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Oscillation conditions for first-order nonlinear advanced differential equations

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Abstract. Our purpose is to analyze a first order nonlinear differential equation with advanced arguments. Then, some sufficient conditions for the oscillatory solutions of this equation are presented. Our results essentially improve two conditions in the paper "Oscillation tests for nonlinear differential equations with several nonmonotone advanced arguments" by N. Kılıç, Ö. Öcalan and U. M. Özkan. Also we give an example to illustrate our results.

Keywords: nonlinear advanced equation; nonmonotone argument; oscillatory solution

Classification: 34C10, 34K06, 34K11

1. Introduction

Numerous situations in the real world where the evolution rate depends on both the present and the future can be modeled using advanced differential equations. In order to account for the influence of hypothetical future actions, which are currently possible and helpful in the decision-making process, an advance argument can be added to the equation. For example, domains like mechanical control engineering, population dynamics, and economic issues are typical ones where such phenomena are considered to arise.

We consider a first order nonlinear differential equation with advanced arguments

$$y'(t) - \sum_{i=1}^{m} p_i(t) g_i\big(y(\varphi_i(t))\big) = 0, \qquad t \geq t_0, \tag{1.1}$$

where the functions $p_i(t)$ and $\varphi_i(t)$ are the functions of nonnegative real numbers for $1 \leq i \leq m$, $\varphi_i(t)$ are not necessarily monotone such that

$$\varphi_i(t) \geq t \quad \text{for } t \geq t_0, \qquad \lim_{t\to\infty} \varphi_i(t) = \infty, \qquad 1 \leq i \leq m, \tag{1.2}$$

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and

$$g_i \in C(\mathbb{R},\mathbb{R}) \quad \text{and} \quad y g_i(y) > 0 \qquad \text{for } y \neq 0,\ 1 \leq i \leq m. \tag{1.3}$$

By a solution of (1.1), we mean continuously differentiable function defined on $[\varphi_i(T_0), \infty)$ for some $T_0 \geq t_0$ such that (1.1) holds for $t \geq T_0$. A solution of (1.1) is said to be *oscillatory* if it has arbitrarily large zeroes. Otherwise, it is called *nonoscillatory*.

When $g(y) = y$, we have the following equation which is the linear form of (1.1)

$$y'(t) - \sum_{i=1}^{m} p_i(t) y(\varphi_i(t)) = 0, \qquad t \geq t_0. \tag{1.4}$$

The question of obtaining new sufficient criteria for the oscillatory solutions of (1.4) has been investigated by researchers. See, for example, [1], [2], [3], [5], [6], [9], [12].

Also, when $m = 1$, (1.1) reduces to

$$y'(t) - p(t) g\big(y(\varphi(t))\big) = 0, \qquad t \geq t_0. \tag{1.5}$$

N. Fukagai and T. Kusano in [7] gave the oscillation conditions for the solutions of (1.5) with nondecreasing argument. Then, Ö. Öcalan et al. in [10] analyzed (1.5) with nonmonotone argument and they obtained some oscillation criteria. Also, in 1987 G. S. Ladde et al. in [9] studied (1.1) with strictly increasing arguments.

Now, we define the following functions.

$$\delta_i(t) := \inf_{s \geq t} \{\varphi_i(s)\} \quad \text{and} \quad \delta(t) = \min_{1 \leq i \leq m} \{\delta_i(t)\}, \qquad t \geq 0. \tag{1.6}$$

Certainly, $\delta_i(t)$ are nondecreasing and $\delta(t) \leq \delta_i(t) \leq \varphi_i(t)$ for all $t \geq 0$, $1 \leq i \leq m$.

Also, suppose that g_i in (1.1) for $1 \leq i \leq m$ satisfy the below one.

$$\limsup_{|y| \to \infty} \frac{y}{g_i(y)} = \widetilde{N}_i, \qquad 0 \leq \widetilde{N}_i < \infty. \tag{1.7}$$

Finally, in 2021 N. Kılıç et al. in [8] examined (1.1) with nonmonotone arguments and presented the following criteria.

Theorem 1.1 ([8], Theorem 1 and Theorem 2)**.** *Assume that* (1.2), (1.3), (1.6) *and* (1.7) *hold. If* $\varphi_i(t)$ *are not necessarily monotone for* $1 \leq i \leq m$ *and*

$$\liminf_{t \to \infty} \int_t^{\varphi(t)} \sum_{i=1}^{m} p_i(s)\, \mathrm{d}s > \frac{\widetilde{N}_*}{\mathrm{e}}, \qquad 0 \leq \widetilde{N}_* < \infty, \tag{1.8}$$

or

$$\limsup_{t\to\infty} \int_t^{\delta(t)} \sum_{i=1}^{m} p_i(s)\,\mathrm{d}s > \widetilde{N}_*, \qquad 0 < \widetilde{N}_* < \infty, \tag{1.9}$$

where $\varphi(t) = \min_{1\le i\le m}\{\varphi_i(t)\}$ *and* $\widetilde{N}_* = \max_{1\le i\le m}\left\{\widetilde{N}_i\right\}$, *then all solutions of* (1.1) *are oscillatory.*

There exists a broad literature on the oscillation theory of differential equations with advanced type. Furthermore, there are a lot of papers about linear advanced differential equations, but there are only a few articles about nonlinear differential equations with advance arguments. Especially, as far as we know, there are only two criteria for the oscillatory solutions of (1.1) with nonmonotone arguments in the literature. In view of this, an interesting question that arises in the case $\varphi_i(t)$ are not necessarily monotone for $1 \le i \le m$ and (1.8) and (1.9) do not hold, is whether we can obtain new oscillation criteria for (1.1). In this article, we will answer to this question in a positive way. So, our purpose is to essentially improve the conditions given above and to present new sufficient conditions for the oscillation of all solutions of (1.1) by using Grönwall inequality.

In the paper we establish some new conditions involving lim sup and lim inf for the all oscillatory solutions of (1.1). We present example to confirm the importance of the main results.

2. Main results

Some sufficient conditions for the oscillatory behaviour of (1.1) are presented in this section, when $\varphi_i(t)$ are not necessarily monotone for $1 \le i \le m$.

The following lemmas help us to prove the main theorems.

Lemma 2.1 (Grönwall inequality)**.** *Assume that* $y(t)$ *is a positive solution of* $y'(t) - \sum_{i=1}^{m} p_i(t)y(t) \ge 0$. *Then, we have*

$$y(s) \ge y(t)\exp\left\{\int_t^s \sum_{i=1}^{m} p_i(r)\,\mathrm{d}r\right\}, \qquad s \ge t. \tag{2.1}$$

By using the same justifications as in the proof of Lemma 2.2 in [11], the following conclusion can be established.

Lemma 2.2. *Assume that* (1.6) *holds and*

$$\liminf_{t\to\infty} \int_t^{\varphi(t)} \sum_{i=1}^{m} p_i(s)\,\mathrm{d}s = L. \tag{2.2}$$

Then, we get

$$\liminf_{t\to\infty}\int_t^{\delta(t)}\sum_{i=1}^m p_i(s)\,\mathrm{d}s = L. \tag{2.3}$$

Lemma 2.3. *Suppose that* (1.2), (1.3) *and* (1.7) *hold and* $y(t)$ *is an eventually positive solution of* (1.1). *If*

$$\limsup_{t\to\infty}\int_t^{\delta(t)}\sum_{i=1}^m \frac{p_i(s)}{N_i}\exp\left\{\int_{\delta(t)}^{\varphi_i(s)}\sum_{j=1}^m \frac{p_j(r)}{N_j}\,\mathrm{d}r\right\}\mathrm{d}s > 0, \tag{2.4}$$

where $\delta(t)$ *is given by* (1.6), *then* $\lim_{t\to\infty} y(t) = \infty$.

Also, suppose that $y(t)$ *is an eventually negative solution of* (1.1). *If* (2.4) *holds, then* $\lim_{t\to\infty} y(t) = -\infty$.

Proof: Suppose that (2.4) holds. Let $y(t)$ be an eventually positive solution of (1.1). Then, there is $t_1 > t_0$ such that $y(t)$, $y(\varphi_i(t)) > 0$ for all $t \ge t_1$ and $1 \le i \le m$. So, from (1.1), we have

$$y'(t) = \sum_{i=1}^m p_i(t) g_i\big(y(\varphi_i(t))\big) \ge 0$$

for all $t \ge t_1$, which shows that $y(t)$ is nondecreasing and has a limit $k > 0$ or $k = \infty$. Now, we claim that $\lim_{t\to\infty} y(t) = \infty$. Otherwise, $\lim_{t\to\infty} y(t) = k > 0$. Then, integrating (1.1) from t to $\delta(t)$, we have

$$y(\delta(t)) - y(t) - \int_t^{\delta(t)}\sum_{i=1}^m p_i(s) g_i\big(y(\varphi_i(s))\big)\,\mathrm{d}s = 0. \tag{2.5}$$

From (1.7), we can choose N_i with $\widetilde{N}_i < N_i$ for $1 \le i \le m$ such that

$$g_i\big(y(\varphi_i(t))\big) \ge \frac{1}{N_i}\, y(\varphi_i(t)). \tag{2.6}$$

Using the inequality (2.6) in (2.5), we have

$$y(\delta(t)) - y(t) - \int_t^{\delta(t)}\sum_{i=1}^m \frac{p_i(s)}{N_i}\, y(\varphi_i(s))\,\mathrm{d}s \ge 0. \tag{2.7}$$

Also, by using Lemma 2.1 in (2.7), we obtain

$$y(\delta(t)) - y(t) - y(\delta(t))\int_t^{\delta(t)}\sum_{i=1}^m \frac{p_i(s)}{N_i}\exp\left\{\int_{\delta(t)}^{\varphi_i(s)}\sum_{j=1}^m \frac{p_j(r)}{N_j}\,\mathrm{d}r\right\}\mathrm{d}s \ge 0. \tag{2.8}$$

Moreover, (2.4) implies that there exists at least one sequence $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \int_{t_n}^{\delta(t_n)} \sum_{i=1}^{m} \frac{p_i(s)}{N_i} \exp\left\{ \int_{\delta(t_n)}^{\varphi_i(s)} \sum_{j=1}^{m} \frac{p_j(r)}{N_j}\,\mathrm{d}r \right\} \mathrm{d}s > 0. \tag{2.9}$$

By $t \to t_n$ and taking limit $n \to \infty$ in (2.8), we get

$$-k \lim_{n\to\infty} \int_{t_n}^{\delta(t_n)} \sum_{i=1}^{m} \frac{p_i(s)}{N_i} \exp\left\{ \int_{\delta(t_n)}^{\varphi_i(s)} \sum_{j=1}^{m} \frac{p_j(r)}{N_j}\,\mathrm{d}r \right\} \mathrm{d}s \geq 0, \tag{2.10}$$

which contradicts with (2.9).

Using the same procedure, it is simple to obtain that when $y(t)$ is an eventually negative solution of (1.1) and (2.4) holds, $\lim_{t\to\infty} y(t) = -\infty$. □

Theorem 2.4. *Suppose that* (1.2), (1.3), (1.6) *and* (1.7) *hold. If*

$$\liminf_{t\to\infty} \int_{t}^{\varphi(t)} \sum_{i=1}^{m} \frac{p_i(s)}{N_i} \exp\left\{ \int_{\delta(s)}^{\varphi_i(s)} \sum_{j=1}^{m} \frac{p_j(r)}{N_j}\,\mathrm{d}r \right\} \mathrm{d}s > \frac{1}{\mathrm{e}}, \tag{2.11}$$

where $\varphi(t) = \min_{1\leq i\leq m}\{\varphi_i(t)\}$ *and* N_i *are constants with* $\widetilde{N}_i < N_i$ *for* $1 \leq i \leq m$, *then all solutions of* (1.1) *are oscillatory.*

Proof: Assume, for the sake of contradiction, that there is an eventually positive solution $y(t)$ of (1.1). If $y(t)$ is an eventually negative solution of (1.1), the proof can be done in similar way. Then, there is $t_1 > t_0$ such that $y(t)$, $y(\varphi_i(t))$, $y(\delta_i(t))$, $y(\delta(t)) > 0$ for all $t \geq t_1$ and $1 \leq i \leq m$. So, from (1.1) we get

$$y'(t) = \sum_{i=1}^{m} p_i(t) g_i\big(y(\varphi_i(t))\big) \geq 0$$

for all $t \geq t_1$, which implies that $y(t)$ is nondecreasing function. Condition (2.11) implies (2.4), so from Lemma 2.3, we have $\lim_{t\to\infty} y(t) = \infty$.

Then, from (1.7), we can choose $t_2 > t_1$ and there are N_i with $\widetilde{N}_i < N_i$ for $1 \leq i \leq m$ such that

$$g_i\big(y(\varphi_i(t))\big) \geq \frac{1}{N_i}\, y(\varphi_i(t)) \qquad \text{for } 1 \leq i \leq m, \tag{2.12}$$

for $t \geq t_2$. By using (2.12) in (1.1), we obtain

$$y'(t) - \sum_{i=1}^{m} \frac{p_i(t)}{N_i}\, y(\varphi_i(t)) \geq 0. \tag{2.13}$$

Then, using that $y(t)$ is nondecreasing and $t \le \varphi_i(t)$ for $1 \le i \le m$, we get

$$y'(t) - y(t)\sum_{i=1}^{m}\frac{p_i(t)}{N_i} \ge 0. \tag{2.14}$$

Hence, by Lemma 2.1,

$$y(\varphi_i(s)) \ge y(\delta(s))\exp\left\{\int_{\delta(s)}^{\varphi_i(s)}\sum_{j=1}^{m}\frac{p_j(r)}{N_j}\,\mathrm{d}r\right\}. \tag{2.15}$$

Also, from (2.11) and Lemma 2.2, there is a constant $c > 0$ such that

$$\int_t^{\delta(t)}\sum_{i=1}^{m}\frac{p_i(s)}{N_i}\,\exp\left\{\int_{\delta(s)}^{\varphi_i(s)}\sum_{j=1}^{m}\frac{p_j(r)}{N_j}\,\mathrm{d}r\right\}\mathrm{d}s \ge c > \frac{1}{\mathrm{e}}, \qquad t \ge t_3 \ge t_2. \tag{2.16}$$

Moreover, from (2.16) there is a real number $t^* \in (t, \delta(t))$ for all $t \ge t_3$ such that

$$\int_t^{t^*}\sum_{i=1}^{m}\frac{p_i(s)}{N_i}\,\exp\left\{\int_{\delta(s)}^{\varphi_i(s)}\sum_{j=1}^{m}\frac{p_j(r)}{N_j}\,\mathrm{d}r\right\}\mathrm{d}s > \frac{1}{2\mathrm{e}} \tag{2.17}$$

and

$$\int_{t^*}^{\delta(t)}\sum_{i=1}^{m}\frac{p_i(s)}{N_i}\,\exp\left\{\int_{\delta(s)}^{\varphi_i(s)}\sum_{j=1}^{m}\frac{p_j(r)}{N_j}\,\mathrm{d}r\right\}\mathrm{d}s > \frac{1}{2\mathrm{e}}. \tag{2.18}$$

Integrating (2.13) from t to t^*, by using $y(t)$ and $\delta(t)$ are nondecreasing and (2.15), we obtain

$$y(t^*) - y(t) - \int_t^{t^*}\sum_{i=1}^{m}\frac{p_i(s)}{N_i}\,y(\varphi_i(s))\,\mathrm{d}s \ge 0,$$

$$y(t^*) - y(t) - \int_t^{t^*}\sum_{i=1}^{m}\frac{p_i(s)}{N_i}\,y(\delta(s))\exp\left\{\int_{\delta(s)}^{\varphi_i(s)}\sum_{j=1}^{m}\frac{p_j(r)}{N_j}\,\mathrm{d}r\right\}\mathrm{d}s \ge 0$$

or

$$y(t^*) - y(t) - y(\delta(t))\int_t^{t^*}\sum_{i=1}^{m}\frac{p_i(s)}{N_i}\,\exp\left\{\int_{\delta(s)}^{\varphi_i(s)}\sum_{j=1}^{m}\frac{p_j(r)}{N_j}\,\mathrm{d}r\right\}\mathrm{d}s \ge 0$$

and by using (2.17) we obtain

$$y(t^*) > \frac{1}{2\mathrm{e}}\,y(\delta(t)). \tag{2.19}$$

Integrating (2.13) from t^* to $\delta(t)$, by using the same facts, we have

$$y(\delta(t)) - y(t^*) - \int_{t^*}^{\delta(t)} \sum_{i=1}^{m} \frac{p_i(s)}{N_i}\, y(\varphi_i(s))\, \mathrm{d}s \geq 0,$$

$$y(\delta(t)) - y(t^*) - \int_{t^*}^{\delta(t)} \sum_{i=1}^{m} \frac{p_i(s)}{N_i}\, y(\delta(s)) \exp\left\{ \int_{\delta(s)}^{\varphi_i(s)} \sum_{j=1}^{m} \frac{p_j(r)}{N_j}\, \mathrm{d}r \right\} \mathrm{d}s \geq 0$$

or

$$y(\delta(t)) - y(t^*) - y(\delta(t^*)) \int_{t^*}^{\delta(t)} \sum_{i=1}^{m} \frac{p_i(s)}{N_i} \exp\left\{ \int_{\delta(s)}^{\varphi_i(s)} \sum_{j=1}^{m} \frac{p_j(r)}{N_j}\, \mathrm{d}r \right\} \mathrm{d}s \geq 0$$

and by using (2.18)

$$y(\delta(t)) > \frac{1}{2\mathrm{e}}\, y(\delta(t^*)). \tag{2.20}$$

Considering (2.19) and (2.20) together, we obtain

$$y(t^*) > \frac{1}{2\mathrm{e}}\, y(\delta(t)) > \frac{1}{(2\mathrm{e})^2}\, y(\delta(t^*)). \tag{2.21}$$

Let

$$\lambda := \liminf_{t\to\infty} \frac{y(\delta(t))}{y(t)} \tag{2.22}$$

and because of $1 \leq \lambda \leq (2\mathrm{e})^2$, λ is finite.

Dividing (1.1) with $y(t)$ and integrating from t to $\delta(t)$, we get

$$\int_t^{\delta(t)} \frac{y'(s)}{y(s)}\, \mathrm{d}s - \int_t^{\delta(t)} \sum_{i=1}^{m} p_i(s) \frac{g_i\big(y(\varphi_i(s))\big)}{y(s)}\, \mathrm{d}s = 0$$

or

$$\ln \frac{y(\delta(t))}{y(t)} - \int_t^{\delta(t)} \sum_{i=1}^{m} p_i(s)\, \frac{g_i\big(y(\varphi_i(s))\big)}{y(\varphi_i(s))}\, \frac{y(\varphi_i(s))}{y(s)}\, \mathrm{d}s = 0. \tag{2.23}$$

By using (2.12) and (2.15) in (2.23), we have

$$\ln \frac{y(\delta(t))}{y(t)} - \int_t^{\delta(t)} \sum_{i=1}^{m} \frac{p_i(s)}{N_i}\, \frac{y(\delta(s))}{y(s)} \exp\left\{ \int_{\delta(s)}^{\varphi_i(s)} \sum_{j=1}^{m} \frac{p_j(r)}{N_j}\, \mathrm{d}r \right\} \mathrm{d}s \geq 0$$

and also there is a ζ such that $t \leq \zeta \leq \delta(t)$. Then, we get

$$\ln \frac{y(\delta(t))}{y(t)} \geq \frac{y(\delta(\zeta))}{y(\zeta)} \int_t^{\delta(t)} \sum_{i=1}^{m} \frac{p_i(s)}{N_i} \exp\left\{ \int_{\delta(s)}^{\varphi_i(s)} \sum_{j=1}^{m} \frac{p_j(r)}{N_j}\, \mathrm{d}r \right\} \mathrm{d}s. \tag{2.24}$$

Now, we take limit inferior on both sides of (2.24), we have

$$\liminf_{t\to\infty} \ln\Big(\frac{y(\delta(t))}{y(t)}\Big) > \liminf_{t\to\infty} \frac{y(\delta(\zeta))}{y(\zeta)}\,\frac{1}{\mathrm{e}}, \tag{2.25}$$

where we use that $\liminf(h(t)k(t)) \geq \liminf(h(t))\liminf(k(t))$. Therefore, from (2.22), (2.25) and $\ln\big(\liminf(y(\delta(t))/y(t))\big) \geq \liminf\big(\ln(y(\delta(t))/y(t))\big)$ we have

$$\ln\lambda > \frac{\lambda}{\mathrm{e}},$$

which is not possible for any positive number λ, so it completes the proof. □

Theorem 2.5. *Assume that* (1.2), (1.3), (1.6) *and* (1.7) *hold. If*

$$\limsup_{t\to\infty} \int_t^{\delta(t)} \sum_{i=1}^{m} \frac{p_i(s)}{N_i} \exp\Big\{\int_{\delta(t)}^{\varphi_i(s)} \sum_{j=1}^{m} \frac{p_j(r)}{N_j}\,\mathrm{d}r\Big\}\,\mathrm{d}s > 1, \tag{2.26}$$

where N_i are constants with $\widetilde{N}_i < N_i$ for $1 \leq i \leq m$, then all solutions of (1.1) *are oscillatory.*

Proof: Assume, for the sake of contradiction, that there is an eventually positive solution $y(t)$ of (1.1). Since (2.26) implies (2.4), by Lemma 2.3, $\lim_{t\to\infty} y(t) = \infty$. As the proof of Theorem 2.1, we have Lemma 2.1. So, from Lemma 2.1, we obtain

$$y(\varphi_i(s)) \geq y(\delta(t)) \exp\Big\{\int_{\delta(t)}^{\varphi_i(s)} \sum_{j=1}^{m} \frac{p_j(r)}{N_j}\,\mathrm{d}r\Big\}. \tag{2.27}$$

Integrating (2.13) from t to $\delta(t)$, we get

$$y(\delta(t)) - y(t) - \int_t^{\delta(t)} \sum_{i=1}^{m} \frac{p_i(s)}{N_i}\, y(\varphi_i(s))\,\mathrm{d}s \geq 0$$

and also by (2.27)

$$y(\delta(t)) - y(t) - y(\delta(t)) \int_t^{\delta(t)} \sum_{i=1}^{m} \frac{p_i(s)}{N_i} \exp\Big\{\int_{\delta(t)}^{\varphi_i(s)} \sum_{j=1}^{m} \frac{p_j(r)}{N_j}\,\mathrm{d}r\Big\}\,\mathrm{d}s \geq 0.$$

Dividing the last inequality by $y(\delta(t))$, we have

$$1 - \frac{y(t)}{y(\delta(t))} \geq \int_t^{\delta(t)} \sum_{i=1}^{m} \frac{p_i(s)}{N_i} \exp\Big\{\int_{\delta(t)}^{\varphi_i(s)} \sum_{j=1}^{m} \frac{p_j(r)}{N_j}\,\mathrm{d}r\Big\}\,\mathrm{d}s,$$

which implies

$$\int_t^{\delta(t)} \sum_{i=1}^{m} \frac{p_i(s)}{N_i} \exp\left\{ \int_{\delta(t)}^{\varphi_i(s)} \sum_{j=1}^{m} \frac{p_j(r)}{N_j}\,\mathrm{d}r \right\} \mathrm{d}s < 1$$

for sufficiently large t. Therefore,

$$\limsup_{t\to\infty} \int_t^{\delta(t)} \sum_{i=1}^{m} \frac{p_i(s)}{N_i} \exp\left\{ \int_{\delta(t)}^{\varphi_i(s)} \sum_{j=1}^{m} \frac{p_j(r)}{N_j}\,\mathrm{d}r \right\} \mathrm{d}s \le 1,$$

but this contradicts with (2.26), so this completes the proof. □

Example 2.6. We consider the following first order nonlinear advanced differential equation

$$\begin{aligned} y'(t) &- 0.26y(\varphi_1(t)) \ln\left(\mathrm{e}^{-|y(\varphi_1(t))|} + 2\right) \\ &- 0.25y(\varphi_2(t)) \ln\left(\mathrm{e}^{-|y(\varphi_2(t))|} + 3\right) = 0, \qquad t \ge 0, \end{aligned} \tag{2.28}$$

where

$$\varphi_1(t) = \begin{cases} 4t - 6a - 2, & t \in [2a+1, 2a+2] \\ -2t + 6a + 10, & t \in [2a+2, 2a+3] \end{cases}, \qquad a \in \mathbb{N}_0,$$
$$\varphi_2(t) = \varphi_1(t) + 1,$$

and

$$\delta_1(t) := \inf_{s \ge t}\{\varphi_1(s)\} = \begin{cases} 4t - 6a - 2, & t \in [2a+1, 2a+1.5] \\ 2a + 4, & t \in [2a+1.5, 2a+3] \end{cases}, \qquad a \in \mathbb{N}_0,$$
$$\delta_2(t) = \delta_1(t) + 1,$$

then,

$$\varphi(t) = \min_{1 \le i \le m}\{\varphi_i(t)\} = \varphi_1(t).$$

Also, we find

$$\widetilde{N}_1 = \limsup_{|y|\to\infty} \frac{y(\varphi_1(t))}{y(\varphi_1(t)) \ln(\mathrm{e}^{-|y(\varphi_1(t))|} + 2)} = \frac{1}{\ln 2} \cong 1.44269$$

and

$$\widetilde{N}_2 = \limsup_{|y|\to\infty} \frac{y(\varphi_2(t))}{y(\varphi_2(t)) \ln(\mathrm{e}^{-|y(\varphi_2(t))|} + 3)} = \frac{1}{\ln 3} \cong 0.91023,$$

then,

$$\widetilde{N}_* = \max_{1 \le i \le m}\left\{\widetilde{N}_i\right\} = \widetilde{N}_1 \cong 1.44269.$$

So, we have

$$\liminf_{t\to\infty}\int_t^{\varphi(t)}\sum_{i=1}^m p_i(s)\,\mathrm{d}s=\liminf_{t\to\infty}\int_{2a+3}^{2a+4}(0.26+0.25)\,\mathrm{d}s=0.51\not>\frac{\widetilde{N}_*}{\mathrm{e}}\cong 0.53073$$

and

$$\limsup_{t\to\infty}\int_t^{\delta(t)}\sum_{i=1}^m p_i(s)\,\mathrm{d}s=\limsup_{t\to\infty}\int_{2k+1.5}^{2k+4}(0.26+0.25)\,\mathrm{d}s$$
$$\cong 1.275\not>\widetilde{N}_*\cong 1.44269$$

that is, (1.8) and (1.9) are not satisfied.

However, when $N_1=1.45$ and $N_2=0.92$, we observe that

$$\begin{aligned}
&\liminf_{t\to\infty}\int_t^{\varphi(t)}\sum_{i=1}^m\frac{p_i(s)}{N_i}\exp\left\{\int_{\delta(s)}^{\varphi_i(s)}\sum_{j=1}^m\frac{p_j(r)}{N_j}\,\mathrm{d}r\right\}\mathrm{d}s\\
&=\liminf_{t\to\infty}\int_{2a+3}^{2a+4}\left[\frac{0.26}{1.45}\exp\left\{\int_{2a+4}^{2a+5}\left(\frac{0.26}{1.45}+\frac{0.25}{0.92}\right)\mathrm{d}r\right\}\right.\\
&\quad\left.+\frac{0.25}{0.92}\exp\left\{\int_{2a+4}^{2a+5}\left(\frac{0.26}{1.45}+\frac{0.25}{0.92}\right)\mathrm{d}r\right\}\right]\mathrm{d}s\\
&\cong 0.60592>\frac{1}{\mathrm{e}}\cong 0.36787,
\end{aligned}$$

then, all conditions of Theorem 2.4 are satisfied and all solutions of (2.28) are oscillatory.

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