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FOURIER DIFFRACTION THEOREM FOR THE TENSOR FIELDS

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Abstract. The paper is devoted to the electromagnetic inverse scattering problem for a dielectric anisotropic and magnetically isotropic media. The properties of an anisotropic medium with respect to electromagnetic waves are defined by the tensors, which give the relation between the inductions and the fields. The tensor Fourier diffraction theorem derived in the paper can be considered a useful tool for studying tensor fields in inverse problems of electromagnetic scattering. The method is based on the first Born approximation.

Keywords: diffraction tomography; tensor Green's function; Born approximation; Fourier transform; inverse scattering

MSC 2020: 78A46, 37K15, 35R30

1. INTRODUCTION

The electromagnetic inverse scattering problem is defined as the problem of determining the size, shape and electromagnetic properties of a scatterer for the given incident and scattered electromagnetic fields [5]. An inverse scattering has many important applications in such diverse fields as radar imaging processing [2], bio-medical diagnostics [17], [8], nondestructive testing [26], geophysical exploration [6]. The reviews on tomographic diffractive microscopy and inverse problems in the scattering theory are given in [24], [3], [25].

The mathematics of wave propagation and scattering essentially differs for various types of waves. Acoustic waves are considered strictly scalar and are used for studying scalar fields, whereas electromagnetic waves are used for investigation of

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vector and tensor fields and, as a consequence, various inversion techniques are used. A quite successful approach based on the linearization of the inverse scattering problem within the first Born or Rytov approximations leads to diffraction tomography [21], [16], [12]. The total field within the Born approach or, equivalently, the complex phase within the Rytov approach are measured over one or several plane surfaces outside the support volume of the object (scatterer, scattering potential) [12], [7].

The relationship between the Fourier transform of the scattered field and the Fourier transform of the scatterer is known as the Fourier Diffraction Theorem (FDT) [12], page 218, [22], or the generalized projection-slice theorem [7], page 372. In the scalar case, the method based on the FDT in the 3D space is essentially reduced to the decomposition of the scattered field into plane waves (Fourier spectral decomposition), and then selection of those wave vectors, which lie on the Ewald sphere, and the interpolation onto a regular sampling grid in Cartesian coordinates. However, many physical quantities such as stress, permittivity, permeability are often presented by the second rank tensor fields, and for determination of these properties more complicated techniques are needed. The possibility of symmetric strain tensor field recovering from integral data in three dimensions is considered in [18], [13] and the nonsymmetric one in [10]. The nonsymmetric tensor fields in dimension more than three are given in [18], [1].

In this paper, the electromagnetic inverse scattering problem is considered for determination of dielectric permittivity $\bar{\epsilon}(\mathbf{r})$ of an inhomogeneous anisotropic object (target, scatterer), which is a second rank tensor, the magnetic permeability $\bar{\mu}$ is assumed to be $\bar{\mu} = \mu \bar{\mathbf{I}}$. The object is immersed in a homogeneous and infinite medium (free-space scattering) characterized by constant ϵ_0 and μ_0 .

The source of the scattered field is an incident electromagnetic field $(\mathbf{E}_i, \mathbf{H}_i)$, which interacts with the object in bounded region D of $\mathbb{R}^3$ with parameters $\bar{\epsilon}(\mathbf{r})$ and $\bar{\mu}(\mathbf{r})$. The incident field $(\mathbf{E}_i, \mathbf{H}_i)$ in the source-free region is expected to satisfy the homogeneous Helmholtz equations:

$$\nabla^2 \mathbf{E}_i(\mathbf{r}) + k^2 \mathbf{E}_i(\mathbf{r}) = 0, \quad \nabla^2 \mathbf{H}_i(\mathbf{r}) + k^2 \mathbf{H}_i(\mathbf{r}) = 0, \quad k = \omega \sqrt{\epsilon_0 \mu_0}.$$

The scattering field $(\mathbf{E}_s, \mathbf{H}_s)$ is supposed to be measured in region $D_1 = \mathbb{R}^3 \setminus D$. Assume for simplicity that region D_1 is nonmagnetic ($\mu = \mu_0$) and isotropic with spatially homogeneous permittivity ϵ_0 . The problem is to determine the unknown electrical properties of the object (scatterer) in region D_1 by using different directions of incident field $\mathbf{E}_i$.

The total field anywhere in D_1 is the sum of the incident and scattered fields according to [7], Chapt. 11.9,

$$(1.1) \quad \mathbf{E}(\mathbf{r}) = \mathbf{E}_i(\mathbf{r}) + \mathbf{E}_s(\mathbf{r}), \quad \mathbf{H}(\mathbf{r}) = \mathbf{H}_i(\mathbf{r}) + \mathbf{H}_s(\mathbf{r}).$$

The general equations for the electromagnetic field in an anisotropic inhomogeneous media writes by the Maxwell equations [4], page 17,

$$(1.2) \quad \nabla \times \mathbf{E}(\mathbf{r}) = i\omega\mu_0\overline{\boldsymbol{\mu}}(\mathbf{r})\mathbf{H}(\mathbf{r}), \quad \nabla \times \mathbf{H}(\mathbf{r}) = -i\omega\varepsilon_0\overline{\boldsymbol{\varepsilon}}(\mathbf{r})\mathbf{E}(\mathbf{r}),$$

where μ_0 and ε_0 are a permeability and permittivity in vacuum, respectively.

Throughout the paper the following notations are used: bold letter means the vector-values variable, the overlined bold letter means tensor-values variable, ordinary letter means scalar or scalar function. The notation type $\mathbf{AB}$ means tensor product of vectors $\mathbf{A}$ and $\mathbf{B}$, i.e., it is an equivalent to $\mathbf{A} \otimes \mathbf{B}$. A short introduction to tensor notations is given in appendix (A.1).

The organization of this paper is as follows: In Section 2, the integral equation for the scattered field is written. In Sections 3, 4, the spectral representation of the scalar and tensor Green's functions in the mixed space $\mathbb{R}^3 = (\mathbb{R}^2, \mathbb{R}^1)$ with coordinates $(\mathbf{q}_\perp, z)$ are derived. The Fourier representations of the tensor Green's functions in these coordinates are presented, as well. The inversion procedure with the first Born approximation in the coordinate-free form is considered in Section 5. The appendix includes the tensor (dyadic) product definition, inverse formulas for some special dyadics, the mixed form of the Fourier transform, which is used for representation of the tensor Green's function. The appendix contains also the procedure of evaluation of contour integral in the complex plane.

2. THE INTEGRAL EQUATION FOR SCATTERING POTENTIAL

As mentioned earlier, only dielectric scatterer will be considered with the tensor permittivity $\overline{\boldsymbol{\varepsilon}}(\mathbf{r})$ and permeability $\overline{\boldsymbol{\mu}} = \mu\overline{\mathbf{I}}$, where μ is constant.

Equations (1.2) rewrites then as

$$(2.1) \quad \nabla \times \mathbf{E}(\mathbf{r}) = i\omega\mu_0\mu\overline{\mathbf{I}} \cdot \mathbf{H}(\mathbf{r}),$$

$$(2.2) \quad \nabla \times \mathbf{H}(\mathbf{r}) = -i\omega\varepsilon_0\overline{\boldsymbol{\varepsilon}}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}).$$

For a dielectric anisotropic medium (scalar μ), it is possible to obtain an equation where only $\mathbf{E}$ appears. Taking the curl of above Maxwell's (2.1) and replacing $\nabla \times \mathbf{H}$ with (2.2), the following equation for the electric field $\mathbf{E}$ in region D_1 is derived:

$$(2.3) \quad \nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - k^2\overline{\boldsymbol{\varepsilon}}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) = 0, \quad \mathbf{r} \in D_1,$$

where $k = \omega\sqrt{\varepsilon_0\mu_0\mu} = 2\pi/\lambda$ is the medium propagation constant associated with wavelength λ .

Equation (2.3) can be rewritten in the form [14]

$$(2.4) \quad \nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - k^2\mathbf{E}(\mathbf{r}) = \overline{\mathbf{V}}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}), \quad \overline{\mathbf{V}}(\mathbf{r}) = k^2[\overline{\boldsymbol{\varepsilon}}(\mathbf{r}) - \overline{\mathbf{I}}], \quad \mathbf{r} \in D_1,$$

where $\overline{\mathbf{V}}(\mathbf{r})$ can be considered as the scattering potential or the object which has to be reconstructed, $\overline{\mathbf{I}}$ is the unit dyad. In Cartesian coordinates $\overline{\mathbf{I}} = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z}$.

The solution of equation (2.4) is given by the vector Lippmann-Schwinger equation [7],

$$(2.5) \quad \mathbf{E}(\mathbf{r}) = \mathbf{E}_i(\mathbf{r}) + \int_D \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot \overline{\mathbf{V}}(\mathbf{r}') \cdot \mathbf{E}(\mathbf{r}') d\mathbf{r}', \quad \mathbf{r}' \in D,$$

$$\mathbf{E}_i(\mathbf{r}) = \hat{\mathbf{E}}_0 e^{i\mathbf{k} \cdot \mathbf{r}} \text{ is the incident field,}$$

where tensor function $\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}')$ is the dyadic Green's function and its representation is given in the next section, $\hat{\mathbf{E}}_0$ is a unit polarization vector.

The vector $\mathbf{k} = k_x \hat{x} + k_y \hat{y} + k_z \hat{z}$ gives the standard coordinates on $\mathbb{R}^3$ in the Fourier space (reciprocal space).

When studying the interaction of an electromagnetic field with various structures, one often deals with flat registration surfaces and interfaces, and it is convenient to represent $\mathbb{R}^3$ as $\mathbb{R}^3 = \mathbb{R}^{2+1}$ interchangeably, then $\mathbf{r} = \mathbf{r}_\perp + r_\parallel \hat{z}$ in a real space and $\mathbf{q} = \mathbf{q}_\perp + q_\parallel \hat{z}$ in the reciprocal space, that is, $\mathbf{r}_\perp, \mathbf{q}_\perp \in \mathbb{R}^2$ and $r_\parallel, q_\parallel \in \mathbb{R}^1$.

3. SPECTRAL DECOMPOSITION OF THE SCALAR GREEN'S FUNCTIONS

It is well known that the Fourier representation of the scalar Green's function for Helmholtz equation is

$$(\nabla^2 + k^2)G(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}'), \quad \tilde{G}(\mathbf{q}) = \frac{1}{q^2 - k^2},$$

$$q^2 = |\mathbf{q}|^2, \quad q_\perp^2 = |\mathbf{q}_\perp|^2, \quad q^2 = q_\perp^2 + q_\parallel^2,$$

where “ $\sim$ ” means 1D, 2D or 3D Fourier transforms depending on the context.

The spectral decomposition of the scalar Green's function in $(\mathbf{q}_\perp, z)$ coordinates can be obtained as one-dimensional inverse Fourier transform of $\tilde{G}(\mathbf{q})$ over $q_\parallel$ in the form (see in the appendix (A.10)–(A.12))

$$(3.1) \quad \tilde{G}(\mathbf{q}_\perp, z) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \frac{e^{iq_\parallel z} dq_\parallel}{q_\perp^2 + q_\parallel^2 - k^2} = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \frac{e^{iq_\parallel z} dq_\parallel}{q_\parallel^2 - (k^2 - q_\perp^2)}.$$

The theory of evaluation of such kind of integrals is well known, see for example [7], page 11, [20]. The integration performs in the complex plane, and it is briefly described in the appendix (A.15), since further a similar procedure can be used to

evaluate the spectral representation of the tensor Green's function:

$$(3.2) \quad \tilde{G}(\mathbf{q}_\perp, z) = \begin{cases} \sqrt{\frac{\pi}{2}} \frac{1}{(q_\perp^2 - k^2)^{1/2}} e^{-(q_\perp^2 - k^2)^{1/2}|z|}, & k^2 < q_\perp^2, \\ \sqrt{\frac{\pi}{2}} \frac{i}{(k^2 - q_\perp^2)^{1/2}} e^{i(k^2 - q_\perp^2)^{1/2}|z|}, & k^2 > q_\perp^2, \end{cases} = \sqrt{\frac{\pi}{2}} \frac{i}{p} e^{ip|z|},$$

$$p = \begin{cases} i(q_\perp^2 - k^2)^{1/2}, & k^2 < q_\perp^2, \\ (k^2 - q_\perp^2)^{1/2}, & k^2 > q_\perp^2. \end{cases}$$

The two-dimensional inverse Fourier transform of (3.2) leads to the well-known Weyl representation [11]

$$(3.3) \quad G(\mathbf{r}_\perp, z) = \frac{i}{2(2\pi)^{1/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{ip|z|} e^{i\mathbf{q}_\perp \cdot \mathbf{r}_\perp}}{p} d\mathbf{q}_\perp, \quad \mathbf{r}_\perp = (x - x', y - y').$$

4. SPECTRAL DECOMPOSITION OF THE TENSOR GREEN'S FUNCTIONS

The electric tensor Green's function $\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}')$ is a solution of the wave equation

$$(4.1) \quad \nabla \times \nabla \times \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') - k^2 \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = \overline{\mathbf{I}} \delta(\mathbf{r} - \mathbf{r}')$$

and is given as [19], [23], [4], [9],

$$(4.2) \quad \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = \left(\overline{\mathbf{I}} + \frac{\nabla \nabla}{k^2} \right) G(\mathbf{r}, \mathbf{r}'), \quad G(\mathbf{r}, \mathbf{r}') = \frac{\exp(ik|\mathbf{r} - \mathbf{r}'|)}{4\pi|\mathbf{r} - \mathbf{r}'|}.$$

Using the vector identity $\nabla \times \nabla \times = \nabla \nabla - \nabla^2$, equation (4.1) writes in the dyadic form as

$$(4.3) \quad [\nabla \nabla - (\nabla^2 + k^2) \overline{\mathbf{I}}] \cdot \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = \overline{\mathbf{I}} \delta(\mathbf{r} - \mathbf{r}').$$

Equation (4.3) in the Fourier space writes

$$(4.4) \quad [\overline{\mathbf{I}}(k^2 - |\mathbf{q}|^2) + \mathbf{q} \cdot \mathbf{q}] \cdot \widetilde{\overline{\mathbf{G}}}(\mathbf{q}) = -\overline{\mathbf{I}}, \quad \mathbf{q} = (q_x, q_y, q_z).$$

By use of relations (A.7) in the appendix, the Fourier transform of tensor Green's function writes as

$$(4.5) \quad \widetilde{\overline{\mathbf{G}}}(\mathbf{q}) = [\overline{\mathbf{I}}(k^2 - |\mathbf{q}|^2) + \mathbf{q} \cdot \mathbf{q}]^{-1} (-\overline{\mathbf{I}}) = \frac{\overline{\mathbf{I}}k^2 - \mathbf{q} \cdot \mathbf{q}}{k^2(|\mathbf{q}|^2 - k^2)}.$$

By making a transformation to $(\mathbf{q}_\perp, q_\parallel)$ coordinates and taking into account (A.11) in appendix, equation (4.5) rewrites as

$$\widetilde{\overline{\mathbf{G}}}(\mathbf{q}_\perp, z) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \frac{\overline{\mathbf{I}}k^2 - \mathbf{q} \cdot \mathbf{q}}{k^2(|\mathbf{q}|^2 - k^2)} e^{iq_\parallel z} dq_\parallel.$$

The integral above can possibly be evaluated using contour integration in the complex plane $q_{\parallel}$ but it is easier to use the explicit expression for the scalar Green's function (3.2). Firstly, we rewrite the operators ∇ and $\nabla\nabla$ in the form

$$\begin{aligned}\nabla &= \nabla_{\perp} + \hat{z} \frac{\partial}{\partial z}, \\ \nabla\nabla &= \left(\nabla_{\perp} + \hat{z} \frac{\partial}{\partial z} \right) \left(\nabla_{\perp} + \hat{z} \frac{\partial}{\partial z} \right) = \nabla_{\perp} \nabla_{\perp} + \hat{z} \hat{z} \frac{\partial^2}{\partial z^2} + (\nabla_{\perp} \hat{z} + \hat{z} \nabla_{\perp}) \frac{\partial}{\partial z}.\end{aligned}$$

In the mixed $(\bar{\mathbf{q}}_{\perp}, z)$ coordinates in the Fourier space, the operator $\nabla\nabla$ is written as (see (A.13), (A.14) in Appendix)

$$(4.6) \quad -\mathbf{q}_{\perp} \mathbf{q}_{\perp} + \hat{z} \hat{z} \frac{\partial^2}{\partial z^2} + i(\mathbf{q}_{\perp} \hat{z} + \hat{z} \mathbf{q}_{\perp}) \frac{\partial}{\partial z},$$

which yields the following representation for $\tilde{\mathbf{G}}(\mathbf{q}_{\perp}, z)$:

$$(4.7) \quad \tilde{\mathbf{G}}(\mathbf{q}_{\perp}, z) = \sqrt{\frac{\pi}{2}} \frac{e^{ip|z|}}{pk^2} [\bar{\mathbf{I}} k^2 - \mathbf{q}_{\perp} \mathbf{q}_{\perp} - (\hat{z} \mathbf{q}_{\perp} + \mathbf{q}_{\perp} \hat{z}) p \operatorname{sgn}(z) - \hat{z} \hat{z} p^2],$$

where p is the same variable as in (3.2).

If on the plane $\mathbf{q}_{\perp}$ perpendicular to $\hat{z}$ a unit vector is chosen, for instance $\hat{\mathbf{q}}_{\perp}^{(1)}$, another unit vector is defined as $\hat{\mathbf{q}}_{\perp}^{(2)} = \hat{z} \times \hat{\mathbf{q}}_{\perp}^{(1)}$, then the unit dyad $\bar{\mathbf{I}}$ in equation (4.7) has the form

$$\bar{\mathbf{I}} = \hat{z} \hat{z} + \hat{\mathbf{q}}_{\perp}^{(1)} \hat{\mathbf{q}}_{\perp}^{(1)} + \hat{z} \times \hat{\mathbf{q}}_{\perp}^{(1)} \hat{\mathbf{q}}_{\perp}^{(1)} \times \hat{z}.$$

5. INVERSION WITH THE FIRST BORN APPROXIMATION

In the case when the contrast of the scatterer is weak, so that the second term on the right of equation (2.5) is small compared to the first term, the approximation $\mathbf{E}(\mathbf{r}) \simeq \mathbf{E}_i(\mathbf{r})$ can be applied. This is known as the first order Born approximation [15]. Then the scattered field in Born approximation with taking into account (2.5) writes as

$$(5.1) \quad \mathbf{E}_{sc}(\mathbf{r}) = \hat{\mathbf{E}}_0 \int_D \bar{\mathbf{G}}(\mathbf{r} - \mathbf{r}') \cdot \bar{\mathbf{V}}(\mathbf{r}') \cdot e^{i\mathbf{k} \cdot \mathbf{r}'} d\mathbf{r}', \quad \mathbf{r}' \in D.$$

Applying the convolution theorem, the Fourier transform of scattered field writes as

$$(5.2) \quad \tilde{\mathbf{E}}_{sc}(\mathbf{q}) = (2\pi)^{3/2} \hat{\mathbf{E}}_0 \cdot \tilde{\mathbf{G}}(\mathbf{q}) \cdot \tilde{\mathbf{V}}(\mathbf{q} - \mathbf{k}).$$

Formula (5.2) expresses the 3D Fourier representation of the scattered field in terms of 3D Fourier spatial transform of scattering potential $\bar{\mathbf{V}}(\mathbf{r})$. However, it is well

known [21] that the Fourier components of the scalar scattering potential may be determined from the knowledge of two-dimensional Fourier transform data of the scattered field in the two planes $z = z^+ > z_0$ and $z = z^- < 0$.

Theorem 5.1 (Tensor Fourier Diffraction Theorem). *Let the scattered field $\mathbf{E}_{sc}(\mathbf{r}_\perp, r_\parallel) \in (L^2(\mathbb{R}^{2+1}))^3$ and the tensor potential $\bar{\mathbf{V}}(\mathbf{r}_\perp, r_\parallel) \in (L^2(\mathbb{R}^{2+1}))^{3 \times 3}$. Then the two-dimensional Fourier transform of the scattered field on the plane perpendicular to z axis at the distance $z = z_0$ relates to the Fourier transform of $\bar{\mathbf{V}}(\mathbf{r}_\perp, r_\parallel)$ as*

$$(5.3) \quad \tilde{E}_{sc}(\mathbf{q}_\perp, z_0) = \frac{1}{2\pi^2} e^{iz_0 p} \frac{i}{pk^2} \hat{\mathbf{E}}_0 \cdot \bar{\mathbf{B}} \cdot \tilde{\mathbf{V}}(\mathbf{q}_\perp - \mathbf{k}_\perp, p - k_\parallel),$$

where $\hat{\mathbf{E}}_0$ is the polarization unit vector of the incident field

$$p = \begin{cases} i(q_\perp^2 - k^2)^{1/2}, & k^2 < q_\perp^2, \\ (k^2 - q_\perp^2)^{1/2}, & k^2 > q_\perp^2, \end{cases}$$

$$\bar{\mathbf{B}} = \bar{\mathbf{I}}k^2 - \mathbf{q}_\perp \mathbf{q}_\perp - (\hat{\mathbf{z}} \mathbf{q}_\perp + \mathbf{q}_\perp \hat{\mathbf{z}}) p \operatorname{sgn}(z) - \hat{\mathbf{z}} \hat{\mathbf{z}} p^2.$$

Proof. We rewrite equation (5.1) in $\mathbf{r} = \mathbf{r}_\perp + r_\parallel \hat{\mathbf{z}}$ and $\mathbf{q} = \mathbf{q}_\perp + q_\parallel \hat{\mathbf{z}}$ coordinates and take the two-dimensional Fourier transform over variable $\mathbf{r}_\perp$ on both sides. Using the convolution theorem, one can get

$$(5.4) \quad \begin{aligned} & \tilde{E}_{sc}(\mathbf{q}_\perp, r_\parallel) \\ &= \frac{1}{(2\pi)} \hat{\mathbf{E}}_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{\mathbf{G}}(\mathbf{r}_\perp - \mathbf{r}'_\perp, r_\parallel - r'_\parallel) \bar{\mathbf{V}}(\mathbf{r}'_\perp, r'_\parallel) e^{i\mathbf{k} \cdot \mathbf{r}'} e^{-i\mathbf{q}_\perp \cdot \mathbf{r}_\perp} d\mathbf{r}_\perp d\mathbf{r}'_\perp dr'_\parallel \\ &= 2\pi \hat{\mathbf{E}}_0 \int_{-\infty}^{\infty} \tilde{\bar{\mathbf{G}}}(\mathbf{q}_\perp, r_\parallel - r'_\parallel) \tilde{\bar{\mathbf{V}}}(\mathbf{q}_\perp - \mathbf{k}_\perp, r'_\parallel) e^{ik_\parallel r'_\parallel} dr'_\parallel. \end{aligned}$$

Let now suppose that the measurements are performed on the plane $z = r_\parallel = z_0$ and $z_0 > r'_\parallel$. Then the scattered field writes as

$$(5.5) \quad \begin{aligned} \tilde{E}_{sc}(\mathbf{q}_\perp, z_0) &= 2\pi \hat{\mathbf{E}}_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{\bar{\mathbf{G}}}(\mathbf{q}_\perp, r_\parallel - r'_\parallel) \tilde{\bar{\mathbf{V}}}(\mathbf{q}_\perp - \mathbf{k}_\perp, r'_\parallel) \delta(r_\parallel - z_0) e^{ik_\parallel r'_\parallel} dr_\parallel dr'_\parallel, \\ &= 2\pi \hat{\mathbf{E}}_0 \int_{-\infty}^{\infty} \tilde{\bar{\mathbf{G}}}(\mathbf{q}_\perp, z_0 - r'_\parallel) \tilde{\bar{\mathbf{V}}}(\mathbf{q}_\perp - \mathbf{k}_\perp, r'_\parallel) e^{ik_\parallel r'_\parallel} dr'_\parallel. \end{aligned}$$

The Fourier transform of the tensor Green's function is (4.7)

$$(5.6) \quad \tilde{\bar{\mathbf{G}}}(\mathbf{q}_\perp, z) = \sqrt{\frac{\pi}{2}} \frac{ie^{ip|z|}}{pk^2} \bar{\mathbf{B}}, \quad \bar{\mathbf{B}} = \bar{\mathbf{I}}k^2 - \mathbf{q}_\perp \mathbf{q}_\perp - (\hat{\mathbf{z}} \mathbf{q}_\perp + \mathbf{q}_\perp \hat{\mathbf{z}}) p \operatorname{sgn}(z) - \hat{\mathbf{z}} \hat{\mathbf{z}} p^2,$$

$$p = \begin{cases} i(q_\perp^2 - k^2)^{1/2}, & k^2 < q_\perp^2, \\ (k^2 - q_\perp^2)^{1/2}, & k^2 > q_\perp^2. \end{cases}$$

Finally, after substitution of (5.6) with $z = z_0 - r'_\parallel$ into (5.5) and integration, the tensor Fourier diffraction theorem writes as

$$(5.7) \quad \tilde{E}_{sc}(\mathbf{q}_\perp, z_0) = 2\pi^2 e^{iz_0 p} \frac{i}{pk^2} \hat{\mathbf{E}}_0 \cdot \overline{\mathbf{B}} \cdot \tilde{\mathbf{V}}(\mathbf{q}_\perp - \mathbf{k}_\perp, p - k_\parallel).$$

The tensor potential is defined by the following procedure:

$$(5.8) \quad \tilde{\mathbf{V}}(\mathbf{q}_\perp - \mathbf{k}_\perp, p - k_\parallel) = \frac{1}{2\pi^2} e^{-iz_0 p} (-i) p k^2 \overline{\mathbf{B}}^{-1} (\hat{\mathbf{E}}_0 \cdot \tilde{E}_{sc}(\mathbf{q}_\perp, z_0)),$$

$$(5.9) \quad \overline{\mathbf{B}}^{-1} = [\overline{\mathbf{I}}(k^2 - p^2) + (\hat{\mathbf{z}}\mathbf{q}_\perp + \mathbf{q}_\perp\hat{\mathbf{z}})p + \hat{\mathbf{z}}\hat{\mathbf{z}}(p^2 - q_\perp^2) - (\hat{\mathbf{z}} \times \mathbf{q}_\perp)(\hat{\mathbf{z}} \times \mathbf{q}_\perp)].$$

The evaluation of relation (5.9) is given in appendix (A.8)–(A.9). $\square$

6. CONCLUSION

This paper has represented a coordinate-independent dyadic formulation of the Fourier Diffraction Theorem (FDT) for the tensor-valued functions (5.7), (5.8). The tensor (dyadic) Green's function has been used as an important instrument for representation of FDT. The mixed representation for the dyadic Green's functions is obtained in the reciprocal space. In the process of representation of the tensor FDT, the auxiliary integrals have been evaluated. By direct matrix calculation some useful matrix identities have been obtained in the appendix (A.5), (A.7), (A.9).

APPENDIX A

1. Dyadic product. A tensor (dyadic) product of the two vectors $\mathbf{A}$ and $\mathbf{B}$ in Cartesian coordinates is denoted as adjacent to each other without dot or cross in terms, that is,

$$(A.1) \quad \mathbf{AB} = (A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3)(B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3) = \sum_{i=1}^3 \sum_{j=1}^3 A_i B_j \mathbf{e}_i \mathbf{e}_j.$$

2. Inversion of the special dyadic. Let vector $\mathbf{q}$ be represented as $\mathbf{q} = (q_x, q_y, q_z)$, and matrix $\overline{\mathbf{A}}$ have the form

$$(A.2) \quad \overline{\mathbf{A}} = \overline{\mathbf{I}}k^2 - \mathbf{q} \cdot \mathbf{q} = \begin{pmatrix} k^2 - q_1^2 & -q_1 q_2 & -q_1 q_3 \\ -q_2 q_1 & k^2 - q_2^2 & -q_2 q_3 \\ -q_3 q_1 & -q_3 q_2 & k^2 - q_3^2 \end{pmatrix}, \quad \overline{\mathbf{I}} \text{ is a unit matrix.}$$

Then the determinant of the matrix writes

$$(A.3) \quad \det \overline{\mathbf{A}} = k^4(k^2 - |\mathbf{q}|^2).$$

The adjoint matrix $\text{adj } \overline{\mathbf{A}}$ for the matrix $\overline{\mathbf{A}}$ is defined as

$$\begin{aligned}
 (\text{A.4}) \quad \text{adj } \overline{\mathbf{A}} &= \begin{pmatrix} A_{22}A_{33} - A_{23}A_{32} & A_{32}A_{13} - A_{33}A_{12} & A_{12}A_{23} - A_{13}A_{22} \\ A_{31}A_{23} - A_{21}A_{33} & A_{11}A_{33} - A_{13}A_{31} & A_{13}A_{21} - A_{11}A_{23} \\ A_{21}A_{32} - A_{31}A_{22} & A_{12}A_{31} - A_{11}A_{32} & A_{11}A_{22} - A_{12}A_{21} \end{pmatrix} \\
 &= k^2 \begin{pmatrix} k^2 - q_y^2 - q_z^2 & q_x q_y & q_x q_z \\ q_y q_x & k^2 - q_x^2 - q_z^2 & q_y q_z \\ q_z q_x & q_z q_y & k^2 - q_x^2 - q_y^2 \end{pmatrix} \\
 &= k^2 [\overline{\mathbf{I}}(k^2 - |\mathbf{q}|^2) + \mathbf{q} \cdot \mathbf{q}].
 \end{aligned}$$

An inverse matrix $\overline{\mathbf{A}}^{-1}$ from equations (A.3), (A.4) has the form

$$(\text{A.5}) \quad \overline{\mathbf{A}}^{-1} = \frac{\text{adj } \overline{\mathbf{A}}}{\det \overline{\mathbf{A}}} = \frac{\overline{\mathbf{I}}(k^2 - |\mathbf{q}|^2) + \mathbf{q} \cdot \mathbf{q}}{k^2(k^2 - |\mathbf{q}|^2)}.$$

Let dyadic $\overline{\mathbf{A}}$ have the form

$$(\text{A.6}) \quad \overline{\mathbf{A}} = \overline{\mathbf{I}}\beta + \mathbf{q} \cdot \mathbf{q} = \begin{pmatrix} \beta + q_1^2 & q_1 q_2 & q_1 q_3 \\ q_2 q_1 & \beta + q_2^2 & q_2 q_3 \\ q_3 q_1 & q_3 q_2 & \beta + q_3^2 \end{pmatrix}.$$

Then the determinant and the adjoint matrix are calculated directly and are equal:

$$\begin{aligned}
 (\text{A.7}) \quad \det \overline{\mathbf{A}} &= \beta^2(\beta + |\mathbf{q}|^2), \quad |\mathbf{q}|^2 = q_1^2 + q_2^2 + q_3^2, \\
 \text{adj } \overline{\mathbf{A}} &= \beta[(\beta + |\mathbf{q}|^2)\overline{\mathbf{I}} - \mathbf{q} \cdot \mathbf{q}], \\
 (\overline{\mathbf{I}}\beta + \mathbf{q} \cdot \mathbf{q})^{-1} &= \overline{\mathbf{A}}^{-1} = \frac{\text{adj } \overline{\mathbf{A}}}{\det \overline{\mathbf{A}}} = \frac{\overline{\mathbf{I}}(\beta + |\mathbf{q}|^2) - \mathbf{q} \cdot \mathbf{q}}{\beta(\beta + |\mathbf{q}|^2)}.
 \end{aligned}$$

Let dyadic $\overline{\mathbf{B}}$ have the form

$$\begin{aligned}
 (\text{A.8}) \quad \overline{\mathbf{B}} &= [\overline{\mathbf{I}}k^2 - \mathbf{q}_\perp \mathbf{q}_\perp - (\hat{\mathbf{z}} \mathbf{q}_\perp + \mathbf{q}_\perp \hat{\mathbf{z}})p - \hat{\mathbf{z}} \hat{\mathbf{z}} p^2] \\
 &= \begin{pmatrix} k^2 - q_x^2 & -q_x q_y & -q_x p \\ -q_y q_x & k^2 - q_y^2 & -q_y p \\ -q_x p & -q_y p & k^2 - p^2 \end{pmatrix}, \\
 \text{adj } \overline{\mathbf{B}} &= k^2 \begin{pmatrix} k^2 - p^2 - q_y^2 & q_x q_y & q_x p \\ q_y q_x & k^2 - p^2 - q_x^2 & q_y p \\ q_x p & q_y p & k^2 - q_x^2 - q_y^2 \end{pmatrix} \\
 &= k^2 \overline{\mathbf{I}}(k^2 - p^2) + (\hat{\mathbf{z}} \mathbf{q}_\perp + \mathbf{q}_\perp \hat{\mathbf{z}})p + \hat{\mathbf{z}} \hat{\mathbf{z}}(p^2 - q_\perp^2) - (\hat{\mathbf{z}} \times \mathbf{q}_\perp)(\hat{\mathbf{z}} \times \mathbf{q}_\perp), \\
 \det \overline{\mathbf{B}} &= k^4(k^2 - p^2 - q_x^2 - q_y^2), \\
 (\text{A.9}) \quad \overline{\mathbf{B}}^{-1} &= \frac{\text{adj } \overline{\mathbf{B}}}{\det \overline{\mathbf{B}}} \\
 &= \frac{\overline{\mathbf{I}}(k^2 - p^2) + (\hat{\mathbf{z}} \mathbf{q}_\perp + \mathbf{q}_\perp \hat{\mathbf{z}})p + \hat{\mathbf{z}} \hat{\mathbf{z}}(p^2 - q_\perp^2) - (\hat{\mathbf{z}} \times \mathbf{q}_\perp)(\hat{\mathbf{z}} \times \mathbf{q}_\perp)}{k^2(k^2 - p^2 - |\mathbf{q}_\perp|^2)}.
 \end{aligned}$$

3. Mixed form of the Fourier transform representation. For a vector function $\mathbf{f}(\mathbf{r})$ in a Cartesian coordinates $\mathbf{r} = (x, y, z)$ the Fourier transform with respect only to the two variables x and y can be introduced as

$$(A.10) \quad \tilde{f}(\mathbf{q}_\perp, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{f}(\mathbf{r}) e^{-i\mathbf{q}_\perp \cdot \mathbf{r}_\perp} d\mathbf{r}_\perp, \quad \mathbf{q}_\perp = (q_x, q_y, 0), \quad \mathbf{r}_\perp = (x, y, 0).$$

Alternatively, it can be written

$$(A.11) \quad \tilde{f}(\mathbf{q}_\perp, z) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \mathbf{f}(\mathbf{q}) e^{iq_\parallel z} dq_\parallel, \quad \mathbf{q} = (\mathbf{q}_\perp, q_\parallel).$$

The inverse transformation to (A.10) is

$$(A.12) \quad \mathbf{f}(\mathbf{r}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(\mathbf{q}_\perp, z) e^{i\mathbf{q}_\perp \cdot \mathbf{r}_\perp} d\mathbf{q}_\perp.$$

That is, the relations for the functions in real and reciprocal spaces can be written as

$$\mathbf{f}(\mathbf{r}) \xleftrightarrow{\mathcal{F}_2} \tilde{f}(\mathbf{q}_\perp, z) \quad \text{and} \quad \tilde{f}(\mathbf{q}_\perp, z) \xleftrightarrow{\mathcal{F}_1} \tilde{f}(\mathbf{q}_\perp, q_\parallel),$$

where $\mathcal{F}_2$ and $\mathcal{F}_1$ mean two- and one-dimensional Fourier transforms.

The operators ∇ and $\nabla\nabla$ have, respectively, the following representations:

$$(A.13) \quad \nabla \xleftrightarrow{\mathcal{F}_2} i\mathbf{q}_\perp + \hat{z} \frac{\partial}{\partial z},$$

$$(A.14) \quad \begin{aligned} \nabla\nabla \xleftrightarrow{\mathcal{F}_2} & \left(i\mathbf{q}_\perp + \hat{z} \frac{\partial}{\partial z} \right) \left(i\mathbf{q}_\perp + \hat{z} \frac{\partial}{\partial z} \right) \\ & = -\mathbf{q}_\perp \mathbf{q}_\perp + \hat{z} \hat{z} \frac{\partial^2}{\partial z^2} + i(\mathbf{q}_\perp \hat{z} + \hat{z} \mathbf{q}_\perp) \frac{\partial}{\partial z}. \end{aligned}$$

4. Contour integration. The spectral decomposition of the scalar Green's function in coordinates $(\mathbf{q}_\perp, z)$ can be obtained as one-dimensional inverse Fourier transform of $\tilde{G}(\mathbf{q})$ over $q_\parallel$ in the form

$$(A.15) \quad \begin{aligned} \tilde{G}(\mathbf{q}_\perp, z) &= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \frac{e^{iq_\parallel z} dq_\parallel}{q_\perp^2 + q_\parallel^2 - k^2} = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \frac{e^{iq_\parallel z} dq_\parallel}{q_\parallel^2 - (k^2 - q_\perp^2)} \\ &= \frac{1}{(2\pi)^{1/2}} \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{e^{iq_\parallel z} dq_\parallel}{q_\parallel^2 - p^2 - i\varepsilon}, \end{aligned}$$

where p is introduced as

$$p = \begin{cases} i(q_\perp^2 - k^2)^{1/2}, & k^2 < q_\perp^2, \\ (k^2 - q_\perp^2)^{1/2}, & k^2 > q_\perp^2. \end{cases}$$

In the case $q_{\perp}^2 < k^2$, there are two first-order poles

$$q_{\parallel} = \pm(p^2 + i\varepsilon)^{1/2} = \pm p(1 + i\varepsilon/p^2) \Rightarrow \pm p \pm i\varepsilon,$$

where ε is redefined with $p > 0$, so that it is still small and has the same sign. The pole $q_{\parallel} = -p - i\varepsilon$ does not matter since it lies outside the contour of integration closing in the upper half plane. However, the pole $q_{\parallel} = p + i\varepsilon$ is inside.

If $q_{\perp}^2 > k^2$ in the upper half plane $p = i(q_{\perp}^2 - k^2)^{1/2}$, then the pole lies on the imaginary axis. Using the residue theorem with the closing integration contour in the upper half plane, one finally gets

$$(A.16) \quad \tilde{G}(\mathbf{q}_{\perp}, z) = \begin{cases} \sqrt{\frac{\pi}{2}} \frac{1}{(q_{\perp}^2 - k^2)^{1/2}} e^{-(q_{\perp}^2 - k^2)^{1/2}|z|}, & k^2 < q_{\perp}^2 \\ \sqrt{\frac{\pi}{2}} \frac{i}{(k^2 - q_{\perp}^2)^{1/2}} e^{i(k^2 - q_{\perp}^2)^{1/2}|z|}, & k^2 > q_{\perp}^2 \end{cases} = \sqrt{\frac{\pi}{2}} \frac{i}{p} e^{ip|z|},$$

$$p = \begin{cases} i(q_{\perp}^2 - k^2)^{1/2}, & k^2 < q_{\perp}^2, \\ (k^2 - q_{\perp}^2)^{1/2}, & k^2 > q_{\perp}^2. \end{cases}$$

Taking the two-dimensional inverse Fourier transform of (A.16), the well-known Weyl identity is deduced [11]:

$$(A.17) \quad G(\mathbf{r}_{\perp}, z) = \frac{i}{2(2\pi)^{1/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{ip|z|} e^{i\mathbf{q}_{\perp} \cdot \mathbf{r}_{\perp}}}{p} d\mathbf{q}_{\perp}, \quad \mathbf{r}_{\perp} = (x - x', y - y').$$

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