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EQUATIONS FOR THE SET OF OVERRINGS OF NORMAL RINGS AND RELATED RING EXTENSIONS

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Abstract. We establish several finiteness characterizations and equations for the cardinality and the length of the set of overrings of rings with nontrivial zero divisors and integrally closed in their total ring of fractions. Similar properties are also obtained for related extensions of commutative rings that are not necessarily integral domains. Numerical characterizations are obtained for rings with some finiteness conditions afterwards.

Keywords: total ring of fractions; ring extension; intermediate ring; overring; finite direct product; FIP extension; FCP extension; integrally closed; integral domain; Prüfer domain; valuation domain; normal pair; normal ring; length of ring extension; number of intermediate ring; number of overring

MSC 2020: 13B02, 13B22, 13E15, 13E99, 13F05, 13G05, 13B30

1. Introduction

Several equations for the number and the length of chains of intermediate rings in extensions of integral domains have been recently established. Such results are still in need when we consider extensions of more general rings. We study in this work similar problems for the set of intermediate rings in some ring extensions with nontrivial zero divisors, and the set of overrings of normal rings, see Definition 1. We generalize results about the cardinality and the length of the set of overrings of integrally closed domains to normal rings, when appropriate finiteness conditions are satisfied.

Let R be a commutative ring with unity. A ring T containing R and contained in the total ring of fractions of R is called an *overring* of R. We recall that a ring R

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with a finite set of overrings is called an FO ring. The ring R is called FC ring if each chain of distinct overrings is finite, see [18]. Ring extensions $R \subseteq S$ with only finitely many intermediate rings have been named FIP extensions; and extensions with finite chains of intermediate rings have been named FCP extensions, see [1]. These extensions have been recently studied by several authors, see [3], [5], [10], [11], [13], [14], [24]. Approximations for the cardinality and the length of the set of intermediate rings have been already obtained in [2] for normal pairs and more recently in [29]. Normal pairs are extensions of integral domains introduced in [12], where each intermediate ring in the ring extension $R \subseteq S$ is integrally closed in S. Normal pairs with zero-divisors are investigated in [9] and [15].

The number of intermediate rings has been calculated for integrally closed domains and normal pairs; and quite recently for more general extensions of integral domains, see [6] and [8]. An algorithm to compute such number has been established in [23] for integrally closed domains. Also the list of such intermediate rings has been obtained in [25]. More results about the number of intermediate rings have been also established for many other classes of integral domains, see [16], [20], [26], [27], [28] and [30]. This remains an open problem for many other classes of rings.

This work starts in the next second section with the investigation of the structure of normal rings with only finitely many overrings showing that such rings are finite direct products of normal domains satisfying several finiteness conditions such as FO and FC , see Theorem 3. Then, in the third section, several equations concerning the cardinality of the set of intermediate rings are established extending several recently obtained results for integral domains to the more general setting of commutative rings with nontrivial zerodivizors, see Theorem 8 and Corollary 9. These results show that the number of overrings of a normal ring R depends on the ordering of the prime ideals and on the minimal prime ideals of R. The last results of this section deal with the number and nature of components in the decomposition of R as a product of normal domains, see Corollaries 11 and 12. Several examples are provided to present the extent of the obtained results, see Examples 10, 13 and 15. Section 4 deals with the length of the set of overrings, see Propositions 20 and 21. It is shown in particular that the length depends on the number of all primes and minimal primes as in Propositions 20 and 21. Section 5 is reserved for numerical characterizations involving the number of overrings and the length of some normal domains, see Corollaries 23 and 24.

All rings considered in this work are assumed to be commutative and to contain an identity element. Let R be a ring and D the subset of elements which are not zero divisors in R, then the total ring of fractions $D^{-1}R$ is usually denoted $\mathrm{Frac}(R)$ or $T(R)$. This is equal to the field of fractions when R is an integral domain. If T is a subring of S, we always assume that it has the same identity element of S. The set of intermediate rings T, $R \subseteq T \subseteq S$, is usually denoted $[R, S]$. $\mathrm{Spec}(R)$ denotes as usual the set of prime ideals of R, and $\mathrm{Max}(R)$ the set of maximal ideals of R.

The ideal consisting of the zero element of a ring R will be denoted by 0. We recall that an ideal I of $R_1 \times R_2$, the direct product of the commutative rings R_1 and R_2 with identity, is prime if and only if it has the form $I_1 \times R_2$ or $R_1 \times I_2$, where I_i is prime in R_i. Any other notation is standard as in [17].

2. Finiteness conditions for normal rings

An integral domain D is called normal (or integrally closed) if D is integrally closed in its quotient field. According to Grothendick in [19] and Matsumura in [31], page 64, this is extended to more general rings as follows.

Definition 1. A ring R is called *normal* if for every prime $P \subset R$, the localization R_P is a normal domain.

We start with an example of normal rings.

Example 2. Let $D = \mathbb{Z}_{2\mathbb{Z}}$, and define E with the following pullback construction of commutative rings:

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & \mathbb{Z}_{3\mathbb{Z}} \cap \mathbb{Z}_{5\mathbb{Z}} \arrow[d] \\
\mathbb{Q}[x]_{(x)} \arrow[r] & \mathbb{Q}[x]_{(x)}/x\mathbb{Q}[x]_{(x)} \simeq \mathbb{Q}.
\end{tikzcd}$$

The integral domain D is a valuation domain of dimension 1 with $\mathrm{Spec}(D) = \{0, L = 2\mathbb{Z}_{2\mathbb{Z}}\}$. The domain $\mathbb{Z}_{3\mathbb{Z}} \cap \mathbb{Z}_{5\mathbb{Z}}$ is Prüfer as it is an overring of $\mathbb{Z}$. Then $E = \mathbb{Z}_{3\mathbb{Z}} \cap \mathbb{Z}_{5\mathbb{Z}} + x\mathbb{Q}[x]_{(x)}$ is also a Prüfer domain with a Y-graph as spectrum by Theorem 2.1 of [4]. We also have $\mathrm{Spec}(E) = \{0, M, P, N\}$ such that $0 \subset M = x\mathbb{Q}[x]_{(x)} \subset P = x\mathbb{Q}[x]_{(x)} + 3(\mathbb{Z}_{3\mathbb{Z}} \cap \mathbb{Z}_{5\mathbb{Z}})$, and $0 \subset M \subset N = x\mathbb{Q}[x]_{(x)} + 5(\mathbb{Z}_{3\mathbb{Z}} \cap \mathbb{Z}_{5\mathbb{Z}})$. The spectrum of $R = D \times E$ is given by $\mathrm{Spec}(D \times E) = \{0 \times E, L \times E, D \times 0, D \times M, D \times P, D \times N\}$ and is ordered as in Figure 1.

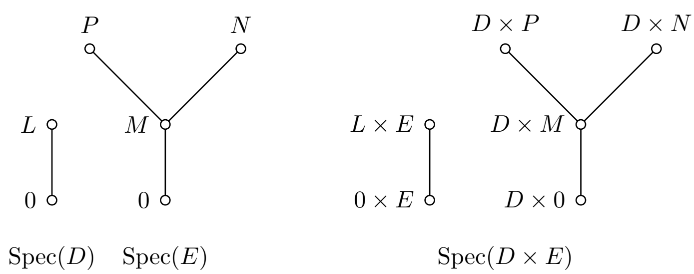


Figure 1.

The localizations $R_{D\times I_2} = (D \times E)_{D\times I_2} = E_{I_2}$ and $R_{I_1\times E} = (D \times E)_{I_1\times E} = D_{I_1}$ confirm that R is a normal ring. The minimal prime ideals of $R = D \times E$ are $0 \times E$ and $D \times 0$. The localizations $R_{D\times 0} = (D \times E)_{D\times 0} = E_0$ and $R_{0\times E} = (D \times E)_{0\times E} = D_0$ show that the total ring of fractions is $\mathrm{Frac}(R) = \mathrm{Frac}(D \times E) = (D \times E)_{0\times E} \times (D \times E)_{D\times 0} = D_0 \times \dot{E}_0 = \mathrm{Frac}(D) \times \mathrm{Frac}(E)$.

Example 2 provides an FO normal ring that is the direct product of FO Prüfer domains. In fact, this is one of several characterizations of such rings as it is shown in the next result.

Theorem 3. *Let R be a ring. Then the following statements are equivalent.*

(1) *R is an* FO *normal ring.*

(2) *R is a finite direct product of* FO *normal domains.*

(3) *R is a finite direct product of* FC *normal domains.*

(4) *R is an* FC *normal ring.*

(5) *R is a finite direct product of Prüfer domains of finite spectrum.*

(6) *R is a finite direct product of Prüfer domains of finite dimension and finite maximal spectrum.*

Proof. (1) $\Rightarrow$ (2): Note that the map $R \to \prod\limits_{m\in\mathrm{Max}(R)} R_m$ is injective by Lemma 10.23.1 of [33]. Then the normal ring R is a reduced ring, as it is a subring of the product of its localizations at all maximal ideals, which is reduced since each localization is a domain. The ring R has necessarily a finite number of minimal primes. Then $R = \prod D_i$, where each D_i is an FO normal domain, by Lemma 10.37.16 of [34].

(2) $\Rightarrow$ (3): Each FO domain is an FC domain.

(3) $\Rightarrow$ (2) $\Rightarrow$ (1) $\Rightarrow$ (4) $\Rightarrow$ (1): Trivial.

(1) $\Rightarrow$ (5): $R = \prod D_i$ and $\mathrm{Spec}(R)$ is isomorphic to the finite union of the finite spectra of D_i.

(5) $\Rightarrow$ (6): Trivial.

(6) $\Rightarrow$ (1): $\mathrm{Spec}(D_i)$ is finite for each D_i in the decomposition of R as a direct product of normal domain. Therefore, each D_i is an FO normal domain implying that R is an FO ring. $\square$

3. The number of overrings of normal rings

In what follows we are going to compute the number and the length of the set of overrings. We need first to recall and state some results and definitions.

Lemma 4 ([2], Lemma 3.1). *Let (R,S) be a normal pair of integral domains R and S, and let $\mathrm{Max}(R) = \{M_i\colon i \in I\}$. Then for each $M_i \in \mathrm{Max}(R)$, there exists a prime R-ideal Q_i such that: $S_{R\setminus M_i} = R_{Q_i}$ and $S = \bigcap_{M_i} \in \mathrm{Max}(R) S_{R\setminus M_i} = \bigcap_{M_i \in \mathrm{Max}(R)} R_{Q_i} = \bigcap_{i\in I} R_{Q_i}$.*

The primes Q_i defined by Lemma 4 are playing a prime role in determining the cardinality and the length of the set of overrings. Indeed, if (R,S) is a normal pair, then according to [21], $\mathrm{Spec}(R,S)$ is defined to be the set $\{P \in \mathrm{Spec}(R)\colon P \not\subset Q_i$, for all $M_i \in \mathrm{Max}(R)\}$. We denote the set of minimal elements of $\mathrm{Spec}(R,S)$ by $\mathrm{MinSpec}(R,S)$.

Definition 5. Let $R = \prod_{i=1}^{n} R_i$ be a direct product of commutative rings, and $i_0 \in \{1,2,\dots,n\}$. For every prime ideal P of R_{i_0} and every set A of prime ideals of R_{i_0}, let

(1) $P^e := \prod_{i=1}^{n} U_i$, where $U_{i_0} = P$ and $U_i = R_i$ for each $i \neq i_0$, and

(2) $A^e := \{P^e\colon P \in A\}$.

Using the fact that each prime ideal of the direct product is a direct product of the form $\prod_{i=1}^{n} U_i$, where $U_{i_0} = P$ for some $i_0 \in \{1,2,\dots,n\}$, and $U_i = R_i$ for each $i \neq i_0$, we obtain the following results.

Proposition 6. *Let $R = \prod_{i=1}^{n} R_i$ be a direct product of commuative rings. Then the following statements hold true.*

(1) $\mathrm{Spec}\Big(\prod_{i=1}^{n} R_i\Big) = \bigcup_{i=1}^{n} (\mathrm{Spec}(R_i))^e$.

(2) $\mathrm{Max}\Big(\prod_{i=1}^{n} R_i\Big) = \bigcup_{i=1}^{n} (\mathrm{Max}(R_i))^e$.

(3) $\mathrm{MinSpec}\Big(\prod_{i=1}^{n} R_i\Big) = \bigcup_{i=1}^{n} (\mathrm{MinSpec}(R_i))^e$.

To state results about the cardinality of the set of intermediate rings, we first need to recall some related definitions and results.

Assume that R is a commutative ring and A a finite set of prime ideals of R. Let P and P' be elements of A. If there is no element Q of A with $P \subset Q \subset P'$, we say that the prime P' covers P in A.

We recall from [23], Theorem 2.3 the function α defined on $\operatorname{Spec}(R)$ by

$$\alpha(P) := \begin{cases} 1 & \text{if } P \text{ is a maximal ideal of } R, \\ \prod\limits_{P' \text{covers } P} (1+\alpha(P')) & \text{if } P \text{ is not a maximal ideal of } R. \end{cases}$$

We usually use α_A instead of α if α is defined on a subset A of $\operatorname{Spec}(R)$. This function is highly involved when we want to compute the number of overrings as it is going to be shown from the next results. Regarding the effect of the function alpha on related spectra we have the following result.

Lemma 7. *Let (R,S) be a normal pair such that $R = \prod\limits_{i=1}^{m} R_i$ and $S = \prod\limits_{i=1}^{m} S_i$ are direct products of integral domains $R_i \subseteq S_i$ for all $i \in \{1,2,\dots,m\}$. Then for each prime ideal P in $\operatorname{Spec}(R_i,S_i)$ we have*

$$\alpha_{\operatorname{Spec}(R_i,S_i)}(P) = \alpha_{\operatorname{Spec}(R,S)}(P^e).$$

Theorem 8. *Let $R_i \subseteq S_i$ be extensions of integral domains for $i = 1,\dots,m$, $R = R_1 \times R_2 \times \ldots \times R_m$ and $S = S_1 \times S_2 \times \ldots \times S_m$. If $R \subseteq S$ is a FIP extension, and (R,S) is a normal pair, then the cardinality of the set $[R,S]$ of intermediate rings is given by:*

$$|[R,S]| = \prod_{P \in \operatorname{MinSpec}(R,S)} \alpha(P).$$

P r o o f. Since $|[R_i,S_i)]| = \prod\limits_{P \in \operatorname{MinSpec}(R_i,S_i)} \alpha(P)$ by [7], and $\Big|\Big[\prod\limits_{i=1}^{m} R_i, \prod\limits_{i=1}^{m} S_i\Big]\Big| = \prod\limits_{i=1}^{m} |[R_i,S_i]|$ by [14], we have

$$\begin{aligned} |[R,S]| &= \prod_{i=1}^{m} |[R_i,S_i]| = \prod_{i=1}^{m} \Big(\prod_{P \in \operatorname{MinSpec}(R_i,S_i)} \alpha_{\operatorname{Spec}(R_i,S_i)}(P) \Big) \\ &= \prod_{i=1}^{m} \Big(\prod_{P \in \operatorname{MinSpec}(R_i,S_i)} \alpha_{\operatorname{Spec}(R,S)}(P) \Big) = \prod_{P \in \operatorname{MinSpec}(R,S)} \alpha(P). \end{aligned}$$

This gives the required result and finishes the proof. □

Using the fact that $\operatorname{Spec}(R, \operatorname{Frac}(R)) = \operatorname{Spec}(R)$, we obtain the following result that provides a generalization of [23], Corollary 2.4 from FO normal domains to FO normal rings.

Corollary 9. *Let R be an* FO *normal ring. Then the number of overrings is given by:*

$$|[R, \mathrm{Frac}(R)]| = \prod_{P \in \mathrm{MinSpec}(R)} \alpha(P).$$

Let R be an FO normal ring, where $R = \prod_{i=1}^{m} R_i$ is the decomposition of R as the product of normal domains. If $R = \prod_{i=1}^{k} D_i \times \prod_{i=k+1}^{m} F_i$, where the D_i are the non-field components in this direct product, then $\prod_{i=k+1}^{m} \alpha_{\mathrm{Spec}(F_i)}(0_{F_i}) = 1$ and the number of overrings is given by $|[R, \mathrm{Frac}(R)]| = \prod_{i=1}^{k} \alpha_{\mathrm{Spec}(D_i)}(\{0\})$.

Let us now have an example of computing the number of overrings of a normal ring.

Example 10. Let D be a valuation domain of dimension 1 with $\mathrm{Spec}(D) = \{0, L\}$, and E a Prüfer domain with a Y-graph as spectrum and $\mathrm{Spec}(E) = \{0, M, P, N\}$ such that $0 \subset M \subset P, N$. As a concrete example, we can take D and E of Example 2. The spectrum of the normal ring $R = D \times E$ is shown in Figure 1. We have

$$\alpha(0 \times E) = 1 + \alpha(L \times E) = 1 + 1 = 2,$$

and

$$\alpha(D \times 0) = 1 + \alpha(D \times M) = 1 + (1 + \alpha(D \times P))(1 + \alpha(D \times N)) = 5.$$

Therefore, the number of overrings of R is given by

$$|[R, \mathrm{Frac}(R)]| = \prod_{P \in \mathrm{Min}(\mathrm{Spec}(R))} \alpha(P) = \alpha(0 \times E)\alpha(D \times 0) = (2)(5) = 10.$$

The sets of overrings are $[D, \mathrm{Frac}(D)] = \{D, D_0 = \mathrm{Frac}(D)\}$, and $[E, \mathrm{Frac}(E)] = \{E = E_P \cap E_N, E_P, E_N, E_M, E_0 = \mathrm{Frac}(E)\}$. They are ordered by the usual inclusion as in Figure 2.

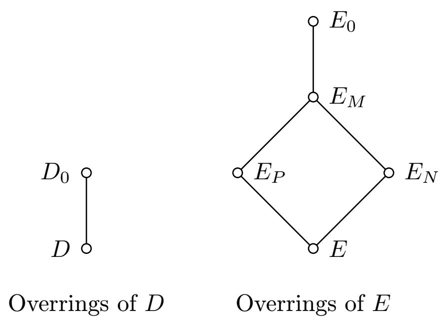
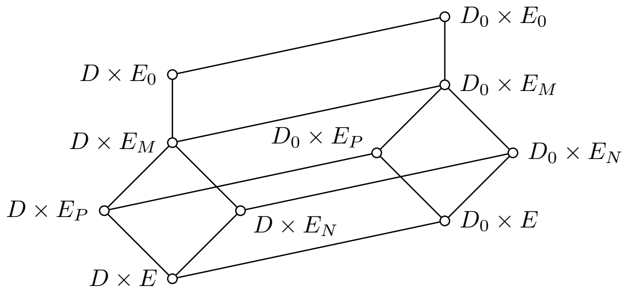


Figure 2.

We have

$$\begin{aligned}[R, \operatorname{Frac}(R)] &= [D, \operatorname{Frac}(D)] \times [E, Frac(E)] \\ &= \{D \times E, D \times E_P, D \times E_N, D \times E_M, D \times E_0, \\ &\quad D_0 \times E, D_0 \times E_P, D_0 \times E_N, D_0 \times E_M, D_0 \times E_0\}.\end{aligned}$$

The set $[R, \operatorname{Frac}(R)]$ of overrings is ordered by inclusion as in Figure 3.



Overrings of $D \times E$

Figure 3.

Corollary 11. *Let $R_i \subseteq S_i$ be ring extensions for $i = 1, \ldots, m$, $R = R_1 \times R_2 \times \ldots \times R_m$ and $S = S_1 \times S_2 \times \ldots \times S_m$. If the ring extension $R \subseteq S$ has exactly $r = \prod_{i=1}^{t} p_i^{u_i}$ intermediate rings, where $\prod_{i=1}^{t} p_i^{u_i}$ is the prime power decomposition of r, then the number of nontrivial ring extensions $R_i \subset S_i$ (i.e., $R_i \neq S_i$) is at most $\sum_{i=1}^{i=t} u_i$.*

P r o o f. The result about the extension $R \subseteq S$ follows from the fact that $|[R, S]| = \prod_{i=1}^{m} |[R_i, S_i]|$ and that $|[R_i, S_i]| > 1$ if and only if $R_i \neq S_i$. $\square$

The next result indicates that the number of factors in the decomposition of R as a direct product of FO normal domains is at most equal to the number of factors in the prime decomposition of the number of overrings of R.

Corollary 12. *If R is an* FO *normal ring with exactly m minimal primes and $r = \prod_{i=1}^{t} p_i^{u_i}$ overrings, then there are d integral domains $D_1, D_2, \ldots, D_d$ that are not fields, and $m - d$ fields $F_1, \ldots, F_{m-d}$ such that*

$$R = \prod_{i=1}^{d} D_i \times \prod_{j=1}^{m-d} F_j, \quad \text{where } d \leqslant \min\Bigl(m, \sum_{i=1}^{i=t} u_i\Bigr).$$

The case, where the number of overrings is a prime or a product of two primes, is particularly interesting as we can see from the next two examples.

Example 13. Let R be an FO normal ring with exactly m minimal primes. Then the number of overrings of R is a prime number p if and only if R is the direct product of a Prüfer (or valuation) domain D with exactly p overrings and the direct product of $m-1$ fields, that is, $R = D \times F_1 \times F_2 \times \ldots \times F_{m-1}$, where $|[D, \mathrm{Frac}(D)]| = p$ and each F_i is a field.

Remark 14. Notice that if $p = 2$, or 3 in the previous example, then D is necessarily a valuation domain as there is no Prüfer non valuation domain with exactly 2 or 3 overrings. Indeed, an FO Prüfer non valuation domain has at least 4 overrings, for if D has two different maximal ideals M and N, then it has at least the following different overrings: $D_M, D_N, D_M \cap D_N$, and $D_0 = \mathrm{Frac}(D)$.

Example 15. If R is an FO normal ring having exactly $r = p_1 p_2$ overrings, where p_1 and p_2 are prime numbers. Then the equations

$$|[R, \mathrm{Frac}(R)]| = \prod_{i=1}^{m} |[D_i, \mathrm{Frac}(D_i)]| = \prod_{i=1}^{m} \alpha_{\mathrm{Spec}(D_i)}(0_{D_i}) = p_1 p_2$$

indicate that there are at most 2 non-field factors in the decomposition of R as a direct product of FO Prüfer domains.

(1) If the number of non-field factors is 1, then $R = D \times F_1 \times F_2 \times \ldots \times F_{m-1}$, where D is a Prüfer or valuation domain with exactly $r = p_1 p_2$ overrings.
(2) If the number of non-field factors is 2, then $R = D_1 \times D_2 \times F_1 \times F_2 \times \ldots \times F_{m-2}$, where D_i is a Prüfer or valuation domain with exactly p_i overrings for $i = 1, 2$.

4. Length of the set of overrings of a normal ring

The previous results showed that counting the cardinality of the set of overrings of a normal ring R involves the minimal prime ideals of R. The next few results show that the length function depends also on the minimal primes of R, however in a different way as shown in the results of this section. We first recall the following definition.

Definition 16 ([21]). A ring extension $R \subseteq S$ is said to be of finite length if there is a nonnegative integer m such that every chain $R = R_0 \subset R_1 \subset \ldots \subset R_k = S$ of intermediate rings is of length $k \leqslant m$. The supremum of lengths of such chains is called the length of the set $[R, S]$ and is denoted $l[R, S]$.

We recall that a ring extension $R \subset S$ is called minimal extension if $|[R,S]| = 2$, that is $R \neq S$ and there is no intermediate ring T such that $R \subset T \subset S$. It is easy to prove the following useful result.

Lemma 17. *Let $R_i \subseteq S_i$ be ring extensions for $i = 1, 2$. Then $R_1 \times R_2 \subset S_1 \times S_2$ is a minimal ring extension if and only if either $R_1 = S_1$ and $R_2 \subset S_2$ is minimal or $R_2 = S_2$ and $R_1 \subset S_1$ is minimal.*

If $\mathcal{C}_1 = \{U_0 \subset U_1 \subset \ldots \subset U_n\}$ is a chain of intermediate rings in the extension $R_1 \subseteq S_1$ and V is an intermediate ring in the extension $R_2 \subseteq S_2$, we will denote by $\mathcal{C}_1 \times V$ the chain $\{U_0 \times V \subset U_1 \times V \subset \ldots \subset U_n \times V\}$ of the extension $R_1 \times R_2 \subseteq S_1 \times S_2$. Similarly, $U \times \mathcal{C}_2$ will denote the chain $\{U \times V_0 \subset U \times V_1 \subset \ldots \subset U \times V_n\}$ of the extension $R_1 \times S_1 \subseteq R_2 \times S_2$, where $\{V_0 \subset V_1 \subset \ldots \subset V_n\}$ is a chain of intermediate rings in the extension $R_2 \subseteq S_2$ and U is an intermediate ring in the extension $R_1 \subseteq S_1$.

Lemma 18. *Let $R_i \subseteq S_i$ be ring extensions, where $i = 1, 2$.*

(1) *If $\mathcal{C}_i$ is a maximal chain of intermediate rings in $R_i \subseteq S_i$ for $i = 1, 2$, then $(\mathcal{C}_1 \times R_2) \cup (S_1 \times C_2)$ is a maximal chain of intermediate rings of length $l(C_1) + l(C_2)$ in $R_1 \times R_2 \subseteq S_1 \times S_2$.*

(2) *Each maximal chain of $[R,S]$ is of length $m \leqslant l[R_1,S_1] + l[R_2,S_2]$.*

Proof. (1) Assume $\mathcal{C}_1 = \{R_1 = U_0 \subset U_1 \subset \ldots \subset U_{m_1} = S_1\}$ and $\mathcal{C}_2 = \{R_2 = V_0 \subset V_1 \subset \ldots \subset V_{m_2} = S_2\}$.

Each inclusion in the chain $(\mathcal{C}_1 \times R_2) \cup (S_1 \times C_2)$: $\{R_1 \times R_2 = U_0 \times R_2 \subset U_1 \times R_2 \subset \ldots \subset U_{m_1} \times R_2 = S_1 \times R_2 = S_1 \times V_0 \subset S_1 \times V_1 \subset \ldots \subset S_1 \times V_{m_2} = S_1 \times S_2\}$ satisfies the conditions of Lemma 17. Indeed in the left half between $R_1 \times R_2$ and $S_1 \times R_2$, the second component is always R_2 and the first component is coming from a maximal chain meaning that this part consists of successive minimal extensions. Similarly for the right half. Therefore, this chain is maximal of length $m_1 + m_2 = l(C_1) + l(C_2)$.

(2) Now consider a maximal chain in $[R_1 \times R_2, S_1 \times S_2]$: $\{R_1 \times R_2 = U_0 \times V_0 \subset U_1 \times V_1 \subset \ldots \subset U_m \times V_m = S_1 \times S_2\}$.

Then $\{U_0 \subseteq U_1 \subseteq \ldots \subseteq U_m\}$ is a chain of non necessarily distinct elements. The "$\subseteq$" in this chain is an "$\subset$" at most $l[R_1,S_1]$ times. If the "$\subseteq$" is an "$=$", this indicates that $V_i \subset V_{i+1}$. This can happen only at most $l[R_2,S_2]$ times. This means that $m \leqslant l[R_1,S_1] + l[R_2,S_2]$. □

Proposition 19. *Let $R_i \subseteq S_i$ be ring extensions for $i = 1, \ldots, m$, $R = R_1 \times R_2 \times \ldots \times R_m$ and $S = S_1 \times S_2 \times \ldots \times S_m$.*

(1) *$R \subseteq S$ is an* FCP *extension if and only if each $R_i \subseteq S_i$ is also an* FCP *extension.*

(2) *If $R \subseteq S$ is an* FCP *extension, then all maximal chains of the extension $R \subseteq S$ are of the same length $l[R, S]$, if and only if all maximal chains are also of the same length $l[R_i, S_i]$ for each extension $R_i \subseteq S_i$.*

P r o o f. It is enough to show the required results for $m = 2$.

(1) $\Rightarrow$ Each chain $\{U_1 \subset U_2 \subset \ldots \subset U_{k_1}\}$ of distinct elements in the extension $R_1 \subseteq S_1$ gives rise to a chain $\{U_1 \times R_2 \subset U_2 \times R_2 \subset \ldots \subset U_{k_1} \times R_2\}$ of distinct elements in $R \subseteq S$, which is finite by assumption. Therefore, the original chain is also finite and $R_1 \subseteq S_1$ is an FCP extension. Similarly for $R_2 \subseteq S_2$.

$\Leftarrow$ Each chain $\{U_0 \times V_0 \subset U_1 \times V_1 \subset \ldots \subset U_m \times V_m\}$ of distinct elements in $R_1 \times R_2 \subset S_1 \times S_2$ gives rise to the chain $\{U_0 \subseteq U_1 \subseteq \ldots \subseteq U_m\}$, whose elements are not necessarily distinct. However, "$\subseteq$" in this chain is an "$\subset$" for just a finite number of occurrences. Moreover, if "$\subseteq$" is an "$=$", this indicates that $V_i \subset V_{i+1}$. This can also happen only for a finite number of occurrences. This indicates that the original chain is finite as required.

(2) $\Rightarrow$ Assume that $\mathcal{C}_1$ and $\mathcal{C}_2$ are maximal chains of $R_1 \subseteq S_1$ and $\mathcal{D}$ is a maximal chain of $R_2 \subseteq S_2$. They give, by Lemma 18, rise to two maximal chains $(\mathcal{C}_1 \times R_2) \cup (S_1 \times \mathcal{D})$ and $(\mathcal{C}_2 \times R_2) \cup (S_1 \times \mathcal{D})$ of $R_1 \times R_2 \subseteq S_1 \times S_2$. The fact that these two new chains are of the same length implies that $\mathcal{C}_1$ and $\mathcal{C}_2$ are also of the same length. The same applies for maximal chains of $R_2 \subseteq S_2$.

$\Leftarrow$ Each maximal chain $\{R_1 \times R_2 = U_0 \times V_0 \subset U_1 \times V_1 \subset \ldots \subset U_m \times V_m = S_1 \times S_2\}$ of $R_1 \times R_2 \subseteq S_1 \times S_2$ gives rise to two chains of non necessarily distinct elements: $\{U_0 \subseteq U_1 \subseteq \ldots \subseteq U_m\}$ of $R_1 \subseteq S_1$, and $\{V_0 \subseteq V_1 \subseteq \ldots \subseteq V_m\}$ of $R_2 \subseteq S_2$.

Using Lemma 17, we can see that the link "$\subseteq$" at the position k in one chain is a "$\subset$" if and only if the corresponding link is an "$=$" at the other chain and vice versa. Again by the same lemma, we can reduce the chain $U_0 \subseteq U_1 \subseteq \ldots \subseteq U_m$ to a maximal chain of length $l[R_1, S_1]$ by eliminating repetitions. Similarly, the second chain is reduced to a maximal chain of length $l[R_2, S_2]$. Since such chains are always of the same length, this indicates that every maximal chain of $R_1 \times R_2 \subseteq S_1 \times S_2$ is of length $l[R_1, S_1] + l[R_2, S_2] = l[R, S]$. This finishes the proof of this result. $\square$

Proposition 20. *Let $R_i \subseteq S_i$ be extensions of integral domains for $i = 1, \ldots, m$, $R = R_1 \times R_2 \times \ldots \times R_m$ and $S = S_1 \times S_2 \times \ldots \times S_m$. If $R \subseteq S$ is an* FCP *extension, (R, S) is a normal pair, and R is of finite spectrum, then S is also of finite spectrum and*

$$l[R, S] = |\mathrm{Spec}(R)| - |\mathrm{Spec}(S)|.$$

P r o o f. Since $l[R_i, S_i] = |\mathrm{Spec}(R_i)| - |\mathrm{Spec}(S_i)|$ for each $i = 1, \ldots, m$, according to [22], Corollary 3.4, $|\mathrm{Spec}(R)| = \sum\limits_{i=1}^{i=m} |\mathrm{Spec}(R_i)|$ by Proposition 6, and $l[R, S] = \sum\limits_{i=1}^{i=m} l[R_i, S_i]$ by [32], we obtain:

$$\begin{aligned} l[R, S] &= \sum_{i=1}^{i=m} l[R_i, S_i] = \sum_{i=1}^{i=m} (|\mathrm{Spec}(R_i)| - |\mathrm{Spec}(S_i)|) \\ &= \sum_{i=1}^{i=m} |\mathrm{Spec}(R_i)| - \sum_{i=1}^{i=m} |\mathrm{Spec}(S_i)| = |\mathrm{Spec}(R)| - |\mathrm{Spec}(S)|. \end{aligned}$$

□

The following result provides a generalization of [22], Corollary 3.4 from FC normal domains to FC normal rings.

Proposition 21. *Let R be an* FC *normal ring and $R = D_1 \times D_2 \times \ldots \times D_m$ the presentation of R as a direct product of normal domains. Then the length of each maximal chain of overrings $R = R_0 \subseteq R_1 \subseteq \ldots \subseteq R_k = \mathrm{Frac}(R)$ is given by:*

$$l[R, \mathrm{Frac}(R)] = |\mathrm{Spec}(R)| - |\mathrm{MinSpec}(R)|.$$

P r o o f. Since $R = D_1 \times D_2 \times \ldots \times D_m$, $\mathrm{Frac}(R) = \mathrm{Frac}(D_1) \times \mathrm{Frac}(D_2) \times \ldots \times \mathrm{Frac}(D_m)$ and $l[D_i, \mathrm{Frac}(D_i)] = |\mathrm{Spec}(D_i)| - 1$ according to [22], Corollary 3.4, we obtain:

$$\begin{aligned} l[R, \mathrm{Frac}(R)] &= \sum_{i=1}^{i=m} l[D_i, \mathrm{Frac}(D_i)] = \sum_{i=1}^{i=m} (|\mathrm{Spec}(D_i)| - 1) \\ &= -m + \sum_{i=1}^{i=m} |\mathrm{Spec}(D_i)| = |\mathrm{Spec}(R)| - |\mathrm{MinSpec}(R)|. \end{aligned}$$

□

Example 22. Consider the integral domains D, E, and the normal ring $R = D \times E$ from Example 10. The length of $[R, \mathrm{Frac}(R)]$ is given by

$$l[R, \mathrm{Frac}(R)] = |\mathrm{Spec}(R)| - |\mathrm{MinSpec}(R)| = 6 - 2 = 4.$$

The chain $\{R = D \times E \subset D \times E_P \subset D \times E_M \subset D \times E_0 \subset D_0 \times E_0 = \mathrm{Frac}(R)\}$ is an example of a maximal chain of $[R, \mathrm{Frac}(R)]$ of length 4, see Figure 3.

5. Numerical characterizations

We obtain in this section characterizations of normal rings, where the set of overrings is of a certain cardinality or length.

Corollary 23. *Let R be an* FO *normal ring with exactly m minimal primes. Then the following statements are equivalent.*

(1) *R has a unique overring, which is R itself.*
(2) $l[R, \mathrm{Frac}(R)] = 0$.
(3) *R is the direct product of m fields, that is, $R = F_1 \times F_2 \times \ldots \times F_m$, where each F_i is a field.*

P r o o f. Before stating the proof, notice that $R = D_1 \times D_2 \times \ldots \times D_m$ is the direct product of m Prüfer domains.

(1) $\Rightarrow$ (2): This is trivial by the definition of the length.

(2) $\Rightarrow$ (3): The equation $l[R, \mathrm{Frac}(R)] = \sum_{i=1}^{i=m} l[D_i, \mathrm{Frac}(D_i)] = 0$ yields that $l[D_i, \mathrm{Frac}(D_i)] = 0$ for each i. This means that each D_i is a field and that R is the direct product of m fields.

(3) $\Rightarrow$ (1): We have

$$\begin{aligned}\mathrm{Frac}(R) &= \mathrm{Frac}(F_1 \times F_2 \times \ldots \times F_m) = \mathrm{Frac}(F_1) \times \mathrm{Frac}(F_2) \times \ldots \times \mathrm{Frac}(F_m)\\ &= F_1 \times F_2 \times \ldots \times F_m = R.\end{aligned}$$

Therefore, R has a unique overring. □

Corollary 24. *Let R be an* FO *normal ring with exactly m minimal primes. Then the following statements are equivalent.*

(1) *R has exactly 2 overrings.*
(2) $l[R, \mathrm{Frac}(R)] = 1$.
(3) *R is the direct product of a valuation domain D of dimension 1 and the direct product of $m-1$ fields, that is, $R = D \times F_1 \times F_2 \times \ldots \times F_{m-1}$, where D is a valuation domain of dimension 1 and each F_i is a field.*

P r o o f. (1) $\Rightarrow$ (2): $|[R, \mathrm{Frac}(R)]| = 2$ implies that there is $i_0 \in \{1, 2, \ldots, m\}$ such that $|[D_{i_0}, \mathrm{Frac}(D_{i_0})]| = 2$, and $|[D_i, \mathrm{Frac}(D_i)]| = 1$ for every $i \neq i_0$. This implies that $l[D_{i_0}, \mathrm{Frac}(D_{i_0})] = 1$, and $l[D_i, \mathrm{Frac}(D_i)] = 0$ for every $i \neq i_0$. Therefore, $l[R, \mathrm{Frac}(R)] = \sum_{i=1}^{i=m} l[D_i, \mathrm{Frac}(D_i)] = 1$.

(2) $\Rightarrow$ (3): The equation $l[R, \operatorname{Frac}(R)] = \sum_{i=1}^{i=m} l[D_i, \operatorname{Frac}(D_i)] = 1$ implies that $l[D_{i_0}, \operatorname{Frac}(D_{i_0})] = 1$, and $l[D_i, \operatorname{Frac}(D_i)] = 0$ for every $i \neq i_0$. Therefore, $D_{i_0} = D$ is a valuation domain of dimension 1, and D_i is a field for every $i \neq i_0$. Therefore, $R = D \times F_1 \times F_2 \times \ldots \times F_{m-1}$.

(3) $\Rightarrow$ (1): We have

$$|[R, \operatorname{Frac}(R)]| = \prod_{i=1}^{m} |[D_i, \operatorname{Frac}(D_i)]| = |[D, \operatorname{Frac}(D)]| \prod_{i=1}^{m-1} |[F_i, \operatorname{Frac}(F_i)]| = (2)(1) = 2.$$

$\square$

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