Anubhav Sharma; Ayyadurai Sankaranarayanan

On the average behavior of the Fourier coefficients of jth symmetric power L-function over certain sequences of positive integers

Czechoslovak Mathematical Journal, Vol. 73 (2023), No. 3, 885–901

Persistent URL: `http://dml.cz/dmlcz/151780`

Czechoslovak Mathematical Journal, 73 (148) (2023), 885–901

ON THE AVERAGE BEHAVIOR OF THE FOURIER COEFFICIENTS OF jTH SYMMETRIC POWER L-FUNCTION OVER CERTAIN SEQUENCES OF POSITIVE INTEGERS

Anubhav Sharma, Ayyadurai Sankaranarayanan, Hyderabad

Received August 16, 2022. Published online April 27, 2023.

Abstract. We investigate the average behavior of the nth normalized Fourier coefficients of the jth ($j \geqslant 2$ be any fixed integer) symmetric power L-function (i.e., $L(s, \mathrm{sym}^j f)$), attached to a primitive holomorphic cusp form f of weight k for the full modular group $SL(2, \mathbb{Z})$ over certain sequences of positive integers. Precisely, we prove an asymptotic formula with an error term for the sum

$$S_j^* := \sum_{\substack{a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2 \leqslant x \\ (a_1,a_2,a_3,a_4,a_5,a_6) \in \mathbb{Z}^6}} \lambda^2_{\mathrm{sym}^j f}(a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2),$$

where x is sufficiently large, and

$$L(s, \mathrm{sym}^j f) := \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s}.$$

When $j = 2$, the error term which we obtain improves the earlier known result.

Keywords: nonprincipal Dirichlet character; Hölder's inequality; jth symmetric power L-function; holomorphic cusp form

MSC 2020: 11M06, 11F11, 11F30

The first author, Anubhav Sharma, wishes to express his gratitude to the University of Hyderabad for its financial support and IoE's performance based publication incentive towards his Ph.D. Program.

DOI: 10.21136/CMJ.2023.0348-22

1. Introduction

Let $L(s,f)$ be the L-function associated with the primitive holomorphic cusp form f of weight k for the group $SL(2,\mathbb{Z})$ and let $\lambda_f(n)$ be the normalized nth Fourier coefficient of the Fourier expansion of $f(z)$ at the cusp ∞, i.e.,

$$f(z)=\sum_{n=1}^{\infty}\lambda_f(n)n^{(k-1)/2}\mathrm{e}^{2\pi\mathrm{i}nz},$$

where $\Im(z)>0$. Then the L-function attached to $\lambda_f(n)$ is defined as

$$L(s,f)=\sum_{n=1}^{\infty}\frac{\lambda_f(n)}{n^s}$$

for $\Re(s)>1$, where $\lambda_f(n)$ are the eigenvalues of all the Hecke operators T_n.

Let χ^* be the Dirichlet character modulo N. If

$$f\Big(\frac{az+b}{cz+d}\Big)=\chi^*(d)(cz+d)^k f(z)$$

for all $z\in\mathbb{H}$ (upper half plane) and $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_0(N)$, then f is known as a modular form of weight k and level N with Nebentypus χ^*. Here, $\Gamma_0(N)$ is the congruence subgroup, i.e.,

$$\Gamma_0(N)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{Z})\colon\ c\equiv 0 \pmod{N}\right\}.$$

Throughout the paper, we assume that the Nebentypus χ^* is trivial.

In 1974, Deligne in [2] proved that for any prime p there exist complex numbers $\alpha(p)$ and $\beta(p)$ such that

$$\alpha(p)+\beta(p)=\lambda_f(p) \tag{1.1}$$

and

$$|\alpha(p)|=|\beta(p)|=1=\alpha(p)\beta(p). \tag{1.2}$$

Then $L(s,f)$ can be written as

$$L(s,f)=\prod_p\Big(1-\frac{\alpha(p)}{p^s}\Big)^{-1}\Big(1-\frac{\beta(p)}{p^s}\Big)^{-1}.$$

Also, $|\lambda_f(n)|\leqslant d(n)$, where $d(n)$ is the divisor function.

The symmetric square L-function is defined as

$$L(s,\mathrm{sym}^2 f) := \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^2 f}(n)}{n^s} = \prod_p \Big(1-\frac{\alpha^2(p)}{p^s}\Big)^{-1}\Big(1-\frac{\beta^2(p)}{p^s}\Big)^{-1}\Big(1-\frac{1}{p^s}\Big)^{-1}$$

for $\Re(s) > 1$, where $\lambda_{\mathrm{sym}^2 f}(n)$ is multiplicative.

Several authors have studied the average behavior of these Fourier coefficients. In 2006, Fomenko in [3] was able to prove some results for the symmetric square L-functions. He showed that

$$\sum_{n\leqslant x} \lambda_{\mathrm{sym}^2 f}(n) \ll x^{1/2}\log^2 x,$$

and further he could establish that

$$\sum_{n\leqslant x} \lambda^2_{\mathrm{sym}^2 f}(n) = cx + O(x^{\theta}),$$

where $\theta < 1$. For more related results, see [6], [12], [13] and [20].

In 2013, Zhai in [22] proved an asymptotic formula for

$$\sum_{\substack{a^2+b^2\leqslant x\\ (a,b)\in\mathbb{Z}^2}} \lambda_f^l(a^2+b^2)$$

for $x \geqslant 1$, and $3 \leqslant l \leqslant 8$. For results related to mean square and higher moments of the coefficients of symmetric square L-functions on a certain sequence of positive integers, see [18] and [19].

One of the most interesting problems of number theory involves the number of lattice points in a k-dimensional hypersphere. For integers $n \geqslant 0$, let Let $r_k(n) := \#\{(n_1,n_2,\ldots,n_k)\in\mathbb{Z}^k\colon\ n_1^2+n_2^2+\ldots+n_k^2 = n\}$. Then the formula

$$\sum_{0\leqslant n\leqslant x} r_k(n)$$

defines the lattice point number of a compact, origin-centered ball of radius $\sqrt{x}$.

Let $r_{k,s}(n)$ denote the number of representations of n as the sum of k sth powers. Due to Hardy-Littlewood method (see [21]), the following asymptotic formula holds for $r_{k,s}(n)$:

$$r_{k,s}(n) = (\mathfrak{S}_k(n)+o(1))\Gamma\Big(1+\frac{1}{s}\Big)^k\Gamma\Big(\frac{k}{s}\Big)^{-1}n^{k/s-1},$$

where $\mathfrak{S}_k(n)$ is called the *singular series* and is defined by

$$\mathfrak{S}_k(n) = \sum_{q=1}^{\infty}\sum_{\substack{a=1\\(a,q)=1}}^{q} (S(q,a)/q)^k e(-an/q) \quad \text{with} \quad S(q,a) = \sum_{m=1}^{q} e(am^s/q),$$

$e(x) = \exp(2\pi \mathrm{i}x)$ and n sufficiently large.

For spheres of dimensions $k \geqslant 4$, the situation is much easier and better understood. In 2000, Krätzel in [11], pages 227–231 proved that

$$P_4(x) := \sum_{0 \leqslant n \leqslant x} r_4(n) - \frac{\pi^2}{2}x^2 = O(x \log x), \quad P_4(x) = \Omega(x \log_2 x).$$

For instance, when k is an even positive integer not less than 4, then we have

$$\sum_{n \leqslant x} r_k(n) = c_k x^{k/2} + P_k(x).$$

For dimensions $k \geqslant 5$, the exact order of the lattice point discrepancy $P_k(x)$ of an origin-centered k-dimensional ball of radius $\sqrt{x}$ has been known for a long time (see [11], pages 227 ff), namely

$$P_k(x) = O(x^{k/2-1}), \quad P_k(x) = \Omega(x^{k/2-1}).$$

For more interesting results pertaining to the asymptotic formula in Waring's problem, one can see [14].

In an earlier paper [17], we considered the sum

$$S_2^* := \sum_{\substack{a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2 \leqslant x \\ (a_1,a_2,a_3,a_4,a_5,a_6) \in \mathbb{Z}^6}} \lambda^2_{\mathrm{sym}^2 f}(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2)$$

for a sufficiently large x and established the asymptotic formula

$$S_2^* = c^* x^3 + O(x^{14/5+\varepsilon}), \tag{1.3}$$

where c^* is an effective constant, see Theorem 1 of [17].

The main aim of this paper is to generalize and improve the result obtained in [17] by using recent celebrated work (see [15], [16]) of Newton and Thorne, and better subconvexity bounds for the related L-functions. Consider the summatory function

$$S_j^* := \sum_{\substack{a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2 \leqslant x \\ (a_1,a_2,a_3,a_4,a_5,a_6) \in \mathbb{Z}^6}} \lambda^2_{\mathrm{sym}^j f}(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2),$$

where $j \geqslant 2$ is any integer. More precisely, we prove:

Theorem 1.1. *Let $j \geqslant 2$ be any fixed integer. For a sufficiently large x, and $\varepsilon > 0$ any small constant, we have*

$$S_j^* = c(j)x^3 + O(x^{3-6/(3(j+1)^2+1)+\varepsilon}),$$

where $c(j)$ is an effective constant defined by

$$c(j) = \frac{16}{3} L(3, \chi) \prod_{n=1}^{j} L(1, \mathrm{sym}^{2n} f) L(3, \mathrm{sym}^{2n} f \otimes \chi) H_j(3),$$

and χ is the nonprincipal Dirichlet character modulo 4.

Remark 1.2. When $j = 2$, Theorem 1.1 gives the error term $O(x^{39/14+\varepsilon})$, which improves the error term in (1.3).

2. Preliminaries and some important lemmas

Let $r_k(n) := \#\{(n_1, n_2, \ldots, n_k) \in \mathbb{Z}^k \colon n_1^2 + n_2^2 + \ldots + n_k^2 = n\}$ allowing zeros, distinguishing signs, and order. We will be concerned with the function $r_6(n)$.

Lemma 2.1. *For any positive integer n we have*

$$r_6(n) = 16 \sum_{d \mid n} \chi(d') d^2 - 4 \sum_{d \mid n} \chi(d) d^2, \tag{2.1}$$

where $dd' = n$, and χ is the nonprincipal Dirichlet character modulo 4, i.e.,

$$\chi(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 4, \\ -1 & \text{if } n \equiv -1 \pmod 4, \\ 0 & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

Proof. See, for instance, Lemma 1 of [17]. □

We can reframe equation (2.1) as

$$r_6(n) = 16 \sum_{d \mid n} \chi(d) \frac{n^2}{d^2} - 4 \sum_{d \mid n} \chi(d) d^2 =: 16 l(n) - 4 v(n).$$

We write $l_1(n) = 16 l(n)$, and $v_1(n) = 4 v(n)$.

The functions $\chi(d)$ and n^2/d^2 are completely multiplicative functions. This implies that $\chi(d) n^2/d^2$ is multiplicative. If $g(d)$ is any multiplicative function, then $\sum_{d \mid n} g(d)$ is also multiplicative. Therefore, $l(n)$ is a multiplicative function. Similarly, $v(n)$ is also multiplicative. Note that

$$l(p) = p^2 + \chi(p), \quad l(p^2) = p^4 + p^2 \chi(p) + \chi(p^2),$$

and

$$v(p) = 1 + p^2 \chi(p), \quad v(p^2) = 1 + p^2 \chi(p) + p^4 \chi(p^2).$$

Also, we can write

$$
\begin{aligned}
(2.2)\qquad S_j^* &= \sum_{\substack{a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2\leqslant x\\ (a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}} \lambda^2_{\mathrm{sym}^j f}(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2)\\
&= \sum_{n\leqslant x}\lambda^2_{\mathrm{sym}^j f}(n) \sum_{\substack{n=a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2\\ (a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}} 1 = \sum_{n\leqslant x}\lambda^2_{\mathrm{sym}^j f}(n) r_6(n)\\
&= \sum_{n\leqslant x}\lambda^2_{\mathrm{sym}^j f}(n)(l_1(n)-v_1(n))\\
&= 16\sum_{n\leqslant x}\lambda^2_{\mathrm{sym}^j f}(n) l(n) - 4\sum_{n\leqslant x}\lambda^2_{\mathrm{sym}^j f}(n) v(n),
\end{aligned}
$$

where $l(n)=\sum_{d|n}\chi(d)n^2/d^2$, and $v(n)=\sum_{d|n}\chi(d)d^2$.

For $j\geqslant 2$ (an integer), and $\Re(s)>1$, the jth symmetric power L-function of f is defined as

$$
(2.3)\qquad L(s,\mathrm{sym}^j f) := \sum_{n=1}^{\infty}\frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s} = \prod_p\prod_{i=0}^{j}\Big(1-\frac{\alpha^{j-i}(p)\beta^i(p)}{p^s}\Big)^{-1}.
$$

Observe that

$$
(2.4)\qquad \lambda_{\mathrm{sym}^j f}(p) = \sum_{m=0}^{j}\alpha^{j-m}(p)\beta^m(p).
$$

Since $\lambda_{\mathrm{sym}^j f}(n)$ is a multiplicative function, and $|\lambda_{\mathrm{sym}^j f}(n)|\leqslant d_{j+1}(n)$ (from (1.1) and (1.2)), where $d_{j+1}(n)$ is the number of ways of expressing n as a product of $j+1$ factors), we can write the Euler product of $L(s,\mathrm{sym}^j f)$ as

$$
(2.5)\qquad \prod_p\Big(1+\frac{\lambda_{\mathrm{sym}^j f}(p)}{p^s}+\ldots+\frac{\lambda_{\mathrm{sym}^j f}(p^l)}{p^{ls}}+\ldots\Big).
$$

Comparing (2.3) and (2.5), we get (2.4).

Also, due to Hecke

$$
(2.6)\qquad \lambda_{\mathrm{sym}^j f}(p) = \lambda_f(p^j).
$$

Further, note that

$$
(2.7)\qquad \lambda_f^2(p^j) = 1+\sum_{l=1}^{j}\lambda_f(p^{2l}),
$$

since

$$\lambda_f^2(p^j) = \Big(\sum_{m=0}^{j} \alpha^{j-m}(p)\beta^m(p)\Big)^2 = \Big(\sum_{m=0}^{j} \alpha^{j-m}(p)\beta^m(p)\Big)\Big(\sum_{m'=0}^{j} \alpha^{j-m'}(p)\beta^{m'}(p)\Big)$$
$$= \sum_{m=0}^{j}\sum_{m'=0}^{j} (\alpha^{2j-(m+m')}(p))(\beta^{(m+m')}(p))$$

(We put $m+m'=t$. Observe that for every fixed integer t in the interval $[0,2l]$ and for every fixed integer m in the interval $[0,l]$, there is a unique integer m' in the interval $[0,l]$ satisfying $m+m'=t$ and thus,)

$$= \sum_{l=0}^{j}\Big(\sum_{t=0}^{2l} \alpha^{2j-t}(p)\beta^t(p)\Big) = 1 + \sum_{l=1}^{j}\Big(\sum_{t=0}^{2l} \alpha^{2j-t}(p)\beta^t(p)\Big) = 1 + \sum_{l=1}^{j} \lambda_f(p^{2l}).$$

Lemma 2.2. *Let f be a normalized primitive holomorphic cusp form of weight k for $SL(2,\mathbb{Z})$, and let $\lambda_{\mathrm{sym}^j f}(n)$ be the nth normalized Fourier coefficient of the jth symmetric power L-function associated to f. If*

$$F_j(s) = \sum_{n=1}^{\infty} \frac{\lambda^2_{\mathrm{sym}^j f}(n)l(n)}{n^s}$$

for $\Re(s) > 3$, then

$$F_j(s) = G_j(s)H_j(s),$$

where

$$G_j(s) := \zeta(s-2)L(s,\chi)\prod_{n=1}^{j} L(s-2,\mathrm{sym}^{2n} f)L(s,\mathrm{sym}^{2n} f\otimes\chi),$$

and χ is the nonprincipal character modulo 4. Here, $H_j(s)$ is a Dirichlet series which converges uniformly, and absolutely in the half plane $\Re(s) > \frac{5}{2}$, and $H_j(s) \neq 0$ on $\Re(s) = 3$.

P r o o f. We observe that $\lambda^2_{\mathrm{sym}^j f}(n)l(n)$ is multiplicative, and hence

$$F_j(s) = \prod_p \Big(1 + \frac{\lambda^2_{\mathrm{sym}^j f}(p)l(p)}{p^s} + \ldots + \frac{\lambda^2_{\mathrm{sym}^j f}(p^m)l(p^m)}{p^{ms}} + \ldots\Big). \tag{2.8}$$

Using (2.6) and (2.7), we note that

$$\begin{aligned}\lambda^2_{\mathrm{sym}^2 f}(p)l(p) &= \lambda_f^2(p^j)(p^2+\chi(p)) = \Big(1+\sum_{l=1}^{j}\lambda_f(p^{2l})\Big)(p^2+\chi(p)) \\ &= \Big(1+\sum_{l=1}^{j}\lambda_{\mathrm{sym}^{2l} f}(p)\Big)(p^2+\chi(p)) \\ &= p^2+\chi(p)+\sum_{l=1}^{j}\lambda_{\mathrm{sym}^{2l} f}(p)p^2+\sum_{l=1}^{j}\lambda_{\mathrm{sym}^{2l} f}(p)\chi(p) =: b(p).\end{aligned}$$

From the structure of $b(p)$, we define the coefficients $b(n)$ as

$$\sum_{n=1}^{\infty}\frac{b(n)}{n^s} = \zeta(s-2)L(s,\chi)\prod_{n=1}^{j}L(s-2,\mathrm{sym}^{2n} f)L(s,\mathrm{sym}^{2n} f\otimes\chi),$$

which is absolutely convergent in $\Re(s)>3$. We also note that

$$\begin{aligned}&\prod_p\Big(1+\frac{b(p)}{p^s}+\ldots+\frac{b(p^m)}{p^{ms}}+\ldots\Big) \\ &\quad = \zeta(s-2)L(s,\chi)\prod_{n=1}^{j}L(s-2,\mathrm{sym}^{2n} f)L(s,\mathrm{sym}^{2n} f\otimes\chi) =: G_j(s)\end{aligned}$$

for $\Re(s)>3$. Observe that $b(n)\ll_\varepsilon n^{2+\varepsilon}$ for any small positive constant ε.

Now, we note that

$$\begin{aligned}&\left|\frac{b(p)}{p^s}+\frac{b(p^2)}{p^{2s}}+\ldots+\frac{b(p^m)}{p^{ms}}+\ldots\right| \\ &\qquad \ll \sum_{m=1}^{\infty}\frac{p^{(2+\varepsilon)m}}{p^{m\sigma}} \leqslant \sum_{m=1}^{\infty}\frac{p^{(2+\varepsilon)m}}{p^{(3+2\varepsilon)m}} \quad (\text{in } \Re(s)\geqslant 3+2\varepsilon) \\ &\qquad = \sum_{m=1}^{\infty}\frac{1}{p^{(1+\varepsilon)m}} = \frac{1/p^{1+\varepsilon}}{1-1/p^{1+\varepsilon}} = \frac{1}{p^{1+\varepsilon}-1} < 1.\end{aligned}$$

Let us write

$$A = \frac{\lambda^2_{\mathrm{sym}^j f}(p)l(p)}{p^s}+\ldots+\frac{\lambda^2_{\mathrm{sym}^j f}(p^m)l(p^m)}{p^{ms}}+\ldots$$

and

$$B = \frac{b(p)}{p^s}+\ldots+\frac{b(p^m)}{p^{ms}}+\ldots$$

From the above calculations we observe that $|B|<1$ in $\Re(s)\geqslant 3+2\varepsilon$.

Note that

$$\begin{aligned}\frac{1+A}{1+B} &= (1+A)(1-B+B^2-B^3+\ldots) \quad (\text{in } \Re(s) \geqslant 3+2\varepsilon)\\ &= 1+A-B-AB+\text{higher terms}\\ &= 1+\frac{\lambda^2_{\mathrm{sym}^j f}(p^2)l(p^2)-b(p^2)}{p^{2s}}+\ldots+\frac{c_m(p^m)}{p^{ms}}+\ldots\end{aligned}$$

with $c_m(n) \ll_\varepsilon n^{2+\varepsilon}$. So,

$$\begin{aligned}\prod_p\Big(\frac{1+A}{1+B}\Big) &= \prod_p\Big(1+\frac{\lambda^2_{\mathrm{sym}^j f}(p^2)l(p^2)-b(p^2)}{p^{2s}}+\ldots+\frac{c_m(p^m)}{p^{ms}}+\ldots\Big)\\ &\ll_\varepsilon 1 \quad (\text{in } \Re(s) > \tfrac{5}{2}).\end{aligned}$$

Thus,

$$H_j(s) := \frac{F_j(s)}{G_j(s)} = \prod_p\Big(\frac{1+A}{1+B}\Big) \ll_\varepsilon 1 \quad (\text{in } \Re(s) > \tfrac{5}{2})$$

and also $H_j(s) \neq 0$ on $\Re(s) = 3$. □

Lemma 2.3. *Let f be a normalized primitive holomorphic cusp form of weight k for $SL(2,\mathbb{Z})$, and let $\lambda_{\mathrm{sym}^j f}(n)$ be the nth normalized Fourier coefficient of the jth symmetric power L-function associated to f. If*

$$\widetilde{F}_j(s) = \sum_{n=1}^{\infty}\frac{\lambda^2_{\mathrm{sym}^j f}(n)v(n)}{n^s}$$

for $\Re(s) > 3$, then

$$\widetilde{F}_j(s) = \widetilde{G}_j(s)\widetilde{H}_j(s),$$

where

$$\widetilde{G}_j(s) := \zeta(s)L(s-2,\chi)\prod_{n=1}^{j} L(s,\mathrm{sym}^{2n}f)L(s-2,\mathrm{sym}^{2n}f\otimes\chi),$$

and χ is the nonprincipal character modulo 4. Here $\widetilde{H}_j(s)$ is a Dirichlet series which converges uniformly, and absolutely in the half plane $\Re(s) > \frac{5}{2}$, and $\widetilde{H}_j(s) \neq 0$ on $\Re(s) = 3$.

P r o o f. We observe that $\lambda^2_{\mathrm{sym}^j f}(n)v(n)$ is multiplicative, and hence

$$\widetilde{F}_j(s) = \prod_p\Big(1+\frac{\lambda^2_{\mathrm{sym}^j f}(p)v(p)}{p^s}+\ldots+\frac{\lambda^2_{\mathrm{sym}^j f}(p^m)v(p^m)}{p^{ms}}+\ldots\Big). \tag{2.9}$$

Using (2.6) and (2.7), we note that

$$\begin{aligned}\lambda^2_{\mathrm{sym}^2 f}(p)v(p) &= \lambda_f^2(p^j)(1+p^2\chi(p)) = \Big(1+\sum_{l=1}^{j}\lambda_f(p^{2l})\Big)(1+p^2\chi(p))\\ &= \Big(1+\sum_{l=1}^{j}\lambda_{\mathrm{sym}^{2l} f}(p)\Big)(1+p^2\chi(p))\\ &= 1+p^2\chi(p)+\sum_{l=1}^{j}\lambda_{\mathrm{sym}^{2l} f}(p)+\sum_{l=1}^{j}\lambda_{\mathrm{sym}^{2l} f}(p)p^2\chi(p) =: h(p).\end{aligned}$$

From the structure of $h(p)$, we define the coefficients $h(n)$ as

$$\sum_{n=1}^{\infty}\frac{h(n)}{n^s} = \zeta(s)L(s-2,\chi)\prod_{n=1}^{j}L(s,\mathrm{sym}^{2n} f)L(s-2,\mathrm{sym}^{2n} f\otimes\chi),$$

which is absolutely convergent in $\Re(s)>3$. We also note that

$$\begin{aligned}&\prod_p\Big(1+\frac{h(p)}{p^s}+\ldots+\frac{h(p^m)}{p^{ms}}+\ldots\Big)\\ &\quad= \zeta(s)L(s-2,\chi)\prod_{n=1}^{j}L(s,\mathrm{sym}^{2n} f)L(s-2,\mathrm{sym}^{2n} f\otimes\chi) =: \widetilde{G}_j(s)\end{aligned}$$

for $\Re(s)>3$. Observe that $h(n)\ll_\varepsilon n^{2+\varepsilon}$ for any small positive constant ε.

Now, we note that

$$\Big|\frac{h(p)}{p^s}+\frac{h(p^2)}{p^{2s}}+\ldots+\frac{h(p^m)}{p^{ms}}+\ldots\Big| \ll \sum_{m=1}^{\infty}\frac{p^{(2+\varepsilon)m}}{p^{m\sigma}} < 1 \quad (\text{in } \Re(s)\geqslant 3+2\varepsilon).$$

Let us write

$$\widetilde{A} = \frac{\lambda^2_{\mathrm{sym}^j f}(p)v(p)}{p^s}+\ldots+\frac{\lambda^2_{\mathrm{sym}^j f}(p^m)v(p^m)}{p^{ms}}+\ldots$$

and

$$\widetilde{B} = \frac{h(p)}{p^s}+\ldots+\frac{h(p^m)}{p^{ms}}+\ldots.$$

From the above calculations, we observe that $|\widetilde{B}|<1$ in $\Re(s)\geqslant 3+2\varepsilon$.

Note that

$$\begin{aligned}\frac{1+\widetilde{A}}{1+\widetilde{B}} &= (1+\widetilde{A})(1-\widetilde{B}+\widetilde{B}^2-\widetilde{B}^3+\ldots) \quad (\text{in } \Re(s)\geqslant 3+2\varepsilon)\\ &= 1+\widetilde{A}-\widetilde{B}-\widetilde{A}\widetilde{B}+\text{higher terms}\\ &= 1+\frac{\lambda^2_{\mathrm{sym}^j f}(p^2)v(p^2)-h(p^2)}{p^{2s}}+\ldots+\frac{\widetilde{c}_m(p^m)}{p^{ms}}+\ldots\end{aligned}$$

with $\widetilde{c}_m(n) \ll_\varepsilon n^{2+\varepsilon}$. So,

$$\prod_p \Big(\frac{1+\widetilde{A}}{1+\widetilde{B}}\Big) = \prod_p \Big(1 + \frac{\lambda^2_{\mathrm{sym}^j f}(p^2)v(p^2) - h(p^2)}{p^{2s}} + \ldots + \frac{\widetilde{c}_m(p^m)}{p^{ms}} + \ldots\Big)$$
$$\ll_\varepsilon 1 \quad (\text{in } \Re(s) > \tfrac{5}{2}).$$

Thus,

$$\widetilde{H}_j(s) := \frac{\widetilde{F}_j(s)}{\widetilde{G}_j(s)} = \prod_p \Big(\frac{1+\widetilde{A}}{1+\widetilde{B}}\Big) \ll_\varepsilon 1 \quad (\text{in } \Re(s) > \tfrac{5}{2})$$

and also $\widetilde{H}_j(s) \neq 0$ on $\Re(s) = 3$. □

Lemma 2.4. *For any $\varepsilon > 0$ we have*

$$\int_1^T \Big|L\Big(\frac{1}{2} + \mathrm{i}t, f\Big)\Big|^6 \,\mathrm{d}t \ll T^{2+\varepsilon} \tag{2.10}$$

uniformly for $T \geqslant 1$, and

$$L(\sigma + \mathrm{i}t) \ll_\varepsilon (1 + |t|)^{(1+\varepsilon-\sigma)/3+\varepsilon} \tag{2.11}$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 1 + \varepsilon$, and $|t| \geqslant t_0$, where t_0 is sufficiently large.

P r o o f. Proof of (2.10) is given by Jutila, for instance, see [10], and using maximum-modulus principle in a suitable rectangle, we get (2.11), for instance, see [4]. □

Lemma 2.5. *For any $\varepsilon > 0$ we have*

$$\int_1^T \Big|\zeta\Big(\frac{1}{2} + \mathrm{i}t\Big)\Big|^{12} \,\mathrm{d}t \ll T^{2+\varepsilon} \tag{2.12}$$

uniformly for $T \geqslant 1$, and

$$\zeta(\sigma + \mathrm{i}t) \ll_\varepsilon (1 + |t|)^{\max\{13(1-\sigma)/42, 0\}+\varepsilon} \tag{2.13}$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 1 + \varepsilon$, and $|t| \geqslant 1$.

P r o o f. For the proof of (2.12), see [7] and (2.13) is due to Bourgain, for instance, see [1]. □

Lemma 2.6. *Let χ be a primitive character modulo q and $\mathfrak{L}^d_{m,n}(s,\chi)$ be a general L-function of degree $2A$. For any $\varepsilon > 0$ we have*

$$\int_T^{2T} |\mathfrak{L}^d_{m,n}(\sigma+\mathrm{i}t,\chi)|^2 \,\mathrm{d}t \ll (qT)^{2A(1-\sigma)+\varepsilon} \tag{2.14}$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 1+\varepsilon$, and $T \geqslant 1$. Also,

$$\mathfrak{L}^d_{m,n}(\sigma+\mathrm{i}t,\chi) \ll (q(1+|t|))^{\max\{A(1-\sigma),0\}+\varepsilon} \tag{2.15}$$

uniformly for $-\varepsilon \leqslant \sigma \leqslant 1+\varepsilon$.

Proof. For the proof of (2.14) and (2.15), see [9]. $\square$

3. Proof of Theorem 1.1

From (2.2), we can write

$$\sum_{n\leqslant x} \lambda^2_{\mathrm{sym}^j f}(n) r_6(n) = \sum_{n\leqslant x} \lambda^2_{\mathrm{sym}^j f}(n) l_1(n) - \sum_{n\leqslant x} \lambda^2_{\mathrm{sym}^j f}(n) v_1(n).$$

Firstly, we consider the sum $\sum\limits_{n\leqslant x} \lambda^2_{\mathrm{sym}^j f}(n) l_1(n)$. We begin by applying the Perron's formula (see Chapter 2.4 of [5]) to $F_j(s)$ with $\eta = 3+\varepsilon$ and $10 \leqslant T \leqslant x$. Thus, we have

$$\sum_{n\leqslant x} \lambda^2_{\mathrm{sym}^j f}(n) l_1(n) = 16 \sum_{n\leqslant x} \lambda^2_{\mathrm{sym}^j f}(n) l(n) = \frac{16}{2\pi\mathrm{i}} \int_{\eta-\mathrm{i}T}^{\eta+\mathrm{i}T} F_j(s) \frac{x^s}{s} \,\mathrm{d}s + O\Big(\frac{x^{3+3\varepsilon}}{T}\Big).$$

We move the line of integration to $\Re(s) = \frac{5}{2}+\varepsilon$, and by Cauchy's residue theorem there is only one simple pole at $s=3$ coming from the factor $\zeta(s-2)$. This contributes a residue, which is $c(j)x^3$, where $c(j)$ is an effective constant depending on the values of various L-functions appearing in $G_j(s)$ at $s=3$.

More precisely,

$$c(j) = 16 \lim_{s\to 3} (s-3) \frac{F_j(s)}{s} = \frac{16}{3} L(3,\chi) \prod_{n=1}^{j} L(1, \mathrm{sym}^{2n} f) L(3, \mathrm{sym}^{2n} f \otimes \chi) H_j(3).$$

So, we obtain

$$\begin{aligned} &\sum_{n\leqslant x} \lambda^2_{\mathrm{sym}^j f}(n) l_1(n) \\ &\quad = c(j)x^3 + \frac{16}{2\pi\mathrm{i}} \Big\{ \int_{5/2+\varepsilon-\mathrm{i}T}^{5/2+\varepsilon+\mathrm{i}T} + \int_{3+\varepsilon-\mathrm{i}T}^{5/2+\varepsilon-\mathrm{i}T} + \int_{5/2+\varepsilon+\mathrm{i}T}^{3+\varepsilon+\mathrm{i}T} \Big\} F_j(s) \frac{x^s}{s} \,\mathrm{d}s + O\Big(\frac{x^{3+3\varepsilon}}{T}\Big) \\ &\quad =: c(j)x^3 + \frac{16}{2\pi\mathrm{i}} (J_1 + J_2 + J_3) + O\Big(\frac{x^{3+3\varepsilon}}{T}\Big). \end{aligned}$$

The contribution of horizontal line integrals (J_2 and J_3) in absolute value (using Lemmas 2.2, 2.5 and 2.6) is

$$\begin{aligned}
&\ll \int_{5/2+\varepsilon}^{3+\varepsilon} \frac{\left|\zeta(\sigma-2+\mathrm{i}T)\prod_{n=1}^{j} L(\sigma-2+\mathrm{i}T,\mathrm{sym}^{2n} f)\right|}{T} x^{\sigma}\,\mathrm{d}\sigma \\
&\ll \int_{1/2+\varepsilon}^{1+\varepsilon} \frac{\left|\zeta(\sigma+\mathrm{i}T)\prod_{n=1}^{j} L(\sigma+\mathrm{i}T,\mathrm{sym}^{2n} f)\right|}{T} x^{\sigma+2}\,\mathrm{d}\sigma \\
&\ll \left(\frac{x^2}{T}\right) \max_{1/2+\varepsilon\leqslant\sigma\leqslant 1+\varepsilon} x^{\sigma} T^{((j+1)^2/2-4/21)(1-\sigma)+\varepsilon} \\
&\ll \left(\frac{x^{2+2\varepsilon}}{T}\right) \max_{1/2+\varepsilon\leqslant\sigma\leqslant 1+\varepsilon} \left(\frac{x}{T^{((j+1)^2/2-4/21)}}\right)^{\sigma} T^{((j+1)^2/2-4/21)}.
\end{aligned}$$

Clearly, $(x/T^{((j+1)^2/2-4/21)})^{\sigma}$ is monotonic as a function of σ for $\frac{1}{2}+\varepsilon \leqslant \sigma \leqslant 1+\varepsilon$, and hence, the maximum is attained at the extremities of the interval $[\frac{1}{2}+\varepsilon, 1+\varepsilon]$. Thus,

$$\begin{aligned}
J_2+J_3 &\ll x^{2+2\varepsilon}\left(x^{1/2+\varepsilon} T^{((j+1)^2/4-2/21-1)} + \frac{x^{1+\varepsilon}}{T}\right) \\
&\ll x^{5/2+3\varepsilon} T^{((j+1)^2/4-23/21+\varepsilon)} + \frac{x^{3+3\varepsilon}}{T}.
\end{aligned}$$

The contribution of the left vertical line integral (J_1) in absolute value (using Lemmas 2.2, 2.5, 2.6 and Hölder's inequality) is

$$\begin{aligned}
&\ll \int_{5/2+\varepsilon-\mathrm{i}T}^{5/2+\varepsilon+\mathrm{i}T} \frac{\left|\zeta(\frac{1}{2}+\varepsilon+\mathrm{i}t)\prod_{n=1}^{j} L(\frac{1}{2}+\varepsilon+\mathrm{i}t,\mathrm{sym}^{2n} f)\right|}{|\frac{5}{2}+\varepsilon+\mathrm{i}t|} x^{5/2+\varepsilon}\,\mathrm{d}t \\
&\ll x^{5/2+\varepsilon} + x^{5/2+\varepsilon}\frac{1}{T}\left\{\int_{10\leqslant|t|\leqslant T}\left|\zeta\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t\Big)\right|^{12}\mathrm{d}t\right\}^{1/12} \\
&\quad\times\left\{\int_{10\leqslant|t|\leqslant T}\left|L\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t,\mathrm{sym}^2 f\Big)\right|^{2}\mathrm{d}t\right\}^{1/2} \\
&\quad\times\left\{\max_{10\leqslant|t|\leqslant T}\left|\prod_{n=2}^{j} L\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t,\mathrm{sym}^{2n} f\Big)\right|^{2/5}\right. \\
&\quad\times\left.\left(\int_{10\leqslant|t|\leqslant T}\left|\prod_{n=2}^{j} L\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t,\mathrm{sym}^{2n} f\Big)\right|^{2}\mathrm{d}t\right)\right\}^{5/12} \\
&\ll x^{5/2+\varepsilon} + x^{5/2+\varepsilon}\big(T^{-1+2/12+3/4+((j+1)^2-4)10/240+((j+1)^2-4)5/24}\big) \\
&\ll x^{5/2+\varepsilon} T^{((j+1)^2/4-13/12+\varepsilon)}.
\end{aligned}$$

Note that $10 \leqslant T \leqslant x$. Thus, we obtain

$$\sum_{n\leqslant x} \lambda^2_{\mathrm{sym}^j f}(n) l_1(n) = c(j)x^3 + O(x^{5/2+\varepsilon} T^{((j+1)^2/4-13/12+\varepsilon)}) + O\Big(\frac{x^{3+3\varepsilon}}{T}\Big).$$

We choose T such that $x^{5/2} T^{((j+1)^2/4-13/12)} \asymp x^3/T$ i.e., $T^{(3(j+1)^2-1/12)} \asymp x^{1/2}$. Therefore, $T \asymp x^{6/(3(j+1)^2-1)}$. Thus, we get

$$\sum_{n\leqslant x} \lambda^2_{\mathrm{sym}^j f}(n) l_1(n) = c(j)x^3 + O(x^{3-6/(3(j+1)^2-1)+3\varepsilon}). \tag{3.1}$$

Similarly, we apply Perron's formula (see chapter 2.4 of [5]) to $\widetilde{F}_j(s)$ with $\eta = 3+\varepsilon$ and $10 \leqslant T \leqslant x$. Thus, we have

$$\sum_{n\leqslant x} \lambda^2_{\mathrm{sym}^j f}(n) v_1(n) = 4\sum_{n\leqslant x} \lambda^2_{\mathrm{sym}^j f}(n) v(n) = \frac{4}{2\pi \mathrm{i}} \int_{\eta-\mathrm{i}T}^{\eta+\mathrm{i}T} \widetilde{F}_j(s) \frac{x^s}{s}\, \mathrm{d}s + O\Big(\frac{x^{3+3\varepsilon}}{T}\Big).$$

We move the line of integration to $\Re(s) = \frac{5}{2}+\varepsilon$. Note that there is no singularity in the rectangle obtained, and the function $\widetilde{F}_j(s)x^s/s$ is analytic in this region. Thus, using Cauchy's theorem for rectangle pertaining to analytic functions, we get

$$\begin{aligned}\sum_{n\leqslant x} \lambda^2_{\mathrm{sym}^j f}(n) v_1(n) &= \frac{4}{2\pi \mathrm{i}} \Big\{ \int_{5/2+\varepsilon-\mathrm{i}T}^{5/2+\varepsilon+\mathrm{i}T} + \int_{3+\varepsilon-\mathrm{i}T}^{5/2+\varepsilon-\mathrm{i}T} + \int_{5/2+\varepsilon+\mathrm{i}T}^{3+\varepsilon+\mathrm{i}T} \Big\} \widetilde{F}_j(s) \frac{x^s}{s}\, \mathrm{d}s \\ &\quad + O\Big(\frac{x^{3+3\varepsilon}}{T}\Big) \\ &=: \frac{4}{2\pi \mathrm{i}} (J_1' + J_2' + J_3') + O\Big(\frac{x^{3+3\varepsilon}}{T}\Big).\end{aligned}$$

The contribution of horizontal line integrals (J_2' and J_3') in absolute value (using Lemmas 2.3, 2.4 and 2.6) is

$$\begin{aligned} &\ll \int_{5/2+\varepsilon}^{3+\varepsilon} \frac{\big| L(\sigma-2+\mathrm{i}T) \prod_{n=1}^{j} L(\sigma-2+\mathrm{i}T, \mathrm{sym}^{2n} f \otimes \chi)\big|}{T} x^\sigma\, \mathrm{d}\sigma \\ &\ll \int_{1/2+\varepsilon}^{1+\varepsilon} \frac{\big| L(\sigma+\mathrm{i}T) \prod_{n=1}^{j} L(\sigma+\mathrm{i}T, \mathrm{sym}^{2n} f \otimes \chi)\big|}{T} x^{\sigma+2}\, \mathrm{d}\sigma \\ &\ll \Big(\frac{x^2}{T}\Big) \max_{1/2+\varepsilon\leqslant\sigma\leqslant 1+\varepsilon} x^\sigma T^{((j+1)^2/2-1/6)(1-\sigma)+\varepsilon} \\ &\ll \Big(\frac{x^{2+2\varepsilon}}{T}\Big) \max_{1/2+\varepsilon\leqslant\sigma\leqslant 1+\varepsilon} \Big(\frac{x}{T^{((j+1)^2/2-1/6)}}\Big)^\sigma T^{((j+1)^2/2-1/6)}.\end{aligned}$$

Clearly, $(x/T^{((j+1)^2/2-1/6)})^\sigma$ is monotonic as a function of σ for $\frac{1}{2}+\varepsilon \leqslant \sigma \leqslant 1+\varepsilon$, and hence, the maximum is attained at the extremities of the interval $[\frac{1}{2}+\varepsilon, 1+\varepsilon]$. Thus,

$$\begin{aligned} J_2' + J_3' &\ll x^{2+2\varepsilon}\Big(x^{1/2+\varepsilon}T^{((j+1)^2/4-1/12-1+\varepsilon)} + \frac{x^{1+\varepsilon}}{T}\Big) \\ &\ll x^{5/2+3\varepsilon}T^{((j+1)^2/4-13/12+\varepsilon)} + \frac{x^{3+3\varepsilon}}{T}. \end{aligned}$$

The contribution of the left vertical line integral (J_1') in absolute value (using Lemmas 2.3, 2.4, 2.6 and Hölder's inequality) is

$$\begin{aligned} &\ll \int_{5/2+\varepsilon-\mathrm{i}T}^{5/2+\varepsilon+\mathrm{i}T} \frac{\Big|L(\frac{1}{2}+\varepsilon+\mathrm{i}t)\prod_{n=1}^{j} L(\frac{1}{2}+\varepsilon+\mathrm{i}t, \mathrm{sym}^{2n} f\otimes\chi)\Big|}{|\frac{5}{2}+\varepsilon+\mathrm{i}t|} x^{5/2+\varepsilon}\,\mathrm{d}t \\ &\ll x^{5/2+\varepsilon} + x^{5/2+\varepsilon}\frac{1}{T}\Big\{\int_{10\leqslant|t|\leqslant T}\Big|L\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t\Big)\Big|^6\,\mathrm{d}t\Big\}^{1/6} \\ &\quad\times\Big\{\int_{10\leqslant|t|\leqslant T}\Big|L\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t, \mathrm{sym}^2 f\otimes\chi\Big)\Big|^2\,\mathrm{d}t\Big\}^{1/2} \\ &\quad\times\Big\{\max_{10\leqslant|t|\leqslant T}\Big|\prod_{n=2}^{j} L\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t, \mathrm{sym}^{2n} f\otimes\chi\Big)\Big| \\ &\quad\times\Big(\int_{10\leqslant|t|\leqslant T}\Big|\prod_{n=2}^{j} L(\frac{1}{2}+\varepsilon+\mathrm{i}t, \mathrm{sym}^{2n} f\otimes\chi)\Big|^2\,\mathrm{d}t\Big)\Big\}^{1/3} \\ &\ll x^{5/2+\varepsilon} + x^{5/2+\varepsilon}(T^{-1+2/6+3/4+((j+1)^2-4)/12+((j+1)^2-4)/6}) \\ &\ll x^{5/2+\varepsilon}T^{((j+1)^2/4-11/12)}. \end{aligned}$$

Note that $10 \leqslant T \leqslant x$. Thus, we obtain

$$\sum_{n\leqslant x} \lambda^2_{\mathrm{sym}^j f}(n)v_1(n) = O(x^{5/2+\varepsilon}T^{((j+1)^2/4-11/12)}) + O\Big(\frac{x^{3+3\varepsilon}}{T}\Big).$$

We choose T such that $x^{5/2}T^{((j+1)^2/4-11/12)} \asymp x^3/T$, i.e., $T^{((j+1)^2/4+1/12)} \asymp x^{1/2}$. Therefore, $T \asymp x^{6/(3(j+1)^2+1)}$. Thus, we get

$$\sum_{n\leqslant x} \lambda^2_{\mathrm{sym}^j f}(n)v_1(n) = O(x^{3-6/(3(j+1)^2+1)+\varepsilon}). \tag{3.2}$$

Combining (3.1) and (3.2), we get

$$\sum_{n\leqslant x} \lambda^2_{\mathrm{sym}^j f}(n)r_6(n) = c(j)x^3 + O(x^{3-6/(3(j+1)^2+1)+\varepsilon}),$$

where $c(j)$ is an effective constant given by

$$c(j) = \frac{16}{3} L(3,\chi) \prod_{n=1}^{j} L(1, \mathrm{sym}^{2n} f) L(3, \mathrm{sym}^{2n} f \otimes \chi) H_j(3),$$

and χ is the nonprincipal Dirichlet character modulo 4.

This proves the theorem. $\square$

Concluding remarks. Note that we have the expected upper bounds, namely,

$$\int_T^{2T} \left|\zeta\left(\frac{5}{7} + \mathrm{i}t\right)\right|^{12} \mathrm{d}t \ll T^{1+\varepsilon} \quad \text{and} \quad \int_T^{2T} \left|L\left(\frac{5}{8} + \mathrm{i}t, f\right)\right|^{4} \mathrm{d}t \ll T^{1+\varepsilon}$$

uniformly for $T \geqslant 1$ (see [4], [8]). Even if we move the line of integration to $\Re(s) = \frac{5}{7}$ and $\Re(s) = \frac{5}{8}$ pertaining to $l_1(n)$ and $v_1(n)$ respectively, and using the arguments of this paper, we end up with the same error term as stated in Theorem 1.1.

Acknowledgements. The authors are also grateful to the anonymous referee for some fruitful comments.

References

[1] *J. Bourgain*: Decoupling, exponential sums and the Riemann zeta function. J. Am. Math. Soc. *30* (2017), 205–224. zbl MR doi
[2] *P. Deligne*: La conjecture de Weil. I. Publ. Math., Inst. Hautes Étud. Sci. *43* (1974), 273–307. (In French.) zbl MR doi
[3] *O. M. Fomenko*: Identities involving coefficients of automorphic L-functions. J. Math. Sci., New York *133* (2006), 1749–1755. zbl MR doi
[4] *A. Good*: The square mean of Dirichlet series associated with cusp forms. Mathematika *29* (1982), 278–295. zbl MR doi
[5] *A. Granville, K. Soundararajan*: Multiplicative number theory: The pretentious approach. Available at `https://dms.umontreal.ca/~andrew/PDF/BookChaps1n2.pdf`.
[6] *X. He*: Integral power sums of Fourier coefficients of symmetric square L-functions. Proc. Am. Math. Soc. *147* (2019), 2847–2856. zbl MR doi
[7] *D. R. Heath-Brown*: The twelfth power moment of the Riemann-function. Q. J. Math., Oxf. II. Ser. *29* (1978), 443–462. zbl MR doi
[8] *A. Ivić*: Exponent pairs and the zeta-function of Riemann. Stud. Sci. Math. Hung. *15* (1980), 157–181. zbl MR
[9] *Y. Jiang, G. Lü*: On the higher mean over arithmetic progressions of Fourier coefficients of cusp forms. Acta Arith. *3* (2014), 231–252. zbl MR doi
[10] *M. Jutila*: Lectures on a Method in the Theory of Exponential Sums. Lectures on Mathematics and Physics. Mathematics. Tata Institute of Fundamental Research 80. Springer, Berlin, 1987. zbl MR
[11] *E. Krätzel*: Analytische Funktionen in der Zahlentheorie. Teubner-Texte zur Mathematik 139. B. G. Teubner, Stuttgart, 2000. zbl MR doi
[12] *H. Lao*: On the fourth moment of coefficients of symmetric square L-function. Chin. Ann. Math., Ser. B *33* (2012), 877–888. zbl MR doi

[13] *Y.-K. Lau, G. Lü*: Sums of Fourier coefficients of cusp forms. Q. J. Math. *62* (2011), 687–716. zbl MR doi
[14] *W. K. A. Loh*: Limitation to the asymptotic formula in Waring's problem. Acta Arith. *74* (1996), 1–15. zbl MR doi
[15] *J. Newton, J. A. Thorne*: Symmetric power functoriality for holomorphic modular forms. Publ. Math., Inst. Hautes Étud. Sci. *134* (2021), 1–116. zbl MR doi
[16] *J. Newton, J. A. Thorne*: Symmetric power functoriality for holomorphic modular forms. II. Publ. Math., Inst. Hautes Étud. Sci. *134* (2021), 117–152. zbl MR doi
[17] *A. Sharma, A. Sankaranarayanan*: Average behavior of the Fourier coefficients of symmetric square L-function over some sequence of integers. Integers *22* (2022), Article ID A74, 17 pages. zbl MR
[18] *A. Sharma, A. Sankaranarayanan*: Discrete mean square of the coefficients of symmetric square L-functions on certain sequence of positive numbers. Res. Number Theory *8* (2022), Article ID 19, 13 pages. zbl MR doi
[19] *A. Sharma, A. Sankaranarayanan*: Higher moments of the Fourier coefficients of symmetric square L-functions on certain sequence. Rend. Circ. Mat. Palermo (2) (2023), 1399–1416. zbl MR doi
[20] *H. Tang*: Estimates for the Fourier coefficients of symmetric square L-functions. Arch. Math. *100* (2013), 123–130. zbl MR doi
[21] *R. C. Vaughan*: The Hardy-Littlewood Method. Cambridge Tracts in Mathematics 125. Cambridge University Press, Cambridge, 1997. zbl MR doi
[22] *S. Zhai*: Average behavior of Fourier coefficients of cusp forms over sum of two squares. J. Number Theory *133* (2013), 3862–3876. zbl MR doi

Authors' addresses: Anubhav Sharma (corresponding author), Ayyadurai Sankaranarayanan, School of Mathematics and Statistics, University of Hyderabad Central University, P.O., Prof. C. R. Rao Road, Gachibowli, Hyderabad-500046, India, e-mail: `19mmpp02@uohyd.ac.in`, `sank@uohyd.ac.in`.