function of $\vec{\xi}$ which is unique up to null sets in $\mathbb{R}^n$.

Since $\{\mathcal{F}_n\}$ is an increasing sequence of σ-fields of Wiener-measurable sets, for $F \in L_1(B)$, $E[|E(F \mid \mathcal{F}_n)|] \leqslant E[|F|]$ for every $n \in \mathbb{N}$. Hence, by [38], Remark 9.4.4,

$$\lim_{n \to \infty} E(F \mid \mathcal{F}_n) = E(F \mid \mathcal{F}) \tag{3.5}$$

almost surely and for every $A \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$,

$$\lim_{n\to\infty} \int_A E(F \mid \mathcal{F}_n)(x)\,\mathrm{d}\nu(x) = \int_A E(F \mid \mathcal{F})(x)\,\mathrm{d}\nu(x)$$

by [38], Lemma 9.4.3. From this and (3.4), it follows that for every $U \in \bigcup_{n=1}^{\infty} \mathcal{B}(\mathbb{R}^n)$,

$$\begin{aligned} &\int_U E(F \mid \gamma_j = \xi_j,\ j = 1,2,\ldots)\,\mathrm{d}P_{X_\infty}(\vec{\xi}) \\ &\qquad = \lim_{n\to\infty} \int_U E(F \mid \gamma_j = \xi_j,\ j = 1,\ldots,n)\,\mathrm{d}P_{X_n}(\vec{\xi}) \end{aligned} \tag{3.6}$$

where

$$\mathrm{d}P_{X_n}(\vec{\xi}) = \prod_{j=1}^{n} \Big[(2\pi)^{-1/2} \exp\Big\{-\frac{\xi_j^2}{2}\Big\}\,\mathrm{d}\xi_j\Big]$$

and

$$\mathrm{d}P_{X_\infty}(\vec{\xi}) = \prod_{j=1}^{\infty} \Big[(2\pi)^{-1/2} \exp\Big\{-\frac{\xi_j^2}{2}\Big\}\,\mathrm{d}\xi_j\Big].$$

In equation (3.6), we used the convention that if $U \in \mathcal{B}(\mathbb{R}^n)$, then $U \in \mathcal{B}(\mathbb{R}^{n+k})$ by identifying U and $U \times \mathbb{R}^k$ in $\mathcal{B}(\mathbb{R}^{n+k})$ for $k = 1,2,\ldots$ Thus, if $U \in \bigcup_{n=1}^{\infty} \mathcal{B}(\mathbb{R}^n)$, then there exists $N \in \mathbb{N}$ such that $U \in \mathcal{B}(\mathbb{R}^n)$ for all $n \geqslant N$, and hence, it follows that

$$\begin{aligned} &\int_U E(F \mid \gamma_j = \xi_j,\ j = 1,2,\ldots)\,\mathrm{d}P_{X_\infty}(\vec{\xi}) \\ &\qquad = \lim_{n\to\infty} \int_U E(F \mid \gamma_j = \xi_j,\ j = 1,\ldots,n)\,\mathrm{d}P_{X_n}(\vec{\xi}), \end{aligned}$$

for all $n \geqslant N$, from which (3.6) follows.

4. Evaluation formula for conditional Wiener integral

In this section, we develop quite simple formulas for converting conditional Wiener integrals of the types $E(F \mid X_n = \vec{\xi}) = E(F \mid \gamma_j = \xi_j,\ j = l,\ldots,n)$ and $E(F \mid \gamma_j = \xi_j,\ j = l,2,\ldots)$ into ordinary Weiner integrals.

Let $\mathcal{H}$, $\{g_1, g_2, \ldots\}$, $\{\gamma_1(x), \gamma_2(x), \ldots\}$, and $\mathcal{H}_n$ be as in the previous section. Define projection maps $\mathcal{P}$ and $\mathcal{P}_n$ from H into $\mathcal{H}$ and $\mathcal{H}_n$, respectively, by

$$\mathcal{P}h = \sum_{j=1}^{\infty} \langle h, g_j\rangle g_j \in \mathcal{H} \quad \text{and} \quad \mathcal{P}_n h = \sum_{j=1}^{n} \langle h, g_j\rangle g_j \in \mathcal{H}_n.$$

For every $x \in B$ and $\vec{\xi} = (\xi_1, \xi_2, \dots) \in \mathbb{R}^{\mathbb{N}}$, let

$$(4.1)\quad x_\infty = \sum_{j=1}^{\infty} \gamma_j(x) g_j, \quad x_n = \sum_{j=1}^{n} \gamma_j(x) g_j, \quad \vec{\xi}_\infty = \sum_{j=1}^{\infty} \xi_j g_j \quad \text{and} \quad \vec{\xi}_n = \sum_{j=1}^{n} \xi_j g_j.$$

Our first lemma plays a key role throughout this paper.

Lemma 4.1. *Let $\{g_1, g_2, \dots\}$ be the orthonormal basis of the subspace $\mathcal{H}$ of H.*

(i) *Let Y_∞ and Z_∞ be random transforms from B to itself given by $Y_\infty(x) = x - x_\infty$ and $Z_\infty(x) = x_\infty$, respectively. Then Y_∞ and Z_∞ are independent.*

(ii) *Let Y_n and Z_n be random transforms from B to itself given by $Y_n(x) = x - x_n$ and $Z_n(x) = x_n$, respectively. Then Y_n and Z_n are independent.*

P r o o f. In order to verify assertion (i), it suffices to show that $(y_1, Y_\infty(x))$ and $(y_2, Z_\infty(x))$ are independent real-valued random variables for all $y_1, y_2 \in B^*$, where $(\cdot,\cdot)$ denotes the B^*-B pairing. But using (3.1) and (2.2), it follows that

$$\begin{aligned}
&E_x[(y_1, Y_\infty(x))(y_2, Z_\infty(x))] \\
&\quad = E_x\Bigg[\Bigg((y_1, x) - \sum_{j=1}^{\infty} \gamma_j(x)(y_1, g_j)\Bigg) \sum_{j=1}^{\infty} \gamma_j(x)(y_2, g_j)\Bigg] \\
&\quad = \sum_{j=1}^{\infty} (y_2, g_j) E_x[(y_1, x)^\sim (g_j, x)^\sim] - \sum_{j=1}^{\infty}\sum_{l=1}^{\infty} (y_1, g_j)(y_2, g_l) E_x[(g_j, x)^\sim (g_l, x)^\sim] = 0.
\end{aligned}$$

Using the same method, one can see that assertion (ii) holds true. □

The following theorem is one of our main results. For notational convenience, we write the conditional expectation $E(F \mid X = \vec{\xi})$ by $E(F(x) \mid X(x) = \vec{\xi})$ as used in [42], [53].

Theorem 4.2. *Let $F \in L_1(B)$. Then*

$$(4.2)\qquad E(F(x) \mid \gamma_j(x) = \xi_j,\ j = 1, 2, \dots) = E_x[F(x - x_\infty + \vec{\xi}_\infty)]$$

and

$$(4.3)\qquad E(F(x) \mid \gamma_j(x) = \xi_j,\ j = 1, \dots, n) = E_x[F(x - x_n + \vec{\xi}_n)].$$

P r o o f. Since $x - x_\infty$ and x_∞ are independent processes by Lemma 4.1, it follows by [2], Corollary 4.38 that

$$\begin{aligned}
E(F(x) \mid \gamma_j(x) = \xi_j,\ j = 1, 2, \dots) &= E(F(x - x_\infty + x_\infty) \mid \gamma_j(x) = \xi_j,\ j = 1, 2, \dots) \\
&= E_x[F(x - x_\infty + \vec{\xi}_\infty)].
\end{aligned}$$

Equation (4.3) follows by the same method. □

The following corollary is a simple consequence of equation (4.3).

Corollary 4.3. *Let $F \in L_1(B)$. Then for every $U \in \mathcal{B}(\mathbb{R}^n)$,*

$$\int_{X_n^{-1}(U)} F(x)\,\mathrm{d}\mu(x) = \int_U E_x[F(x - x_n + \vec{\xi}_n)]\,\mathrm{d}P_{X_n}(\vec{\xi}).$$

Remark 4.4. Using equations in (4.1), we can rewrite equations (4.2) and (4.3) as

$$\text{(4.4)}\qquad \begin{aligned} E(F \mid \gamma_j = \xi_j,\ j = 1,2,\dots) &\equiv E(F(x) \mid \gamma_j(x) = \xi_j,\ j = 1,2,\dots) \\ &= E_x\biggl[F\biggl(x - \sum_{j=1}^{\infty}(g_j,x)^{\sim} g_j + \sum_{j=1}^{\infty}\xi_j g_j\biggr)\biggr] \end{aligned}$$

and as

$$\text{(4.5)}\qquad \begin{aligned} E(F \mid \gamma_j = \xi_j,\ j = 1,\dots,n) &\equiv E(F(x) \mid \gamma_j(x) = \xi_j,\ j = 1,\dots,n) \\ &= E_x\biggl[F\biggl(x - \sum_{j=1}^{n}(g_j,x)^{\sim} g_j + \sum_{j=1}^{n}\xi_j g_j\biggr)\biggr], \end{aligned}$$

respectively. Furthermore, it follows by (3.5), (4.4) and (4.5) that

(4.6)

$$E_x\biggl[F\biggl(x - \sum_{j=1}^{\infty}(g_j,x)^{\sim} g_j + \sum_{j=1}^{\infty}\xi_j g_j\biggr)\biggr] = \lim_{n\to\infty} E_x\biggl[F\biggl(x - \sum_{j=1}^{n}(g_j,x)^{\sim} g_j + \sum_{j=1}^{n}\xi_j g_j\biggr)\biggr].$$

5. Conditional analytic Fourier-Feynman transform

As an application of the result obtained in Section 4, we introduce the concept of the CFFT of functionals on $(B, \mathcal{W}(B), \nu)$. We then provide explicit formulas for CFFTs of functionals F in the Kallianpur and Bromley Fresnel class $\mathcal{F}(B)$.

Let X be an $\mathbb{R}^n$ (or $\mathbb{R}^{\mathbb{N}}$)-valued transform on B and let F be a complex-valued $\mathcal{W}(B)$-measurable functional such that the integral $E_x[F(\lambda^{-1/2}x)]$ exists as a finite number for all $\lambda > 0$. For $\lambda > 0$ and $\vec{\xi}$ in $\mathbb{R}^n$ (or $\mathbb{R}^{\mathbb{N}}$), let

$$J_F(\lambda;\vec{\xi}) = E(F(\lambda^{-1/2}\cdot) \mid X(\lambda^{-1/2}\cdot))(\vec{\xi})$$

denote the conditional Wiener integral of $F(\lambda^{-1/2}\cdot)$ given $X(\lambda^{-1/2}\cdot)$. If for a.e. $\vec{\xi}$ in $\mathbb{R}^n$ (or $\mathbb{R}^{\mathbb{N}}$), there exists a function $J_F^*(\lambda;\vec{\xi})$ analytic in λ on $\mathbb{C}_+$ such that $J_F^*(\lambda;\vec{\xi}) = J_F(\lambda;\vec{\xi})$ for all $\lambda > 0$, then $J_F^*(\lambda;\cdot)$ is defined to be the conditional analytic Wiener integral of F over B given X with parameter λ, and for $\lambda \in \mathbb{C}_+$, we write

$$E^{\mathrm{an}\,w_\lambda}(F \mid X = \vec{\xi}) \equiv E^{\mathrm{an}\,w_\lambda}(F(x) \mid X(x) = \vec{\xi}) = J_F^*(\lambda;\vec{\xi}).$$

If for a fixed real $q \in \mathbb{R} \setminus \{0\}$, the limit

$$\lim_{\substack{\lambda \to -iq \\ \lambda \in \mathbb{C}_+}} E^{\mathrm{an}\, w_\lambda}(F \mid X = \vec{\xi})$$

exists for a.e. $\vec{\xi}$ in $\mathbb{R}^n$ (or $\mathbb{R}^{\mathbb{N}}$), then we denote the value of this limit by $E^{\mathrm{an}\, f_q}(F \mid X = \vec{\xi})$ and call it the conditional analytic Feynman integral of F over B given X with parameter q.

To define the CFFT, we consider conditioning functions X_n and X_∞ given by (3.2) and (3.3). Let $F\colon B \to \mathbb{C}$ be a $\mathcal{W}(B)$-measurable functional on B such that the integral $E_x[F(y + \lambda^{-1/2}x)]$ exists as a finite number for all $\lambda > 0$. Then one can easily see from (4.4) and (4.5) that for all $\lambda > 0$,

$$\begin{aligned} &E(F(\lambda^{-1/2}x) \mid \gamma_j(\lambda^{-1/2}x) = \xi_j,\ j = 1, 2, \ldots) \\ &\qquad = E_x\left[F\left(\lambda^{-1/2}x - \lambda^{-1/2}\sum_{j=1}^{\infty}(g_j, x)^{\sim} g_j + \sum_{j=1}^{\infty} \xi_j g_j\right)\right] \end{aligned}$$

and

$$\begin{aligned} &E(F(\lambda^{-1/2}x) \mid \gamma_j(\lambda^{-1/2}x) = \xi_j,\ j = 1, \ldots, n) \\ &\qquad = E_x\left[F\left(\lambda^{-1/2}x - \lambda^{-1/2}\sum_{j=1}^{n}(g_j, x)^{\sim} g_j + \sum_{j=1}^{n} \xi_j g_j\right)\right], \end{aligned} \tag{5.1}$$

respectively. Thus, we have that
(5.2)

$$E^{\mathrm{an}\, w_\lambda}(F(x) \mid \gamma_j(x) = \xi_j,\ j = 1, 2, \ldots) = E_x^{\mathrm{an}\, w_\lambda}\left[F\left(x - \sum_{j=1}^{\infty}(g_j, x)^{\sim} g_j + \sum_{j=1}^{\infty} \xi_j g_j\right)\right],$$

$$E^{\mathrm{an}\, f_q}(F(x) \mid \gamma_j(x) = \xi_j,\ j = 1, 2, \ldots) = E_x^{\mathrm{an}\, f_q}\left[F\left(x - \sum_{j=1}^{\infty}(g_j, x)^{\sim} g_j + \sum_{j=1}^{\infty} \xi_j g_j\right)\right],$$

$$E^{\mathrm{an}\, w_\lambda}(F(x) \mid \gamma_j(x) = \xi_j,\ j = 1, \ldots, n) = E_x^{\mathrm{an}\, w_\lambda}\left[F\left(x - \sum_{j=1}^{n}(g_j, x)^{\sim} g_j + \sum_{j=1}^{n} \xi_j g_j\right)\right],$$

and

$$E^{\mathrm{an}\, f_q}(F(x) \mid \gamma_j(x) = \xi_j,\ j = 1, \ldots, n) = E_x^{\mathrm{an}\, f_q}\left[F\left(x - \sum_{j=1}^{n}(g_j, x)^{\sim} g_j + \sum_{j=1}^{n} \xi_j g_j\right)\right].$$

We are now ready to state the definition of CFFT of functionals F on B.

Definition 5.1. Let $F\colon B \to \mathbb{C}$ be a $\mathcal{W}(B)$-measurable functional on B such that the integral $E_x[F(y+\lambda^{-1/2}x)]$ exists as a finite number for all $\lambda > 0$. For $\lambda \in \mathbb{C}_+$ and $y \in B$, $T_\lambda(F \mid X_n)(y,\vec{\xi})$ denotes the conditional analytic Wiener integral of $F(y+\cdot)$ given $X_n(x) = (\gamma_1(x),\dots,\gamma_n(x))$, that is to say,

$$\begin{aligned} T_\lambda(F \mid X_n)(y,\vec{\xi}) &= E^{\mathrm{an}\, w_\lambda}(F(y+x) \mid \gamma_j(x) = \xi_j,\ j = 1,\dots,n) \\ &= E_x^{\mathrm{an}\, w_\lambda}\biggl[F\biggl(y + x - \sum_{j=1}^{n}(g_j,x)^\sim g_j + \sum_{j=1}^{n}\xi_j g_j\biggr)\biggr]. \end{aligned}$$

We define the L_1 analytic CFFT $T_q^{(1)}(F \mid X_n)(y,\vec{\xi})$ of F given X_n by the formula

$$\begin{aligned} T_q^{(1)}(F \mid X_n)(y,\vec{\xi}) &= \lim_{\substack{\lambda\to -\mathrm{i}q\\ \lambda\in\mathbb{C}_+}} T_\lambda(F \mid X_n)(y,\vec{\xi}) \\ &= E_x^{\mathrm{an}\, f_q}\biggl[F\biggl(y + x - \sum_{j=1}^{n}(g_j,x)^\sim g_j + \sum_{j=1}^{n}\xi_j g_j\biggr)\biggr]. \end{aligned}$$

Remark 5.2. For the definition of CFFT of F given X_∞, a similar statement is understood when the conditioning function X_n given by (3.2) is replaced by the conditioning function X_∞ given by (3.3).

Lemma 5.3. *For every $h \in H$ and any $\varrho > 0$, it follows that*

$$E_x[\exp\{\mathrm{i}\varrho(h,x)^\sim\}] = \exp\{-\varrho^2|h|^2\}. \tag{5.3}$$

From the linearity of the stochastic inner product $(\cdot,\cdot)^\sim$ and equation (5.3), we have the following lemma.

Lemma 5.4. *Let $\{g_1,\dots,g_n\}$ be an orthonormal set in H. Then for every $h \in H$ and any $\varrho > 0$, it follows that*

$$\begin{aligned} E_x\biggl[\exp\biggl\{\mathrm{i}\varrho\biggl(h, x - \sum_{j=1}^{n}\gamma_j(x)g_j\biggr)^\sim\biggr\}\biggr] &= \exp\biggl\{-\frac{\varrho^2}{2}\biggl|h - \sum_{j=1}^{n}\langle h,g_j\rangle g_j\biggr|^2\biggr\} \\ &= \exp\biggl\{-\frac{\varrho^2}{2}\biggl[|h|^2 - \sum_{j=1}^{n}\langle h,g_j\rangle^2\biggr]\biggr\}. \end{aligned} \tag{5.4}$$

In our next theorem, we obtain explicit formulas for CFFT of functionals F in the Kallianpur and Bromley Fresnel class $\mathcal{F}(B)$.

Theorem 5.5. *Let $F \in \mathcal{F}(B)$ be given by equation* (2.3), *and let X_n be given by equation* (3.2). *Then for a.e. $\vec{\xi} \in \mathbb{R}$, it follows that*

(5.5)

$$T_\lambda(F \mid X_n)(y,\vec{\xi}) = \int_H \exp\left\{ \mathrm{i}(h,y)^\sim - \frac{1}{2\lambda}\left[|h|^2 - \sum_{j=1}^n \langle h, g_j\rangle^2\right] + \mathrm{i}\sum_{j=1}^n \xi_j \langle h, g_j\rangle \right\} \mathrm{d}\sigma(h)$$

for all $\lambda \in \mathbb{C}_+$, and

(5.6)

$$T_q^{(1)}(F \mid X_n)(y,\vec{\xi}) = \int_H \exp\left\{ \mathrm{i}(h,y)^\sim - \frac{\mathrm{i}}{2q}\left[|h|^2 - \sum_{j=1}^n \langle h, g_j\rangle^2\right] + \mathrm{i}\sum_{j=1}^n \xi_j \langle h, g_j\rangle \right\} \mathrm{d}\sigma(h)$$

for all real $q \in \mathbb{R} \setminus \{0\}$.

P r o o f. Using (2.3), (5.1) with F replaced by $F(y+\cdot)$, the Fubini theorem, (5.4) with $\varrho = \lambda^{-1/2}$, it follows that for $(\lambda, \vec{\xi}) \in (0,\infty) \times \mathbb{R}^n$,

$$\begin{aligned}
&E(F(y+\lambda^{-1/2}\cdot) \mid X_n(\lambda^{-1/2}\cdot) = \vec{\xi})\\
&\quad\equiv E(F(y+\lambda^{-1/2}x) \mid \gamma(\lambda^{-1/2}x) = \xi_j,\ j=1,\dots,n)\\
&\quad= E_x\left[F\left(y + \lambda^{-1/2}x - \lambda^{-1/2}\sum_{j=1}^n (g_j,x)^\sim g_j + \sum_{j=1}^n \xi_j g_j\right)\right]\\
&\quad= \int_H \exp\left\{\mathrm{i}(h,y)^\sim + \mathrm{i}\sum_{j=1}^n \xi_j (h,g_j)^\sim\right\}\\
&\qquad\quad \times E_x\left[\exp\left\{\mathrm{i}\lambda^{-1/2}\left(h, x - \sum_{j=1}^n (g_j,x)^\sim g_j\right)^\sim\right\}\right] \mathrm{d}\sigma(h)\\
&\quad= \int_H \exp\left\{\mathrm{i}(h,y)^\sim - \frac{1}{2\lambda}\left[|h|^2 - \sum_{j=1}^n \langle h,g_j\rangle^2\right] + \mathrm{i}\sum_{j=1}^n \xi_j \langle h, g_j\rangle\right\} \mathrm{d}\sigma(h).
\end{aligned}$$

But the last expression above is an analytic function of λ throughout $\mathbb{C}_+$ and is a continuous function of λ in $\widetilde{\mathbb{C}}_+$ since σ is a finite Borel measure on $\mathcal{B}(H)$, the σ-field of Borel sets in H. Thus, equations (5.5) and (5.6) are established in view of Definition 5.1. □

In the next theorem we also establish an evaluation formula for the CFFT of the $F \in \mathcal{F}(B)$ given infinite dimensional conditioning function X_∞.

Theorem 5.6. *Let $F \in \mathcal{F}(B)$ be given by* (2.3), *let an orthonormal sequence $\{g_1, g_2, \dots\}$ be given which spans an infinite dimensional subspace $\mathcal{H}$ of H, and let X_∞ be given by* (3.3). *Then for a.e. $\vec{\xi} \in \mathbb{R}$, it follows that*

$$T_q(F \mid X_\infty)(y,\vec{\xi}) = \int_H \exp\left\{ \mathrm{i}(h,y)^\sim - \frac{\mathrm{i}}{2q}\left[|h|^2 - \sum_{j=1}^\infty \langle h, g_j\rangle^2\right] + \mathrm{i}\sum_{j=1}^\infty \xi_j \langle h, g_j\rangle \right\} \mathrm{d}\sigma(h)$$

for all real $q \in \mathbb{R} \setminus \{0\}$.

P r o o f. By the definition of the analytic CFFT, $T_q^{(1)}(F \mid X_\infty)$, of F given X_∞ (see Remark 5.2 above), we see that

$$T_q^{(1)}(F \mid X_\infty)(y,\vec{\xi}) = E_x^{\mathrm{an}\, f_q}\Big[F\Big(y + x - \sum_{j=1}^{\infty}(g_j,x)^\sim g_j + \sum_{j=1}^{\infty}\xi_j g_j\Big)\Big]$$

for s-a.e. $y \in B$. Thus, using the same method as in the proof of Theorem 5.5, and applying equations (5.2) with F replaced with $F(y+\cdot)$ and (4.6) with x replaced with $\lambda^{-1/2}x$, it follows that

$$\begin{aligned}
&T_q^{(1)}(F \mid X_\infty)(y,\vec{\xi})\\
&\quad= E_x^{\mathrm{an}\, f_q}\Big[F\Big(y + x - \sum_{j=1}^{\infty}(g_j,x)^\sim g_j + \sum_{j=1}^{\infty}\xi_j g_j\Big)\Big]\\
&\quad= \lim_{n\to\infty} E_x^{\mathrm{an}\, f_q}\Big[F\Big(y + x - \sum_{j=1}^{n}(g_j,x)^\sim g_j + \sum_{j=1}^{n}\xi_j g_j\Big)\Big]\\
&\quad= \lim_{n\to\infty} T_q^{(1)}(F \mid X_n)(y,\vec{\xi})\\
&\quad= \lim_{n\to\infty}\int_H \exp\Big\{\mathrm{i}(h,y)^\sim - \frac{\mathrm{i}}{2q}\Big[|h|^2 - \sum_{j=1}^{n}\langle h,g_j\rangle^2\Big] + \mathrm{i}\sum_{j=1}^{n}\xi_j(h,g_j)^\sim\Big\}\,\mathrm{d}\sigma(h).
\end{aligned}$$

Since σ is a finite Borel measure on $\mathcal{B}(H)$, by the bounded convergence theorem, it also follows that

$$\begin{aligned}
&T_q^{(1)}(F \mid X_\infty)(y,\vec{\xi})\\
&\quad= \int_H \lim_{n\to\infty}\exp\Big\{\mathrm{i}(h,y)^\sim - \frac{\mathrm{i}}{2q}\Big[|h|^2 - \sum_{j=1}^{n}\langle h,g_j\rangle^2\Big] + \mathrm{i}\sum_{j=1}^{n}\xi_j(h,g_j)^\sim\Big\}\,\mathrm{d}\sigma(h)\\
&\quad= \int_H \exp\Big\{\mathrm{i}(h,y)^\sim - \frac{\mathrm{i}}{2q}\Big[|h|^2 - \sum_{j=1}^{\infty}\langle h,g_j\rangle^2\Big] + \mathrm{i}\sum_{j=1}^{\infty}\xi_j\langle h,g_j\rangle\Big\}\,\mathrm{d}\sigma(h)
\end{aligned}$$

as desired. □

By Parseval's identity, we have the following corollary.

Corollary 5.7. *Let $F \in \mathcal{F}(B)$ be given by equation* (2.3)*, let a complete orthonormal basis $\{g_1, g_2, \dots\}$ of H be given, and let X_∞ be given by equation* (3.3)*. Then for a.e. $\vec{\xi} \in \mathbb{R}$, it follows that*

$$T_q(F \mid X_\infty)(y,\vec{\xi}) = \int_H \exp\Big\{\mathrm{i}(h,y)^\sim + \mathrm{i}\sum_{j=1}^{\infty}\xi_j\langle h,g_j\rangle\Big\}\,\mathrm{d}\sigma(h)$$

for all real $q \in \mathbb{R}\setminus\{0\}$.

Let X_∞ be given by (3.3). Note that given a functional $F \in \mathcal{F}(B)$, the L_1 analytic CFFT of F given X_∞, $T_q^{(1)}(F \mid X_\infty)(\cdot, \vec{\xi})$, can be considered as a bounded functional on B. Thus, using the techniques similar to those used in the proofs of Theorems 5.5 and 5.6, we observe that for all nonzero real numbers q_1 and q_2 with $q_1 + q_2 \neq 0$,

$$\begin{aligned}(6.1)\quad & T_{q_2}^{(1)}(T_{q_1}^{(1)}(F \mid X_\infty)(\cdot, \vec{\xi_1}) \mid X_\infty)(y, \vec{\xi_2}) \\ &= \int_H \exp\bigg\{ \mathrm{i}(h,y)^\sim - \frac{\mathrm{i}}{2q_1}\bigg[|h|^2 - \sum_{j=1}^\infty \langle h, g_j\rangle^2\bigg] + \mathrm{i}\sum_{j=1}^\infty \xi_{1j}\langle h, g_j\rangle \\ &\qquad - \frac{\mathrm{i}}{2q_2}\bigg[|h|^2 - \sum_{j=1}^\infty \langle h, g_j\rangle^2\bigg] + \mathrm{i}\sum_{j=1}^\infty \xi_{2j}\langle h, g_j\rangle \bigg\}\, \mathrm{d}\sigma(h) \\ &= \int_H \exp\bigg\{ \mathrm{i}(h,y)^\sim - \frac{\mathrm{i}}{2q_1q_2/(q_1+q_2)}\bigg[|h|^2 - \sum_{j=1}^\infty \langle h, g_j\rangle^2\bigg] \\ &\qquad + \mathrm{i}\sum_{j=1}^\infty (\xi_{1j} + \xi_{2j})\langle h, g_j\rangle \bigg\}\, \mathrm{d}\sigma(h).\end{aligned}$$

Thus, we have the relation

$$T_{q_2}^{(1)}(T_{q_1}^{(1)}(F \mid X_\infty)(\cdot, \vec{\xi_1}) \mid X_\infty)(y, \vec{\xi_2}) = T_{q_1q_2/(q_1+q_2)}^{(1)}(F \mid X_\infty)(y, \vec{\xi_1} + \vec{\xi_2}).$$

From an induction argument and in view of Theorem 5.6, we have the following assertions.

Theorem 6.1. *Let F and X_∞ be as in Theorem* 5.6. *Let $\{q_1, \ldots, q_m\}$ be a finite sequence in $\mathbb{R} \setminus \{0\}$ with*

$$\frac{1}{q_1} + \ldots + \frac{1}{q_k} \neq 0 \quad \textit{for } k \in \{2, \ldots, m\}.$$

Then it follows that

$$\begin{aligned}& T_{q_m}^{(1)}(\ldots(T_{q_2}^{(1)}(T_{q_1}^{(1)}(F \mid X_\infty)(\cdot, \vec{\xi_1}) \mid X_\infty)(\cdot, \vec{\xi_2}))\ldots \mid X_\infty)(y, \vec{\xi_m}) \\ &\qquad = T_{\alpha_m}^{(1)}(F \mid X_\infty)\bigg(y, \sum_{k=1}^m \vec{\xi_k}\bigg), \quad \textit{where} \quad \alpha_m = \bigg(\frac{1}{q_1} + \ldots + \frac{1}{q_m}\bigg)^{-1}.\end{aligned}$$

Remark 6.2. A close examination of (6.1) shows that for any nonzero real number q,

$$\begin{aligned}& T_{-q}^{(1)}(T_q^{(1)}(F \mid X_\infty)/(\cdot, \vec{\xi_1}) \mid X_\infty)(y, \vec{\xi_2}) \\ &\qquad = \int_H \exp\bigg\{ \mathrm{i}(h,y)^\sim + \mathrm{i}\sum_{j=1}^\infty (\xi_{1j} + \xi_{2j})\langle h, g_j\rangle \bigg\}\, \mathrm{d}\sigma(h).\end{aligned}$$

From this, we also have the relation

$$T_{-q}^{(1)}(T_q^{(1)}(F \mid X_\infty)(\cdot,\vec{\xi}) \mid X_\infty)(y,-\vec{\xi}) = F(y)$$

for s-a.e. $y \in B$.

All arguments for the analytic CFFT $T_q^{(1)}(F \mid X_\infty)$ discussed in this section also hold for the analytic CFFT $T_q^{(1)}(F \mid X_n)$ for functionals F in $\mathcal{F}(B)$.

7. An epilogue

In the highly celebrated papers (see [42], [43], [44]), Park and Skoug established simple formulas in order to evaluate the conditional Wiener integral which can be used in heat and Schrödinger equations, and in [45], they founded the concept of CFFT and studied the conditional transform using the simple formula. These fundamental concepts would have been very useful to us in establishing many of the results in [12], [14]–[19]. We feel strongly that the fundamental concept of CFFT given infinite dimensional conditioning functions in this paper will prove to be very useful in future work for ourselves as well as other researchers in this area. For instance, we expect the results such as (1.5) and (1.6) with our infinite-dimensional conditioning functions on abstract Wiener space.

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