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ON POPOV'S EXPLICIT FORMULA AND THE DAVENPORT EXPANSION

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To Professor Dr. Imre Kátai with deep respect and friendship

Abstract. We shall establish an explicit formula for the Davenport series in terms of trivial zeros of the Riemann zeta-function, where by the Davenport series we mean an infinite series involving a PNT (Prime Number Theorem) related to arithmetic function a_n with the periodic Bernoulli polynomial weight $\overline{B}_{\varkappa}(nx)$ and PNT arithmetic functions include the von Mangoldt function, Möbius function and Liouville function, etc. The Riesz sum of order 0 or 1 gives the well-known explicit formula for respectively the partial sum or the Riesz sum of order 1 of PNT functions. Then we may reveal the genesis of the Popov explicit formula as the integrated Davenport series with the Riesz sum of order 1 subtracted. The Fourier expansion of the Davenport series is proved to be a consequence of the functional equation, which is referred to as the Davenport expansion. By the explicit formula for the Davenport series, we also prove that the Davenport expansion for the von Mangoldt function is equivalent to the Kummer's Fourier series up to a formula of Ramanujan and a fortiori is equivalent to the functional equation for the Riemann zeta-function.

Keywords: explicit formula; Davenport expansion; Kummer's Fourier series; Riemann zeta-function; functional equation

MSC 2020: 11M41, 11N05, 11J54

1. Introduction and Popov's explicit formula

Let $\zeta(s)$ be the Riemann zeta-function and let $\Lambda(n)$ be the von Mangoldt function defined by

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}, \quad \sigma := \operatorname{Re} s > 1. \tag{1.1}$$

For basic knowledge on zeta-functions, we refer to [6], [12], [24], [29] etc.

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Let $\{y\} = y - [y]$ denote the fractional part of $y \in \mathbb{R}$ with $[y]$ the integral part of y, i.e., the greatest integer not exceeding y. Let

$$\overline{B}_\varkappa(x) = B_\varkappa(\{x\}) = \sum_{k=0}^{\varkappa} \binom{\varkappa}{k} B_{\varkappa-k}\{x\}^k \tag{1.2}$$

be the $\varkappa$th periodic Bernoulli polynomial, where $\varkappa \in \mathbb{N}$, B_k is the kth Bernoulli number. It has the Fourier expansion

$$\overline{B}_\varkappa(x) = -\frac{\varkappa!}{(2\pi\mathrm{i})^\varkappa} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\mathrm{e}^{2\pi\mathrm{i}nx}}{n^\varkappa}. \tag{1.3}$$

We use the following cases

$$\begin{aligned} \overline{B}_1(x) &= \{x\} - \frac{1}{2} = -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin 2\pi nx, \quad x \notin \mathbb{Z}, \\ \overline{B}_2(x) &= \{x\}^2 - \{x\} + \frac{1}{6} = \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos 2\pi nx. \end{aligned}$$

Patkowski in [19] proved the generalized Popov formula (see [23]):

$$\begin{aligned} &\frac{1}{2} \sum_{n>x} \frac{\Lambda(n)}{n^{r+1}} \Big(\Big\{ \frac{n}{x} \Big\} - \Big\{ \frac{n}{x} \Big\}^2 \Big) \\ &\quad = h_r(x) + \sum_{\varrho} \Big(\frac{r-\varrho+1}{2(r-\varrho-1)(r-\varrho)} - \frac{\zeta(r-\varrho)}{r-\varrho} \Big) \frac{x^{\varrho-r-1}}{r+1-\varrho} \\ &\quad\quad + \sum_{k=1}^{\infty} \Big(\frac{2k+r+1}{2(2k+r-1)(2k+r)} - \frac{\zeta(r+2k)}{r+2k} \Big) \frac{x^{-r-2k-1}}{r+2k+1}, \end{aligned} \tag{1.4}$$

where

$$h_r(x) = \begin{cases} \dfrac{2-\log 2\pi}{2x}, & r = 1, \\ \Big(\dfrac{r}{2(1-r)(2-r)} - \dfrac{\zeta(1-r)}{1-r} \Big) \dfrac{x^{-r}}{r}, & r > 1 \end{cases} \tag{1.5}$$

and the first sum on the right of (1.4) is over all nontrivial zeros ϱ in the critical strip and the second is the sum over trivial zeros $-2k$, $k \in \mathbb{N}$. We assume throughout that all the nontrivial zeros of $\zeta(s)$ are simple.

The argument depends on the following identity. For $r \geqslant 1$, $x > 0$ and $1 < c < 2$,

$$\begin{aligned} &\frac{1}{2} \sum_{n>x} \frac{\Lambda(n)}{n^{r+1}} \Big(\Big\{ \frac{n}{x} \Big\} - \Big\{ \frac{n}{x} \Big\}^2 \Big) \\ &\quad = \frac{1}{2\pi\mathrm{i}} \int_{(c)} \Big(\frac{2-s}{2s(s-1)} - \frac{\zeta(1-s)}{1-s} \Big) \frac{\zeta'}{\zeta}(s+r-1) \frac{x^{s-2}}{2-s} \,\mathrm{d}s. \end{aligned} \tag{1.6}$$

The main result of [21] is Theorem 1.1, which gives an explicit formula for the sum $\frac{1}{2}\sum\limits_{n>x}(f(n)/n^{r+1})(\{n/x\}-\{n/x\}^2)$ similar to (1.4) with $\Lambda(n)$ replaced by $f(n)$. The sets $\{f(n)\}$ are generated by the Dirichlet series $L(s)=\sum\limits_{n=1}^{\infty}f(n)/n^s$ absolutely convergent for $\sigma>1$.

In this paper, we assume $L(s)$ is of the form

$$L(s)=\sum_{n=1}^{\infty}\frac{f(n)}{n^s}=\frac{Z(s)}{\zeta(s)},\quad Z(s)=\sum_{n=1}^{\infty}\frac{g(n)}{n^s} \tag{1.7}$$

absolutely convergent for $\sigma>1$ and $Z(s)$ is a meromorphic function which has no zeros at the zeros of $\zeta(s)$ (this corresponds to simplicity of zeros of the Riemann zeta-function) and such that

$$L(s)=O(\log|s|) \tag{1.8}$$

for $|s|$ large (save for neighborhoods of trivial zeros). Equation (1.8) is satisfied by $L(s)=-\zeta'(s)/\zeta(s)$, see [24], Satz 4.3, page 227, [6], page 108, etc.

We call the (formal) expansion

$$\sum_{n=1}^{\infty}a_n\overline{B}_{\varkappa}(nx)=-\frac{\varkappa!}{(2\pi\mathrm{i})^{\varkappa}}\sum_{l=1}^{\infty}(\mathrm{e}^{2\pi\mathrm{i}lx}+(-1)^{\varkappa}\mathrm{e}^{-2\pi\mathrm{i}lx})c_l,\quad c_l=l^{-\varkappa}\sum_{d|l}a_d \tag{1.9}$$

the *Davenport expansion* and we refer to the left-hand side series as the *Davenport series*, where $\{a_n\}$ are arithmetical functions which appear in PNT. Since $\overline{B}'_{\varkappa}(x)=\varkappa\overline{B}_{\varkappa-1}(x)$, we may work with the integrated Davenport series and shift to the lower order by termwise differentiation as long as the differentiated series is uniformly convergent, cf. [4], [5], [31], etc.

Our aim is to give an explicit formula (in terms of *trivial zeros*) for the Davenport series or the integrated one with respectively

$$a_n=\frac{f(n)}{n^r}\quad\text{or}\quad a_n=\frac{f(n)}{n^{r+1}}, \tag{1.10}$$

where $r\geqslant 1$ and $f(n)$ are generated by (1.7), cf. Theorem 1.2 and Corollary 2.1. We restrict our selves to the cases

$$\begin{aligned} &f(n)=\Lambda(n),\quad (Z(s)=-\zeta'(s)),\quad f(n)=\mu(n),\quad (Z(s)=1),\\ &f(n)=\lambda(n),\quad (Z(s)=\zeta(2s))\end{aligned} \tag{1.11}$$

and illustrate this by Theorem 1.2 for the von Mangoldt function $\Lambda(n)$. Subtracting the explicit formula in Lemma 1.2 from the formula in Lemma 1.1 yields the Popov formula, see Theorem 1.1. Equating the explicit formula with the Davenport expansion (see Theorem 2.1) we deduce Corollary 1.1 to the effect that the Davenport expansion for the von Mangoldt function is equivalent to the functional equation for the Riemann zeta-function, cf. concluding remarks.

Replacing (1.1) by (1.7), (1.6) reads

$$\frac{1}{2}\sum_{n>x}\frac{f(n)}{n^{r+1}}\Big(\Big\{\frac{n}{x}\Big\}-\Big\{\frac{n}{x}\Big\}^2\Big)=\frac{1}{2\pi \mathrm{i}}\int_{(b)}\Big(\frac{s+1}{2s(s-1)}-\frac{\zeta(s)}{s}\Big)L(r-s)\frac{x^{-s-1}}{s+1}\,\mathrm{d}s, \tag{1.12}$$

where $-1<b<0$. This follows from

$$\frac{1}{2}(\{x\}^2-\{x\})=\frac{1}{2\pi \mathrm{i}}\int_{(b)}\Big(\frac{s+1}{2s(s-1)}-\frac{\zeta(s)}{s}\Big)\frac{x^{s+1}}{s+1}\,\mathrm{d}s, \tag{1.13}$$

which is proved in [19], but we cannot follow the proof and we shall use

$$\frac{1}{2}(\{x\}^2-\{x\})=-\frac{1}{2\pi \mathrm{i}}\int_{(b)}\frac{1}{s(s+1)}\zeta(-s-1)x^{-s}\,\mathrm{d}s,\quad -\frac{1}{2}<b<0. \tag{1.14}$$

We shall see the effect of adding

$$I:=-\frac{1}{2}\frac{1}{2\pi \mathrm{i}}\int_{(b)}\frac{1}{s(s+1)}x^{s+1}\,\mathrm{d}s$$

to the identity (1.14). Shifting the integration path to $\sigma=c>0$ in

$$I=\frac{1}{2}\frac{1}{2\pi \mathrm{i}}\int_{(b-1)}\frac{1}{s(s+1)}x^{s+2}\,\mathrm{d}s,$$

we find that

$$-I=\frac{1}{2}\frac{1}{2\pi \mathrm{i}}\int_{(c)}\frac{1}{s(s+1)}x^{s+2}\,\mathrm{d}s-\frac{1}{2}(x^2-x)=\begin{cases}0, & x>1,\\ -\frac{1}{2}(x^2-x), & 0<x\leqslant 1\end{cases}$$

since

$$\frac{1}{2\pi \mathrm{i}}\int_{(c)}\frac{1}{s(s+1)}x^{s+2}\,\mathrm{d}s=\begin{cases}x^2(1-x^{-1}), & x>1,\\ 0, & 0<x\leqslant 1.\end{cases}$$

This corresponds to cancellation of the terms with $0<x<1$ in (1.22) and moving the Riesz sum to the right.

Formula (1.14) must be a special case of Mikolás formula, see [17], Proposition 1 and [18]:

$$\overline{B}_\varkappa(x)=-(-1)^\varkappa\varkappa!\frac{1}{2\pi \mathrm{i}}\int_{(c)}\frac{\Gamma(s)}{\Gamma(s+\varkappa)}\zeta(1-s-\varkappa)x^{-s}\,\mathrm{d}s, \tag{1.15}$$

where $0 < c < 1$, which has been used in the literature, e.g., in [20]. A slightly different but equivalent form of (1.15) is used for $\overline{B}_\varkappa(x)$, cf. [2]:

$$(2\pi\mathrm{i})^\varkappa \overline{B}_\varkappa(x) = -\varkappa! \frac{1}{2\pi\mathrm{i}} \int_{(c)} (\mathrm{e}^{\pi\mathrm{i}s/2} + (-1)^\varkappa \mathrm{e}^{-\pi\mathrm{i}s/2}) \Gamma(s)\zeta(s+\varkappa)(2\pi x)^{-s}\,\mathrm{d}s$$

or

$$(2\pi)^\varkappa \overline{B}_\varkappa(x) = -\varkappa! \frac{1}{2\pi\mathrm{i}} \int_{(c)} 2\cos\frac{\pi}{2}(\varkappa - s)\Gamma(s)\zeta(s+\varkappa)(2\pi x)^{-s}\,\mathrm{d}s. \tag{1.16}$$

For the integral in (1.15) to be absolutely convergent, $-1 < c < -\frac{1}{2}$ must hold in view of [29], (5.1.3), page 81. This corresponds to shifting the integration path to the left to $\sigma = d$, $-1 < d < -\frac{1}{2}$ (the left side limit -1 is for excluding poles) and we encounter a simple pole at $s = 0$ with residue

$$\begin{cases} -\dfrac{1}{2}, & \varkappa = 1, \\ (-1)^\varkappa B_\varkappa, & \varkappa \geqslant 2. \end{cases}$$

Hence, (1.15) leads to

$$\overline{B}_\varkappa(x) - B_\varkappa = -(-1)^\varkappa \varkappa! \frac{1}{2\pi\mathrm{i}} \int_{(d)} \frac{\Gamma(s)}{\Gamma(s+\varkappa)} \zeta(1-s-\varkappa)x^{-s}\,\mathrm{d}s, \tag{1.17}$$

where $-1 < d < -\frac{1}{2}$ and the integral is absolutely convergent. The case $\varkappa = 2$ is consistent with Popov's weight $\{n/x\} - \{n/x\}^2$.

We use the special case of (1.17) with $\varkappa = 2$, which occurs on writing s for $-s-1$ ($b = -d-1$)

$$\frac{1}{2}\overline{B}_2(x) - \frac{1}{2}B_2 = -\frac{1}{2\pi\mathrm{i}} \int_{(b)} \frac{1}{s(s+1)} \zeta(-s-1)x^{-s}\,\mathrm{d}s, \tag{1.18}$$

where

$$-\frac{1}{2} < b < 0 \tag{1.19}$$

and (1.14) follows.

This is reminiscent of Hamburger's result (see [8]) on the Fourier series for $\overline{B}_2(x)$. Formula (4.2) in [2], ($\varkappa > 0$)

$$\frac{1}{\varkappa}(B_\varkappa(x) - \overline{B}_\varkappa(x)) = \sum_{n \leqslant x} (x-n)^{\varkappa-1}$$

explains the addition of I above and suggests the division of the sum on the left of (1.23) into two: $n \leqslant x$ and $n > x$, i.e.,

$$\frac{1}{2}\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^2}\left(\left\{\frac{n}{x}\right\}^2 - \left\{\frac{n}{x}\right\}\right) = \frac{1}{2}\sum_{n \leqslant x} + \frac{1}{2}\sum_{n > x}. \tag{1.20}$$

The first sum on the right is

$$\frac{1}{2}\sum_{n\leqslant x}\frac{\Lambda(n)}{n^2}\Big(\Big(\frac{n}{x}\Big)^2-\frac{n}{x}\Big)=-\frac{1}{2x^2}\sum_{n\leqslant x}\frac{\Lambda(n)}{n}(x-n), \tag{1.21}$$

which is the Riesz sum of order 1 (in Lemma 1.2). Thus, the following principle comes out.

The whole sum over $n=1,2,\ldots$ in (1.22) with periodic Bernoulli polynomial weight consists of $n\leqslant x$ part (Riesz sum of order 1) and $n>x$ part (Popov's formula). The whole sum has two expressions — the (integrated) Davenport expansion which is a consequence of the functional equation (see Theorem 2.1) and the explicit formula involving trivial zeros (see Theorem 1.2). Comparison of them yields Kummer's Fourier series in the case of the von Mangoldt function.

Although (1.12) gives a direct proof of Popov's theorem, Theorem 1.1, it does not give any more information. Thus, in view of (1.14), instead of (1.12) we are to use

$$\begin{aligned}\frac{1}{2}\sum_{n=1}^{\infty}\frac{f(n)}{n^{r+1}}\Big(\Big\{\frac{n}{x}\Big\}^2-\Big\{\frac{n}{x}\Big\}\Big)\\ &= -\frac{1}{2\pi \mathrm{i}}\int_{(b)}\frac{1}{s(s+1)}\zeta(-s-1)L(s+r+1)x^s\,\mathrm{d}s\\ &= -\frac{1}{2\pi \mathrm{i}}\int_{(c)}\frac{1}{(1-s)(2-s)}\zeta(1-s)L(s+r-1)x^{s-2}\,\mathrm{d}s,\end{aligned} \tag{1.22}$$

where $1<c:=b+2<\frac{3}{2}$. As long as the series $J:=\frac{1}{2}\sum\limits_{n=1}^{\infty}f(n)/n^{r+1}$ is convergent, we may think of (1.22) as the Davenport expansion (with $\overline{B}_2$ weight) with the series $\frac{1}{2}B_2J$ subtracted, cf., e.g., (1.32) in Theorem 1.2.

As is stated, instead of going on general lines, we restrict ourselves to the case of the von Mangoldt function and state the explicit formula for the Davenport sum, cf. Theorem 1.2.

Lemma 1.1. *For $x>1$ we have*

$$\begin{aligned}\frac{1}{2}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^2}\Big(\Big\{\frac{n}{x}\Big\}^2-\Big\{\frac{n}{x}\Big\}\Big) &= -\frac{1}{2x}\Big(\Big(\frac{1}{x}-1\Big)\log 2\pi+1+\log x-\gamma\Big)\\ &\quad+\sum_{k=1}^{\infty}\frac{\zeta(2k+1)}{(2k+1)(2k+2)}x^{-2k-2}.\end{aligned} \tag{1.23}$$

P r o o f. We specify $r = 1$ and $L(s) = -\zeta'(s)/\zeta(s)$ and apply the Cauchy residue theorem to the rectangle with vertices at $(c \pm \mathrm{i}T)$, $(-N \pm \mathrm{i}T)$. The horizontals tend to 0 as $T \to \infty$ and the vertical one also tends to 0 as $N \to \infty$ by (1.8). Hence, the integral in (1.22) is the sum of residues at poles in the strip $-N < \sigma < c$, $s = 1$ is a double pole and the residue is

$$\frac{1}{2x}(-\log 2\pi + 1 + \log x + \gamma), \tag{1.24}$$

where we use $\zeta'(0) = -\frac{1}{2}\log 2\pi$ and

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} - \gamma + O(s-1), \tag{1.25}$$

and where γ is the Euler constant, cf., e.g., [6], page 81.

At a zero ϱ, the denominator $\zeta(s)$ has a simple pole and is of form $\zeta'(\varrho)(s-\varrho)+\ldots$ If it is a nontrivial zero, then it is cancelled by the zero of $\zeta(1-s)$. If it is a trivial zero $s = -2k$, $k = 1, 2, \ldots$, the residue is $\zeta(2k+1)/((2k+1)(2k+2))x^{-2k-2}$. Hence, the sum of residues is $\sum\limits_{k=1}^{\infty} \zeta(2k+1)/((2k+1)(2k+2))x^{-2k-2}$. Hence, the assertion follows. □

Lemma 1.2. *For $x \geqslant 1$ we have the explicit formula*

$$\sum_{n\leqslant x} \frac{\Lambda(n)}{n}(x-n) = (-1 + \log x - \gamma)x + \log 2\pi - \sum_{\varrho} \frac{x^{\varrho}}{\varrho(\varrho-1)} - \sum_{k=1}^{\infty} \frac{1}{2k(2k+1)} x^{-2k}. \tag{1.26}$$

P r o o f. Recall Perron's formula

$$\frac{1}{\Gamma(\varkappa+1)} {\sum_{\lambda_k\leqslant x}}' \alpha_k (x-\lambda_n)^{\varkappa} = \frac{1}{2\pi\mathrm{i}} \int_{(c)} \frac{\Gamma(s)\varphi(s)x^{s+\varkappa}}{\Gamma(s+\varkappa+1)}\,\mathrm{d}s, \tag{1.27}$$

where $\varphi(s) = \sum\limits_{k=1}^{\infty} \alpha_n/n^s$, the left-hand side sum is called the Riesz sum of order $\varkappa$ and the right-hand side is the G-function $G^{1,0}_{1,1}G^{1,0}_{1,1}\left(z^{-1}\,\middle|\, {\varkappa \atop 0}\right)$ and becomes $(2\pi\mathrm{i})^{-1}\int_{(c)}(s(s+1)\ldots(s+\varkappa))^{-1}z^{-s}\,\mathrm{d}s$ for $\varkappa$ a positive integer. Here the prime on the summation sign means that the term corresponding to $n = x$ is to be halved in the case $\varkappa = 0$. For Riesz sums, cf. [15], Chapter 6.

We use the case $\varkappa = 1$:

$$\sum_{n\leqslant x} \alpha_n (x-n) = \frac{1}{2\pi\mathrm{i}} \int_{(c)} \frac{1}{s(s+1)}\varphi(s)x^{s+1}\,\mathrm{d}s, \quad c > 1,\ x > 0. \tag{1.28}$$

To treat (1.21), we shall find an explicit formula for

$$\sum_{n\leqslant x}\frac{\Lambda(n)}{n}(x-n)=\frac{1}{2\pi \mathrm{i}}\int_{(c)}\frac{1}{s(s+1)}\Big(-\frac{\zeta'(s+1)}{\zeta(s+1)}\Big)x^{s+1}\,\mathrm{d}s \tag{1.29}$$

in analogy with [12], (14), page 21. The residue at the double pole at $s=0$ is

$$(-1+\log x-\gamma)x \tag{1.30}$$

and the residue at $s=-1$ is $\log 2\pi$. Computation of residues at the zeros of ζ is similar to that in the proof of Lemma 1.1. $\square$

Differentiation of (1.26) leads to the explicit formula for $\sum\limits_{n\leqslant x}\Lambda(n)/n$, see[12], page 81. In what follows we simply say by differentiation without giving details on termwise differentiability of the (second) infinite series. For we may appeal to the well-established results on convergence in symmetric sum $\lim\limits_{X\to\infty}\sum\limits_{|\varrho|<X}x^{\varrho-1}/(\varrho-1)$ and Landau's differencing method, cf., e.g., [30].

Incorporating Lemmas 1.1 and 1.2 in (1.20), we obtain Popov's result.

Theorem 1.1. *For* $|x|>1$ *we have*

$$\begin{aligned}\sum_{n>x}\frac{\Lambda(n)}{n^2}\Big(\Big\{\frac{n}{x}\Big\}-\Big\{\frac{n}{x}\Big\}^2\Big)=\frac{1}{x}(2-\log 2\pi)+\sum_{\varrho}\frac{x^{\varrho-2}}{\varrho(\varrho-1)}\\ +\sum_{k=1}^{\infty}\frac{1}{2k(2k+1)}x^{-2k-2}\\ -\sum_{k=1}^{\infty}\frac{\zeta(2k+1)}{(2k+1)(k+1)}x^{-2k-2}.\end{aligned} \tag{1.31}$$

Lemma 1.1 also reads:

Theorem 1.2. *For* $0<x<1$ *we have*

$$\begin{aligned}\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{\log n}{n^2}\cos 2\pi nx+\frac{1}{12}\frac{\zeta'}{\zeta}(2)&=\frac{1}{2}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^2}(\{nx\}^2-\{nx\})\\ &=-\frac{x}{2}((x-1)\log 2\pi+1-\log x-\gamma)\\ &\quad+\sum_{k=1}^{\infty}\frac{\zeta(2k+1)}{(2k+1)(2k+2)}x^{2k+2}.\end{aligned} \tag{1.32}$$

By differentiation, we have

$$-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\log n}{n}\sin 2\pi nx = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n}\Big(\{nx\}-\frac{1}{2}\Big) \tag{1.33}$$
$$= -x\log 2\pi - \frac{1}{2}(-\log 2\pi - \log x - \gamma) + \sum_{k=1}^{\infty}\frac{\zeta(2k+1)}{2k+1}x^{2k+1}.$$

The first equality in (1.33) is the Davenport expansion, see Theorem 2.1.

To prove Corollary 1.1 below we need

Lemma 1.3 (Ramanujan)**.** *For $0 \leqslant \lambda \in \mathbb{Z}$ and $|z| < |\alpha|$ we have*

$$\sum_{m=2}^{\infty}\frac{\zeta(m,\alpha)}{m+\lambda}z^{m+\lambda} = \sum_{k=0}^{\lambda}\binom{\lambda}{k}\zeta'(-k,\alpha-z)z^{\lambda-k} - \zeta'(-\lambda,\alpha) \tag{1.34}$$
$$-\sum_{k=1}^{\lambda}\frac{1}{k}\zeta(k-\lambda,\alpha)z^{k} + \frac{1}{\lambda+1}(\psi(\alpha)-H_{\lambda})z^{\lambda+1}.$$

Here $\zeta(s,\alpha)$, $\psi(\alpha)$, H_λ denote the Hurwitz zeta-function, the Euler digamma function, harmonic number, respectively.

Lemma 1.3 with $\lambda = 0$ and Lerch's formula

$$\zeta'(0,x) = \log\frac{\Gamma(x)}{\sqrt{2\pi}} \tag{1.35}$$

yield a special case of Wilton's formula (1923):

$$\sum_{n=1}^{\infty}\zeta(n,a)\frac{x^n}{n} = \log\Gamma(a-x) - \log\Gamma(a) + \psi(a)x, \quad |x| < |a|. \tag{1.36}$$

The *odd part* is

$$\sum_{k=1}^{\infty}\zeta(2k+1,a)\frac{x^{2k+1}}{2k+1} = \frac{1}{2}(\log\Gamma(a-x) - \log\Gamma(a+x)) + \psi(a)x \tag{1.37}$$

cf. [27], page 159. Hence, the last sum in (1.33) is

$$\sum_{k=1}^{\infty}\zeta(2k+1)\frac{x^{2k+1}}{2k+1} = \frac{1}{2}(\log\Gamma(1-x) - \log\Gamma(1+x)) - \gamma x, \quad |x| < 1. \tag{1.38}$$

Corollary 1.1. *Theorem* 1.2 *implies Kummer's Fourier series*

$$\log\Gamma(x) = \log\sqrt{\pi} - (\gamma + \log 2\pi)B_1(x) - \frac{1}{2}\log\sin\pi x + \sum_{n=1}^{\infty}\frac{\log n}{\pi n}\sin 2\pi nx, \quad 0 < x < 1. \tag{1.39}$$

The Davenport expansion for the von Mangoldt function is equivalent to the functional equation for the Riemann zeta-function (up to Ramanujan's formula (1.34)*).*

Proof. On using the reciprocal relation $\Gamma(x)\Gamma(1-x) = \pi/\sin\pi x$, (1.38) becomes

$$\sum_{k=1}^{\infty}\zeta(2k+1)\frac{x^{2k+1}}{2k+1} = \frac{1}{2}(-\log\sin\pi x - \log x + \log\pi) - \log\Gamma(x) - \gamma x, \quad |x| < 1. \tag{1.40}$$

Substituting this in (1.33) proves (1.39). The second assertion follows from the fact that Kummer's Fourier series is equivalent to the functional equation and Theorem 2.1. □

2. Davenport expansions

Arithmetical Fourier series (1.9) have been treated in [15], pages 196–209, which are divided into two classes: Diophantine Fourier series considered by [9], [11], and others involving divisor functions, and arithmetical Fourier series involving the PNT functions studied by [4], [5], [6] and others. Here the former is more related to the functional equation and the latter to the zero-free region. Let $\Phi_\varkappa(s,x)$ be the Dirichlet series considered by Hardy and Littlewood, see [9] (by Barnes multiple zeta-function), [11] (by zeta-functions with Grössen characters) [16], page 103:

$$\Phi_\varkappa(s,x) = \sum_{n=1}^{\infty}\frac{\overline{B}_\varkappa(nx)}{n^s}, \tag{2.1}$$

where $\overline{B}_\varkappa(x)$ is the periodic Bernoulli polynomial in (1.2).

The authors in [10], page 116 proved the a.e. convergence of the Fourier series to $\Phi_\varkappa(s,x)$:

$$\Phi_\varkappa(s,x) \sim -\frac{\varkappa!}{(2\pi i)^\varkappa}\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\frac{\sigma_{\varkappa-s}(k)}{k^\varkappa}\mathrm{e}^{2\pi i kx} \tag{2.2}$$

for $\frac{1}{2} < \operatorname{Re} s \leqslant 1$, where $\sigma_\alpha(n)$ is the sum-of-divisors function

$$\sigma_\alpha(n) = \sum_{d|n} d^\alpha.$$

To think of (2.1) as a Davenport series, we must think of n^{-s} as the coefficient and forget about zeta-function aspects. But since using (2.1) as the zeta-function gives a far-reaching and prominent results, we think it proper to distinguish them from the case of PNT coefficients $\{a_n\}$—the Davenport expansion.

The paper [13] treats both Diophantine Fourier series and arithmetical Fourier series as Davenport expansions and is very informative on convergence of the Davenport expansions in various function spaces and relations with fractal geometry. He calls the Davenport series in (1.9) as the Davenport expansion (with respect to the Riesz basis in a Sobolev space) and we use the term in a narrower sense. We refer to [2], [17], [20], [22], which are published subsequently for Diophantine Fourier series. The book [16] is not listed but is an essential source-book. Regarding Davenport expansions, the following are missing although they are essential material: [3], [25] (both are concerned with convergence in L^2), [26] (pointwise but to be regarded as L^2 convergence), [31] (pointwise convergence and share the Kubert function aspect with [25]). The paper [14] also deals with L^2 convergence. The argument is to be made more elaborate taking into account absolute convergence. However, this can be dispensed with, thanks to the following theorem.

Theorem 2.1. *The Davenport expansion for the von Mangoldt function, Möbius function and Liouville function are consequences of the functional equation (for the Riemann zeta-function):*

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{f(n)}{n^{r+1}}\overline{B}_2(nx)=\frac{1}{2\pi^3}\sum_{n=1}^{\infty}\frac{\tilde{f}(n)}{n^2}\cos 2\pi nx, \tag{2.3}$$

where

$$\tilde{f}(n)=\sum_{d\mid n}d^{1-r}f(d). \tag{2.4}$$

The differentiated form is the traditional Davenport expansion

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^r}\Big(\{nx\}-\frac{1}{2}\Big)=-\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{\tilde{f}(n)}{n}\sin 2\pi nx. \tag{2.5}$$

P r o o f. Recall (1.22) in the form

$$\begin{aligned}\frac{1}{2}\sum_{n=1}^{\infty}&\frac{f(n)}{n^{r+1}}(\{nx\}^2-\{nx\})\\&=-\frac{1}{2\pi \mathrm{i}}\int_{(c)}\frac{1}{(1-s)(2-s)}\zeta(1-s)L(s+r-1)x^{2-s}\,\mathrm{d}s,\end{aligned} \tag{2.6}$$

where $1 < c < 2$. We substitute

$$\zeta(1-s)L(s+r-1) = \pi^{1/2-s}\frac{\Gamma(s/2)}{\Gamma((1-s)/2)}\zeta(s)L(s+r-1) \tag{2.7}$$
$$= \pi^{1/2-s}\frac{\Gamma(s/2)}{\Gamma((1-s)/2)}\sum_{n=1}^{\infty}\frac{\tilde{f}(n)}{n^s}$$

in (2.6) and change the order of summation and integration to deduce that
(2.8)

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{f(n)}{n^{r+1}}(\{nx\}^2-\{nx\}) = \frac{1}{\pi^{3/2}}\sum_{n=1}^{\infty}\tilde{f}(n)\frac{1}{2\pi \mathrm{i}}\int_{(c)}\frac{\Gamma(s/2)}{(2-s)\Gamma((3-s)/2)}(\pi nx)^{2-s}\,\mathrm{d}s.$$

In order to evaluate the resulting integral, we use the differentiated form

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^r}\left(\{nx\}-\frac{1}{2}\right) = -\frac{x}{\pi^{3/2}}\sum_{n=1}^{\infty}\tilde{f}(n)\frac{1}{2\pi \mathrm{i}}\int_{(c)}\frac{\Gamma(s/2)}{\Gamma((3-s)/2)}(\pi nx)^{-s}\,\mathrm{d}s. \tag{2.9}$$

The integral is the G-function

$$G_{0,2}^{1,0}\left(z\,\middle|\,\begin{matrix}-\\ b, b-\frac{1}{2}\end{matrix}\right) = \frac{z^{b-1/2}}{\sqrt{\pi}}\sin 2\sqrt{z} \tag{2.10}$$

with $z = (\pi nx)^2$ and $b = 0$. Hence, substituting this, (2.9) leads to (2.5). □

Remark 2.1. In the case $r = 1$ we apply the functional equation as

$$\zeta(1-s)L(s) = \pi^{1/2-s}\frac{\Gamma(s/2)}{\Gamma((1-s)/2)}\sum_{n=1}^{\infty}g(n)n^{-s},$$

say, and (2.9) reads

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^2}\left(\{nx\}-\frac{1}{2}\right) = -\frac{x}{\pi^{3/2}}\sum_{n=1}^{\infty}g(n)\frac{1}{2\pi \mathrm{i}}\int_{(c)}\frac{\Gamma(s/2)}{\Gamma((3-s)/2)}(\pi nx)^{-s}\,\mathrm{d}s. \tag{2.11}$$

From (2.4), we have

$$\sum_{d|n}f(d) = g(n). \tag{2.12}$$

Hence as $f = \Lambda$, $f = \mu$, $f = \lambda$, we have $g(n) = \log n$, $g(n) = \delta_{n1}$ (Kronecker delta), $g(n) = \chi_2(n)$ (the characteristic function of squares), whence we have familiar formulas

$$\sum_{d|n}\Lambda(d) = n, \quad \sum_{d|n}\mu(d) = \delta_{n1}, \quad \sum_{d|n}\lambda(d) = \begin{cases}1, & n = \text{square},\\ 0, & n \neq \text{square}.\end{cases} \tag{2.13}$$

We consider the case of the Möbius function based on [28]. Correspondingly to Theorem 2.1, we have:

Corollary 2.1 ([19], Theorem 2.1)**.**

$$\frac{1}{2\pi^2}\cos 2\pi nx - \frac{1}{12}\frac{1}{\zeta(2)} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{\mu(n)}{n^2}(\{nx\}^2 - \{nx\}) = \frac{1}{2\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k)!}(2\pi x)^{2k}. \tag{2.14}$$

The differentiated form amounts to the classical Davenport expansion

$$-\frac{n\sin 2\pi nx}{\pi} = \sum_{n=1}^{\infty}\frac{\mu(n)}{n}\Big(\{nx\} - \frac{1}{2}\Big) = -\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}(2\pi x)^{2k+1}. \tag{2.15}$$

In computing residues in this case, we apply the formula

$$\zeta'(-2k) = \frac{(-1)^k\zeta(2k+1)(2k)!}{\pi^{2k}2^{2k+1}}, \quad k \in \mathbb{N}. \tag{2.16}$$

Unlike Corollary 1.1, this gives only the power series expansions for the sine and cosine functions.

For the Liouville function defined by

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\lambda(n)}{n^s}, \quad \sigma > 1, \tag{2.17}$$

the Davenport expansion reads

$$\sum_{n=1}^{\infty}\frac{\lambda(n)}{n}\Big(\{nx\} - \frac{1}{2}\Big) = -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin 2\pi n^2 x. \tag{2.18}$$

We cannot give an explicit formula for the left-hand side of (2.17) as in Theorem 1.2 (since residue calculus does not apply) although the explicit formula for the Riesz sum is known by [7]. We hope to return to this elsewhere.

Concluding remarks. The integral expression (1.22) (which in turn depends on (a special case of) Mikolás formula (1.15)) for the Davenport series $\sum_{n=1}^{\infty} a_n\overline{B}_{\varkappa}(nx)$ is the key. The presence of $\zeta(1-s)$ has two effects: by applying the functional equation, it leads to the Davenport expansion (see Theorem 2.1) and by applying the residue theorem, the nontrivial zeros of $\zeta(1-s)$ cancel the poles at nontrivial zeros of $\zeta(s)$, thus yielding an explicit formula (see Theorem 1.2) in terms of trivial zeros. This study provides a vast amount of further research problems including the Davenport series with Clausen function weight. Also note that distributions are considered in [13] and a distribution-theoretic explicit formula has been developed, cf., e.g., [1]. We hope to return to them elsewhere.

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