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ON THE r-FREE VALUES OF THE POLYNOMIAL $x^2+y^2+z^2+k$

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Abstract. Let k be a fixed integer. We study the asymptotic formula of $R(H,r,k)$, which is the number of positive integer solutions $1 \leqslant x,y,z \leqslant H$ such that the polynomial $x^2+y^2+z^2+k$ is r-free. We obtained the asymptotic formula of $R(H,r,k)$ for all $r \geqslant 2$. Our result is new even in the case $r=2$. We proved that $R(H,2,k)=c_kH^3+O(H^{9/4+\varepsilon})$, where $c_k>0$ is a constant depending on k. This improves upon the error term $O(H^{7/3+\varepsilon})$ obtained by G.-L. Zhou, Y. Ding (2022).

Keywords: square-free; Salié sum; asymptotic formula

MSC 2020: 11N25, 11L05, 11L40

1. Introduction

There exists an outstanding conjecture that the polynomial x^2+1 contains infinitely many primes. Iwaniec in [10] proved there are infinitely many n such that n^2+1 has at most two prime factors. So far, various authors considered the square-free values of some specific polynomials with integer coefficients. In 1931, Estermann in [6] showed that

$$\sum_{1\leqslant x\leqslant H}\mu^2(x^2+1)=c_0H+O(H^{2/3+\varepsilon}),$$

where c_0 is an absolute constant. This error term was improved to $O(H^{7/12+\varepsilon})$ by Heath-Brown, see [9]. Carlitz in [2] studied the polynomial x^2+x and established that

$$\sum_{1\leqslant x\leqslant H}\mu^2(x^2+x)=\prod_p\Big(1-\frac{2}{p^2}\Big)H+O(H^{2/3+\varepsilon}).$$

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Later, Heath-Brown in [8] improved the error term $O(H^{2/3+\varepsilon})$ to $O(H^{7/11}\log^7 H)$ which was further improved by Reuss (see [16]) to $O(H^{0.578+\varepsilon})$. Mirsky in [14] studied the square-free values of the polynomial $(x+a_1)(x+a_2)$.

In 2012 Tolev in [17] considered the square-free values of a polynomial in two variables. Let $S_2(H)$ stand for the number of square-free values of x^2+y^2+1 with $1 \leqslant x, y \leqslant H$. Using Weil's estimate for the Kloosterman sum, Tolev proved that

$$S_2(H) = \prod_p \Big(1 - \frac{\lambda_2(p^2)}{p^4}\Big) H^2 + O(H^{4/3+\varepsilon}),$$

where $\lambda_2(q)$ is the number of integer solutions to the congruence equation

$$x^2 + y^2 + 1 \equiv 0 \pmod q, \quad 1 \leqslant x,\ y \leqslant q.$$

In 2022 Zhou and Ding in [18] studied square-free values of the polynomial $x^2+y^2+z^2+k$, where k is a fixed (nonzero) integer. Let

$$S_3(H,k) = \sum_{1 \leqslant x,y,z \leqslant H} \mu^2(x^2+y^2+z^2+k). \tag{1.1}$$

It was proved in [18] that

$$S_3(H,k) = \prod_p \Big(1 - \frac{\lambda(p^2;k)}{p^6}\Big) H^3 + O(H^{7/3+\varepsilon}), \tag{1.2}$$

where $\lambda(q;k)$ is defined as

$$\lambda(q;k) = \sum_{\substack{1 \leqslant x,y,z \leqslant q \\ x^2+y^2+z^2+k \equiv 0 \ (\mathrm{mod}\ q)}} 1. \tag{1.3}$$

Moreover, with the help of the method developed by Tolev (see [17]), many other interesting results were proved. For example, Dimitrov in [4], [5] found asymptotic formulas for consecutive square-free numbers of the form x^2+y^2+1, x^2+y^2+2 and the form x^2+1, x^2+2, respectively. Subsequently Chen in [3] generalized the results of Dimitrov and gave an asymptotic formula for consecutive square-free numbers of the form $x_1^2+\ldots+x_k^2+1$, $x_1^2+\ldots+x_k^2+2$ for $k \geqslant 3$. Further, Jing and Liu in [12] made a slight improvement to the error term in [4] and studied consecutive square-free numbers of the form $xy+1$, $xy+2$.

In addition, some articles are devoted to r-free values of polynomials in one variable. Note that for $r \geqslant 2$, an integer n is r-free if $p^r \nmid n$ for all primes p. Let $S(H,r)$ be the number of integers $x \leqslant H$ such that x^2+x is r-free. In 1932, Carlitz in [2] obtained

$$S(H,r) = \prod_p \Big(1 - \frac{2}{p^r}\Big) H + O(H^{2/(r+1)+\varepsilon}).$$

The above error term was improved to $O(H^{14/(7r+8)+\varepsilon})$ by Brandes (see [1]) generalizing Heath-Brown's method in [8]. Brandes' result was further improved by Reuss, see [16]. Furthermore, Reuss in [16] established a more general result

$$S_{r,h}(H) = c_{r,h}H + O(H^{\upsilon(r)+\varepsilon}),$$

where $S_{r,h}(H)$ stands for the number of integers $x \leqslant H$ such that x and $x+h$ are r-free, $c_{r,h}$ is a constant depending on r and h, and $\upsilon(2) = 0.578$ while $\upsilon(r) = 169/(144r)$ for $r \geqslant 3$. Also, r-free values of the polynomial $(x+a_1)(x+a_2)$ were considered by Mirsky (see [13]) who gave the error term $O(H^{2/(r+1)+\varepsilon})$.

Inspired by the above works, we study the r-free values of $x^2+y^2+z^2+k$. We put

$$R(H,r,k) = \sum_{\substack{1\leqslant x,y,z\leqslant H \\ x^2+y^2+z^2+k \text{ is } r\text{-free}}} 1. \tag{1.4}$$

The main result of this paper is the asymptotic formula of $R(H,r,k)$.

Theorem 1.1. *Let $r \geqslant 2$ be a natural number. Let k be a fixed integer. Let $\varepsilon > 0$ be an arbitrarily small positive number. Then*

(i) *For $k \neq 0$,*

$$R(H,r,k) = \prod_p \Big(1 - \frac{\lambda(p^r;k)}{p^{3r}}\Big)H^3 + O(H^2 + H^{3/2+3/2r+\varepsilon}). \tag{1.5}$$

(ii) *We have*

$$R(H,r,0) = \prod_p \Big(1 - \frac{\lambda(p^r;0)}{p^{3r}}\Big)H^3 + O(H^{2+\varepsilon} + H^{9r/(5r-2)+\varepsilon}). \tag{1.6}$$

Note that $R(H,2,k)$ is exactly $S_3(H,k)$ defined in (1.1). In fact, Theorem 1.1 is also new in the case $r=2$. We have the following theorem.

Theorem 1.2. *Let k be a fixed integer. Let $\varepsilon > 0$ be an arbitrarily small positive number. We have*

$$S_3(H,k) = \prod_p \Big(1 - \frac{\lambda(p^2;k)}{p^6}\Big)H^3 + O(H^{9/4+\varepsilon}).$$

Theorem 1.2 improves upon (1.2) obtained by Zhou and Ding, see [18].

2. Notations and some lemmas

Let H be a sufficiently large positive number. The letters k, m, n, l, a, b, c stand for integers and d, r, h, q, x, y, z, α, ϱ stand for positive integers. The letters η, ξ denote real numbers and the letter p is reserved for primes. By ε we denote an arbitrarily small positive number. Throughout this paper, k and $r \geqslant 2$ are fixed integers, and the implied constants may depend on k, r and ε.

As usual, the functions $\mu(n)$ and $\tau(n)$ represent the Möbius function and the number of positive divisors of n, respectively. We write $(n_1, \dots, n_u)$ for the greatest common divisor of $n_1, \dots, n_u$. Let $\|\xi\|$ be the distance from ξ to its nearest integer. Further let $e(t) = \exp(2\pi i t)$ and $e_q(t) = e(t/q)$. For any q and x such that $(q, x) = 1$ we denote by $\overline{x}_q$ the inverse of x modulo q. If the modulus q is clear from the context then we write $\overline{x}$ for simplicity. For any odd q we denote by $\left(\frac{\cdot}{q}\right)$ the Jacobi symbol.

We introduce the Gauss sum

$$G(q; n, m) = \sum_{1 \leqslant x \leqslant q} e_q(nx^2 + mx), \quad G(q; n) = \sum_{1 \leqslant x \leqslant q} e_q(nx^2). \tag{2.1}$$

First, we introduce some basic properties of the Gauss sum.

Lemma 2.1. *For the Gauss sum we have:*

(i) *See* [7], *Lemma* 7. *If* $(q, n) = d$ *then*

$$G(q; n, m) = \begin{cases} dG\left(\frac{q}{d}; \frac{n}{d}, \frac{m}{d}\right) & \text{if } d \mid m, \\ 0 & \text{if } d \nmid m. \end{cases}$$

(ii) *See* [7], *Lemma* 3. *If* $(q, 2n) = 1$ *then*

$$G(q; n, m) = e_q(-\overline{(4n)}m^2)\left(\frac{n}{q}\right)G(q; 1).$$

(iii) *See* [11], *Lemma* 4.8. *If* $(q, 2) = 1$ *then*

$$G^2(q; 1) = (-1)^{(q-1)/2} q.$$

We introduce the Salié sum

$$S(c; a, b) = \sum_{\substack{1 \leqslant x \leqslant c \\ (x, c) = 1}} \left(\frac{x}{c}\right) e_c(ax + b\overline{x}). \tag{2.2}$$

It is easy to see that

$$S(c; a, b) = S(c; b, a). \tag{2.3}$$

The following result comes from Corollary 4.10 in [11].

Lemma 2.2. *Let p be an odd prime. Let α be a positive integer. If $p \nmid a$ (or $p \nmid b$), then*

$$|S(p^\alpha; a, b)| \leqslant \tau(p^\alpha) p^{\alpha/2}. \tag{2.4}$$

We put

$$T(c; a, \varrho) = \sum_{\substack{1 \leqslant x \leqslant c \\ (x,c)=1}} \left(\frac{x}{c}\right)^\varrho e_c(ax), \quad 2 \nmid c. \tag{2.5}$$

Lemma 2.3. *Let p be an odd prime. Let α and ϱ be positive integers. Suppose that $p \nmid a$. One has*

$$T(p^\alpha; a, \varrho) = 0 \quad \text{for } \alpha \geqslant 2 \quad \text{and} \quad |T(p; a, \varrho)| \leqslant p^{1/2}.$$

Proof. We first consider the case $\alpha = 1$. Note that $T(p; a; \varrho)$ is either the Gauss sum or the Ramanujan sum, and we have $|T(p; a, \varrho)| \leqslant p^{1/2}$.

Now we consider the case $\alpha \geqslant 2$. We write $x = yp + z$ to deduce that

$$\begin{aligned} T(p^\alpha; a, \varrho) &= \sum_{0 \leqslant y \leqslant p^{\alpha-1}-1} \sum_{0 \leqslant z \leqslant p-1} \left(\frac{z}{p^\alpha}\right)^\varrho e_{p^\alpha}(apy + az) \\ &= \sum_{0 \leqslant z \leqslant p-1} \left(\frac{z}{p^\alpha}\right)^\varrho e_{p^\alpha}(az) \sum_{0 \leqslant y \leqslant p^{\alpha-1}-1} e_{p^{\alpha-1}}(ay) = 0. \end{aligned}$$

This completes the proof. □

Lemma 2.4. *Let p be an odd prime and let α be a positive integer. One has*

$$|S(p^\alpha; a, 0)| \leqslant p^{\alpha/2} (p^\alpha, a)^{1/2}. \tag{2.6}$$

Proof. Note that (2.6) holds trivially when $p^\alpha \mid a$. We only need to consider $p^\alpha \nmid a$. We assume that $p^\beta \| a$ and $0 \leqslant \beta \leqslant \alpha - 1$. Then we write $a = p^\beta a'$ with $p \nmid a'$. We have

$$S(p^\alpha; a, 0) = \sum_{\substack{1 \leqslant x \leqslant p^\alpha \\ (x,p)=1}} \left(\frac{x}{p}\right)^\alpha e_{p^{\alpha-\beta}}(a'x).$$

We write $x = yp^{\alpha-\beta} + z$ to deduce that

$$\begin{aligned} S(p^\alpha; a, 0) &= \sum_{0 \leqslant y \leqslant p^\beta - 1} \sum_{\substack{0 \leqslant z \leqslant p^{\alpha-\beta}-1 \\ (z,p)=1}} \left(\frac{z}{p}\right)^\alpha e_{p^{\alpha-\beta}}(a'z) = p^\beta \sum_{\substack{0 \leqslant z \leqslant p^{\alpha-\beta}-1 \\ (z,p)=1}} \left(\frac{z}{p}\right)^\alpha e_{p^{\alpha-\beta}}(a'z) \\ &= p^\beta T(p^{\alpha-\beta}; a', \alpha). \end{aligned}$$

Since $p \nmid a'$, we conclude from Lemma 2.3 that

$$|S(p^\alpha; a, 0)| \leqslant p^{\beta+(\alpha-\beta)/2} = p^{\alpha/2+\beta/2} = p^{\alpha/2}(p^\alpha, a)^{1/2}.$$

This completes the proof. $\square$

Let

$$\lambda(q; n, m, l, k) = \sum_{\substack{1 \leqslant x,y,z \leqslant q \\ x^2+y^2+z^2+k \equiv 0 \pmod q}} e_q(nx + my + lz). \tag{2.7}$$

Lemma 2.5 ([18], Lemma 2.2)**.** *Suppose that* $(q_1, q_2) = 1$. *One has*

$$\lambda(q_1q_2; n, m, l, k) = \lambda(q_1; \overline{q_2}_{q_1} n, \overline{q_2}_{q_1} m, \overline{q_2}_{q_1} l, k)\lambda(q_2; \overline{q_1}_{q_2} n, \overline{q_1}_{q_2} m, \overline{q_1}_{q_2} l, k).$$

In particular, we have

$$\lambda(q_1q_2; k) = \lambda(q_1; k)\lambda(q_2; k).$$

Now we apply Lemmas 2.2, 2.4 and 2.5 to obtain an upper bound of $\lambda(q; n, m, l, k)$.

Lemma 2.6. *Suppose that* $p^r \| q$ *for all primes* $p \mid q$.

(i) *If* $k \neq 0$, *then we have*

$$\lambda(q; n, m, l, k) \ll q^{1+\varepsilon}(q, n, m, l). \tag{2.8}$$

(ii) *One has (for* $k = 0$*)*

$$\lambda(q; n, m, l, 0) \ll q^{1+\varepsilon}(q, n, m, l)(q, n^2 + m^2 + l^2)^{1/2}. \tag{2.9}$$

P r o o f. We write $q = p_1^r p_2^r \dots p_s^r$, where p_i $(1 \leqslant i \leqslant s)$ are distinct primes. By Lemma 2.5, we have

$$\lambda(q; n, m, l, k) = \prod_{i=1}^{s} \lambda(p_i^r; n\overline{q_i}, m\overline{q_i}, l\overline{q_i}, k), \tag{2.10}$$

where $q_i = q/p_i^r$ and $\overline{q_i}$ stands for the inverse of q_i modulo p_i^r.

Let $n_i = n\overline{q_i}$, $m_i = m\overline{q_i}$ and $l_i = l\overline{q_i}$. Note that $(p_i^r, \overline{q_i}) = 1$, so we have $(p_i^r, n_i, m_i, l_i) = (p_i^r, n, m, l)$ and $(p_i^r, n_i^2 + m_i^2 + l_i^2) = (p_i^r, n^2 + m^2 + l^2)$. Therefore, we only need to prove

$$\lambda(p^r; u, v, w, k) \ll p^{r+\varepsilon}(p^r, u, v, w) \quad \text{for } k \neq 0 \tag{2.11}$$

and

$$\lambda(p^r; u, v, w, 0) \ll p^{r+\varepsilon}(p^r, u, v, w)(p^r, u^2 + v^2 + w^2)^{1/2}. \tag{2.12}$$

One has the trivial bound $|\lambda(p^r;u,v,w,k)| \leqslant p^{3r}$. Since the implied constants were allowed to depend on r and k, we assume that p is odd and we further assume that $p \nmid k$ if $k \neq 0$. We have

$$\begin{aligned}\lambda(p^r;u,v,w,k) &= p^{-r} \sum_{1\leqslant x,y,z\leqslant p^r} e_{p^r}(ux+vy+wz) \sum_{1\leqslant h\leqslant p^r} e_{p^r}(h(x^2+y^2+z^2+k)) \\ &= p^{-r} \sum_{1\leqslant h\leqslant p^r} e_{p^r}(hk)G(p^r;h,u)G(p^r;h,v)G(p^r;h,w) \\ &= p^{-r} \sum_{0\leqslant\beta\leqslant r} \sum_{\substack{1\leqslant h\leqslant p^r \\ (h,p^r)=p^\beta}} e_{p^r}(hk)G(p^r;h,u)G(p^r;h,v)G(p^r;h,w).\end{aligned}$$

We write $h = h'p^\beta$ with $1 \leqslant h' \leqslant p^{r-\beta}$ and $(h', p^{r-\beta}) = 1$. Then we have

(2.13)
$$\begin{aligned}&\lambda(p^r;u,v,w,k) \\ &\quad= p^{-r} \sum_{0\leqslant\beta\leqslant r} \sum_{\substack{1\leqslant h'\leqslant p^{r-\beta} \\ (h',p^{r-\beta})=1}} e_{p^{r-\beta}}(h'k)G(p^r;h'p^\beta,u)G(p^r;h'p^\beta,v)G(p^r;h'p^\beta,w).\end{aligned}$$

By Lemma 2.1 (i), we have

$$\begin{aligned}G(p^r;h,u)&G(p^r;h,v)G(p^r;h,w) \\ &= p^{3\beta}G(p^{r-\beta};h',up^{-\beta})G(p^{r-\beta};h',vp^{-\beta})G(p^{r-\beta};h',wp^{-\beta})\end{aligned} \tag{2.14}$$

if $p^\beta \mid (u,v,w)$, and $G(p^r;h,u)G(p^r;h,v)G(p^r;h,w) = 0$ if $p^\beta \nmid (u,v,w)$. When p is odd and $p^\beta \mid (u,v,w)$, by Lemma 2.1 (ii), we have

$$\begin{aligned}G(p^{r-\beta};h',up^{-\beta})&G(p^{r-\beta};h',vp^{-\beta})G(p^{r-\beta};h',wp^{-\beta}) \\ &= \Big(\frac{h'}{p^{r-\beta}}\Big)^3 G(p^{r-\beta},1)^3 e_{p^{r-\beta}}(-\overline{(4h')}_{p^{r-\beta}}(u^2+v^2+w^2)p^{-2\beta}).\end{aligned} \tag{2.15}$$

From (2.13)–(2.15), we obtain

$$\lambda(p^r;u,v,w,k) = p^{-r} \sum_{\substack{0\leqslant\beta\leqslant r \\ p^\beta\mid(u,v,w)}} p^{3\beta}G(p^{r-\beta};1)^3 S\Big(p^{r-\beta};k,-\overline{4}\frac{u^2+v^2+w^2}{p^{2\beta}}\Big). \tag{2.16}$$

We first deal with the case $k \neq 0$. As mentioned above, p is odd and $p \nmid k$, then by Lemmas 2.1 (iii) and 2.2, we have

$$\lambda(p^r;u,v,w,k) \ll p^{-r} \sum_{\substack{0\leqslant\beta\leqslant r \\ p^\beta\mid(p^r,u,v,w)}} p^{3\beta}p^{3/2(r-\beta)}\tau(p^{r-\beta})p^{1/2(r-\beta)} \ll p^{r+\varepsilon}(p^r,u,v,w).$$

This establishes (2.11). Now we consider the case $k=0$. By Lemmas 2.1 (iii) and 2.4, we deduce from (2.16) that

$$\begin{aligned}\lambda(p^r;u,v,w,0) &\ll p^{-r}\sum_{\substack{0\leqslant\beta\leqslant r\\ p^\beta\mid(p^r,u,v,w)}} p^{3\beta}p^{3/2(r-\beta)}p^{1/2(r-\beta)}\Big(p^{r-\beta},\frac{u^2+v^2+w^2}{p^{2\beta}}\Big)^{1/2}\\ &\ll p^{r+\varepsilon}(p^r,u,v,w)(p^r,u^2+v^2+w^2)^{1/2}.\end{aligned}$$

This establishes (2.12). The proof of the lemma is complete. □

Lemma 2.7 ([18], Lemma 2.5)**.** *Let $Q\geqslant 2$. Let $q=d^r$ with d odd and square-free. We put*

$$U_1(Q,q)=\sum_{1\leqslant n\leqslant Q}\frac{|\lambda(q;n,0,0,k)|}{n},\quad U_2(Q,q)=\sum_{1\leqslant n,m\leqslant Q}\frac{|\lambda(q;n,m,0,k)|}{nm},$$
$$U_3(Q,q)=\sum_{1\leqslant n,m,l\leqslant Q}\frac{|\lambda(q;n,m,l,k)|}{nml}.$$

Suppose that $k\neq 0$. For $1\leqslant i\leqslant 3$, we have

$$U_i(Q,q)\ll q^{1+\varepsilon}Q^{\varepsilon}.$$

Lemma 2.8 ([15], Lemma 4.7)**.** *For any real number ξ and all integers N_1, N_2 with $N_1<N_2$,*

$$\sum_{n=N_1+1}^{N_2} e(\xi n)\ll\min\{N_2-N_1,\|\xi\|^{-1}\}.$$

Now we introduce

$$N_1(H,q,k)=\frac{1}{q}\sum_{1\leqslant t\leqslant q-1}\lambda(q;-t,0,0,k)\sum_{1\leqslant h\leqslant H}e_q(ht), \tag{2.17}$$

$$N_2(H,q,k)=\frac{1}{q^2}\sum_{1\leqslant t_1,t_2\leqslant q-1}\lambda(q;-t_1,-t_2,0,k)\prod_{i=1}^{2}\Big(\sum_{1\leqslant h_i\leqslant H}e_q(h_it_i)\Big), \tag{2.18}$$

$$N_3(H,q,k)=\frac{1}{q^3}\sum_{1\leqslant t_1,t_2,t_3\leqslant q-1}\lambda(q;-t_1,-t_2,-t_3,k)\prod_{i=1}^{3}\Big(\sum_{1\leqslant h_i\leqslant H}e_q(h_it_i)\Big). \tag{2.19}$$

Lemma 2.9. *Let $H\geqslant 2$. Let d be odd and square-free. Suppose that $d^r\ll H^2$. If $k\neq 0$, then for $1\leqslant i\leqslant 3$ we have*

$$N_i(H,d^r,k)\ll d^{r+\varepsilon}.$$

P r o o f. By Lemma 2.8, for any $1 \leqslant t \leqslant q-1$, we have

$$\sum_{1\leqslant h\leqslant H} e_{d^r}(ht) \ll \left\| \frac{t}{d^r} \right\|^{-1}.$$

Then we obtain

$$N_i(H, d^r, k) \ll U_i\Big(\frac{d^r-1}{2}, d^r\Big).$$

The desired estimate follows from Lemma 2.7 directly. □

3. Proof of Theorem 1.1

Let

$$D(H,q,k) = \sum_{\substack{1\leqslant x,y,z\leqslant H\\ x^2+y^2+z^2+k\equiv 0 \pmod q}} 1.$$

Lemma 3.1. *Suppose that $q \ll H^2$. Then we have*

$$D(H,q,k) \ll q^{-1}H^{3+\varepsilon}. \tag{3.1}$$

In particular, one has

$$\lambda(q;k) \ll q^{2+\varepsilon}. \tag{3.2}$$

P r o o f. The proof is standard and we include the details for completeness. We have

$$D(H,q,k) = \sum_{|t|\leqslant(3H^2+|k|)/q}\ \sum_{1\leqslant z\leqslant H}\ \sum_{\substack{1\leqslant x,y\leqslant H\\ x^2+y^2=qt-z^2-k}} 1 \ll \sum_{|t|\leqslant(3H^2+|k|)/q}\ \sum_{1\leqslant z\leqslant H} H^\varepsilon$$
$$\ll q^{-1}H^{3+\varepsilon}.$$

This establishes (3.1). Note that $\lambda(q;k) = D(q,q,k)$. The estimate (3.2) follows immediately from (3.1). This completes the proof. □

Now we represent $R(H,r,k)$ in terms of $D(H,q,k)$.

Lemma 3.2. *Suppose that $1 \leqslant \eta \leqslant (3H^2+|k|)^{1/r}$. Then we have*

$$R(H,r,k) = \sum_{d\leqslant\eta} \mu(d)D(H,d^r,k) + O(H^{3+\varepsilon}\eta^{1-r}).$$

P r o o f. The starting point is to apply the identity

$$\sum_{d^r \mid n} \mu(d) = \begin{cases} 1 & \text{if } n \text{ is } r\text{-free}, \\ 0 & \text{if } n \neq 0 \text{ is not } r\text{-free}. \end{cases}$$

Then we deduce that

$$\begin{aligned} R(H,r,k) &= \sum_{\substack{1 \leqslant x,y,z \leqslant H \\ x^2+y^2+z^2+k \neq 0}} \sum_{d^r \mid x^2+y^2+z^2+k} \mu(d) \\ &= \sum_{1 \leqslant d \leqslant (3H^2+|k|)^{1/r}} \mu(d) \sum_{\substack{1 \leqslant x,y,z \leqslant H \\ x^2+y^2+z^2+k \equiv 0 \pmod{d^r} \\ x^2+y^2+z^2+k \neq 0}} 1. \end{aligned}$$

Note that

$$\sum_{\substack{1 \leqslant x,y,z \leqslant H \\ x^2+y^2+z^2+k \equiv 0 \pmod{q} \\ x^2+y^2+z^2+k \neq 0}} 1 = D(H,q,k) + O(1).$$

Now we conclude from the above that

$$\begin{aligned} R(H,r,k) &= \sum_{1 \leqslant d \leqslant (3H^2+|k|)^{1/r}} \mu(d) D(H,d^r,k) + \sum_{1 \leqslant d \leqslant (3H^2+|k|)^{1/r}} O(1) \\ &= \sum_{1 \leqslant d \leqslant (3H^2+|k|)^{1/r}} \mu(d) D(H,d^r,k) + O(H^{2/r}). \end{aligned}$$

Splitting the above summation into two parts, we obtain

$$R(H,r,k) = \sum_{1 \leqslant d \leqslant \eta} \mu(d) D(H,d^r,k) + \sum_{\eta < d \leqslant (3H^2+|k|)^{1/r}} \mu(d) D(H,d^r,k) + O(H^{2/r}).$$

On applying Lemma 3.1, we further obtain

$$R(H,r,k) = \sum_{1 \leqslant d \leqslant \eta} \mu(d) D(H,d^r,k) + O(H^{3+\varepsilon} \eta^{1-r}).$$

This completes the proof. □

Now we deal with $D(H,q,k)$ for $q \leqslant \eta^r$.

Lemma 3.3. *Let* $\lambda(q;k)$ *be defined in* (1.3) *and let* $N_i(H,q,k)$ *be defined in* (2.17)–(2.19). *We have*

$$D(H,q,k) = \frac{H^3}{q^3} \lambda(q;k) + 3\frac{H^2}{q^2} N_1(H,q,k) + 3\frac{H}{q} N_2(H,q,k) + N_3(H,q,k). \tag{3.3}$$

P r o o f. Note that

$$D(H,q,k)=\sum_{\substack{1\leqslant x,y,z\leqslant q\\ x^2+y^2+z^2+k\equiv 0 \pmod q}}\ \sum_{\substack{1\leqslant m,n,l\leqslant H\\ m\equiv x \pmod q\\ n\equiv y \pmod q\\ l\equiv z \pmod q}} 1.$$

By orthogonality, we have

$$\begin{aligned}\sum_{\substack{1\leqslant m\leqslant H\\ m\equiv x \pmod q}} 1 &= q^{-1}\sum_{1\leqslant h\leqslant H}\sum_{1\leqslant t\leqslant q} e_q((h-x)t)=q^{-1}\sum_{1\leqslant t\leqslant q} e_q(-xt)\sum_{1\leqslant h\leqslant H} e_q(ht)\\ &= Hq^{-1}+q^{-1}\sum_{1\leqslant t\leqslant q-1} e_q(-xt)\sum_{1\leqslant h\leqslant H} e_q(ht).\end{aligned}$$

Now we conclude from above that

$$D(H,q,k)=\sum_{\substack{1\leqslant x,y,z\leqslant q\\ x^2+y^2+z^2+k\equiv 0 \pmod q}}\Big(\frac{H^3}{q^3}+3\frac{H^2}{q^2}L_1(x)+3\frac{H}{q}L_2(x,y)+L_3(x,y,z)\Big), \tag{3.4}$$

where

$$L_1(x):=L_1(x;q,H)=\frac{1}{q}\sum_{1\leqslant t\leqslant q-1} e_q(-xt)\sum_{1\leqslant h\leqslant H} e_q(ht),$$

$$L_2(x,y):=L_2(x,y;q,H)=\frac{1}{q^2}\sum_{1\leqslant t_1,t_2\leqslant q-1} e_q(-xt_1-yt_2)\prod_{i=1}^{2}\Big(\sum_{1\leqslant h_i\leqslant H} e_q(h_it_i)\Big),$$

and

$$\begin{aligned}L_3(x,y,z)&:=L_3(x,y,z;q,H)\\ &=\frac{1}{q^3}\sum_{1\leqslant t_1,t_2,t_3\leqslant q-1} e_q(-xt_1-yt_2-zt_3)\prod_{i=1}^{3}\Big(\sum_{1\leqslant h_i\leqslant H} e_q(h_it_i)\Big).\end{aligned}$$

For convenience, we write $L_1(x,y,z;q,H)=L_1(x;q,H)$ and $L_2(x,y,z;q,H)=L_2(x,y;q,H)$. By exchanging the order of summations and recalling the definitions of $\lambda(q;n,m,l,k)$ and $N_i(H,q,k)$, we obtain

$$\sum_{\substack{1\leqslant x,y,z\leqslant q\\ x^2+y^2+z^2+k\equiv 0 \pmod q}} L_i(x,y,z;q,H)=N_i(H,q,k). \tag{3.5}$$

Now (3.3) follows from (3.4) and (3.5). $\square$

Proof of Theorem 1.1. We first deal with the case $k \neq 0$. By Lemmas 2.9 and 3.3, we can get

$$D(H, d^r, k) = H^3 \frac{\lambda(d^r; k)}{d^{3r}} + O(H^2 d^{\varepsilon - r} + H d^{\varepsilon} + d^{r+\varepsilon}).$$

By Lemma 3.2, we find that

$$R(H, r, k) = H^3 \sum_{1 \leqslant d \leqslant \eta} \frac{\mu(d)\lambda(d^r; k)}{d^{3r}} + O(H^2 + H\eta^{1+\varepsilon} + \eta^{r+1+\varepsilon} + H^{3+\varepsilon}\eta^{1-r})$$

and then by (3.2), we have

$$R(H, r, k) = H^3 \sum_{d=1}^{\infty} \frac{\mu(d)\lambda(d^r; k)}{d^{3r}} + O(H^2 + H\eta^{1+\varepsilon} + \eta^{r+1+\varepsilon} + H^{3+\varepsilon}\eta^{1-r}).$$

Now we choose $\eta = H^{3/2r}$ to conclude that

$$R(H, r, k) = H^3 \sum_{d=1}^{\infty} \frac{\mu(d)\lambda(d^r; k)}{d^{3r}} + O(H^2 + H^{3/2+3/2r+\varepsilon}).$$

Since $\lambda(q; k)$ is multiplicative as a function of q, we obtain for $k \neq 0$ that

$$R(H, r, k) = \prod_p \Big(1 - \frac{\lambda(p^r; k)}{p^{3r}}\Big) H^3 + O(H^2 + H^{3/2+3/2r+\varepsilon})$$

and this completes the proof of (1.5).

From now on, we consider the case $k = 0$. We introduce

$$T_1(H, \eta) = \sum_{1 \leqslant d \leqslant \eta} \frac{\mu(d)}{d^{2r}} N_1(H, d^r, 0), \quad T_2(H, \eta) = \sum_{1 \leqslant d \leqslant \eta} \frac{\mu(d)}{d^r} N_2(H, d^r, 0),$$
$$T_3(H, \eta) = \sum_{1 \leqslant d \leqslant \eta} \mu(d) N_3(H, d^r, 0).$$

By Lemma 3.2 and (3.3), we have

$$\begin{aligned} R(H, r, 0) = H^3 \sum_{1 \leqslant d \leqslant \eta} \frac{\mu(d)\lambda(d^r; 0)}{d^{3r}} + 3H^2 T_1(H, \eta) + 3H T_2(H, \eta) \\ + T_3(H, \eta) + O(H^{3+\varepsilon}\eta^{1-r}). \end{aligned} \tag{3.6}$$

Recalling the definition of $N_1(H, d^r, 0)$ in (2.17), we deduce from (2.9) and Lemma 2.8 that

$$\begin{aligned} T_1(H, \eta) &\ll \sum_{1 \leqslant d \leqslant \eta} d^{-2r} \sum_{1 \leqslant |t| \leqslant (d^r - 1)/2} \frac{|\lambda(d^r; t, 0, 0, 0)|}{|t|} \\ &\ll \sum_{1 \leqslant d \leqslant \eta} d^{-r+\varepsilon} \sum_{1 \leqslant |t| \leqslant (d^r - 1)/2} \frac{(d^r, t)(d^r, t^2)^{1/2}}{|t|}. \end{aligned}$$

Since $(d^r, t^2)^{1/2} \leqslant d^{r/2}$, we have

$$T_1(H,\eta) \ll \sum_{1 \leqslant d \leqslant \eta} d^{-r/2+\varepsilon} \sum_{1 \leqslant |t| \leqslant (d^r-1)/2} \frac{(d^r, t)}{t}.$$

From the elementary estimate

$$\sum_{1 \leqslant t \leqslant (d^r-1)/2} \frac{(d^r, t)}{t} \ll d^{\varepsilon},$$

we conclude for $r \geqslant 2$ that

$$T_1(H,\eta) \ll \sum_{1 \leqslant d \leqslant \eta} d^{-r/2+\varepsilon} \ll \eta^{\varepsilon}. \tag{3.7}$$

Recalling (2.18), we deduce from (2.9) and Lemma 2.8 that

$$\begin{aligned} T_2(H,\eta) &\ll \sum_{1 \leqslant d \leqslant \eta} d^{-r} \sum_{1 \leqslant |t_1|, |t_2| \leqslant (d^r-1)/2} \frac{|\lambda(d^r; t_1, t_2, 0, 0)|}{|t_1 t_2|} \\ &\ll \eta^{\varepsilon} \sum_{1 \leqslant d \leqslant \eta} \sum_{1 \leqslant |t_1|, |t_2| \leqslant (d^r-1)/2} \frac{(d^r, t_1^2 + t_2^2)^{1/2} (d^r, t_1, t_2)}{|t_1 t_2|}. \end{aligned}$$

Since $(d^r, t_1^2 + t_2^2)^{1/2} \leqslant d^{(r-2)/2}(d, t_1^2 + t_2^2) \leqslant \eta^{(r-2)/2}(d, t_1^2 + t_2^2)$ and $(d^r, t_1, t_2) \leqslant (t_1, t_2)$, we conclude that

$$T_2(H,\eta) \ll \eta^{(r-2)/2+\varepsilon} \sum_{1 \leqslant d \leqslant \eta} \sum_{1 \leqslant |t_1|, |t_2| \leqslant (d^r-1)/2} (d, t_1^2 + t_2^2) \frac{(t_1, t_2)}{t_1 t_2}.$$

By exchanging the order of summations, we have

$$T_2(H,\eta) \ll \eta^{(r-2)/2+\varepsilon} \sum_{1 \leqslant t_1, t_2 \leqslant \eta^t} \frac{(t_1, t_2)}{t_1 t_2} \sum_{1 \leqslant d \leqslant \eta} (d, t_1^2 + t_2^2). \tag{3.8}$$

Now we easily obtain

$$T_2(H,\eta) \ll \eta^{r/2+\varepsilon}. \tag{3.9}$$

We estimate the sum $T_3(H,\eta)$ in same way, that is

$$T_3(H,\eta) \ll \sum_{1 \leqslant d \leqslant \eta} d^{r+\varepsilon} \sum_{1 \leqslant |t_1|, |t_2|, |t_3| \leqslant (d^r-1)/2} \frac{(d^r, t_1^2 + t_2^2 + t_3^2)^{1/2} (d^r, t_1, t_2, t_3)}{|t_1 t_2 t_3|},$$

and by $(d^r, t_1^2 + t_2^2 + t_3^2)^{1/2} \leqslant d^{(r-2)/2}(d, t_1^2 + t_2^2 + t_3^2) \leqslant \eta^{(r-2)/2}(d, t_1^2 + t_2^2 + t_3^2)$ and $(d^r, t_1, t_2, t_3) \leqslant (t_1, t_2, t_3)$, we have

$$T_3(H,\eta) \ll \eta^{(3r-2)/2+\varepsilon} \sum_{1 \leqslant d \leqslant \eta} \sum_{1 \leqslant |t_1|, |t_2|, |t_3| \leqslant (d^r-1)/2} (d, t_1^2 + t_2^2 + t_3^2) \frac{(t_1, t_2, t_3)}{t_1 t_2 t_3}.$$

By a similar argument as in (3.8), we finally obtain

$$T_3(H,\eta) \ll \eta^{3r/2+\varepsilon}. \tag{3.10}$$

Now we combine (3.6)–(3.10) to conclude that

$$R(H,r,0) = H^3 \sum_{1\leqslant d\leqslant \eta} \frac{\mu(d)\lambda(d^r;0)}{d^{3r}} + O(H^{2+\varepsilon} + H\eta^{r/2+\varepsilon} + \eta^{3r/2+\varepsilon} + H^{3+\varepsilon}\eta^{1-r}).$$

Then by (3.2), we get

$$R(H,r,0) = H^3 \sum_{d=1}^{\infty} \frac{\mu(d)\lambda(d^r;0)}{d^{3r}} + O(H^{2+\varepsilon} + H\eta^{r/2+\varepsilon} + \eta^{3r/2+\varepsilon} + H^{3+\varepsilon}\eta^{1-r}).$$

Since $\lambda(q;0)$ is multiplicative, we obtain by choosing $\eta = H^{6/(5r-2)}$ that

$$R(H,r,0) = \prod_p \Big(1 - \frac{\lambda(p^r;0)}{p^{3r}}\Big) H^3 + O(H^{2+\varepsilon} + H^{9r/(5r-2)+\varepsilon}).$$

This establishes (1.6). The proof of Theorem 1.1 is complete. $\square$

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