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INVERSE RATE-DEPENDENT PRANDTL-ISHLINSKII OPERATORS AND APPLICATIONS

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Abstract. In the past years, we observed an increased interest in rate-dependent hysteresis models to characterize complex time-dependent nonlinearities in smart actuators. A natural way to include rate-dependence to the Prandtl-Ishlinskii model is to consider it as a linear combination of play operators whose thresholds are functions of time. In this work, we propose the extension of the class of rate-dependent Prandtl-Ishlinskii operators to the case of a whole continuum of play operators with time-dependent thresholds. We prove the existence of an analytical inversion formula, and illustrate its applicability in the study of error bounds for inverse compensation.

Keywords: hysteresis; Prandtl-Ishlinskii operator; inverse rate-dependent Prandtl-Ishlinskii operator

MSC 2020: 74N30, 47J40

1. Introduction

The presence of hysteresis nonlinearities in smart systems, such as piezoelectric and magnetostrictive actuators, has been widely associated with various performance limitations, see for instance [4], [15], [17]. Notably, micro/nano-positioning applications are very sensitive to such hysteretic effects, therefore, the characterization and modeling of the hysteresis properties is crucial for designing efficient compensation algorithms. Among the hysteresis models whose inverse have been employed as feedforward compensator, those based on the Preisach operator and the Prandtl-Ishlinskii operator are of particular interest. However, while the usage of the Preisach operator

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relies on numerical methods for obtaining approximate inversions of the model [5], [14], [16], the application of the Prandtl-Ishlinskii operator is more advantageous due to the fact that such operator is analytically invertible [7], [10].

Rate-dependent processes also appear to be an effective alternative in modeling hysteresis in smart materials. The notion of a parameter-dependent Preisach operator introduced in [12], for example, provides a good framework for the study of thermal effects in piezoelectric models [13]. Besides, there is a number of papers featuring the so-called rate-dependent Prandtl-Ishlinskii operator in different applications for hysteresis compensation [2], [3]. These applications benefit from the fact that an explicit representation of the inverse is still available for rate-dependent Prandtl-Ishlinskii model in the discrete case, that is, for the case when the Prandtl-Ishlinskii operator is defined as a finite linear combination of play operators with time-dependent thresholds, see [1], [9].

In practice, the number of active thresholds in the Prandtl-Ishlinskii operator can be very large, and in this case the explicit inversion formula requires a non-negligible computational effort. The question if all the thresholds are really necessary for a sufficiently accurate inversion is therefore legitimate, because a possible reduction of the number of thresholds decreases the computational complexity. We follow the classical concept of continuum mechanics: In order to show that the inversion algorithm is stable with respect to memory discretization, we address the inversion problem for the Prandtl-Ishlinskii model in a broader sense by considering the rate-dependent Prandtl-Ishlinskii operator to be the continuous counterpart to the discrete notion investigated in [1]; for such operators we then establish an inversion formula whose definition, like in the classical case, somehow relies on the inverse of its initial loading curve. The main tool in this investigation is the inversion formula available for discrete rate-dependent Prandtl-Ishlinskii operators, see [1], [9]. The approach consists in considering a discretization of the Prandtl-Ishlinskii model and, making use of the explicit inversion formula, to approximate and validate the proposed inverse. As the next step, we derive error bounds of inverse compensation when a discretized inverse Prandtl-Ishlinskii operator is applied as a feedforward controller.

This paper is organized as follows: In Section 2 we introduce the rate-dependent Prandtl-Ishlinskii operator based on a shape function; a function which accounts for the initial loading curve. Section 3 represents the core of the paper and presents the inverse of the rate-dependent Prandtl-Ishlinskii operator (Theorem 3.1). In Section 4 we investigate the approximate inversion error relying on the inverse of a discretized rate-dependent Prandtl-Ishlinskii operator and accounting for measurement errors of the initial loading curve. An example applying the inverse rate-dependent Prandtl-Ishlinskii as feedforward compensator is given in Section 5.

2. Rate-dependent Prandtl-Ishlinskii operator

In what follows, for $T > 0$ we denote by $C(0,T)$ the space of continuous functions defined in $[0,T]$ endowed with the norm $|z|_{[0,T]} := \max_{t\in[0,T]} |z(t)|$, and by $W^{1,1}(0,T)$ the space of absolutely continuous functions. The spaces $BV_{\text{loc}}(0,\infty)$ of real functions which have bounded variation on every bounded interval, and $W^{1,\infty}(0,\infty)$ of Lipschitz continuous functions are important in our investigation.

Herein, the time-dependent play operator has a central role in the development of our study; thus, we recall its definition in the sequel.

Given a function $\varrho \in C(0,T)$ such that $\varrho(t) \geqslant 0$ for all $t \in [0,T]$, for each function $v \in W^{1,1}(0,T)$ and $x_0 \in [-\varrho(0), \varrho(0)]$, let $\xi \in W^{1,1}(0,T)$ be the solution of the variational inequality

$$\begin{cases} |v(t) - \xi(t)| \leqslant \varrho(t) & \forall\, t \in [0,T], \\ v(0) - \xi(0) = x_0, & \\ \dot{\xi}(t)(v(t) - \xi(t) - \varrho(t)\zeta) \geqslant 0 & \text{a.e. } t \in [0,T] \quad \forall\, |\zeta| \leqslant 1. \end{cases} \tag{2.1}$$

The mapping $\mathfrak{p}_{\varrho(\cdot)}$ which with each initial value x_0 and each input function v associates the solution $\xi = \mathfrak{p}_{\varrho(\cdot)}[x_0, v]$ of (2.1) is called the *time-dependent play operator*. The definition of such an operator can be extended to the case of regulated functions by means of an integral formulation of the variational inequality, see details in [8].

We now introduce the main object of our study:

Definition 2.1. Let $z \in C(0,T)$ be a function such that $z(t) \geqslant 0$ for all $t \in [0,T]$, and fix an arbitrary $R > 0$. Given a constant $a_0 > 0$, let $\varphi \in W^{1,\infty}(0,\infty)$ be a function such that $\varphi' \in BV_{\text{loc}}(0,\infty)$, $\varphi(0) = \varphi'(0) = \varphi'(0+) = 0$, and $\sup_{r>0} |\varphi'(r)| =: \overline{\varphi} < a_0$. Denote by I the identity function in $\mathbb{R}$, i.e. $I(s) = s$ for $s \in \mathbb{R}$. For each input function $v \in W^{1,1}(0,T)$ such that $|v|_{[0,T]} \leqslant R$ and each initial value function $x \in W^{1,\infty}(0,\infty)$ such that

$$|x(0)| \leqslant z(0), \ |x'(r)| \leqslant 1 \text{ a.e.}, \ x(r) = v(0) \quad \text{for } r \geqslant R, \tag{2.2}$$

we define the *rate-dependent Prandtl-Ishlinskii operator* P_φ with shape function $a_0 I + \varphi$ by the integral

$$P_\varphi[x,v](t) = a_0 v(t) - \int_0^\infty \varphi'(r) \frac{\partial}{\partial r} \mathfrak{p}_{r+z(t)}[x(r), v](t)\,\mathrm{d}r, \tag{2.3}$$

where for a parameter value $r > 0$, $\mathfrak{p}_{r+z(\cdot)}$ is the solution of the variational inequality (2.1) with the choice $\varrho(t) = r + z(t)$.

Note that by choosing $a_0 = z(t) = 0$ we retrieve the classical Prandtl-Ishlinskii operator. Furthermore, the generalized Brokate identity ([1], Lemma 1.2) yields

$$\mathfrak{p}_{r+z(t)}[x(r),v](t) = \mathfrak{p}_r[x(r)-x(0),\xi_0](t), \quad \xi_0(t) = \mathfrak{p}_{z(t)}[x(0),v](t), \tag{2.4}$$

so that the formula (2.3) is meaningful by virtue of [11]; while the assumptions on both the initial value function and the input function ensure that we integrate over a bounded interval. Unlike in [11], though, the integral in (2.3) can be interpreted as the Lebesgue integral thanks to the regularity of φ and the fact that the left derivative of $\mathfrak{p}_r$ coincides with its distributional derivative almost everywhere.

It is of practical interest to consider a piecewise linear approximation of the shape function associated with a division $0 = r_0 < r_1 < \ldots < r_m = R < r_{m+1} = \infty$, namely,

$$\varphi_m(r) = \int_0^r \varphi'_m(s)\,\mathrm{d}s, \quad \varphi'_m(r) = \sum_{i=1}^{m+1} \widehat{\varphi}_{i-1}\chi_{[r_{i-1},r_i)}(r) \tag{2.5}$$

with

$$\widehat{\varphi}_{i-1} = \frac{\varphi(r_i)-\varphi(r_{i-1})}{r_i - r_{i-1}} \quad \text{for } i = 1,\ldots,m, \ \widehat{\varphi}_m = \varphi'(r_m+).$$

The symbol χ_A denotes the characteristic function of a set $A \subset [0,\infty)$, that is, $\chi_A(r) = 1$ if $r \in A$ and $\chi_A(r) = 0$ if $r \notin A$. Noting that our hypotheses on x and v imply that

$$\mathfrak{p}_{r+z(t)}[x(r),v](t) = 0 \quad \text{for } r \geqslant R, \tag{2.6}$$

an explicit calculation of the integral in (2.3) shows that the Prandtl-Ishlinskii operator P_{φ_m} with shape function $a_0 I + \varphi_m$ can be written in a particularly simple form

$$P_{\varphi_m}[x,v](t) = a_0 v(t) + \widehat{\varphi}_0 \mathfrak{p}_{z(t)}[x(0),v](t) + \sum_{i=1}^{m} (\widehat{\varphi}_i - \widehat{\varphi}_{i-1})\mathfrak{p}_{r_i+z(t)}[x(r_i),v](t). \tag{2.7}$$

The operator described in (2.7) can be understood as a discretization of the rate-dependent Prandtl-Ishlinskii operator (2.3). This approximation satisfies the following inequality:

Proposition 2.1. *Let φ and φ_m be as above. Then for every z, v, and x satisfying our hypotheses we have*

$$|P_\varphi[x,v] - P_{\varphi_m}[x,v]|_{[0,T]} \leqslant \max_{i=1,\ldots,m} |r_i - r_{i-1}| \operatorname{Var}_{[0,\infty)}\varphi'.$$

P r o o f. By (2.4) we have $|(\partial/\partial r)\mathfrak{p}_{r+z(t)}[x(r),v](t)| \leqslant 1$ a.e. Using (2.6), we obtain the estimate

$$|P_\varphi[x,v](t) - P_{\varphi_m}[x,v](t)| \leqslant \int_0^R |\varphi'(r) - \varphi'_m(r)|\,\mathrm{d}r$$
$$= \sum_{i=1}^m \int_{r_{i-1}}^{r_i} \left|\varphi'(r) - \frac{\varphi(r_i)-\varphi(r_{i-1})}{r_i - r_{i-1}}\right| \mathrm{d}r$$

and the assertion follows. □

3. Inversion formula

In this section we present the main result of the paper, Theorem 3.1, whose aim is to establish the inverse of the rate-dependent Prandtl-Ishlinskii operator relying on the inversion of the shape function and on a particular counterpart of the initial value function.

Note that the operator P_{φ_m} given by (2.7) is a superposition of time-dependent play operators like the Prandtl-Ishlinskii operator investigated in [9], and we can rewrite it as

$$P_{\varphi_m}[x,v](t) = a_0 v(t) + \sum_{i=1}^{m+1} a_i \mathfrak{p}_{\widetilde{r}_i(t)}[\widetilde{x}_i, v](t) \tag{3.1}$$

with $a_{i+1} = \widehat{\varphi}_i - \widehat{\varphi}_{i-1}$, $\widetilde{r}_{i+1}(t) = r_i + z(t)$, and $\widetilde{x}_{i+1} = x(r_i)$ for $i = 0, 1, \ldots, m$, with the convention $\widehat{\varphi}_{-1} = 0$. By [9], Theorem 1.2, for every input w the inverse operator to P_{φ_m} is given by the explicit formula

$$P_{\varphi_m}^{-1}[x,w](t) = b_0 w(t) + \sum_{i=1}^{m+1} b_i \mathfrak{p}_{\widetilde{s}_i(t)}[\widetilde{y}_i, w](t), \tag{3.2}$$

where $b_0 = 1/a_0$, and for $i = 1, \ldots, m+1$ we have

$$b_i = \frac{1}{a_0 + \widehat{\varphi}_{i-1}} - \frac{1}{a_0 + \widehat{\varphi}_{i-2}}, \tag{3.3}$$

$$\widetilde{s}_i(t) = s_{i-1} + a_0 z(t) \tag{3.4}$$

with $s_0 = 0$, $\widetilde{y}_1 = a_0 x(0)$, and for $i = 1, \ldots, m$

$$s_i - s_{i-1} = (a_0 + \widehat{\varphi}_{i-1})(r_i - r_{i-1}), \tag{3.5}$$

$$\widetilde{y}_{i+1} - \widetilde{y}_i = (a_0 + \widehat{\varphi}_{i-1})(\widetilde{x}_{i+1} - \widetilde{x}_i). \tag{3.6}$$

It is known that the inversion of a Prandtl-Ishlinskii operator can be performed analytically by inverting its initial loading curve. Similarly, to establish an inversion formula for the rate-dependent Prandtl-Ishlinskii operator P_φ as defined in (2.3), we consider the function ψ given by

$$\psi(s) = (a_0 I + \varphi)^{-1}(s) - \frac{s}{a_0} \quad \text{for } s \geqslant 0. \tag{3.7}$$

Note that $\psi'(s) \in [1/(a_0 + \overline{\varphi}) - 1/a_0, 1/a_0 - \overline{\varphi} - 1/a_0]$, and additionally,

$$\psi'(s) = \frac{1}{a_0 + \varphi'(r)\big|_{r=(a_0 I+\varphi)^{-1}(s)}} - \frac{1}{a_0}.$$

In particular, for any sequence $0 = \varrho_0 < \varrho_1 < \ldots < \varrho_N$, taking $\sigma_i = a_0\varrho_i + \varphi(\varrho_i)$, $i = 0, \ldots, N$, we have

$$\begin{aligned}\sum_{i=1}^{N} |\psi'(\sigma_i) - \psi'(\sigma_{i-1})| &= \sum_{i=1}^{N} \left| \frac{1}{a_0 + \varphi'(\varrho_i)} - \frac{1}{a_0 + \varphi'(\varrho_{i-1})} \right| \\ &\leqslant \frac{1}{(a_0 - \overline{\varphi})^2} \sum_{i=1}^{N} |\varphi'(\varrho_i) - \varphi'(\varrho_{i-1})|,\end{aligned}$$

hence, the variation of ψ' satisfies

$$\mathrm{Var}_{[0,\infty)}\psi' \leqslant \frac{1}{(a_0 - \overline{\varphi})^2} \mathrm{Var}_{[0,\infty)}\varphi' < \infty. \tag{3.8}$$

With the function ψ defined in (3.7) and with initial value function x as in (2.2) we associate the inverse initial value function y given by the formula

$$y(s) = a_0 x(0) + \int_0^{a_0/s+\psi(s)} x'(r)(a_0 + \varphi'(r))\,\mathrm{d}r. \tag{3.9}$$

Note that we have $y'(s) = x'(r)|_{r=a_0/s+\psi(s)}$ for a.e. $s > 0$, hence $|y'(s)| \leqslant 1$ a.e. For the sequence $\{s_i\}$ as in (3.5) we define the piecewise linear approximation ψ_m of ψ by the formula

$$\psi_m(s) = \int_0^s \psi_m'(\sigma)\,\mathrm{d}\sigma, \quad \psi_m'(s) = \sum_{i=1}^{m+1} \widehat{\psi}_{i-1}\chi_{[s_{i-1},s_i)}(s), \tag{3.10}$$

with

$$\widehat{\psi}_{i-1} = \frac{\psi(s_i) - \psi(s_{i-1})}{s_i - s_{i-1}} \quad \text{for } i = 1, \ldots, m,\ \widehat{\psi}_m = \psi'(s_m+) \text{ and } s_{m+1} = \infty.$$

We now define the Prandtl-Ishlinskii operator with shape function $b_0 I + \psi$ by a formula analogous to (2.3), namely,

$$P_\psi[y,w](t) = \frac{1}{a_0} w(t) - \int_0^\infty \psi'(s) \frac{\partial}{\partial s} \mathfrak{p}_{s+a_0 z(t)}[y(s), w](t)\,\mathrm{d}s, \tag{3.11}$$

and we consider its discrete approximation as a counterpart to (2.7),

$$P_{\psi_m}[y,w](t) = \frac{1}{a_0} w(t) + \sum_{i=0}^{m} (\widehat{\psi}_i - \widehat{\psi}_{i-1}) \mathfrak{p}_{s_i + a_0 z(t)}[y(s_i), w](t), \tag{3.12}$$

with the convention $\widehat{\psi}_{-1} = 0$. By (3.5) we have $s_i - s_{i-1} = a_0(r_i - r_{i-1}) + \varphi(r_i) - \varphi(r_{i-1})$. Since $r_0 = s_0 = 0$, we conclude that

$$s_i = a_0 r_i + \varphi(r_i), \quad r_i = \frac{1}{a_0} s_i + \psi(s_i)$$

for all $i = 1, \ldots, m$. Noting that $\psi(0) = 0$, the identities above also imply that

$$\frac{1}{a_0} + \widehat{\psi}_i = \frac{1}{a_0 + \widehat{\varphi}_i} \tag{3.13}$$

for all $i = 0, 1, \ldots, m$, so that comparing (3.12) with (3.3)–(3.4) we see that the operator P_{ψ_m} can be rewritten as

$$P_{\psi_m}[y,w](t) = b_0 w(t) + \sum_{i=1}^{m+1} b_i \mathfrak{p}_{\widetilde{s}_i(t)}[y(s_{i-1}), w](t). \tag{3.14}$$

We see that it differs from $P^{-1}_{\varphi_m}$ only in the initial condition $y(s_i)$. By (3.6) for $i = 1, \ldots, m$ we have

$$\widetilde{y}_{i+1} - \widetilde{y}_i = (a_0 + \widehat{\varphi}_{i-1})(x(r_i) - x(r_{i-1})) = \int_{r_{i-1}}^{r_i} x'(r)(a_0 + \varphi'(\varrho_i))\,\mathrm{d}r$$

for some $\varrho_i \in [r_{i-1}, r_i]$, and

$$y(s_i) - y(s_{i-1}) = \int_{r_{i-1}}^{r_i} x'(r)(a_0 + \varphi'(r))\,\mathrm{d}r,$$

hence, using (2.2), we get

$$|y(s_i) - \widetilde{y}_{i+1} - y(s_{i-1}) + \widetilde{y}_i| \leqslant \int_{r_{j-1}}^{r_j} |\varphi'(r) - \varphi'(\varrho_j)|\,\mathrm{d}r. \tag{3.15}$$

Thus, it follows that

$$|y(s_i) - \widetilde{y}_{i+1}| \leqslant \sum_{j=1}^{i} \int_{r_{i-1}}^{r_i} |\varphi'(r) - \varphi'(\varrho_i)| \, \mathrm{d}r + |y(s_0) - \widetilde{y}_1|, \quad i = 1, \ldots, m.$$

Since $y(0) = a_0 x(0) = \widetilde{y}_1$, summing up the inequality above over i we thus obtain

$$\max_{i=1,\ldots,m+1} |y(s_{i-1}) - \widetilde{y}_i| \leqslant \max_{i=1,\ldots,m} |r_i - r_{i-1}| \mathrm{Var}_{[0,\infty)} \varphi'. \tag{3.16}$$

Recall that the dependence of each individual play on the initial condition is Lipschitz with constant 1, in particular, for $i = 1, \ldots, m+1$,

$$|\mathfrak{p}_{\widetilde{s}_i(t)}[y(s_{i-1}), w](t) - \mathfrak{p}_{\widetilde{s}_i(t)}[\widetilde{y}_i, w](t)| \leqslant |y(s_{i-1}) - \widetilde{y}_i|$$

for every input w and every $t \in [0, T]$. Hence,

$$\begin{aligned} |P_{\psi_m}[y, w](t) - P_{\varphi_m}^{-1}[x, w](t)| &\leqslant \max_{i=1,\ldots,m+1} |y(s_{i-1}) - \widetilde{y}_i| \sum_{j=1}^{m+1} |b_j| \\ &\leqslant \max_{i=1,\ldots,m} |r_i - r_{i-1}| \mathrm{Var}_{[0,\infty)} \varphi' \mathrm{Var}_{[0,\infty)} \psi', \end{aligned} \tag{3.17}$$

where the last inequality is a consequence of (3.13) and (3.3).

We now easily prove the following inversion theorem.

Theorem 3.1. *Let $a_0 > 0$ be given and let z, x, φ be functions satisfying the assumptions of Definition* 2.1. *Assume that ψ and y are the functions given by* (3.7) *and* (3.9), *respectively. Then the operators P_φ and P_ψ defined by* (2.3) *and* (3.11) *are mutually inverse.*

Proof. Given an input function $v \in W^{1,1}(0,T)$ with $|v|_{[0,T]} \leqslant R$, denote $w = P_\varphi[x, v]$. For $m \in \mathbb{N}$, consider an arbitrary division $0 = r_0 < r_1 < \ldots < r_m = R$ and let $w_m = P_{\varphi_m}[x, v]$ be the corresponding discretized Prandtl-Ishlinskii operator as in (3.1). Clearly $v = P_{\varphi_m}^{-1}[x, w_m]$, thus for each $t \in [0, T]$ we can write

$$\begin{aligned} |P_\psi[y, w](t) - v(t)| &\leqslant |P_\psi[y, w](t) - P_{\psi_m}[y, w](t)| \\ &+ |P_{\psi_m}[y, w](t) - P_{\varphi_m}^{-1}[x, w](t)| \\ &+ |P_{\varphi_m}^{-1}[x, w](t) - P_{\varphi_m}^{-1}[x, w_m](t)|. \end{aligned} \tag{3.18}$$

Recalling that the play operator is Lipschitz with respect to the input function, it follows from the definition of $P_{\varphi_m}^{-1}$ and Proposition 2.2 that

$$\begin{aligned} |P_{\varphi_m}^{-1}[x, w](t) - P_{\varphi_m}^{-1}[x, w_m](t)| &\leqslant |w(t) - w_m(t)| \sum_{i=0}^{m+1} |b_i| \\ &\leqslant \max_{i=1,\ldots,m} |r_i - r_{i-1}| \mathrm{Var}_{[0,\infty)} \varphi' \mathrm{Var}_{[0,\infty)} \psi'. \end{aligned} \tag{3.19}$$

Moreover, by reasoning like in Proposition 2.2 we can see that

$$\begin{aligned}|P_\psi[y,w](t) - P_{\psi_m}[y,w](t)| &\leqslant \sum_{i=1}^{m} \int_{s_{i-1}}^{s_i} |\psi'(s) - \widehat{\psi}_{i-1}|\,\mathrm{d}s \\ &\leqslant \max_{i=1,\dots,m} |s_i - s_{i-1}| \mathrm{Var}_{[0,\infty)}\psi',\end{aligned}$$

which together with (3.5) yields

$$|P_\psi[y,w](t) - P_{\psi_m}[y,w](t)| \leqslant (a_0 + \overline{\varphi}) \max_{i=1,\dots,m} |r_i - r_{i-1}| \mathrm{Var}_{[0,\infty)}\psi'.$$

Using (3.17) and the inequalities above, we conclude that

$$|P_\psi[y,w](t) - v(t)| \leqslant \max_{i=1,\dots,m} |r_i - r_{i-1}|(a_0 + \overline{\varphi} + 2\mathrm{Var}_{[0,\infty)}\varphi')\mathrm{Var}_{[0,\infty)}\psi'.$$

Since the inequality above holds for any division of the interval $[0,R]$, by taking a division fine enough, the assertion follows. □

4. Approximate inversion error and error bounds

In this section we study the inversion error using the inverse rate-dependent Prandtl-Ishlinskii operator in hysteresis compensation while considering approximate measured shape function.

In practice, we cannot measure the whole initial loading curve φ. Instead, for a given division $0 = r_0 < r_1 < \ldots < r_m = R$ we determine the approximate value of $\varphi(r_i)$ with a measurement error ε. In other words, the approximate measured value, φ_i^*, satisfies

$$|\varphi(r_i) - \varphi_i^*| \leqslant \varepsilon \quad \text{for } i = 0,1,\dots,m. \tag{4.1}$$

Consider the function $\varphi^*\colon [0,R] \to \mathbb{R}$ defined by

$$\varphi^*(r) = \int_0^r (\varphi^*)'(s)\,\mathrm{d}s, \quad (\varphi^*)'(r) = \sum_{i=1}^{m} \widehat{\varphi}_{i-1}^* \chi_{[r_{i-1},r_i)}(r) \tag{4.2}$$

with $\widehat{\varphi}_{i-1}^* = (\varphi_i^* - \varphi_{i-1}^*)/(r_i - r_{i-1})$ for $i = 1,\dots,m$. The corresponding Prandtl-Ishlinskii operator can be written as

$$P_{\varphi^*}[x,v](t) = a_0 v(t) + \sum_{i=1}^{m+1} a_i^* \mathfrak{p}_{\widetilde{r}_i(t)}[\widetilde{x}_i, v](t) \tag{4.3}$$

with

$$a_{i+1}^* = \widehat{\varphi}_i^* - \widehat{\varphi}_{i-1}^*, \quad \text{and} \quad \widetilde{x}_{i+1} = x(r_i) \quad \text{for } i = 0,1,\dots,m,$$

with the convention $\widehat{\varphi}_{-1}^* = 0$.

We have the following approximation statement.

Proposition 4.1. *Let φ^* be the function in* (4.2) *and φ_m be the piecewise linear approximation of φ as in* (2.5). *Then for every v, z, and x satisfying our hypotheses we have*

$$|P_{\varphi^*}[x,v]-P_{\varphi_m}[x,v]|_{[0,T]}\leqslant \frac{2\varepsilon R}{\min\limits_{i=1,\dots,m}|r_i-r_{i-1}|}.$$

P r o o f. Similiarly as in the proof of Proposition 2.2 and using (4.1), we can estimate

$$\begin{aligned}|P_{\varphi^*}[x,v](t)-P_{\varphi_m}[x,v](t)| &\leqslant \int_0^R |(\varphi^*)'(r)-\varphi_m'(r)|\,\mathrm{d}r\\ &=\sum_{i=1}^m\int_{r_{i-1}}^{r_i}\Big|\frac{\varphi_i^*-\varphi_{i-1}^*}{r_i-r_{i-1}}-\frac{\varphi(r_i)-\varphi(r_{i-1})}{r_i-r_{i-1}}\Big|\,\mathrm{d}r<2m\varepsilon.\end{aligned}$$

The result is then a consequence of the fact that $R\geqslant m\min\limits_{i=1,\dots,m}|r_i-r_{i-1}|$. □

We know that P_{φ^*} admits an explicit inverse of the form (3.2), and for functions v and w,

$$|P_{\varphi^*}^{-1}[x,v](t)-P_{\varphi^*}^{-1}[x,w]|\leqslant|v(t)-w(t)|\sum_{i=0}^{m+1}|b_i^*|, \tag{4.4}$$

where

$$b_0^*=\frac{1}{a_0},\quad b_i^*=\frac{1}{a_0+\widehat{\varphi}_{i-1}^*}-\frac{1}{a_0+\widehat{\varphi}_{i-2}^*},\quad i=1,\dots,m+1. \tag{4.5}$$

Having this in mind, we can define the approximate compensation error as

$$E=\sup|u-P_{\varphi^*}^{-1}[x,P_\varphi[x,u]]|_{[0,T]}, \tag{4.6}$$

where the supremum is taken over $u\in W^{1,1}(0,T)$ with $|u|_{[0,T]}\leqslant R$.

Observing that $u(t)=P_{\varphi^*}^{-1}[x,P_{\varphi^*}[x,u]](t)$ and using (4.4) together with Proposition 4.1, we obtain

$$\begin{aligned}&|u(t)-P_{\varphi^*}^{-1}[x,P_\varphi[x,u]](t)|\\ &\quad\leqslant|P_{\varphi^*}[x,u](t)-P_\varphi[x,u](t)|\sum_{i=0}^{m+1}|b_i^*|\\ &\quad\leqslant\Big(\frac{2\varepsilon R}{\min\limits_{i=1,\dots,m}|r_i-r_{i-1}|}+|P_{\varphi_m}[x,u](t)-P_\varphi[x,u](t)|\Big)\sum_{i=0}^{m+1}|b_i^*|.\end{aligned} \tag{4.7}$$

Finally, from Proposition 2.2 it follows that

$$E \leqslant \Big(\frac{2\varepsilon R}{\min\limits_{i=1,\dots,m}|r_i - r_{i-1}|} + \max_{i=1,\dots,m}|r_i - r_{i-1}|\mathrm{Var}_{[0,\infty)}\varphi'\Big)\sum_{i=0}^{m+1}|b_i^*|. \tag{4.8}$$

This leads to the following result:

Theorem 4.1. *The approximate inversion error, when the operator with continuous thresholds is replaced with an operator with discrete thresholds, is given by formula* (4.8). *Moreover, if the division points are sufficiently equidistant, the optimal theoretical error is of order* $\sqrt{\varepsilon}$*.*

In the particular case of a discrete rate-dependent model (i.e. when the Prandtl-Ishlinskii operator corresponds to a linear combination of play operators with time-dependent thresholds), the above theorem provides a hint on how to approximate inverse Prandtl-Ishlinskii operators with a large number of thresholds by using Prandtl-Ishlinskii operators with a smaller number of thresholds while keeping the inversion error under control.

5. Applications in memory-discrete compensation

To illustrate the applicability of the obtained results in the analysis of compensation error, we consider an operator of the form

$$P_{\varphi_n}[u](t) = a_0 u(t) + \sum_{j=1}^{100} a_j \mathfrak{p}_{\widetilde{\varrho}_j(t)}[u](t), \tag{5.1}$$

where $\widetilde{\varrho}_j(t) = 0.01j + 3\times 10^{-4}|\dot{u}(t)|$, $a_j = 0.5\,\mathrm{e}^{-0.1j}$, $j = 1,\dots,100$, and $a_0 = 1.5$. For such an operator we study how a reduction in the number of active play operators may impact the compensation error bounds. To this end, we consider approximate rate-dependent Prandtl-Ishlinskii operators $P_{\varphi_m}[u](t)$ as defined in (3.1), with weights $a_0, a_1, \dots, a_{m+1}$ and threshold functions $\widetilde{r}_i(t) = r_i + z(t)$, $j = 1,\dots,m$, where $z(t) = \beta|\dot{u}(t)|$, $\beta > 0$ and the number of thresholds $m \in \{10, 19, 30\}$. The parameters of the operator $P_{\varphi_m}[u](t)$, represented by a vector $X = \{\beta, r_1, r_2, \dots, r_{m+1}, a_0, a_1, a_2, \dots, a_{m+1}\}$, are identified via the Real Coded Genetic Algorithm—for more details on such parameter identification technique, see [6]. The parameters of the inverse rate-dependent Prandtl-Ishlinskii operator are obtained using (3.3) and (3.4). The estimated thresholds r_i and weights a_i at different values of m are depicted in Figure 1.

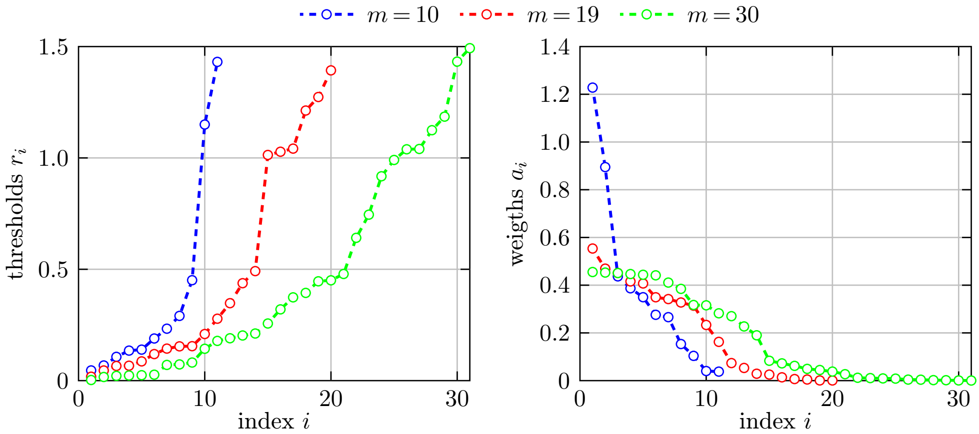

Figure 1. (a) The estimated thresholds r_i, and (b) the estimated weights a_i with thresholds number of $m = 10$, 19, and 30.

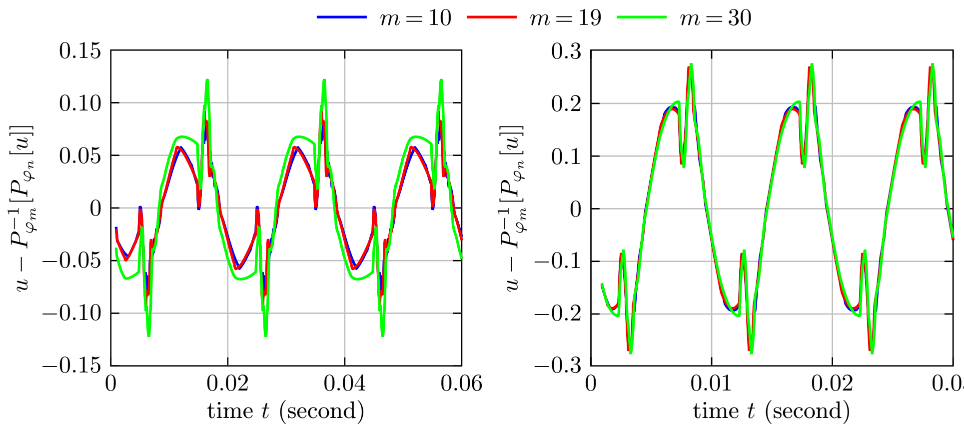

Figure 2. The time history of the compensation error $(u - P_{\varphi_m}^{-1}[P_{\varphi_n}[u]])$ for a desired input of $u(t) = 8.9\sin(2\pi ft)$ under frequencies $f = 50$ and $f = 100$ Hz.

The evolution in time of the compensation error $(u - P_{\varphi_m}^{-1}[P_{\varphi_n}[u]])$ is illustrated in Figure 2 for particular input $u(t) = 8.9\sin(2\pi ft)$ in the case of frequencies $f = 50$ and $f = 100$ Hz.

Figure 3 shows the maximum compensation error in (4.6) and the error bound calculated based on (4.8) for the approximate rate-dependent Prandtl-Ishlinskii operator $P_{\varphi_m}^{-1}$ when different excitation frequencies are considered, namely, 10, 50, 100, and 150 Hz. The results show that increasing the number of thresholds does not yield better performance in hysteresis compensation; on the other hand, by selecting sufficient division points, even a small number of thresholds can keep the compensation error E under control as stated in Theorem 4.2.

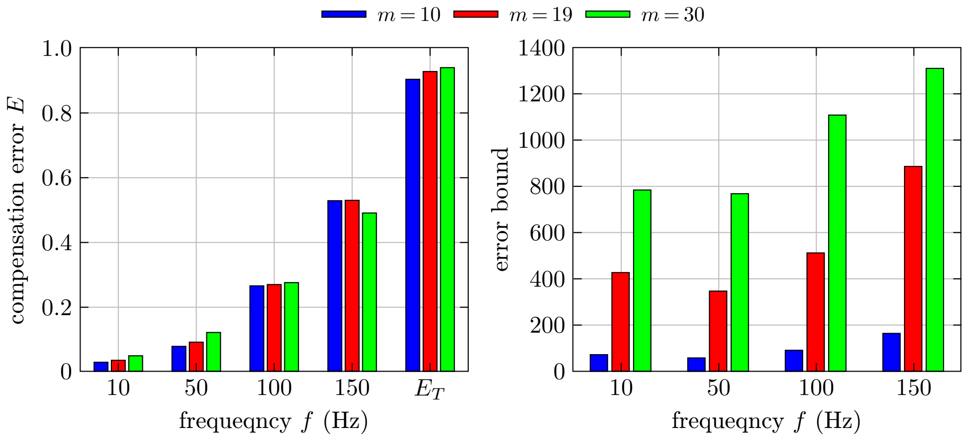

Figure 3. (a) The maximum compensation error in (4.6) under excitation frequencies of 10, 50, 100, and 150 Hz, (b) the error bound calculated based on (4.8) for the approximate rate-dependent Prandtl-Ishlinskii operator $P_{\varphi_m}^{-1}$.

References

[1] *M. Al Janaideh, P. Krejčí*: An inversion formula for a Prandtl-Ishlinskii operator with time dependent thresholds. Phys. B *406* (2011), 1528–1532. doi

[2] *M. Al Janaideh, M. Rakotondrabe*: Precision motion control of a piezoelectric cantilever positioning system with rate-dependent hysteresis nonlinearities. Nonlinear Dyn. *104* (2021), 3385–3405. doi

[3] *M. Al Janaideh, R. Xu, X. Tan*: Adaptive estimation of play radii for a Prandtl-Ishlinskii hysteresis operator. IEEE Trans. Control Syst. Technol. *29* (2021), 2687–2695. doi

[4] *D. Davino, A. Giustiniani, C. Visone*: Magnetoelastic energy harvesting: Modeling and experiments. Smart Actuation and Sensing Systems: Recent Advances and Future Challenges. IntechOpen, London, 2012, pp. 487–512. doi

[5] *R. V. Iyer, X. Tan, P. S. Krishnaprasad*: Approximate inversion of the Preisach hysteresis operator with application to control of smart actuators. IEEE Trans. Autom. Control *50* (2005), 798–810. zbl MR doi

[6] *H. J. Khasawneh, Z. S. Abo-Hammour, M. I. Al Saaideh, S. M. Momani*: Identification of hysteresis models using real-coded genetic algorithms. Eur. Phys. J. Plus *134* (2019), Article ID 507, 17 pages. doi

[7] *P. Krejčí*: Hysteresis, Convexity and Dissipation in Hyperbolic Equations. GAKUTO International Series. Mathematical Sciences and Applications 8. Gakkotosho, Tokyo, 1996. zbl MR

[8] *P. Krejčí*: The Kurzweil integral and hysteresis. J. Phys., Conf. Ser. *55* (2006), 144–154. doi

[9] *P. Krejčí, M. Al Janaideh, F. Deasy*: Inversion of hysteresis and creep operators. Phys. B *407* (2012), 1354–1356. doi

[10] *P. Krejčí, K. Kuhnen*: Inverse control of systems with hysteresis and creep. IEE Proc., Control Theory Appl. *148* (2001), 185–192. doi

[11] *P. Krejčí, H. Lamba, S. Melnik, D. Rachinskii*: Kurzweil integral representation of interacting Prandtl-Ishlinskii operators. Discrete Contin. Dyn. Syst., Ser. B *20* (2015), 2949–2965. zbl MR doi

[12] *P. Krejčí, G. A. Monteiro*: Inverse parameter-dependent Preisach operator in thermo-piezoelectricity modeling. Discrete Contin. Dyn. Syst., Ser. B *24* (2019), 3051–3066. zbl MR doi

[13] *P. Krejčí, G. A. Monteiro*: Oscillations of a temperature-dependent piezoelectric rod. Nonlinear Anal., Real World Appl. *46* (2019), 403–420. zbl MR doi

[14] *K. Kuhnen, P. Krejčí*: Compensation of complex hysteresis and creep effects in piezoelectrically actuated systems: A new Preisach modeling approach. IEEE Trans. Autom. Control *54* (2009), 537–550. zbl MR doi

[15] *K. K. Leang, S. Devasia*: Design of hysteresis-compensating iterative learning control for piezo-positioners: Application to atomic force microscopes. Mechatron. *16* (2006), 141–158. doi

[16] *X. Tan, J. S. Baras*: Modeling and control of hysteresis in magnetostrictive actuators. Automatica *40* (2004), 1469–1480. zbl MR doi

[17] *C. Visone*: Hysteresis modelling and compensation for smart sensors and actuators. J. Phys., Conf. Ser. *138* (2008), Article ID 012028, 24 pages. doi

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