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SPATIAL DECAY ESTIMATES FOR THE FORCHHEIMER FLUID EQUATIONS IN A SEMI-INFINITE CYLINDER

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Abstract. The spatial behavior of solutions is studied in the model of Forchheimer equations. Using the energy estimate method and the differential inequality technology, exponential decay bounds for solutions are derived. To make the decay bounds explicit, we obtain the upper bound for the total energy. We also extend the study of spatial behavior of Forchheimer porous material in a saturated porous medium.

Keywords: spatial behavior; Forchheimer equations; energy estimate bounds; upper bound; porous medium

MSC 2020: 35B40, 35Q30, 76D05

1. Introduction

In this paper, we introduce a semi-infinite cylinder whose generators are assumed to be parallel to the x_3 axis (see Figure 1), i.e.,

$$R = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in D,\ x_3 \geqslant 0\},$$

where D is a bounded region in $x_1 O x_2$ and has smooth boundary ∂D.

Let R_z and D_z denote the sub-region of R and the cross section of R at $x_3 = z$, respectively. We introduce the notations

$$R_z = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in D,\ x_3 > z \geqslant 0\},$$
$$D_z = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in D,\ x_3 = z \geqslant 0\},$$

where z is a variable running along the x_3 axis.

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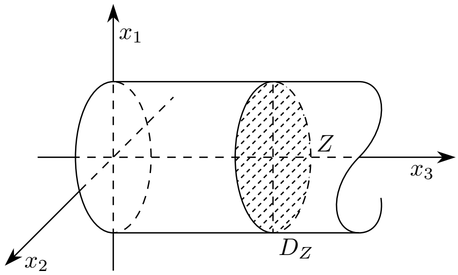

Figure 1. Cylindrical pipe R.

Let $\boldsymbol{u} = (u_1, u_2, u_3)$ and p denote the average fluid velocity and pressure in the porous medium, respectively. The Forchheimer equations we consider in this paper are (see [7])

$$u_{i,t} = -au_i - b|\boldsymbol{u}|u_i - c|\boldsymbol{u}|^2u_i - p_{,i} \quad \text{in } R \times (0,\infty), \tag{1.1}$$

$$u_{i,i} = 0 \quad \text{in } R \times (0,\infty), \tag{1.2}$$

where a is the Darcy coefficient (viscosity divided by permeability), b and c are the Forchheimer coefficients. Throughout this paper, the usual summation convention is employed with repeated Latin subscripts summed from 1 to 3 and repeated Greek subscripts summed from 1 to 2. The comma is used to indicate partial differentiation, i.e., $u_{i,j} = \partial u_i/\partial x_j$, $\varphi_{\alpha,\alpha} = \sum_{\alpha=1}^{2} \partial\varphi_\alpha/\partial x_\alpha$. Firdaouss et al. [6] and Giorgi [9] have given the derivations of the equations (1.1) and (1.2). When $b = 0$ but c is nonzero, Néel [31] considered the fluid by employing (1.1) and a temperature field. Payne and Straughan [36] discussed the effect of the Forchheimer terms in (1.1) (b and c terms) in the context of nonlinear stability in thermal convection. Franchi and Straughan [7] obtained the continuous dependence on the Forchheimer coefficient if equations (1.1) and (1.2) are defined in a bounded region.

The aim of this paper is to assume that the nonhomogeneous boundary conditions are satisfied at the finite end of the main body, and prove that the solution decays exponentially as $z \to \infty$. This type of decay results can be regarded as the category of the Saint-Venant principle. Saint-Venant's principle is a famous mathematical and mechanical principle which was conjectured by de Saint-Venant in his paper [5]. Early works on Saint-Venant's principle primarily focused on the decay results for the initial-boundary value problems for elliptic equation (see, e.g., [10]–[12]). After the origin work of Boley [1], extensive attention has been paid to parabolic problems. We mention in particular the papers of Knowles [16], [17], Quintanilla et al. [37], [38], Knops and Quintanilla [14], [15], [30], [42], Liu et al. [25], [27]–[29], [39], Li et al. [4], [19]–[23]. For more papers one can see [2], [3], [13], [18], [24], [26].

A major motivation of the present paper is to derive the spatial decay bounds of solutions to Forchheimer equations in a semi-infinite cylinder. There is little attention in the literature on the spatial properties of models with nonlinear terms. We also extend this type of research to Forchheimer porous materials. The results of this paper lay a foundation for further study of the structural stability of the solutions to (1.1)–(1.2) in a cylinder.

Equations (1.1)–(1.2) also satisfy the initial-boundary conditions

$$u_i = 0 \quad \text{on } \partial D \times \{x_3 > 0\} \times (0,\infty), \tag{1.3}$$

$$u_3 = f(x_1, x_2, t) \quad \text{on } D \times (0,\infty), \tag{1.4}$$

$$u_i(x_1, x_2, 0) = 0 \quad \text{in } R, \tag{1.5}$$

$$|\boldsymbol{u}|, u_3, p = o(x_3^{-1}), \tag{1.6}$$

where f is a differentiable function which is assumed to satisfy appropriate compatibility conditions.

In this paper, we will frequently use the following lemmas.

Lemma 1.1 ([33]). *Assume that D is a plane domain with sufficiently smooth boundary ∂D, and v a sufficiently smooth function defined on the closure of D. If $v|_{\partial D} = 0$, then*

$$\lambda_1 \int_D v^2 \,\mathrm{d}A \leqslant \int_D v_{,\alpha} v_{,\alpha} \,\mathrm{d}A,$$

where λ_1 is the smallest positive eigenvalue of

$$\varphi_{,\alpha\alpha} + \lambda\varphi = 0 \quad \text{in } D, \quad \varphi = 0 \quad \text{on } \partial D.$$

Lemma 1.2 ([33]). *If $v|_{\partial D} = 0$, then there exists a positive k_1 such that*

$$\int_D v^4 \,\mathrm{d}A \leqslant k_1 \int_D v^2 \,\mathrm{d}A \int_D v_{,\alpha} v_{,\alpha} \,\mathrm{d}A.$$

Lemma 1.3 ([33]). *If $\int_D w \,\mathrm{d}A = 0$, then there exists a vector function $\boldsymbol{v} = (v_1, v_2)$ such that*

$$v_{\alpha,\alpha} = w \quad \text{in } D, \quad v_\alpha = 0 \quad \text{on } \partial D,$$

and a positive constant k_2 depending only on the geometry of D such that

$$\int_D v_{\alpha,\beta} v_{\alpha,\beta} \,\mathrm{d}A \leqslant k_2 \int_D v_{\alpha,\alpha}^2 \,\mathrm{d}A. \tag{1.7}$$

2. Main results

In this paper, we establish the "energy" function

$$F(z,t)=\int_0^t\int_{D_z}\mathrm{e}^{-\omega\eta}pu_{3,\eta}\,\mathrm{d}A\,\mathrm{d}\eta, \tag{2.1}$$

where ω is a positive constant to be determined later.

Using the divergence theorem and the equations (1.1)–(1.6), from (2.1) we have

$$\begin{aligned} F(z,t)&=-\int_0^t\int_{R_z}\mathrm{e}^{-\omega\eta}p_{,i}u_i\,\mathrm{d}x\,\mathrm{d}\eta\\ &=\int_0^t\int_{R_z}\mathrm{e}^{-\omega\eta}\Big[u_{i,\eta}+au_i+b|\boldsymbol{u}|u_i+c|\boldsymbol{u}|^2u_i\Big]u_{i,\eta}\,\mathrm{d}x\,\mathrm{d}\eta\\ &=\int_0^t\int_{R_z}\mathrm{e}^{-\omega\eta}\Big[u_{i,\eta}u_{i,\eta}+\frac12a\omega|\boldsymbol{u}|^2+\frac13b\omega|\boldsymbol{u}|^3+\frac14c\omega|\boldsymbol{u}|^4\Big]\,\mathrm{d}x\,\mathrm{d}\eta\\ &\quad+\mathrm{e}^{-\omega t}\int_{R_z}\Big[\frac12a|\boldsymbol{u}|^2+\frac13b|\boldsymbol{u}|^3+\frac14c|\boldsymbol{u}|^4\Big]\,\mathrm{d}x. \end{aligned} \tag{2.2}$$

Therefore,

$$\begin{aligned} -\frac{\partial}{\partial z}F(z,t)&=\int_0^t\int_{D_z}\mathrm{e}^{-\omega\eta}\Big[u_{i,\eta}u_{i,\eta}+\frac12a\omega|\boldsymbol{u}|^2+\frac13b\omega|\boldsymbol{u}|^3+\frac14c\omega|\boldsymbol{u}|^4\Big]\,\mathrm{d}A\,\mathrm{d}\eta\\ &\quad+\mathrm{e}^{-\omega t}\int_{D_z}\Big[\frac12a|\boldsymbol{u}|^2+\frac13b|\boldsymbol{u}|^3+\frac14c|\boldsymbol{u}|^4\Big]\,\mathrm{d}A. \end{aligned} \tag{2.3}$$

We have the following result.

Theorem 2.1. *Assume that $\boldsymbol{u}$ is the solution of (1.1)–(1.6) with $\int_D f\,\mathrm{d}A=0$. Then $\boldsymbol{u}$ decays exponentially as $z\to\infty$. Specifically,*

$$\begin{aligned} &\int_0^t\int_{R_z}\mathrm{e}^{-\omega\eta}\Big[u_{i,\eta}u_{i,\eta}+\frac12a\omega|\boldsymbol{u}|^2+\frac13b\omega|\boldsymbol{u}|^3+\frac14c\omega|\boldsymbol{u}|^4\Big]\,\mathrm{d}x\,\mathrm{d}\eta\\ &\qquad+\mathrm{e}^{-\omega t}\int_{R_z}\Big[\frac12a|\boldsymbol{u}|^2+\frac13b|\boldsymbol{u}|^3+\frac14c|\boldsymbol{u}|^4\Big]\,\mathrm{d}x\\ &\qquad\leqslant m_6Q_1^2(0,t)\mathrm{e}^{-2z/m_5}+m_5Q_1(0,t)\mathrm{e}^{-z/m_5}, \end{aligned} \tag{2.4}$$

where m_5 and m_6 are positive constants, and $Q_1(0,t)$ is defined as

$$Q_1(0,t)=\bigg[\sqrt{\frac{F(0,t)}{m_6}+\frac{m_5^2}{4m_6^2}}-\frac{m_5}{2m_6}\bigg]\mathrm{e}^{2m_6/m_5\sqrt{F(0,t)/m_6+m_5^2/(4m_6^2)}}. \tag{2.5}$$

Remark 2.1. From the definition of $Q_1(0,t)$ in (2.5), we find that $Q_1(0,t)$ depends on $F(0,t)$. To make the decay estimate explicit, we have to derive the upper bounds for $F(0,t)$. We have the following theorem.

Theorem 2.2. *Assume that $\boldsymbol{u}$ is the solution of* (1.1)–(1.6) *with* $\int_D f\,\mathrm{d}A = 0$ *and* $f \in C(R)$. *Then*

$$\int_0^t \int_R \mathrm{e}^{-\omega\eta}\Big[u_{i,\eta}u_{i,\eta} + \frac{1}{2}a\omega|\boldsymbol{u}|^2 + \frac{1}{3}b\omega|\boldsymbol{u}|^3 + \frac{1}{4}c\omega|\boldsymbol{u}|^4\Big]\,\mathrm{d}x\,\mathrm{d}\eta + \mathrm{e}^{-\omega t}\int_R \Big[\frac{1}{2}a|\boldsymbol{u}|^2 + \frac{1}{3}b|\boldsymbol{u}|^3 + \frac{1}{4}c|\boldsymbol{u}|^4\Big]\,\mathrm{d}x \leqslant s(t). \tag{2.6}$$

3. Proof of Theorem 2.1

We note that

$$\begin{aligned}\int_{D_z} u_3\,\mathrm{d}A &= \int_{D_0} u_3\,\mathrm{d}A + \int_0^z \int_{D_\xi} u_{3,3}\,\mathrm{d}A\,\mathrm{d}\xi \\ &= \int_{D_0} u_3\,\mathrm{d}A - \int_0^z \int_{D_\xi} u_{\alpha,\alpha}\,\mathrm{d}A\,\mathrm{d}\xi = \int_{D_0} f\,\mathrm{d}A.\end{aligned}$$

Since $\int_{D_0} f\,\mathrm{d}A = 0$, we have $\int_{D_z} u_3\,\mathrm{d}A = 0$. According to Lemma 1.3, there exists a vector function v such that

$$v_{\alpha,\alpha} = u_3 \quad \text{in } D, \quad v_\alpha = 0 \quad \text{on } \partial D.$$

Therefore, using the divergence theorem and the equations (1.1)–(1.6), we have

$$\begin{aligned}F(z,t) &= \int_0^t \int_{D_z} \mathrm{e}^{-\omega\eta} p u_{3,\eta}\,\mathrm{d}A\,\mathrm{d}\eta = \int_0^t \int_{D_z} \mathrm{e}^{-\omega\eta} p v_{\alpha,\alpha\eta}\,\mathrm{d}A\,\mathrm{d}\eta \\ &= -\int_0^t \int_{D_z} \mathrm{e}^{-\omega\eta} p_{,\alpha} v_{\alpha,\eta}\,\mathrm{d}A\,\mathrm{d}\eta \\ &= \int_0^t \int_{D_z} \mathrm{e}^{-\omega\eta}[u_{\alpha,\eta} + a u_\alpha + b|\boldsymbol{u}|u_\alpha + c|\boldsymbol{u}|^2 u_\alpha] v_{\alpha,\eta}\,\mathrm{d}A\,\mathrm{d}\eta \\ &\doteq I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{3.1}$$

Using the Hölder inequality, the Young inequality, Lemma 1.1 and Lemma 1.3, we obtain

$$\begin{aligned}I_1 &\leqslant \Big[\int_0^t \int_{D_z} \mathrm{e}^{-\omega\eta} u_{\alpha,\eta}u_{\alpha,\eta}\,\mathrm{d}A\,\mathrm{d}\eta \int_0^t \int_{D_z} \mathrm{e}^{-\omega\eta} v_{\alpha,\eta}v_{\alpha,\eta}\,\mathrm{d}A\,\mathrm{d}\eta\Big]^{1/2} \\ &\leqslant \frac{1}{\sqrt{\lambda_1}}\Big[\int_0^t \int_{D_z} \mathrm{e}^{-\omega\eta} u_{\alpha,\eta}u_{\alpha,\eta}\,\mathrm{d}A\,\mathrm{d}\eta \int_0^t \int_{D_z} \mathrm{e}^{-\omega\eta} v_{\alpha,\beta\eta}v_{\alpha,\beta\eta}\,\mathrm{d}A\,\mathrm{d}\eta\Big]^{1/2} \\ &\leqslant \frac{\sqrt{k_2}}{\sqrt{\lambda_1}}\Big[\int_0^t \int_{D_z} \mathrm{e}^{-\omega\eta} u_{\alpha,\eta}u_{\alpha,\eta}\,\mathrm{d}A\,\mathrm{d}\eta \int_0^t \int_{D_z} \mathrm{e}^{-\omega\eta} v_{\alpha,\alpha\eta}v_{\alpha,\alpha\eta}\,\mathrm{d}A\,\mathrm{d}\eta\Big]^{1/2}\end{aligned} \tag{3.2}$$

$$\leqslant \frac{\sqrt{k_2}}{\sqrt{\lambda_1}}\bigg[\int_0^t\int_{D_z} \mathrm{e}^{-\omega\eta}u_{\alpha,\eta}u_{\alpha,\eta}\,\mathrm{d}A\,\mathrm{d}\eta\int_0^t\int_{D_z} \mathrm{e}^{-\omega\eta}u_{3,\eta}^2\,\mathrm{d}A\,\mathrm{d}\eta\bigg]^{1/2}$$

$$\leqslant \frac{\sqrt{k_2}}{2\sqrt{\lambda_1}}\int_0^t\int_{D_z} \mathrm{e}^{-\omega\eta}u_{i,\eta}u_{i,\eta}\,\mathrm{d}A\,\mathrm{d}\eta,$$

$$(3.3)\qquad I_2 \leqslant a\bigg[\int_0^t\int_{D_z} \mathrm{e}^{-\omega\eta}u_\alpha u_\alpha\,\mathrm{d}A\,\mathrm{d}\eta\int_0^t\int_{D_z} \mathrm{e}^{-\omega\eta}v_{\alpha,\eta}v_{\alpha,\eta}\,\mathrm{d}A\,\mathrm{d}\eta\bigg]^{1/2}$$

$$\leqslant \frac{\sqrt{k_2}a}{\sqrt{\lambda_1}}\bigg[\int_0^t\int_{D_z} \mathrm{e}^{-\omega\eta}u_\alpha u_\alpha\,\mathrm{d}A\,\mathrm{d}\eta\int_0^t\int_{D_z} \mathrm{e}^{-\omega\eta}u_{3,\eta}^2\,\mathrm{d}A\,\mathrm{d}\eta\bigg]^{1/2}$$

$$\leqslant \frac{\sqrt{2ak_2}}{2\sqrt{\omega\lambda_1}}\bigg[\frac{1}{2}a\omega\int_0^t\int_{D_z} \mathrm{e}^{-\omega\eta}|\boldsymbol{u}|^2\,\mathrm{d}A\,\mathrm{d}\eta+\int_0^t\int_{D_z} \mathrm{e}^{-\omega\eta}u_{3,\eta}^2\,\mathrm{d}A\,\mathrm{d}\eta\bigg],$$

$$(3.4)\qquad I_3 \leqslant b\bigg[\int_0^t\int_{D_z} \mathrm{e}^{-\omega\eta}|\boldsymbol{u}|^4\,\mathrm{d}A\,\mathrm{d}\eta\int_0^t\int_{D_z} \mathrm{e}^{-\omega\eta}v_{\alpha,\eta}v_{\alpha,\eta}\,\mathrm{d}A\,\mathrm{d}\eta\bigg]^{1/2}$$

$$\leqslant \frac{\sqrt{k_2}}{\sqrt{c\omega\lambda_1}}b\bigg[\frac{1}{4}c\omega\int_0^t\int_{D_z} \mathrm{e}^{-\omega\eta}|\boldsymbol{u}|^4\,\mathrm{d}A\,\mathrm{d}\eta+\int_0^t\int_{D_z} \mathrm{e}^{-\omega\eta}u_{3,\eta}^2\,\mathrm{d}A\,\mathrm{d}\eta\bigg].$$

Noting Lemma 1.2, we have

$$(3.5)\qquad I_4 \leqslant c\bigg[\int_0^t\int_{D_z} \mathrm{e}^{-\omega\eta}|\boldsymbol{u}|^4\,\mathrm{d}A\,\mathrm{d}\eta\bigg]^{1/2}\bigg[\int_0^t\int_{D_z} \mathrm{e}^{-\omega\eta}(u_\alpha u_\alpha)^2\,\mathrm{d}A\,\mathrm{d}\eta\bigg]^{1/4}$$

$$\times\bigg[\int_0^t\int_{D_z} \mathrm{e}^{-\omega\eta}(v_{\alpha,\eta}v_{\alpha,\eta})^2\,\mathrm{d}A\,\mathrm{d}\eta\bigg]^{1/4}$$

$$\leqslant c\sqrt[4]{k_1}\bigg[\int_0^t\int_{D_z} \mathrm{e}^{-\omega\eta}|\boldsymbol{u}|^4\,\mathrm{d}A\,\mathrm{d}\eta\bigg]^{3/4}\bigg[\int_0^t\int_{D_z} \mathrm{e}^{-\omega\eta}v_{\alpha,\eta}v_{\alpha,\eta}\,\mathrm{d}A\,\mathrm{d}\eta\bigg]^{1/4}$$

$$\times\bigg[\int_0^t\int_{D_z} \mathrm{e}^{-\omega\eta}v_{\alpha,\beta\eta}v_{\alpha,\beta\eta}\,\mathrm{d}A\,\mathrm{d}\eta\bigg]^{1/4}$$

$$\leqslant c\sqrt[4]{\frac{k_1}{\lambda_1}}\bigg[\int_0^t\int_{D_z} \mathrm{e}^{-\omega\eta}|\boldsymbol{u}|^4\,\mathrm{d}A\,\mathrm{d}\eta\bigg]^{3/4}\bigg[\int_0^t\int_{D_z} \mathrm{e}^{-\omega\eta}v_{\alpha,\beta\eta}v_{\alpha,\beta\eta}\,\mathrm{d}A\,\mathrm{d}\eta\bigg]^{1/2}$$

$$\leqslant \frac{12^{3/4}2^{1/2}}{5^{5/4}}\sqrt[4]{\frac{ck_1k_2^2}{\lambda_1\omega^3}}\bigg[\frac{5}{12}c\omega\int_0^t\int_{D_z} \mathrm{e}^{-\omega\eta}|\boldsymbol{u}|^4\,\mathrm{d}A\,\mathrm{d}\eta\bigg]^{3/4}$$

$$\times\bigg[\frac{5}{2}\int_0^t\int_{D_z} \mathrm{e}^{-\omega\eta}u_{3,\eta}^2\,\mathrm{d}A\,\mathrm{d}\eta\bigg]^{1/2}.$$

In (3.5), using the inequality

$$a^{3/4}b^{1/2}\leqslant\Big(\frac{3}{5}a+\frac{2}{5}b\Big)^{5/4},\quad a,b>0,$$

we have

(3.6)
$$I_4 \leqslant \frac{12^{3/4}2^{1/2}}{5^{5/4}} \sqrt[4]{\frac{ck_1k_2^2}{\lambda_1\omega^3}} \Big[\frac{1}{4}c\omega \int_0^t \int_{D_z} \mathrm{e}^{-\omega\eta}|\boldsymbol{u}|^4 \,\mathrm{d}A\,\mathrm{d}\eta + \int_0^t \int_{D_z} \mathrm{e}^{-\omega\eta} u_{3,\eta}^2 \,\mathrm{d}A\,\mathrm{d}\eta\Big]^{5/4}.$$

Inserting (3.2)–(3.4) and (3.6) into (3.1), we find that

$$F(z,t) \leqslant m_1\Big[-\frac{\partial}{\partial z}F(z,t)\Big] + m_2\Big[-\frac{\partial}{\partial z}F(z,t)\Big]^{5/4}, \tag{3.7}$$

where

$$m_1 = \frac{\sqrt{k_2}}{2\sqrt{\lambda_1}} + \frac{\sqrt{2ak_2}}{2\sqrt{\omega\lambda_1}} + \frac{\sqrt{k_2}}{\sqrt{c\omega\lambda_1}}b, \quad m_2 = \frac{12^{3/4}2^{1/2}}{5^{5/4}} \sqrt[4]{\frac{ck_1k_2^2}{\lambda_1\omega^3}}.$$

Using the Young inequality again, we have

$$\begin{aligned}\Big[-\frac{\partial F}{\partial z}(z,t)\Big]^{5/4} &= \Big[-\frac{\partial F}{\partial z}(z,t)\Big]^{1\cdot 3/4}\Big[-\frac{\partial F}{\partial z}(z,t)\Big]^{2\cdot 1/4} \\ &\leqslant \frac{3}{4}\Big[-\frac{\partial F}{\partial z}(z,t)\Big] + \frac{1}{4}\Big[-\frac{\partial F}{\partial z}(z,t)\Big]^2.\end{aligned} \tag{3.8}$$

Inserting (3.8) into (3.7), we obtain

$$F(z,t) \leqslant m_5\Big[-\frac{\partial}{\partial z}F(z,t)\Big] + m_6\Big[-\frac{\partial}{\partial z}F(z,t)\Big]^2, \tag{3.9}$$

where

$$m_5 = m_1 + \frac{3}{4}m_2, \quad m_6 = \frac{1}{4}m_2.$$

Thus, from (3.9) we get

$$-\frac{\partial F}{\partial z}(z,t) \geqslant \sqrt{\frac{F(z,t)}{m_6} + \frac{m_5^2}{4m_6^2}} - \frac{m_5}{2m_6}.$$

Therefore, we have

(3.10)
$$\Big\{2m_6 + m_5\Big(\sqrt{\frac{F(z,t)}{m_6} + \frac{m_5^2}{4m_6^2}}\Big)^{-1} - \frac{m_5}{2m_6}\Big\}\,\mathrm{d}\Big\{\sqrt{\frac{F(z,t)}{m_6} + \frac{m_5^2}{4m_6^2}} - \frac{m_5}{2m_6}\Big\} \leqslant -1.$$

Integrating (3.10) from 0 to z, we obtain

(3.11)
$$\begin{aligned}&2m_6\Big[\sqrt{\frac{F(z,t)}{m_6} + \frac{m_5^2}{4m_6^2}} - \sqrt{\frac{F(0,t)}{m_6} + \frac{m_5^2}{4m_6^2}}\Big] \\ &\quad + m_5\ln\Big[\sqrt{\frac{F(z,t)}{m_6} + \frac{m_5^2}{4m_6^2}} - \frac{m_5}{2m_6}\Big] - m_5\ln\Big[\sqrt{\frac{F(0,t)}{m_6} + \frac{m_5^2}{4m_6^2}} - \frac{m_5}{2m_6}\Big] \leqslant -z.\end{aligned}$$

Dropping the first term on the left-hand side of (3.11), we get

$$m_5 \ln\Bigl[\sqrt{\frac{F(z,t)}{m_6}+\frac{m_5^2}{4m_6^2}}-\frac{m_5}{2m_6}\Bigr]$$
$$\leqslant -z+2m_6\sqrt{\frac{F(0,t)}{m_6}+\frac{m_5^2}{4m_6^2}}+m_5\ln\Bigl[\sqrt{\frac{F(0,t)}{m_6}+\frac{m_5^2}{4m_6^2}}-\frac{m_5}{2m_6}\Bigr].$$

Therefore, we have

$$\sqrt{\frac{F(z,t)}{m_6}+\frac{m_5^2}{4m_6^2}} \leqslant Q_1(0,t)\mathrm{e}^{-z/m_5}+\frac{m_5}{2m_6}. \tag{3.12}$$

Squaring (3.12), we find that

$$F(z,t) \leqslant m_6 Q_1^2(0,t)\mathrm{e}^{-2z/m_5}+m_5 Q_1(0,t)\mathrm{e}^{-z/m_5}. \tag{3.13}$$

We can complete the proof of Theorem 2.1 by combining (2.2) and (3.14).

R e m a r k 3.1. If $b=0$ in (1.1), Theorem 2.1 also holds.

R e m a r k 3.2. If $c=0$ in (1.1), we recalculate (3.4) to obtain

$$\begin{aligned}
I_3 &\leqslant b\Bigl[\int_0^t\int_{D_z}\mathrm{e}^{-\omega\eta}|\boldsymbol{u}|^3\,\mathrm{d}A\,\mathrm{d}\eta\Bigr]^{2/3}\Bigl[\int_0^t\int_{D_z}\mathrm{e}^{-\omega\eta}(v_{\alpha,\eta}v_{\alpha,\eta})^{3/2}\,\mathrm{d}A\,\mathrm{d}\eta\Bigr]^{1/3}\\
&\leqslant b\Bigl[\int_0^t\int_{D_z}\mathrm{e}^{-\omega\eta}|\boldsymbol{u}|^3\,\mathrm{d}A\,\mathrm{d}\eta\Bigr]^{2/3}\Bigl[\int_0^t\int_{D_z}\mathrm{e}^{-\omega\eta}(v_{\alpha,\eta}v_{\alpha,\eta})^{2}\,\mathrm{d}A\,\mathrm{d}\eta\Bigr]^{1/6}\\
&\quad\times\Bigl[\int_0^t\int_{D_z}\mathrm{e}^{-\omega\eta}v_{\alpha,\eta}v_{\alpha,\eta}\,\mathrm{d}A\,\mathrm{d}\eta\Bigr]^{1/6}\\
&\leqslant b\sqrt[6]{k_1}\Bigl[\int_0^t\int_{D_z}\mathrm{e}^{-\omega\eta}|\boldsymbol{u}|^3\,\mathrm{d}A\,\mathrm{d}\eta\Bigr]^{2/3}\Bigl[\int_0^t\int_{D_z}\mathrm{e}^{-\omega\eta}v_{\alpha,\beta\eta}v_{\alpha,\beta\eta}\,\mathrm{d}A\,\mathrm{d}\eta\Bigr]^{1/6}\\
&\quad\times\Bigl[\int_0^t\int_{D_z}\mathrm{e}^{-\omega\eta}v_{\alpha,\eta}v_{\alpha,\eta}\,\mathrm{d}A\,\mathrm{d}\eta\Bigr]^{1/3}\\
&\leqslant \frac{b\sqrt[6]{k_1}}{\sqrt[3]{\lambda_1}}\Bigl[\int_0^t\int_{D_z}\mathrm{e}^{-\omega\eta}|\boldsymbol{u}|^3\,\mathrm{d}A\,\mathrm{d}\eta\Bigr]^{2/3}\Bigl[\int_0^t\int_{D_z}\mathrm{e}^{-\omega\eta}v_{\alpha,\beta\eta}v_{\alpha,\beta\eta}\,\mathrm{d}A\,\mathrm{d}\eta\Bigr]^{1/2}\\
&\leqslant \frac{12^{2/3}\sqrt{3k_2}\sqrt[3]{b}\sqrt[6]{k_1}}{7^{6/7}\sqrt[3]{\lambda_1\omega^2}}\Bigl[\frac{7}{12}b\omega\int_0^t\int_{D_z}\mathrm{e}^{-\omega\eta}|\boldsymbol{u}|^3\,\mathrm{d}A\,\mathrm{d}\eta\Bigr]^{2/3}\\
&\quad\times\Bigl[\frac{7}{3}\int_0^t\int_{D_z}\mathrm{e}^{-\omega\eta}u_{3,\eta}^2\,\mathrm{d}A\,\mathrm{d}\eta\Bigr]^{1/2}.
\end{aligned}\tag{3.14}$$

Noting that

$$a^{2/3}b^{1/2} \leqslant \Big[\frac{4}{7}a + \frac{3}{7}b\Big]^{7/6}, \quad a, b > 0,$$

we have from (3.14)

(3.15)

$$I_3 \leqslant \frac{12^{2/3}\sqrt{2k_2}\sqrt[3]{b}\sqrt[6]{k_1}}{7^{6/7}\sqrt[3]{\lambda_1\omega^2}}\Big[\frac{1}{4}b\omega\int_0^t\int_{D_z} \mathrm{e}^{-\omega\eta}|\boldsymbol{u}|^3\,\mathrm{d}A\,\mathrm{d}\eta + \int_0^t\int_{D_z} \mathrm{e}^{-\omega\eta}u_{3,\eta}^2\,\mathrm{d}A\,\mathrm{d}\eta\Big]^{7/6}.$$

Combining (3.1), (3.2), (3.3) and (3.15), we arrive at

$$F(z,t) \leqslant b_1\Big[-\frac{\partial}{\partial z}F(z,t)\Big] + b_2\Big[-\frac{\partial}{\partial z}F(z,t)\Big]^{7/6}, \tag{3.16}$$

where

$$b_1 = \frac{\sqrt{k_2}}{2\sqrt{\lambda_1}} + \frac{\sqrt{2ak_2}}{2\sqrt{\omega\lambda_1}}, \quad b_2 = \frac{12^{2/3}\sqrt{2k_2}\sqrt[3]{b}\sqrt[6]{k_1}}{7^{6/7}\sqrt[3]{\lambda_1\omega^2}}.$$

Using the Young inequality, we get

$$\begin{aligned}\Big[-\frac{\partial F}{\partial z}(z,t)\Big]^{7/6} &= \Big[-\frac{\partial F}{\partial z}(z,t)\Big]^{1\cdot 5/6}\Big[-\frac{\partial F}{\partial z}(z,t)\Big]^{2\cdot 1/6} \\ &\leqslant \frac{5}{6}\Big[-\frac{\partial F}{\partial z}(z,t)\Big] + \frac{1}{6}\Big[-\frac{\partial F}{\partial z}(z,t)\Big]^2.\end{aligned} \tag{3.17}$$

Inserting (3.17) into (3.16), we have

$$F(z,t) \leqslant m_9\Big[-\frac{\partial}{\partial z}F(z,t)\Big] + m_{10}\Big[-\frac{\partial}{\partial z}F(z,t)\Big]^2,$$

where $m_9 = \frac{5}{6}b_2 + b_1$, $m_{10} = \frac{1}{6}b_2$.

Similarly to (3.9), we have the following theorem.

Theorem 3.1. *Assume that $\boldsymbol{u}$ is the solution of (1.1)–(1.6) with $\int_D f\,\mathrm{d}A = 0$ and $c = 0$. Then $\boldsymbol{u}$ decays exponentially as $z \to \infty$. Specifically,*

$$\begin{aligned}\int_0^t\int_{R_z} \mathrm{e}^{-\omega\eta}\Big[u_{i,\eta}u_{i,\eta} + \frac{1}{2}a\omega|\boldsymbol{u}|^2 + \frac{1}{3}b\omega|\boldsymbol{u}|^3\Big]\,\mathrm{d}x\,\mathrm{d}\eta \\ + \mathrm{e}^{-\omega t}\int_{R_z}\Big[\frac{1}{2}a|\boldsymbol{u}|^2 + \frac{1}{3}b|\boldsymbol{u}|^3\Big]\,\mathrm{d}x \\ \leqslant m_{10}Q_2^2(0,t)\mathrm{e}^{-2z/m_9} + m_9Q_2(0,t)\mathrm{e}^{-z/m_9},\end{aligned} \tag{3.18}$$

where m_9 and m_{10} are positive constants, and $Q_2(0,t)$ is a positive function which depends on $F(0,t)$.

R e m a r k 3.3. Clearly, Theorem 3.1 also holds for the Brinkman-Forchheimer equations

$$\begin{aligned} u_{i,t} - \Delta u_i &= -a u_i - b|\boldsymbol{u}|u_i - p_{,i} \quad &&\text{in } R \times (0,\infty),\\ u_{i,i} &= 0 &&\text{in } R \times (0,\infty), \end{aligned}$$

whose convergence, continuous dependence and decay in a bounded region have been studied by Payne and Straughan [35].

In this case, we can establish a new function

$$\mathcal{F}(z,t) = \int_0^t \int_{D_z} \mathrm{e}^{-\omega\eta} p u_{3,\eta} \,\mathrm{d}A\,\mathrm{d}\eta + \int_0^t \int_{D_z} \mathrm{e}^{-\omega\eta} u_{i,3} u_{i,\eta} \,\mathrm{d}A\,\mathrm{d}\eta. \tag{3.19}$$

We have

$$\begin{aligned} -\frac{\partial}{\partial z}\mathcal{F}(z,t) = &\int_0^t \int_{D_z} \mathrm{e}^{-\omega\eta}\Big[u_{i,\eta}u_{i,\eta} + \frac{1}{2}\omega|\nabla\boldsymbol{u}|^2 + \frac{1}{2}a\omega|\boldsymbol{u}|^2 + \frac{1}{3}b\omega|\boldsymbol{u}|^3\Big]\,\mathrm{d}A\,\mathrm{d}\eta\\ &+ \mathrm{e}^{-\omega t}\int_{D_z}\Big[\frac{1}{2}a|\boldsymbol{u}|^2 + \frac{1}{2}|\nabla\boldsymbol{u}|^2 + \frac{1}{3}b|\boldsymbol{u}|^3\Big]\,\mathrm{d}A. \end{aligned} \tag{3.20}$$

Note that

$$\begin{aligned} &\int_0^t \int_{D_z} \mathrm{e}^{-\omega\eta} u_{i,3}u_{i,\eta}\,\mathrm{d}A\,\mathrm{d}\eta\\ &\quad \leqslant \frac{\sqrt{2}}{2\sqrt{\omega}}\Big[\frac{1}{2}\omega\int_0^t \int_{D_z} \mathrm{e}^{-\omega\eta} u_{i,3}u_{i,3}\,\mathrm{d}A\,\mathrm{d}\eta + \int_0^t \int_{D_z} \mathrm{e}^{-\omega\eta} u_{i,\eta}u_{i,\eta}\,\mathrm{d}A\,\mathrm{d}\eta\Big]. \end{aligned}$$

Using the analysis presented in this paper, we have a result similar to Theorem 2.1.

R e m a r k 3.4. The Forchheimer system including temperature can be written as (see [41])

$$\begin{aligned} u_{i,t} &= -b|\boldsymbol{u}|u_i - p_{,i} + g_i T \quad &&\text{in } R \times (0,\infty),\\ u_{i,i} &= 0 &&\text{in } R \times (0,\infty),\\ T_{,t} + u_i T_{,i} &= \Delta T &&\text{in } R \times (0,\infty). \end{aligned}$$

Using the technique of Theorem 3.1, the continuous dependence on b can be also obtained.

4. Proof of Theorem 2.2

Noting (1.2) and $\int_D f\,\mathrm{d}A = 0$, we see that there exists a vector function ϕ such that

$$\phi_{i,i} = 0 \quad \text{in } R\times(0,\infty), \qquad \phi_i n_i = u_i n_i \quad \text{on } \partial R\times(0,\infty). \tag{4.1}$$

Clearly, $\phi_i(x_1,x_2,x_3,t)$ has the same boundary conditions with u_i. We note that (4.1) can be satisfied easily. For example, we can choose

$$\phi_3(x_1,x_2,x_3,t) = f(x_1,x_2,t)\mathrm{e}^{-\sigma x_3},$$

where σ is an arbitrary positive constant. Since $\phi_{i,i}=0$, we obtain

$$\phi_{\alpha,\alpha}(x_1,x_2,x_3,t) = \sigma f(x_1,x_2,t)\mathrm{e}^{-\sigma x_3}.$$

We choose $z=0$ in (2.1) and (2.2) to get

$$F(0,t) = \int_0^t\int_D \mathrm{e}^{-\omega\eta} p u_{3,\eta}\,\mathrm{d}A\,\mathrm{d}\eta = \int_0^t\int_D \mathrm{e}^{-\omega\eta} p\phi_{3,\eta}\,\mathrm{d}A\,\mathrm{d}\eta, \tag{4.2}$$

and

$$\begin{aligned} F(0,t) = {}& \int_0^t\int_R \mathrm{e}^{-\omega\eta}\Big[u_{i,\eta}u_{i,\eta} + \frac{1}{2}a\omega|\boldsymbol{u}|^2 + \frac{1}{3}b\omega|\boldsymbol{u}|^3 + \frac{1}{4}c\omega|\boldsymbol{u}|^4\Big]\,\mathrm{d}x\,\mathrm{d}\eta \\ & + \mathrm{e}^{-\omega t}\int_R \Big[\frac{1}{2}a|\boldsymbol{u}|^2 + \frac{1}{3}b|\boldsymbol{u}|^3 + \frac{1}{4}c|\boldsymbol{u}|^4\Big]\,\mathrm{d}x. \end{aligned} \tag{4.3}$$

From (4.2), we have

$$\begin{aligned} F(0,t) &= -\int_0^t\int_{\partial R} \mathrm{e}^{-\omega\eta} p\phi_{i,\eta} n_i dS\,\mathrm{d}\eta = -\int_0^t\int_R \mathrm{e}^{-\omega\eta} p_{,i}\phi_{i,\eta}\,\mathrm{d}x\,\mathrm{d}\eta \\ &= \int_0^t\int_R \mathrm{e}^{-\omega\eta}[u_{i,\eta} + au_i + b|\boldsymbol{u}|u_i + c|\boldsymbol{u}|^2u_i]\phi_{i,\eta}\,\mathrm{d}x\,\mathrm{d}\eta \\ &:= J_1 + J_2 + J_3 + J_4. \end{aligned} \tag{4.4}$$

Using the Hölder inequality and the Young inequality, we obtain

$$\begin{aligned} J_1 &\leqslant \frac{1}{2}\int_0^t\int_R \mathrm{e}^{-\omega\eta} u_{i,\eta}u_{i,\eta}\,\mathrm{d}x\,\mathrm{d}\eta + \frac{1}{2}\int_0^t\int_R \mathrm{e}^{-\omega\eta}\phi_{i,\eta}\phi_{i,\eta}\,\mathrm{d}x\,\mathrm{d}\eta, \\ J_2 &\leqslant \frac{1}{2}\int_0^t\int_R \mathrm{e}^{-\omega\eta}\frac{1}{2}a\omega|\boldsymbol{u}|^2\,\mathrm{d}x\,\mathrm{d}\eta + \frac{a}{\omega}\int_0^t\int_R \mathrm{e}^{-\omega\eta}\phi_{i,\eta}\phi_{i,\eta}\,\mathrm{d}x\,\mathrm{d}\eta, \\ J_3 &\leqslant \frac{2}{\sqrt[3]{3}}\varepsilon_1\int_0^t\int_R \mathrm{e}^{-\omega\eta}\frac{1}{3}b\omega|\boldsymbol{u}|^3\,\mathrm{d}x\,\mathrm{d}\eta + \frac{b}{\sqrt[3]{3}\varepsilon_1^2\omega^2}\int_0^t\int_R \mathrm{e}^{-\omega\eta}(\phi_{i,\eta}\phi_{i,\eta})^{3/2}\,\mathrm{d}x\,\mathrm{d}\eta, \\ J_4 &\leqslant \frac{3}{4}\varepsilon_2\int_0^t\int_R \mathrm{e}^{-\omega\eta}\frac{1}{4}c\omega|\boldsymbol{u}|^4\,\mathrm{d}x\,\mathrm{d}\eta + \frac{c}{16\omega^3\varepsilon_2^3}\int_0^t\int_R \mathrm{e}^{-\omega\eta}(\phi_{i,\eta}\phi_{i,\eta})^2\,\mathrm{d}x\,\mathrm{d}\eta, \end{aligned} \tag{4.5}$$

where ε_1 and ε_2 are positive constants. Choosing $\varepsilon_1 = \frac{1}{4}\sqrt[3]{3}$ and $\varepsilon_2 = \frac{2}{3}$, inserting (4.5)–(4.8) into (4.4) and combining it with (4.3), we find that

$$F(0,t) \leqslant \frac{1}{2}F(0,t) + \frac{1}{2}s(t), \tag{4.6}$$

where

$$\begin{aligned}\frac{1}{2}s(t) = {}& \frac{1}{2}\int_0^t\int_R \mathrm{e}^{-\omega\eta}\phi_{i,\eta}\phi_{i,\eta}\,\mathrm{d}x\,\mathrm{d}\eta + \frac{a}{\omega}\int_0^t\int_R \mathrm{e}^{-\omega\eta}\phi_{i,\eta}\phi_{i,\eta}\,\mathrm{d}x\,\mathrm{d}\eta \\ & + \frac{b}{\sqrt[3]{3}\varepsilon_1^2\omega^2}\int_0^t\int_R \mathrm{e}^{-\omega\eta}(\phi_{i,\eta}\phi_{i,\eta})^{3/2}\,\mathrm{d}x\,\mathrm{d}\eta \\ & + \frac{c}{16\omega^3\varepsilon_2^3}\int_0^t\int_R \mathrm{e}^{-\omega\eta}(\phi_{i,\eta}\phi_{i,\eta})^2\,\mathrm{d}x\,\mathrm{d}\eta.\end{aligned}$$

From (4.3) and (4.9), we can complete the proof of Theorem 2.2.

5. Forchheimer porous material

In 1998, Payne and Straughan [34] proposed a momentum equation in saturated material of Forchheimer type which had the form

$$u_i + c|\boldsymbol{u}|^2u_i = -p_{,i} + g_iT + h_iT^2 + l_iT^3 \quad \text{in } R\times(0,\infty), \tag{5.1}$$

$$u_{i,i} = 0 \quad \text{in } R\times(0,\infty), \tag{5.2}$$

$$T_{,t} + u_iT_{,i} = \Delta T \quad \text{in } R\times(0,\infty), \tag{5.3}$$

where g_i, h_i and l_i are given positive functions. Without loss of generality, we suppose that g_ig_i, h_ih_i, $l_il_i \leqslant 1$. The form of temperature dependence in (5.1) can be found in [40]. Gentile and Straughan [8] derived the structural stability of (5.1)–(5.3) in a bounded region.

Equations (5.1)–(5.3) also satisfy the initial-boundary conditions

$$u_i = 0,\ T = 0 \quad \text{on } \partial D\times\{x_3>0\}\times(0,\infty), \tag{5.4}$$

$$u_3 = f(x_1,x_2,t),\ T = F(x_1,x_2,t) \quad \text{on } D\times(0,\infty), \tag{5.5}$$

$$T(x_1,x_2,0) = 0 \quad \text{in } R, \tag{5.6}$$

$$|\boldsymbol{u}|, u_3, T, T_{,3}, p = o(x_3^{-1}), \tag{5.7}$$

where F is a continuously differentiable function.

Now, we establish a new function

$$E(z,t)=\int_0^t\int_{R_z}\mathrm{e}^{-\omega\eta}\Big[|\boldsymbol{u}|^2+c|\boldsymbol{u}|^4+\frac{1}{2}\delta_1\omega T^2+\frac{1}{4}\omega\delta_2T^4+\delta_1|\nabla T|^2+3\delta_2|\nabla T^2|^2\Big]\,\mathrm{d}x\,\mathrm{d}\eta+\mathrm{e}^{-\omega t}\int_{R_z}\Big[\frac{1}{2}\delta_1T^2+\frac{1}{4}\delta_2T^4\Big]\,\mathrm{d}x, \tag{5.8}$$

where δ_1 and δ_2 are positive constants to be determined later. From (5.8), we have

$$-\frac{\partial}{\partial z}E(z,t)=\int_0^t\int_{D_z}\mathrm{e}^{-\omega\eta}\Big[|\boldsymbol{u}|^2+c|\boldsymbol{u}|^4+\frac{1}{2}\delta_1\omega T^2+\frac{1}{4}\omega\delta_2T^4+\delta_1|\nabla T|^2+3\delta_2|\nabla T^2|^2\Big]\,\mathrm{d}A\,\mathrm{d}\eta+\mathrm{e}^{-\omega t}\int_{D_z}\Big[\frac{1}{2}\omega\delta_1T^2+\frac{1}{4}\omega\delta_2T^4\Big]\,\mathrm{d}A. \tag{5.9}$$

We obtain the following theorem.

Theorem 5.1. *Assume that $(\boldsymbol{u},T)$ is the solution of (5.1)–(5.7) with $\int_D f\,\mathrm{d}A=0$. Then $(\boldsymbol{u},T)$ decays exponentially as $z\to\infty$. Specifically,*

$$E(z,t)\leqslant m_6Q_2^2(0,t)\mathrm{e}^{-2z/m_5}+m_5Q_2(0,t)\mathrm{e}^{-z/m_5},$$

where m_5 and m_6 are positive constants, and $Q_2(0,t)$ is defined as

$$Q_2(0,t)=\bigg[\sqrt{\frac{E(0,t)}{m_6}+\frac{m_5^2}{4m_6^2}}-\frac{m_5}{2m_6}\bigg]\mathrm{e}^{2m_6/m_5\sqrt{E(0,t)/m_6+m_5^2/(4m_6^2)}}.$$

Remark 5.1. By using the methods of Section 4, we can also obtain the upper bound for $E(0,t)$ in terms of known data.

Proof. We multiply (5.1) by $\mathrm{e}^{-\omega t}u_i$ and integrate in $R_z\times(0,t)$ to obtain

$$\int_0^t\int_{R_z}\mathrm{e}^{-\omega\eta}[u_i+c|\boldsymbol{u}|^2u_i-p_{,i}-g_iT-h_iT^2-l_iT^3]u_i\,\mathrm{d}x\,\mathrm{d}\eta=0. \tag{5.10}$$

Using the divergence theorem and (5.1)–(5.7), from (5.1) we have

$$\begin{aligned}\int_0^t\int_{R_z}\mathrm{e}^{-\omega\eta}[|\boldsymbol{u}|^2+c|\boldsymbol{u}|^4]\,\mathrm{d}x\,\mathrm{d}\eta&=\int_0^t\int_{R_z}\mathrm{e}^{-\omega\eta}g_iu_iT\,\mathrm{d}x\,\mathrm{d}\eta+\int_0^t\int_{R_z}\mathrm{e}^{-\omega\eta}h_iu_iT^2\,\mathrm{d}x\,\mathrm{d}\eta\\&\quad+\int_0^t\int_{R_z}\mathrm{e}^{-\omega\eta}l_iu_iT^3\,\mathrm{d}x\,\mathrm{d}\eta+\int_0^t\int_{D_z}\mathrm{e}^{-\omega\eta}u_3p\,\mathrm{d}x\,\mathrm{d}\eta\\&:=K_1+K_2+K_3+K_4.\end{aligned} \tag{5.11}$$

Using the Hölder inequality and the Young inequality, we get

$$K_1 \leqslant \frac{1}{4}\int_0^t\int_{R_z} \mathrm{e}^{-\omega\eta}|\boldsymbol{u}|^2 \,\mathrm{d}x\,\mathrm{d}\eta + \int_0^t\int_{R_z} \mathrm{e}^{-\omega\eta}T^2 \,\mathrm{d}x\,\mathrm{d}\eta, \tag{5.12}$$

$$K_2 \leqslant \frac{1}{4}\int_0^t\int_{R_z} \mathrm{e}^{-\omega\eta}|\boldsymbol{u}|^2 \,\mathrm{d}x\,\mathrm{d}\eta + \int_0^t\int_{R_z} \mathrm{e}^{-\omega\eta}T^4 \,\mathrm{d}x\,\mathrm{d}\eta, \tag{5.13}$$

$$K_3 \leqslant \frac{1}{4}c\int_0^t\int_{R_z} \mathrm{e}^{-\omega\eta}|\boldsymbol{u}|^4 \,\mathrm{d}x\,\mathrm{d}\eta + \frac{3}{4}c^{-1/3}\int_0^t\int_{R_z} \mathrm{e}^{-\omega\eta}T^4 \,\mathrm{d}x\,\mathrm{d}\eta. \tag{5.14}$$

Using a method similar to (3.1), we have for K_4

$$\begin{aligned} K_4 &= \int_0^t\int_{D_z} \mathrm{e}^{-\omega\eta}[u_\alpha + c|\boldsymbol{u}|^2u_\alpha - g_\alpha T - h_\alpha T^2 - l_\alpha T^3]v_\alpha \,\mathrm{d}A\,\mathrm{d}\eta \\ &\doteq K_{41} + K_{42} + K_{43} + K_{44} + K_{45}. \end{aligned} \tag{5.15}$$

It is clear that

$$K_{41} \leqslant \frac{\sqrt{k_2}}{2\sqrt{\lambda_1}}\int_0^t\int_{D_z} \mathrm{e}^{-\omega\eta}|\boldsymbol{u}|^2 \,\mathrm{d}A\,\mathrm{d}\eta, \tag{5.16}$$

$$\begin{aligned} K_{42} \leqslant \frac{4^{3/4}2^{1/2}}{5^{5/4}}\sqrt[4]{\frac{ck_1k_2^2}{\lambda_1}}\Big[c\int_0^t\int_{D_z} &\mathrm{e}^{-\omega\eta}|\boldsymbol{u}|^4 \,\mathrm{d}A\,\mathrm{d}\eta \\ &+ \int_0^t\int_{D_z} \mathrm{e}^{-\omega\eta}|\boldsymbol{u}|^2 \,\mathrm{d}A\,\mathrm{d}\eta\Big]^{5/4}, \end{aligned} \tag{5.17}$$

$$K_{43} \leqslant \frac{\sqrt{k_2}}{\sqrt{2\lambda_1\delta_1\omega}}\Big[\frac{1}{2}\delta_1\omega\int_0^t\int_{D_z} \mathrm{e}^{-\omega\eta}T^2 \,\mathrm{d}A\,\mathrm{d}\eta + \int_0^t\int_{D_z} \mathrm{e}^{-\omega\eta}|\boldsymbol{u}|^2 \,\mathrm{d}A\,\mathrm{d}\eta\Big], \tag{5.18}$$

$$K_{44} \leqslant \frac{\sqrt{k_2}}{\sqrt{\lambda_1\delta_2\omega}}\Big[\frac{1}{4}\delta_2\omega\int_0^t\int_{D_z} \mathrm{e}^{-\omega\eta}T^4 \,\mathrm{d}A\,\mathrm{d}\eta + \int_0^t\int_{D_z} \mathrm{e}^{-\omega\eta}|\boldsymbol{u}|^2 \,\mathrm{d}A\,\mathrm{d}\eta\Big], \tag{5.19}$$

$$\begin{aligned} K_{45} \leqslant \frac{12^{3/4}2^{1/2}}{5^{5/4}}\sqrt[4]{\frac{ck_1k_2^2}{\lambda_1\delta_2^3\omega^3}}\Big[\frac{1}{4}\delta_2\omega\int_0^t\int_{D_z} &\mathrm{e}^{-\omega\eta}T^4 \,\mathrm{d}A\,\mathrm{d}\eta \\ &+ \int_0^t\int_{D_z} \mathrm{e}^{-\omega\eta}|\boldsymbol{u}|^2 \,\mathrm{d}A\,\mathrm{d}\eta\Big]^{5/4}. \end{aligned} \tag{5.20}$$

Inserting (5.16) and (5.20) into (5.15), we have

$$K_4 \leqslant n_1\Big[-\frac{\partial}{\partial z}E(z,t)\Big] + n_2\Big[-\frac{\partial}{\partial z}E(z,t)\Big]^{5/4}, \tag{5.21}$$

where

$$n_1 = \frac{\sqrt{k_2}}{2\sqrt{\lambda_1}} + \frac{\sqrt{k_2}}{\sqrt{2\lambda_1\delta_1\omega}} + \frac{\sqrt{k_2}}{\sqrt{\lambda_1\delta_2\omega}}, \quad n_2 = \frac{4^{3/4}2^{1/2}}{5^{5/4}}\sqrt[4]{\frac{ck_1k_2^2}{\lambda_1}} + \frac{12^{3/4}2^{1/2}}{5^{5/4}}\sqrt[4]{\frac{ck_1k_2^2}{\lambda_1\delta_2^3\omega^3}}.$$

Combining (5.12), (5.13), (5.14), (5.21) and (5.11), we obtain

$$
\begin{aligned}
(5.22)\qquad &\int_0^t \int_{R_z} \mathrm{e}^{-\omega\eta}\Big[\frac{1}{2}|\boldsymbol{u}|^2 + \frac{3}{4}c|\boldsymbol{u}|^4\Big]\,\mathrm{d}x\,\mathrm{d}\eta \\
&\quad \leqslant \int_0^t \int_{R_z} \mathrm{e}^{-\omega\eta}T^2\,\mathrm{d}x\,\mathrm{d}\eta + \Big[1 + \frac{3}{4}c^{-1/3}\Big]\int_0^t \int_{R_z} \mathrm{e}^{-\omega\eta}T^4\,\mathrm{d}x\,\mathrm{d}\eta \\
&\quad\quad + n_1\Big[-\frac{\partial}{\partial z}E(z,t)\Big] + n_2\Big[-\frac{\partial}{\partial z}E(z,t)\Big]^{5/4}.
\end{aligned}
$$

To bound $\int_0^t \int_{R_z} \mathrm{e}^{-\omega\eta}T^2\,\mathrm{d}x\,\mathrm{d}\eta$ and $\int_0^t \int_{R_z} \mathrm{e}^{-\omega\eta}T^4\,\mathrm{d}x\,\mathrm{d}\eta$, we multiply (5.3) by $\mathrm{e}^{-\omega t}T$ and integrate in $R_z \times (0,t)$ to obtain

$$
\int_0^t \int_{R_z} \mathrm{e}^{-\omega\eta}[T_{,\eta} + u_i T_{,i} - \Delta T]T\,\mathrm{d}x\,\mathrm{d}\eta = 0.
$$

Therefore, we get

$$
\begin{aligned}
(5.23)\quad &\int_0^t \int_{R_z} \mathrm{e}^{-\omega\eta}\omega T^2\,\mathrm{d}x\,\mathrm{d}\eta + \mathrm{e}^{-\omega t}\int_{R_z} T^2\,\mathrm{d}x + 2\int_0^t \int_{R_z} \mathrm{e}^{-\omega\eta}|\nabla T|^2\,\mathrm{d}x\,\mathrm{d}\eta \\
&= \int_0^t \int_{D_z} \mathrm{e}^{-\omega\eta}T^2 u_3\,\mathrm{d}A\,\mathrm{d}\eta + 2\int_0^t \int_{D_z} \mathrm{e}^{-\omega\eta}TT_{,3}\,\mathrm{d}A\,\mathrm{d}\eta \\
&\leqslant \frac{2}{\sqrt{\omega\delta_2}}\Big[\frac{1}{4}\omega\delta_2\int_0^t \int_{D_z} \mathrm{e}^{-\omega\eta}T^4\,\mathrm{d}A\,\mathrm{d}\eta + \int_0^t \int_{D_z} \mathrm{e}^{-\omega\eta}|\boldsymbol{u}|^2\,\mathrm{d}A\,\mathrm{d}\eta\Big] \\
&\quad + \frac{2}{\sqrt{2\omega}\delta_1}\Big[\frac{1}{2}\omega\delta_1\int_0^t \int_{D_z} \mathrm{e}^{-\omega\eta}T^2\,\mathrm{d}A\,\mathrm{d}\eta + \delta_1\int_0^t \int_{D_z} \mathrm{e}^{-\omega\eta}T_{,3}^2\,\mathrm{d}A\,\mathrm{d}\eta\Big].
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
(5.24)\quad &\int_0^t \int_{R_z} \mathrm{e}^{-\omega\eta}\omega T^4\,\mathrm{d}x\,\mathrm{d}\eta + \mathrm{e}^{-\omega t}\int_{R_z} T^4\,\mathrm{d}x + 3\int_0^t \int_{R_z} \mathrm{e}^{-\omega\eta}|\nabla T^2|^2\,\mathrm{d}x\,\mathrm{d}\eta \\
&= \int_0^t \int_{D_z} \mathrm{e}^{-\omega\eta}T^4 u_3\,\mathrm{d}A\,\mathrm{d}\eta + 2\int_0^t \int_{D_z} \mathrm{e}^{-\omega\eta}T^2(T^2)_{,3}\,\mathrm{d}A\,\mathrm{d}\eta \\
&\leqslant \frac{F_M^2}{\sqrt{\omega\delta_2}}\Big[\frac{1}{4}\omega\delta_2\int_0^t \int_{D_z} \mathrm{e}^{-\omega\eta}T^4\,\mathrm{d}A\,\mathrm{d}\eta + \int_0^t \int_{D_z} \mathrm{e}^{-\omega\eta}|\boldsymbol{u}|^2\,\mathrm{d}A\,\mathrm{d}\eta\Big] \\
&\quad + \frac{2}{\sqrt{3\omega}\delta_2}\Big[\frac{1}{4}\omega\delta_2\int_0^t \int_{D_z} \mathrm{e}^{-\omega\eta}T^4\,\mathrm{d}A\,\mathrm{d}\eta + 3\delta_2\int_0^t \int_{D_z} \mathrm{e}^{-\omega\eta}|\nabla T^2|^2\,\mathrm{d}A\,\mathrm{d}\eta\Big],
\end{aligned}
$$

where F_M^2 is the maximum value of F in $D \times (0,\infty)$.

Combing (5.22)–(5.24) and in view of (5.9), we have

$$\begin{aligned}(5.25)\quad &\int_0^t\int_{R_z} \mathrm{e}^{-\omega\eta}\Big[\frac{1}{2}|\boldsymbol{u}|^2+\frac{3}{4}c|\boldsymbol{u}|^4+\omega\delta_1T^2+\omega\delta_2T^4+2\delta_1|\nabla T|^2+3\delta_2|\nabla T^2|^2\Big]\,\mathrm{d}x\,\mathrm{d}\eta\\
&\quad+\mathrm{e}^{-\omega t}\int_{R_z}[\delta_1T^2+\delta_2T^4]\,\mathrm{d}x\\
&\leqslant \int_0^t\int_{R_z}\mathrm{e}^{-\omega\eta}T^2\,\mathrm{d}x\,\mathrm{d}\eta+\Big[1+\frac{3}{4}c^{-1/3}\Big]\int_0^t\int_{R_z}\mathrm{e}^{-\omega\eta}T^4\,\mathrm{d}x\,\mathrm{d}\eta\\
&\quad+n_3\Big[-\frac{\partial}{\partial z}E(z,t)\Big]+n_2\Big[-\frac{\partial}{\partial z}E(z,t)\Big]^{5/4},\end{aligned}$$

where we have chosen that $\delta_1>2/\omega$, $\delta_2>2/\omega[1+\frac{3}{4}c^{-1/3}]$ and let

$$n_3=n_1+\max\Big\{\frac{2\delta_1}{\sqrt{\omega\delta_2}}+\frac{F_M^2\sqrt{\delta_2}}{\sqrt{\omega}},\frac{\sqrt{2}}{\sqrt{\omega}}\Big\}.$$

In view of (5.8), we can conclude that the "energy expression" $E(z,t)$ also satisfies the inequality (3.7). Using a method similar to Section 3, we to have Theorem 5.1. □

6. Conclusion

In this paper, the equations (1.1)–(1.2) are reconsidered in a semi-infinite cylinder and the spatial decay bounds of solutions are obtained. In three different types of a two-dimensional pipe, Payne and Schaefer [32] obtained Phragmén-Lindelöf alternative results for biharmonic equation. As far as we know, there are few results for this type of three-dimensional cylinder region (see Figure 2).

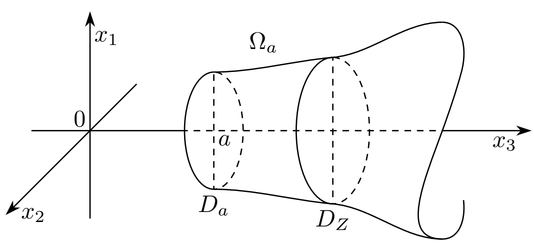

Figure 2. Cylindrical pipe Ω_a.

Therefore, it would be very interesting to replace the pipe R by

$$\Omega_a=\{(x_1,x_2,x_3)\mid (x_1,x_2)\in D_{x_3},\ x_3>a>0\},$$

where D_{x_3} is defined as

$$D_{x_3}=\Big\{(x_1,x_2,x_3)\mid \frac{x_1^2}{m^2}+\frac{x_2^2}{n^2}=x_3^{\gamma},\ x_3>a>0,\ m,\ n>0,\ 0<\gamma\leqslant 1\Big\}.$$

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