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LARGE TIME BEHAVIOUR OF A CONSERVATION LAW REGULARISED BY A RIESZ-FELLER OPERATOR: THE SUB-CRITICAL CASE

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Abstract. We study the large time behaviour of the solutions of a nonlocal regularisation of a scalar conservation law. This regularisation is given by a fractional derivative of order $1+\alpha$, with $\alpha \in (0,1)$, which is a Riesz-Feller operator. The nonlinear flux is given by the locally Lipschitz function $|u|^{q-1}u/q$ for $q>1$. We show that in the sub-critical case, $1<q<1+\alpha$, the large time behaviour is governed by the unique entropy solution of the scalar conservation law. Our proof adapts the proofs of the analogous results for the local case (where the regularisation is the Laplacian) and, more closely, the ones for the regularisation given by the fractional Laplacian with order larger than one, see L. I. Ignat and D. Stan (2018). The main difference is that our operator is not symmetric and its Fourier symbol is not real. We can also adapt the proof and obtain similar results for general Riesz-Feller operators.

Keywords: large time asymptotic; regularisation of conservation law; Riesz-Feller operator

MSC 2020: 35B40, 47J35, 26A33

1. Introduction

In this paper, we study the large time asymptotic behaviour of nonnegative solutions to the convection-diffusion equation

$$(1.1)\qquad \begin{cases} \partial_t u(t,x) + |u(t,x)|^{q-1}\partial_x u(t,x) = \partial_x \mathcal{D}^{\alpha}[u(t,\cdot)](x), & t>0,\ x\in\mathbb{R},\\ u(0,x) = u_0(x), & x\in\mathbb{R}, \end{cases}$$

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where $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $q > 1$. The operator $\mathcal{D}^\alpha[\cdot]$, acting on x, has $\alpha \in (0,1)$ and is defined by means of

$$(1.2) \qquad \mathcal{D}^\alpha[g](x) = d_{\alpha+1} \int_{-\infty}^0 \frac{g(x+z) - g(x)}{|z|^{\alpha+1}} \, \mathrm{d}z \quad \text{for } 0 < \alpha < 1, \ d_{\alpha+1} = \frac{1}{\Gamma(-\alpha)}.$$

The operator $\partial_x \mathcal{D}^\alpha[\cdot]$ is of Riesz-Feller type, as we shall see below. The operator $\mathcal{D}^\alpha[\cdot]$ can also be seen as a right Weyl-Marchaud fractional derivative (see [29], [37]) of order α. The nonlinear flux $f(u) = |u|^{q-1}u/q$ is considered here as a paradigm locally Lipschitz function.

The equation in (1.1) is a modified Burgers' equation, and appears in [34] as a model of viscoelastic waves with $\alpha = \frac{1}{2}$. There are other models of physical phenomena, where this kind of nonlocal operator appears, such as problems in fluid dynamics, see for instance the references listed in [3].

The models that motivate our study describe the internal structure of hydraulic jumps in a shallow water model. The general form being, for $C_1, C_2 \geqslant 0$,

$$\partial_t u + \partial_x (f(u)) = C_1 \partial_x \mathcal{D}^{1/3}[u] + C_2 \partial_x^3 u, \quad t > 0, \ x \in \mathbb{R},$$

see [24] and [35], where the flux might be u^2 or u^3 and the dispersive term might or might not be relevant, depending on the asymptotic regime considered.

Another example where such operators appear (although not as a regularising term) can be found in [20], where a model for dune formation is presented.

Formally, the study of the large time behaviour can be transferred to a limit problem by the appropriate scaling; for any $\lambda > 0$, let the change of variables be

$$(1.3) \qquad t = \lambda^q s, \quad x = \lambda y$$

and the function be

$$(1.4) \qquad u_\lambda(s, y) := \lambda u(\lambda^q s, \lambda y).$$

Then, if u is a solution of (1.1), u_λ satisfies

$$(1.5) \qquad \begin{cases} \partial_s u_\lambda + |u_\lambda|^{q-1} \partial_y u_\lambda = \lambda^{q-1-\alpha} \partial_y \mathcal{D}^\alpha[u_\lambda(s, \cdot)](y), & s > 0, \ y \in \mathbb{R}, \\ u_\lambda(0, y) = \lambda u_0(\lambda y), & y \in \mathbb{R}. \end{cases}$$

Observe that when $t \to \infty$, if we keep s of order one, then $\lambda \to \infty$, and in the new variables this means that, depending on the sign of the exponent $q - 1 - \alpha$, different terms dominate the limit behaviour. According to this heuristic argument, we can distinguish three different regimes: The sub-critical case $1 < q < 1 + \alpha$

(the conservation law formally dominates), the critical case $q = 1+\alpha$ (one expects self-similar behaviour associated to the balance of all terms) and the super-critical case $1+\alpha < q$ (the nonlocal heat equation formally dominates).

In this paper we focus on the sub-critical case for nonnegative solutions, henceforth we assume that $1 < q < 1+\alpha$. Our main theorem is:

Theorem 1.1. *Let $1 < q < 1+\alpha$ and $1 \leqslant p < \infty$. Given $u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ with $\int_{\mathbb{R}} u_0(x)\,\mathrm{d}x = M > 0$ and $u_0(x) \geqslant 0$ for all $x \in \mathbb{R}$, then there exists a unique solution* (1.1) *with*

(i) *for $p = 1$, $\partial_t u \in C((0,\infty), L^1(\mathbb{R}))$, $u \in C((0,\infty), C_b(\mathbb{R}) \cap C^1(\mathbb{R}))$ and $u \in C([0,\infty), L^1(\mathbb{R}))$,*

(ii) *for $1 < p < \infty$, $\partial_t u \in C((0,\infty), L^p(\mathbb{R}))$ and $u \in C((0,\infty), L^p(\mathbb{R}) \cap \dot{H}^{\theta,p}(\mathbb{R}))$ for any $0 < \theta < 1+\alpha+\min\{\alpha, q-1\}$.*

Moreover, u satisfies

$$\lim_{t\to\infty} t^{(1-1/p)/q} \|u(t,\cdot) - U_M(t,\cdot)\|_{L^p(\mathbb{R})} = 0, \tag{1.6}$$

where U_M is the unique entropy solution of

$$\begin{cases} \partial_t U_M + \partial_x(|U_M|^{q-1}U_M/q) = 0, & t>0,\ x\in\mathbb{R}, \\ U_M(0,x) = M\delta_0, & x \in \mathbb{R}. \end{cases} \tag{1.7}$$

Here the notation $\dot{H}^{\theta,p}(\mathbb{R})$ is for the homogeneous Sobolev spaces (see e.g., [4]) of order $\theta > 0$

$$\dot{H}^{\theta,p}(\mathbb{R}) := \{g \in \mathcal{S}'(\mathbb{R})\colon\ \mathcal{F}^{-1}[|\xi|^\theta \mathcal{F}(g)] \in L^p(\mathbb{R})\} \tag{1.8}$$

and $\mathcal{F}$ denotes the Fourier transform

$$\mathcal{F}(g(x))(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(x) \mathrm{e}^{-\mathrm{i}\xi x}\,\mathrm{d}x. \tag{1.9}$$

We also recall here the definition of entropy solution of (1.7), see [25]:

Definition 1.1. Let U_M be a weak solution of (1.7) such that

$$U_M \in L^\infty((0,\infty), L^1(\mathbb{R})) \cap L^\infty((\tau,\infty)\times\mathbb{R}) \quad \forall\, \tau \in (0,\infty).$$

Then U_M is said to be an entropy solution of (1.7) if and only if the following inequality holds for every $k \in \mathbb{R}$ and $\varphi \in C_c^\infty((0,\infty)\times\mathbb{R})$ nonnegative:

$$\int_0^\infty \int_{\mathbb{R}} (|U_M - k|\partial_t\varphi + \operatorname{sgn}(U_M - k)(f(U_M) - f(k))\partial_x\varphi)\,\mathrm{d}x\,\mathrm{d}t \geqslant 0 \tag{1.10}$$

with $f(u) = |u|^{q-1}u/q$, and for any $\psi \in C_b(\mathbb{R})$

$$\lim_{t\downarrow 0} \operatorname{ess} \int_{\mathbb{R}} U_M(t,x)\psi(x)\,\mathrm{d}x = M\psi(0). \tag{1.11}$$

We recall that this unique entropy solution is given by the N-wave profile, see [27].

The proof follows the method developed by Kamin and Vázquez in [23], namely that with the rescaling (1.3)–(1.4), (1.6) is formally equivalent to

$$\lim_{\lambda\to\infty} \|u_\lambda(s_0,\cdot) - U_M(s_0,\cdot)\|_{L^p(\mathbb{R})} = 0 \tag{1.12}$$

for some $s_0 > 0$ fixed. We recall that the local case in $\mathbb{R}^N$ for $N \in \mathbb{N}$ and for all $q > 1$ has been analysed by Escobedo, Vázquez and Zuazua, see [17], [18], [19].

The sub-critical case, for the fractional Laplacian as viscous term, in dimension one has been studied by Ignat and Stan in [22], and these are the results that we shall adapt. We also mention that Biler, Karch and Woyczyński in [6], [7] study the critical and super-critical cases for a more general Lévy operator as viscous term, showing the expected asymptotic behaviour; this being given by the self-similar solution and by the fractional heat kernel, respectively. This operator is defined by means of a Fourier multiplier, such that the symbol can be represented by the Lévy-Khintchine formula in the Fourier variable, see [5], Chapter 1, Theorem 1. The fractional Laplacian serves as a particular example of such operators. In all these results the non-negativity of the symbol and the symmetry of the operator are necessary conditions. Operator (1.2) does not satisfy these conditions.

For completeness, we also consider the case of a general Riesz-Feller operator. Thus, we show how the analogous to Theorem 1.1 follows if the term $\partial_x\mathcal{D}^\alpha[u]$ is replaced by the Riesz-Feller operator term $D^\beta_\gamma[u]$ of order $\beta \in (1,2)$ and skewness γ. We recall that such an operator can be defined by means of a Fourier multiplier operator (see, e.g., [28]) and we choose the definition

$$\mathcal{F}(D^\beta_\gamma[g])(\xi) = \psi^\beta_\gamma(\xi)\mathcal{F}(g)(\xi), \tag{1.13}$$

where $\beta \in (0,2]$ and $|\gamma| \leqslant \min\{\beta, 2-\beta\}$ and the symbol satisfies

$$\psi^\beta_\gamma(\xi) = -|\xi|^\beta \mathrm{e}^{-\mathrm{i}\,\mathrm{sgn}(\xi)\gamma\pi/2}. \tag{1.14}$$

In particular, we observe that the derivative of (1.2) is of this form with $\beta = 1+\alpha$ and $\gamma = 1-\alpha$, as we explain in the next section. We recall that the definition we use here for Riesz-Feller operators differs from the usual one: the symbol we obtain is the complex conjugate of the one with the standard definition. This is simply because such a definition uses the complex conjugate of (1.9) (up to a scaling factor) as the Fourier transform.

Finally, we mention that we focus on nonnegative solutions that might be considered as certain density functions. Results for sign-changing solutions for the local case and the one regularised by the fractional Laplacian appear in [18], Section 3 and [9], Section 6, respectively.

The paper is organised as follows. Section 2 contains preliminary results and is divided into three sections. In the first one, we recall some properties of the nonlocal operator (1.2) and its derivative. These include some equivalent integral representations, the computation of their Fourier symbol and the definition and properties of certain *dual* operators. In the second part we recall the linear problem and derive some estimates of the fundamental solutions that will be necessary later. We end the section be defining mild solutions for problem (1.1) and giving existence and regularity results for this. We also recall the viscous entropy inequality, and derive a comparison principle. Most of the results appear in [15] or can be adapted from [22] with the ingredients given in the previous sections, thus, we shall only prove whatever does not appear elsewhere.

In Section 3 we derive necessary *a priori* estimates, namely an Oleinik type entropy inequality and an energy type estimate. In order to do this we first consider the problem with a positive initial condition (which makes the nonlinear flux regular, since positivity is preserved, by the comparison principle). In this case we can show the desired estimates, which are then preserved in the limit to a nonnegative initial condition. The proofs are similar to those in [22], we only give the details where necessary.

In Section 4 we prove Theorem 1.1. First, we translate the results of Section 3 into the rescaled problem (1.5). Additionally, before proving the limit $\lambda \to \infty$, we have to take care of the behaviour of u_λ for large $|y|$. With these estimates we prove the limit (1.12). We finish Section 5 by showing how the proofs generalise to Riesz-Feller operators.

We recall that we do not give the proofs that are analogous to those given in [22], the interested reader is also referred to Diez-Izagirre's PhD thesis, see [14], Chapter 3 for details.

2. Preliminary results

In this section we recast the necessary lemmas that are needed to adapt the results of [22].

2.1. Derivation and integration by parts rules. In this section we derive integration by parts rules for (1.2).

First we need the notation for the following operator:

$$\overline{\mathcal{D}^{\alpha}}[g](x) = -d_{\alpha+1} \int_0^{\infty} \frac{g(x+z) - g(x)}{|z|^{\alpha+1}} \, \mathrm{d}z. \tag{2.1}$$

Then the following integration by parts follows, see [15].

Lemma 2.1. *Let $\alpha \in (0,1)$, $u \in C_b^2(\mathbb{R})$ and $\varphi \in C_c^\infty(\mathbb{R})$. Then for all $t > 0$*

$$\int_{\mathbb{R}} \varphi(x)\partial_x \mathcal{D}^\alpha[u(t,\cdot)](x)\,\mathrm{d}x = \int_{\mathbb{R}} \partial_x \overline{\mathcal{D}^\alpha}[\varphi](x) u(t,x)\,\mathrm{d}x.$$

We also recall the following representation of the operators, see also [15].

Lemma 2.2 (Integral representations). *If $\alpha \in (0,1)$, then for all $g \in C_b^1(\mathbb{R})$ and all $x \in \mathbb{R}$,*

$$\partial_x \mathcal{D}^\alpha[g](x) = d_{\alpha+2} \int_{-\infty}^0 \frac{g(x+z) - g(x) - g'(x)z}{|z|^{\alpha+2}}\,\mathrm{d}z$$

and

$$\partial_x \overline{\mathcal{D}^\alpha}[g](x) = d_{\alpha+2} \int_0^\infty \frac{g(x+z) - g(x) - g'(x)z}{|z|^{\alpha+2}}\,\mathrm{d}z.$$

With equation (1.9) we obtain, formally, the Fourier symbol of $\mathcal{D}^\alpha[\cdot]$ (see, e.g., [31], Chapter 7):

$$\mathcal{F}(\mathcal{D}^\alpha[g](x))(\xi) = (\mathrm{i}\xi)^\alpha \mathcal{F}(g)(\xi), \tag{2.2}$$

and that of $\overline{\mathcal{D}^\alpha}[\cdot]$,

$$\mathcal{F}(\overline{\mathcal{D}^\alpha}[g](x))(\xi) = -(-\mathrm{i}\xi)^\alpha \mathcal{F}(g)(\xi). \tag{2.3}$$

In particular, $\mathcal{D}^\alpha[\cdot]$ is not of Riesz-Feller type because its symbol has $\beta = \alpha$ and $\gamma = 2 - \alpha$, but $\overline{\mathcal{D}^\alpha}[\cdot]$ belongs to this class since $\beta = \alpha$ and $\gamma = \alpha$. We also observe that their symbols satisfy $(\mathrm{i}\xi)^\alpha = -\overline{(-(-\mathrm{i}\xi)^\alpha)}$, where the bar on the right-hand side denotes complex conjugation.

We can then conclude that

$$\mathcal{F}(\partial_x \mathcal{D}^\alpha[g])(\xi) = (\mathrm{i}\xi)^{\alpha+1} \mathcal{F}(g)(\xi).$$

We observe, writing

$$(\mathrm{i}\xi)^{\alpha+1} = -|\xi|^{\alpha+1}\Big(\cos\Big((1-\alpha)\frac{\pi}{2}\Big) - \mathrm{i}\,\mathrm{sgn}(\xi)\sin\Big((1-\alpha)\frac{\pi}{2}\Big)\Big),$$

that $\partial_x \mathcal{D}^\alpha[\cdot]$ is an operator of Riesz-Feller type with $\beta = 1 + \alpha$ and $\gamma = 1 - \alpha$, see (1.13)–(1.14). Then we also get:

$$\mathcal{F}(\partial_x \overline{\mathcal{D}^\alpha}[g])(\xi) = (-\mathrm{i}\xi)^{\alpha+1}\ \mathcal{F}(g)(\xi) \quad \text{for } 0 < \alpha < 1.$$

With this Fourier representation formulas we can now prove the following integration by parts rule:

Lemma 2.3. *Let $0 < \alpha < 1$ and $\frac{1}{2} < \theta_1, \theta_2 < 1$ such that $1+\alpha = \theta_1+\theta_2$. Assume also that $g \in H^2(\mathbb{R})$, so that $\partial_x \mathcal{D}^\alpha[g], \mathcal{D}^{\theta_1}[g], \overline{\mathcal{D}^{\theta_2}}[h] \in L^2(\mathbb{R})$, and let $h \in L^2(\mathbb{R})$. Then*

$$\int_{\mathbb{R}} \partial_x \mathcal{D}^\alpha[g](x) h(x)\,\mathrm{d}x = -\int_{\mathbb{R}} \mathcal{D}^{\theta_1}[g](x)\overline{\mathcal{D}^{\theta_2}}[h](x)\,\mathrm{d}x.$$

Proof. Since $\partial_x \mathcal{D}^\alpha[g]$, $h \in L^2(\mathbb{R})$, then Plancherel's theorem yields

$$\begin{aligned}
\int_{\mathbb{R}} \partial_x \mathcal{D}^\alpha[g](x) h(x)\,\mathrm{d}x &= \int_{\mathbb{R}} (\mathrm{i}\xi)^{1+\alpha} \mathcal{F}(g)(\xi)\overline{\mathcal{F}(h)(\xi)}\,\mathrm{d}\xi \\
&= -\int_{\mathbb{R}} (\mathrm{i}\xi)^{\theta_1} \mathcal{F}(g)(\xi)\overline{(-(-\mathrm{i}\xi)^{\theta_2})\mathcal{F}(h)(\xi)}\,\mathrm{d}\xi \\
&= -\int_{\mathbb{R}} \mathcal{F}(\mathcal{D}^{\theta_1}[g])(\xi)\overline{\mathcal{F}(\overline{\mathcal{D}^{\theta_2}}[h])(\xi)}\,\mathrm{d}\xi \\
&= -\int_{\mathbb{R}} \mathcal{D}^{\theta_1}[g](x)\overline{\mathcal{D}^{\theta_2}}[h](x)\,\mathrm{d}x
\end{aligned}$$

for $\frac{1}{2} < \theta_1$, $\theta_2 < 1$ such that $1+\alpha = \theta_1+\theta_2$. □

We now recall some facts about the fractional Laplacian that we need later to conclude L^p-regularity of the solution.

There are several equivalent definitions of the fractional Laplacian, see [26]. Here we consider the one given by the Fourier symbol for $0 < \theta < 2$, namely,

$$|D|^\theta[g](x) := (-\Delta)^{\theta/2}[g](x) := \mathcal{F}^{-1}(|\xi|^\theta \mathcal{F}(g)(\xi))(x). \tag{2.4}$$

We observe that applying Plancherel's theorem $\||D|^\theta[g]\|_{L^2(\mathbb{R})} = \|g\|_{\dot{H}^{\theta,2}(\mathbb{R})}$ and also

$$\|\mathcal{D}^\theta[g]\|_{L^2(\mathbb{R})} = \|\overline{\mathcal{D}^\theta}[g]\|_{L^2(\mathbb{R})} = \|g\|_{\dot{H}^{\theta,2}(\mathbb{R})} \quad \text{for } 0 < \theta < 1 \tag{2.5}$$

since

$$\|\mathcal{D}^\theta[g]\|_{L^2(\mathbb{R})} = \|(\mathrm{i}\cdot)^\theta \mathcal{F}(g)(\cdot)\|_{L^2(\mathbb{R})} = \||\cdot|^\theta \mathcal{F}(g)(\cdot)\|_{L^2(\mathbb{R})} = \||D|^\theta[g]\|_{L^2(\mathbb{R})}.$$

For comparison purposes, we give an equivalent definition of the fractional Laplacian of order $\theta > 1$ for functions $g \in C_b^2(\mathbb{R})$, see [16]:

$$(-\Delta)^{\theta/2}[g] = c_\theta \int_{\mathbb{R}} \frac{g(x+z) - g(x) - g'(x)z}{|z|^{\theta+1}}\,\mathrm{d}z \quad \text{with} \quad \theta \in (1,2) \tag{2.6}$$

where

$$c_\theta = \frac{2^\theta \Gamma((\theta+1)/2)}{\pi^{1/2}\Gamma(-\theta/2)}.$$

2.2. Linear fractional diffusion equation. In this section we recall some results concerning the linear problem

$$\begin{cases} \partial_t U(t,x) - \partial_x \mathcal{D}^\alpha[U(t,\cdot)](x) = 0, & t > 0,\ x \in \mathbb{R}, \\ U(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{2.7}$$

For initial data $u_0 \in L^\infty(\mathbb{R})$ the solution of (2.7) can be represented as

$$U(t,x) = (K(t,\cdot) * u_0)(x) = \int_{\mathbb{R}} K(t,x-y) u_0(y)\,\mathrm{d}y$$

such that the kernel $K(t,x)$ is defined by means of

$$K(t,x) = \mathcal{F}^{-1}(\mathrm{e}^{(\mathrm{i}\xi)^{\alpha+1} t})(x) \quad \forall\, t > 0,\ x \in \mathbb{R}, \tag{2.8}$$

which can be formally obtained using Fourier transform, see [2] for the proof. Some pertinent properties of the kernel are derived in [1] and [15]. In particular, we recall that

$$K(t,x) = \frac{1}{t^{1/(1+\alpha)}} K\Big(1, \frac{x}{t^{1/(1+\alpha)}}\Big) \quad \forall\, t > 0,\ x \in \mathbb{R}, \tag{2.9}$$

where K is nonnegative and $K(t,x) \in C^\infty((0,\infty) \times \mathbb{R})$. It also preserves mass and has the semi-group property.

Since the regularity of solutions is established with respect to derivation with the fractional Laplacian, we need the following estimates:

Lemma 2.4 (Time behaviour of K). *For all $\alpha, \theta \in (0,1)$ and $1 \leqslant p \leqslant \infty$, $K(t,x)$ satisfies the following estimates for any $t > 0$:*

$$\|K(t,\cdot)\|_{L^p(\mathbb{R})} = Ct^{-(1-1/p)/(1+\alpha)}, \quad \|\partial_x K(t,\cdot)\|_{L^p(\mathbb{R})} \lesssim t^{-(1-1/p)/(1+\alpha) - 1/(1+\alpha)},$$
$$\||D|^\theta [K(t,\cdot)]\|_{L^p(\mathbb{R})} \lesssim t^{-(1-1/p)/(1+\alpha) - \theta/(1+\alpha)},$$
$$\||D|^\theta [\partial_x K(t,\cdot)]\|_{L^p(\mathbb{R})} \lesssim t^{-(1-1/p)/(1+\alpha) - (1+\theta)/(1+\alpha)}$$

for a constant $C > 0$.

P r o o f. The first and second identities follow from (2.9); namely, the mass-conservation property of K and $\partial_x K$ and from their boundedness on $(0,T) \times \mathbb{R}$ for any $T > 0$.

For the third estimate we first use (2.9), then we rescale the fractional Laplacian:

$$\begin{aligned} ||D|^\theta [K(t,\cdot)](x)| &= \frac{1}{t^{1/(1+\alpha)}} \Big| |D|^\theta \Big[K\Big(1, \frac{\cdot}{t^{1/(1+\alpha)}}\Big)\Big](x) \Big| \\ &= \frac{1}{t^{(1+\theta)/(1+\alpha)}} \Big| |D|^\theta [K(1,\cdot)] \Big(\frac{x}{t^{1/(1+\alpha)}}\Big) \Big|. \end{aligned} \tag{2.10}$$

Now, when computing the L^p-norm, we apply the change of variable $X = x/t^{1/(1+\alpha)}$:
(2.11)

$$\||D|^\theta[K(t,\cdot)](x)\|_{L^p(\mathbb{R})} = \frac{1}{t^{(1+\theta)/(1+\alpha)}}\left(\int_{\mathbb{R}} \left||D|^\theta[K(1,\cdot)]\left(\frac{x}{t^{1/(1+\alpha)}}\right)\right|^p \mathrm{d}x\right)^{1/p}$$
$$= \frac{1}{t^{(1+\theta)/(1+\alpha)}} t^{1/(p(1+\alpha))}\left(\int_{\mathbb{R}} ||D|^\theta[K(1,\cdot)](X)|^p \,\mathrm{d}X\right)^{1/p}.$$

It only remains to prove that the L^p-norm of $|D|^\theta[K(1,\cdot)]$ is finite. In order to show this, we first observe that using (2.4), the integrand of (2.11) is bounded:

$$||D|^\theta[K(1,\cdot)](X)| = \frac{1}{\sqrt{2\pi}}\left|\int_{\mathbb{R}} |\xi|^\theta \mathrm{e}^{(\mathrm{i}\xi)^{1+\alpha}} \mathrm{e}^{\mathrm{i}X\xi}\,\mathrm{d}\xi\right| \leqslant \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} |\xi|^\theta \mathrm{e}^{-|\xi|^{1+\alpha}\sin(\alpha\pi/2)}\,\mathrm{d}\xi < \infty. \tag{2.12}$$

Next we show that $||D|^\theta[K(1,\cdot)(X)|^p$ is integrable for large $|X|$. We will apply Lemma 2 of [30], which gives that for large X

$$\int_0^\infty \mathrm{e}^{-\mathrm{i}X\xi}\mathrm{e}^{-\sigma\xi^r}\xi^\theta\,\mathrm{d}\xi = O(|X|^{-1-\theta})$$

with $r \in (0,2)$, $\theta \geqslant 0$ and σ such that

$$\sigma \in \left\{a + \mathrm{i}b \in \mathbb{C}\colon\ -\cos\left(\frac{r\pi}{2}\right) \leqslant a \leqslant 1,\ |b| \leqslant -\tan\left(\frac{r\pi}{2}\right)\right\}. \tag{2.13}$$

We first write

$$\begin{aligned}|D|^\theta[K(1,\cdot)](X) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} |\xi|^\theta \mathrm{e}^{(\mathrm{i}\xi)^{1+\alpha}}\mathrm{e}^{\mathrm{i}X\xi}\,\mathrm{d}\xi\\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} |\xi|^\theta \mathrm{e}^{-|\xi|^{1+\alpha}(\sin(\alpha\pi/2)-\mathrm{i}\,\mathrm{sgn}(\xi)\cos(\alpha\pi/2))}\mathrm{e}^{\mathrm{i}X\xi}\,\mathrm{d}\xi\\ &= \frac{1}{\sqrt{2\pi}}\int_0^\infty \xi^\theta \mathrm{e}^{-\xi^{1+\alpha}(\sin(\alpha\pi/2)-\mathrm{i}\cos(\alpha\pi/2))}\mathrm{e}^{-\mathrm{i}(-X)\xi}\,\mathrm{d}\xi\\ &\quad+ \frac{1}{\sqrt{2\pi}}\int_0^\infty \xi^\theta \mathrm{e}^{-\xi^{1+\alpha}(\sin(\alpha\pi/2)+\mathrm{i}\cos(\alpha\pi/2))}\mathrm{e}^{-\mathrm{i}X\xi}\,\mathrm{d}\xi.\end{aligned}$$

Note that we have applied the change of variables $\xi \to -\xi$ in the second integral. Then, applying Lemma 2 of [30] in both integral terms, with $r = 1+\alpha$, implies

$$||D|^\theta[K(1,\cdot)](X)| \lesssim \frac{1}{|X|^{1+\theta}}, \quad |X| \gg 1. \tag{2.14}$$

Observe that condition (2.13), since $\theta > 0$, is satisfied for, in this case,

$$\sigma = \sin\Big(\frac{\alpha\pi}{2}\Big) - \mathrm{i}\cos\Big(\frac{\alpha\pi}{2}\Big).$$

and its complex conjugate.

Then (2.12) and (2.14) imply $|D|^\theta[K(1,\cdot)](X) \in L^p(\mathbb{R})$ for $p \geqslant 1$, which together with (2.11) implies the third estimate.

Finally, the fourth estimate is obtained in a similar way. The main difference is that we have to differentiate the kernel first, which gives a factor $\mathrm{i}\xi$ in the integrand, but we can still apply Lemma 2 of [30] to get

$$||D|^\theta[\partial_X K(1,\cdot)](X)| = \frac{1}{\sqrt{2\pi}}\left|\int_{\mathbb{R}} |\xi|^\theta (\mathrm{i}\xi)\mathrm{e}^{(\mathrm{i}\xi)^{1+\alpha}}\mathrm{e}^{\mathrm{i}X\xi}\,\mathrm{d}\xi\right| \lesssim \frac{1}{|X|^{2+\theta}} \quad \text{for } |X| \gg 1.$$

Then, with this and (2.9), we can argue as for (2.10) to conclude the proof. □

2.3. Mild formulation, existence and regularity results. We now define the mild formulation associated to (1.1):

Definition 2.1 (Mild solution). Given $T \in (0,\infty]$ and $u_0 \in L^\infty(\mathbb{R})$, we say that a mild solution of (1.1) on $(0,T)\times\mathbb{R}$ is a function $u \in C_b((0,T)\times\mathbb{R})$ which satisfies

$$u(t,x) = K(t,\cdot) * u_0(x) - \int_0^t \partial_x K(t-s,\cdot) * f(u(s,\cdot))(x)\,\mathrm{d}s \tag{2.15}$$

in a.e. $(t,x) \in (0,T)\times\mathbb{R}$, where $f(u) = |u|^{q-1}u/q$ with $q > 1$.

Regarding existence and uniqueness of mild solutions, we have the following:

Theorem 2.1 (Existence and uniqueness). *Let $u_0 \in L^\infty(\mathbb{R})$. Then there exists a unique global mild solution u to the initial value problem (1.1) with $u \in C((0,\infty), C^1(\mathbb{R})) \cap C_b((0,\infty)\times\mathbb{R})$ and such that*

$$\operatorname{ess\,inf}\{u_0\} \leqslant u(t,x) \leqslant \operatorname{ess\,sup}\{u_0\}, \quad t > 0,\ x \in \mathbb{R}. \tag{2.16}$$

If $u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, then also $u \in C([0,\infty), L^1(\mathbb{R}))$ and

$$\|u(t,\cdot)\|_{L^1(\mathbb{R})} \leqslant \|u_0\|_{L^1(\mathbb{R})} \quad \forall\, t > 0. \tag{2.17}$$

P r o o f. The existence and uniqueness result, the upper bound of (2.16) and (2.17) have already been proved in [15] for a regular flux function. We observe that in order to obtain existence and regularity, we can proceed as in [15], Propositions 4–5, but since the flux function is only continuous with bounded first derivative, we can only apply two steps of the argument. This means that we can only gain C^1 regularity in x and continuity for $t \in (0,T)$ for any $T > 0$, thus $u \in C((0,T), C_b^1(\mathbb{R}))$.

Now, in order to prove global existence, uniqueness and regularity as well as the upper bound analogous to the one in (2.16), we first regularise the flux function (to at least a C^2 function) and apply the results in the previous reference. The lower bound analogous to the one in (2.16) is proved by following the proof of [15], Lemma 3, Proposition 6 while changing the role of the supremum and the infimum. Then we have to pass to the limit to get the results for the original flux.

We define the function f_δ by means of

$$f_\delta(v) := (\delta^2 + v^2)^{(q-1)/2} \frac{(v+\delta)}{q}. \tag{2.18}$$

Notice that the function f_δ is C^2 for $\delta > 0$ and converges to $f(v) = |v|^{q-1}v/q$ with $|f_\delta(v) - f(v)| \leqslant \delta C |v|^{q-1}$ as $\delta \to 0$, for a positive constant C.

Let u_δ be the solution of

$$\begin{cases} \partial_t u_\delta(t,x) + \partial_x(f_\delta(u_\delta(t,x))) = \partial_x \mathcal{D}^\alpha[u_\delta(t,\cdot)](x), & t > 0,\ x \in \mathbb{R}, \\ u_\delta(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{2.19}$$

As a consequence, we obtain the global existence in time for (2.19). Therefore, by continuity in $t > 0$ and the uniqueness result we can extend the solution for $t \in (0,\infty)$ and it satisfies $\operatorname{ess\,inf} u_0 \leqslant u_\delta(t,x) \leqslant \operatorname{ess\,sup} u_0$ for all $(t,x) \in (0,\infty) \times \mathbb{R}$.

Now, we extend the result of (2.19) to (1.1). First, we prove that for any $T > 0$, u_δ converges uniformly to u as $\delta \to 0$ in $(t,x) \in (0,T) \times \mathbb{R}$, where u is a mild solution of (1.1). We compute:

$$\begin{aligned}
&\|u_\delta(t,\cdot) - u(t,\cdot)\|_{L^\infty(\mathbb{R})} \\
&\quad = \left\| \int_0^t \partial_x K(t-s,\cdot) * (f_\delta(u_\delta) - f(u))(x)\,\mathrm{d}s \right\|_{L^\infty(\mathbb{R})} \\
&\quad \leqslant \int_0^t \|\partial_x K(t-s,\cdot)\|_{L^1(\mathbb{R})} \|f_\delta(u_\delta(s,\cdot)) - f(u(s,\cdot))\|_{L^\infty(\mathbb{R})}\,\mathrm{d}s \\
&\quad \leqslant C \int_0^t (t-s)^{-1/(1+\alpha)} \|f_\delta(u_\delta(s,\cdot)) - f(u_\delta(s,\cdot))\|_{L^\infty(\mathbb{R})}\,\mathrm{d}s \\
&\qquad + C \int_0^t (t-s)^{-1/(1+\alpha)} \|f(u_\delta(s,\cdot)) - f(u(s,\cdot))\|_{L^\infty(\mathbb{R})}\,\mathrm{d}s \\
&\quad = \delta C(\|u_0\|_\infty) t^{\alpha/(1+\alpha)} + C(T) \int_0^t (t-s)^{-1/(1+\alpha)} \|u_\delta(s,\cdot) - u(s,\cdot)\|_{L^\infty(\mathbb{R})}\,\mathrm{d}s,
\end{aligned}$$

where the constant $C(T)$ depends linearly on $\sup_{t\in(0,T]} \|u(t,\cdot)\|_\infty$, which is finite for all T, see [15]. Here we have used the second estimate with $p=1$ of Lemma 2.4, the convergence $f_\delta \to f$ as $\delta \to 0$ and the local Lipschitz continuity of f.

Since $1+\alpha > 1$, we can apply the fractional Gronwall Lemma (see [8], Lemma 2.4) to obtain that for any $T>0$ and $\delta>0$ there exists a positive constant $C(T)$ such that

$$\|u_\delta(t,\cdot) - u(t,\cdot)\|_{L^\infty(\mathbb{R})} \leqslant \delta C(T) \quad \forall\, t \in [0,T].$$

In particular, this implies that u_δ converges uniformly on compact sets of t and point-wise (since $u \in C((0,T), C^1(\mathbb{R}))$. Thus, we can conclude the global existence of u and (2.16).

Finally, (2.17) follows as in [15], Theorem 2. □

As a corollary we obtain positivity of the solutions for positive initial conditions:

Corollary 2.1. *Let $u_0 \in L^\infty(\mathbb{R})$ with $u_0(x) \geqslant \varepsilon > 0$, then the unique mild solution of* (1.1) *satisfies*

$$\varepsilon \leqslant u(t,x) \leqslant \|u_0\|_\infty \quad \forall\, t>0,\ x \in \mathbb{R}. \tag{2.20}$$

Moreover, $u \in C_b^\infty((0,\infty)\times\mathbb{R})$.

P r o o f. Estimate (2.20) is a direct consequence of (2.16).

Since now u is positive, it means that $|u| = u$, so the flux is $f(u) = u^q/q$ and belongs to $C^\infty([\varepsilon, \|u_0\|_\infty])$. This implies that then $u \in C_b^\infty((0,\infty)\times\mathbb{R})$, see [15]. □

We now give some L^p-regularity of the mild solution.

Proposition 2.1 (L^p-regularity)**.** *Let $1 < p < \infty$ and u be the unique mild solution of* (1.1) *with $u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$. Then $\partial_t u \in C((0,\infty), L^p(\mathbb{R}))$ and $u \in C((0,\infty), L^p(\mathbb{R}) \cap \dot{H}^{\theta,p}(\mathbb{R}))$ for any $\theta < 1+\alpha+\min\{\alpha, q-1\}$.*

The proof is based on applying the fractional Laplacian to the mild formulation (2.15) followed by the bootstrap argument used in [22], Proposition 3.1. We use the fractional Laplacian to study regularity, because we can readily apply the chain and product rules available from [21], Theorem 3, Corollary of Theorem 5, [10], Proposition 3.1 and [36], Proposition A.1. This is also the reason why we need Lemma 2.4 (the estimates needed to gain regularity by applying fractional derivatives on the kernel are obtained by convenience applying the fractional Laplacian instead of (1.2)). With this ingredients one can just mimic the proof given in [22].

Finally, we give another auxiliary result that we will need later on. Namely, the mild solution of (1.1) satisfies a weak entropy inequality for the Kružkov's entropies, see [15], Theorem 1:

Theorem 2.2 (Weak viscous entropy inequality). *For all $k \in \mathbb{R}$, let $\eta_k(v) = |v-k| \in C(\mathbb{R})$ be a convex entropy function and $u \in C((0,\infty), C^1(\mathbb{R})) \cap C_b((0,\infty)\times\mathbb{R})$ a solution of* (1.1). *Then for all non-negative $\varphi \in C_c^\infty((0,\infty)\times\mathbb{R})$*

$$\int_0^\infty \int_{\mathbb{R}} (|u(t,x)-k|\partial_t\varphi + \operatorname{sgn}(u(t,x)-k)(f(u(t,x))-f(k))\partial_x\varphi + |u(t,x)-k|\partial_x\overline{\mathcal{D}^\alpha}[\varphi(t,\cdot)](x))\,\mathrm{d}x\,\mathrm{d}t \geqslant 0, \tag{2.21}$$

where $f(u) = |u|^{q-1}u/q$.

We remark that the proof of this result is as in [15], but for the problem with the regularised flux (2.18), as above. The proof is completed by passing to the limit $\delta \to 0$. We omit the details here.

3. Oleinik type inequality for nonnegative solutions

In this section we derive an Oleinik type inequality. We prove it for nonnegative solutions by first deriving the inequality for positive ones (for which the flux is regular).

Let $u_0 \in L^\infty(\mathbb{R})$ be nonnegative, then we consider the following approximating problem:

$$\begin{cases} \partial_t u_\varepsilon(t,x) + (u_\varepsilon)^{q-1}\partial_x u_\varepsilon(t,x) = \partial_x\mathcal{D}^\alpha[u_\varepsilon(t,\cdot)](x), & t>0,\ x\in\mathbb{R}, \\ u_\varepsilon(0,x) = u_0(x)+\varepsilon, & x\in\mathbb{R}. \end{cases} \tag{3.1}$$

The existence and uniqueness of this problem is guaranteed by Theorem 2.1 and Corollary 2.1 implies that $u_\varepsilon(x,t) > 0$ for all (x,t). Then the following holds:

Lemma 3.1. *Let $u(t,x)$ be the solution of* (1.1) *with $0 \leqslant u_0 \in L^\infty(\mathbb{R})$ and let $u_\varepsilon(t,x)$ be the solution of* (3.1). *Then for every $T>0$,*

$$\max_{t\in[0,T]} \|u_\varepsilon(t,\cdot) - u(t,\cdot)\|_{L^\infty(\mathbb{R})} \to 0 \quad \text{as } \varepsilon \to 0.$$

The proof is analogous to that in [22], we do not prove it here. It is based on the comparison of the mild solutions in norm and on the application of the fractional Gronwall Lemma [8].

We can now obtain an Oleinik type entropy inequality.

Proposition 3.1 (Oleinik entropy inequality). *Let $u_0 \in L^\infty(\mathbb{R})$. Then for any $\varepsilon > 0$, the solution u_ε of* (3.1) *satisfies*

$$\partial_x(u_\varepsilon^{q-1}(t,x)) \leqslant \frac{1}{t} \quad \forall\, t>0,\ x\in\mathbb{R}.$$

The proof is analogous to that in [22]; we briefly give the idea here since this proposition is key to the proof of Theorem 1.1: First, one writes the equation for $w = (u_\varepsilon^{q-1})_x$ as

$$w_t + w^2 + u_\varepsilon^{q-1} w_x + N(w, u_e) = 0, \tag{3.2}$$

where N is the resulting nonlinear and nonlocal term, and shows that $W = \sup_{x\in\mathbb{R}} w(\cdot, x)$ satisfies

$$W'(t) + W^2(t) + Q(t)W(t) \leqslant 0, \quad Q(t) > 0,$$

which, by ODEs arguments, implies the result. The estimates of the nonlocal term of (3.2) can be adapted from the proof in [22]. This requires splitting the integral, using the representation (2.6), near zero and away from zero, but this can be done similarly using the integral representations of Lemma 2.2 in our case.

We can show properties of the family u_ε to the solution of (1.1) by taking the limit $\varepsilon \to 0$. Namely:

Lemma 3.2. *Let u be the solution of problem* (1.1) *with non-negative initial data $u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$. Then the following estimates hold:*

(i) *(Mass conservation) $\int_{\mathbb{R}} u(t,x)\,\mathrm{d}x = M$ for all $t > 0$, where M is defined as $M = \int_{\mathbb{R}} u_0(x)\,\mathrm{d}x$.*

(ii) *(Oleinik entropy condition) $\partial_x(u^{q-1}(t,x)) \leqslant 1/t$ for all $t > 0$ in a weak distributional sense.*

(iii) *(Upper bound) $0 \leqslant u(t,x) \leqslant (qM/(q-1))^{1/q} t^{-1/q}$ for all $t > 0$ and $x \in \mathbb{R}$.*

(iv) *(Decay in L^p-norm) for $1 \leqslant p \leqslant \infty$,*

$$\|u(t,\cdot)\|_{L^p(\mathbb{R})} \leqslant \Big(\frac{q}{q-1}\Big)^{(p-1)/(pq)} M^{(p-1)/pq+1/p} t^{-(1-1/p)/q} \quad \forall\, t > 0.$$

(v) *(Decay of the spatial derivative) $\partial_x u(t,x) \leqslant C(q) M^{(2-q)/q} t^{-2/q}$ for all $t > 0$ and a.e. $x \in \mathbb{R}$.*

(vi) *($W^{1,1}_{\mathrm{loc}}(\mathbb{R})$ estimate) For any $R > 0$,*

$$\int_{|x|<R} |\partial_x u(t,x)|\,\mathrm{d}x \leqslant 2RC(q) M^{(2-q)/q} t^{-2/q} + 2\Big(\frac{q}{q-1}M\Big)^{1/q} t^{-1/q} \quad \forall\, t > 0.$$

(vii) *(Energy estimate) For every $0 < \tau < T$,*

$$\int_\tau^T \int_{\mathbb{R}} |\mathcal{D}^{(\alpha+1)/2}[u(t,\cdot)](x)|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{1}{2}\Big(\frac{q}{q-1}\Big)^{1/q} \tau^{-1/q} M^{(q+1)/q}.$$

Proof. The proof of (i) is as in [22].

We recall the proof of (ii): First we recall that $u \in C((0,\infty), L^p(\mathbb{R}) \cap \dot{H}^{\theta,p}(\mathbb{R}))$ for any $\theta < 1+\alpha+\min\{\alpha, q-1\}$ and $u_\varepsilon \in C_b^\infty((0,\infty)\times\mathbb{R})$, then using Lemma 3.1, $u_\varepsilon \to u$ as $\varepsilon \to 0$ point-wise for all $t>0$ and $x \in \mathbb{R}$. As a result, (ii) holds by taking the limit $\varepsilon \to 0$ and Proposition 3.1. With this, one can again proceed as in [22] to conclude (iii) and (iv). The proofs of (v) and (vi) follow as in [22] too, they do not depend on the form of the nonlocal operator.

We now prove (vii). First, we multiply (1.1) by u and get the following identity, after integrating with respect to x:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} u^2 \,\mathrm{d}x - \int_{\mathbb{R}} u\partial_x \mathcal{D}^\alpha[u](x)\,\mathrm{d}x = 0.$$

We observe that using integration by parts

$$-\int_{\mathbb{R}} u\partial_x \mathcal{D}^\alpha[u](x)\,\mathrm{d}x = \int_{\mathbb{R}} \partial_x u \mathcal{D}^\alpha[u](x)\,\mathrm{d}x \geqslant 0,$$

the last inequality is shown in [12], Lemma 2.2. Then, proceeding as in the proof of Lemma 2.3 with $\theta_1=\theta_2$, using (2.2)–(2.3) with α replaced by $\frac{1}{2}(\alpha+1)$, we have

$$\begin{aligned}0 \leqslant -\int_{\mathbb{R}} u\partial_x \mathcal{D}^\alpha[u](x)\,\mathrm{d}x &= \int_{\mathbb{R}} (\mathrm{i}\xi)\mathcal{F}(u)(\xi)\overline{(\mathrm{i}\xi)^\alpha \mathcal{F}(u)(\xi)}\,\mathrm{d}\xi \\ &= \left|\int_{\mathbb{R}} (-1)^{(1+\alpha)/2+1}(\mathrm{i}\xi)^{(1+\alpha)/2}(-\mathrm{i}\xi)^{(1+\alpha)/2}\mathcal{F}(u)(\xi)\overline{\mathcal{F}(u)(\xi)}\,\mathrm{d}\xi\right| \\ &= \int_{\mathbb{R}} (\mathrm{i}\xi)^{(1+\alpha)/2}\mathcal{F}(u)(\xi)\overline{(\mathrm{i}\xi)^{(1+\alpha)/2}\mathcal{F}(u)(\xi)}\,\mathrm{d}\xi.\end{aligned}$$

Applying again Plancherel's theorem we finally have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} u^2(t,x)\,\mathrm{d}x + \int_{\mathbb{R}} |\mathcal{D}^{(\alpha+1)/2}[u(t,\cdot)](x)|^2\,\mathrm{d}x = 0. \tag{3.3}$$

The conditions for Plancherel's theorem to be applied follow from (iv) and Proposition 2.1.

Finally, we integrate (3.3) over (τ, T) for some $\tau > 0$ to get

$$\frac{1}{2}\int_{\mathbb{R}} u^2(T,x)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}} u^2(\tau,x)\,\mathrm{d}x + \int_\tau^T\int_{\mathbb{R}} |\mathcal{D}^{(\alpha+1)/2}[u(t,\cdot)](x)|^2\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Then, (vii) follows taking into account that the first term is nonnegative and applying (iv) for $p=2$. □

4. Asymptotic behaviour

In this section we prove Theorem 1.1. As we have mentioned, this is equivalent to proving limit (1.12) for some fixed s. Henceforth, we consider for $\lambda > 1$, u_λ be defined by means of (1.3)–(1.4). We follow the proof in [22], we use Lemma 3.2 (translated to u_λ) and the following lemma:

Lemma 4.1 (Tail control estimate). *Let $\lambda > 1$ and u_λ be the solution of* (1.5) *with $0 \leqslant u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$. Then for any $\lambda > 1$ and $R > 0$, there exists a constant $C(M, q) > 0$ such that*

$$\int_{|y|>2R} u_\lambda(s, y)\,\mathrm{d}y \leqslant \int_{|y|>R} u_0(y)\,\mathrm{d}y + C(M, q)\Big(\frac{s\lambda^{q-1-\alpha}}{R^{\alpha+1}} + \frac{s^{1/q}}{R}\Big) \quad \forall\, s > 0.$$

Noticing that $\mathcal{D}^\alpha[u_\lambda(s,\cdot)](y) = \lambda^{\alpha+1}\mathcal{D}^\alpha[u(\lambda^q s,\cdot)](\lambda y)$ and the same scaling works for $\partial_y\overline{\mathcal{D}^\alpha}[\cdot]$, we can just follow [22] to conclude the result.

P r o o f of Theorem 1.1. The existence and the regularity item (i) and (ii) follow from Theorem 2.1 and Proposition 2.1. It remains to show the large time behaviour.

First, we show that (1.6) and (1.12) are equivalent. Without loss of generality we consider $s_0 = 1$, applying the scaling (1.3)–(1.4), and observing that (1.7) is invariant under the rescaling, we get that for any $\lambda > 1$,

$$u_\lambda(1, y) - U_M(1, y) = t^{1/q}(u(t, x) - U_M(t, x)).$$

And performing the change of variables in the integral, we finally get

$$\|u_\lambda(1, y) - U_M(1, y)\|_{L^p(\mathbb{R})} = t^{(1-1/p)/q}\|u(t, x) - U_M(t, x)\|_{L^p(\mathbb{R})}.$$

Let us then prove (1.12). We divide the proof into several steps. Let us first show the convergence of a sub-sequence of $\{u_\lambda\}_{\lambda>1}$. Using Theorem 5 of [33] we shall get that $\{u_\lambda\}_{\lambda>1}$ is relatively compact in $C([s_1, s_2], L^2_{\mathrm{loc}}(\mathbb{R}))$ for any $0 < s_1 < s_2 < \infty$.

We let $B_R = (-R, R)$ and we apply Theorem 5 of [33], to the triple $W^{1,1}(B_R) \hookrightarrow L^2(B_R) \hookrightarrow H^{-1}(B_R)$. Observe that (i) and (iii) in Lemma 3.2 imply that $\{u_\lambda\}_{\lambda>1}$ is uniformly bounded in $L^\infty((s_1, s_2), W^{1,1}(B_R))$, and this gives the first condition of this theorem. Then by [33], Lemma 4 we can conclude that

$$(4.1)\quad \|u_\lambda(s+h,\cdot) - u_\lambda(s,\cdot)\|_{L^\infty((0,T-h),H^{-1}(B_R))} \to 0 \quad \text{as } h \to 0 \text{ uniformly for } \lambda > 1$$

provided that $\{\partial_s u_\lambda\}_{\lambda>1}$ is uniformly bounded in $L^p((s_1, s_2), H^{-1}(B_R))$ for some $p < \infty$. Let us show this with $p = 2$. First, let us choose $\varphi \in C_c((0,\infty) \times B_R)$ and extend it by zero outside B_R. For such φ and $\lambda > 1$ we have

(4.2)

$$
\begin{aligned}
&\left|\int_{s_1}^{s_2}\int_{\mathbb{R}} (\partial_s u_\lambda)\varphi \,\mathrm{d}y\,\mathrm{d}s\right| \\
&\quad\leqslant \left|\int_{s_1}^{s_2}\int_{\mathbb{R}} \partial_y (u_\lambda)^q \varphi \,\mathrm{d}y\,\mathrm{d}s\right| + \lambda^{q-1-\alpha}\left|\int_{s_1}^{s_2}\int_{\mathbb{R}} \partial_y \mathcal{D}^\alpha[u_\lambda]\varphi \,\mathrm{d}y\,\mathrm{d}s\right| \\
&\quad= \left|\int_{s_1}^{s_2}\int_{\mathbb{R}} u_\lambda^q \partial_y \varphi \,\mathrm{d}y\,\mathrm{d}s\right| + \lambda^{q-1-\alpha}\left|\int_{s_1}^{s_2}\int_{\mathbb{R}} \mathcal{D}^{(1+\alpha)/2}[u_\lambda]\overline{\mathcal{D}^{(1+\alpha)/2}}[\varphi] \,\mathrm{d}y\,\mathrm{d}s\right| \\
&\quad\leqslant \|u_\lambda^q\|_{L^2((s_1,s_2),L^2(\mathbb{R}))}\|\varphi\|_{L^2((s_1,s_2),H^1(\mathbb{R}))} \\
&\qquad + \lambda^{(q-1-\alpha)/2}\left(\lambda^{q-1-\alpha}\int_{s_1}^{s_2}\int_{\mathbb{R}} |\mathcal{D}^{(1+\alpha)/2}[u_\lambda]|^2 \,\mathrm{d}y\,\mathrm{d}s\right)^{1/2} \\
&\qquad \times \left(\int_{s_1}^{s_2}\int_{\mathbb{R}} |\overline{\mathcal{D}^{(1+\alpha)/2}}[\varphi]|^2 \,\mathrm{d}y\,\mathrm{d}s\right)^{1/2} \\
&\quad\leqslant C'(M,q,s_1,s_2)\|\varphi\|_{L^2((s_1,s_2),H^1(\mathbb{R}))} \\
&\qquad + \lambda^{(q-1-\alpha)/2}\frac{1}{\sqrt{2}}\Big(\frac{q}{q-1}\Big)^{1/(2q)} M^{(1+q)/(2q)} s_1^{-1/(2q)}\|\varphi\|_{L^2((s_1,s_2),\dot{H}^{(1+\alpha)/2}(\mathbb{R}))} \\
&\quad\leqslant C(M,q,s_1,s_2)\|\varphi\|_{L^2((s_1,s_2),H^1(\mathbb{R}))}.
\end{aligned}
$$

Here, we have applied Lemma 2.3 in the second inequality and the energy estimate Lemma 3.2 (iv), as well as (2.5). All these steps can be performed since conservation of mass and the regularity of u is transferred to u_λ (see Proposition 2.1, in particular) and by the choice of φ. Now, the Riesz representation theorem, see [13], Chapter IV Corollary 4 and (4.2) imply that

$$\|\partial_s u_\lambda\|_{L^2((s_1,s_2),H^{-1}(B_R))} \leqslant C(M,q,s_1,s_2) \quad \forall\,\lambda > 1,$$

and we can conclude (4.1). Hence, we can apply Theorem 5 of [33], this means that $\{u_\lambda\}_{\lambda>1}$ is relatively compact in $C([s_1,s_2],L^2(B_R))$.

As a consequence, there exists $u_\infty \in C([s_1,s_2],L^2(B_R))$ such that, up to a subsequence, $u_\lambda \to u_\infty$ as $\lambda \to \infty$ in $C([s_1,s_2],L^2(B_R))$. By a diagonal argument we can conclude the convergence for any compact set and therefore,

$$u_\lambda \longrightarrow u_\infty \quad \text{as } \lambda \to \infty \text{ in } C([s_1,s_2],L^2_{\mathrm{loc}}(\mathbb{R})). \tag{4.3}$$

We observe that (4.3) implies also that $u_\lambda \to u_\infty$ in $C([s_1,s_2],L^1_{\mathrm{loc}}(\mathbb{R}))$. In order to extend this convergence to $C([s_1,s_2],L^1(\mathbb{R}))$, we use Lemma 4.1, see [22]. Hence,

$$u_\lambda \longrightarrow u_\infty \quad \text{as } \lambda \to \infty \text{ in } C([s_1,s_2],L^1(\mathbb{R})).$$

The next step is to prove that $u_\infty = U_M$, i.e., that it satisfies Definition 1.1. First, we recall that u satisfies (2.21) of Theorem 2.2. Therefore, u_λ satisfies the following inequality for any nonnegative $\varphi \in C_c^\infty((0,\infty)\times\mathbb{R})$, using Lemma 2.2:

$$\int_0^\infty \int_{\mathbb{R}} \Big(|u_\lambda - k|\partial_s\varphi + \frac{1}{q}\operatorname{sgn}(u_\lambda - k)((u_\lambda)^q - k^q)\partial_y\varphi \\ + \lambda^{q-1-\alpha}\partial_y \mathcal{D}^\alpha[|u_\lambda - k|](y)\varphi\Big)\,\mathrm{d}y\,\mathrm{d}s \geqslant 0. \tag{4.4}$$

In what follows we pass to the limit $\lambda \to \infty$ in (4.4). We prove that the last term tends to zero as $\lambda \to \infty$. We split this integral term into two as follows, given $r > 0$:

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}} \partial_y \mathcal{D}^\alpha[|u_\lambda - k|](y)\varphi(s,y)\,\mathrm{d}y\,\mathrm{d}s \\ &= d_{\alpha+2}\int_0^\infty \int_{\mathbb{R}}\int_{-\infty}^{-r} \frac{|u_\lambda(s,y+z)-k| - |u_\lambda(s,y)-k| - \partial_y(|u_\lambda - k|)z}{|z|^{\alpha+2}}\varphi(s,y)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}s \\ &\quad + d_{\alpha+2}\int_0^\infty \int_{\mathbb{R}} |u_\lambda(s,y)-k| \int_0^r \frac{\varphi(s,y+z)-\varphi(s,y)-\partial_y\varphi z}{|z|^{\alpha+2}}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}s. \end{aligned}$$

The second integral term has been obtained by using Fubini's theorem, integration by parts in y in the third term, and the pertinent changes of variables.

Following the ideas of [22], we bound the first and second integral terms applying the regularity of φ and the non-negativity and conservation of mass of u_λ. Then, this means that the last term on the left-hand side of (4.4) goes to zero as $\lambda \to \infty$.

Since $u_\lambda \to u_\infty$ in $C((0,\infty), L^1(\mathbb{R}))$, we can pass to the limit in property (i) of Lemma 3.2, so that $\int_{\mathbb{R}} u_\infty(s,y)\,\mathrm{d}y = M$. Moreover, $u_\lambda \to u_\infty$ a.e. in $(0,\infty)\times\mathbb{R}$, which shows that property (ii) of Lemma 3.2 with $p = \infty$ is transferred to u_∞:

$$\|u_\infty(s,\cdot)\|_{L^\infty(\mathbb{R})} \leqslant C(M)s^{-1/q}.$$

This last inequality is sufficient to prove that $(u_\lambda)^q \to (u_\infty)^q$ as $\lambda \to \infty$ in $C((0,\infty), L^1(\mathbb{R}))$ and, therefore, passage to the limit $\lambda \to \infty$ in (4.4) gives Definition 1.1–(1.10) (with U_M replaced by u_∞) for every constant $k \in \mathbb{R}$ and $\varphi \in C_c^\infty((0,\infty)\times\mathbb{R})$, $\varphi \geqslant 0$.

Finally, we have to check that u_∞ satisfies Definition 1.1, (1.11) for any $\psi \in C_b(\mathbb{R})$. First, one proves it for any $\psi \in C_b^2(\mathbb{R})$, which, by density, can be generalised to $\psi \in H^2(\mathbb{R})$. Finally, the result with $\psi \in C_b(\mathbb{R})$ follows by an approximation argument and Lemma 3.2.

Thus, we have shown that u_∞ satisfies Definition 1.1. Since (1.7) has a unique entropy solution U_M, then $\{u_\lambda\}_{\lambda>1}$ converges to U_M in $C([s_1,s_2], L^1(\mathbb{R}))$ as $\lambda \to \infty$.

In order to finish the proof, one has to extend this convergence to $L^p(\mathbb{R})$ with $1 < p < \infty$. This follows by interpolation as in [22]. □

5. Regularisation by a general Riesz-Feller operator

In this section, we focus on showing how to generalise the previous results of Sections 2, 3 and 4 for the problem

$$
\begin{cases} \partial_t u(t,x) + |u(t,x)|^{q-1}\partial_x(u(t,x)) = D^{\beta}_{\gamma}[u(t,\cdot)](x), & t>0,\ x\in\mathbb{R},\\ u(0,x)=u_0(x), & x\in\mathbb{R}, \end{cases} \tag{5.1}
$$

where the diffusion is given by a general Riesz-Feller operator (1.13)–(1.14). Here β and γ satisfy the assumptions of such definition and $u_0\in L^{\infty}(\mathbb{R})\cap L^1(\mathbb{R})$.

We use the following formulation of the nonlocal operator, given in [2], Proposition 2.3 (or see [11], [28], [32]): for any $0<\beta<2$ and $|\gamma|\leqslant \min\{\beta, 2-\beta\}$,

$$
D^{\beta}_{\gamma}[g](x) = c^1_{\gamma}\int_0^{\infty} \frac{g(x-z)-g(x)+g'(x)z}{z^{1+\beta}}\,\mathrm{d}z + c^2_{\gamma}\int_0^{\infty}\frac{g(x+z)-g(x)-g'(x)z}{z^{1+\beta}}\,\mathrm{d}z
$$

for $1<\beta<2$, where (see e.g., [28])

$$
c^1_{\gamma} = \frac{\Gamma(1+\beta)}{\pi}\sin\Big((\beta-\gamma)\frac{\pi}{2}\Big) \quad\text{and}\quad c^2_{\gamma} = \frac{\Gamma(1+\beta)}{\pi}\sin\Big((\beta+\gamma)\frac{\pi}{2}\Big),
$$

in particular, $c^1_{\gamma}+c^2_{\gamma}>0$.

Using Lemma 2.2 it is easy to show that

$$
D^{\beta}_{\gamma}[g](x) = \frac{1}{d_{\beta+1}}(c^1_{\gamma}\partial_x\mathcal{D}^{\beta-1}[g](x) + c^2_{\gamma}\partial_x\overline{\mathcal{D}^{\beta-1}}[g](x)). \tag{5.2}
$$

Existence and regularity results for (5.1) are proved similarly by defining mild solutions as in Definition 2.1 with the kernel

$$
K^{\beta}_{\gamma}(t,x) := \mathcal{F}^{-1}(\mathrm{e}^{t\psi^{\beta}_{\gamma}(\xi)})(x). \tag{5.3}
$$

These steps have already been explained in [15], Section 6. This is because K^{β}_{γ} satisfies similar properties as K does, the proofs are given in e.g., [2], Lemma 2.1. Thus, we can say that Theorem 2.1 and Corollary 2.1 hold unchanged for (5.1).

In order to generalise Proposition 2.1 for (5.1), we need the following lemma (analogous to Lemma 2.4):

Lemma 5.1 (Time behaviour of K^{β}_{γ})**.** *For all $\theta\in(0,1)$ and $1\leqslant p\leqslant\infty$, the kernel $K^{\beta}_{\gamma}(t,x)$, such that $\beta\in(1,2)$ and $|\gamma|\leqslant\min\{\beta,2-\beta\}$, satisfies the following estimates for any $t>0$:*

$$
\|K^{\beta}_{\gamma}(t,\cdot)\|_{L^p(\mathbb{R})} = Ct^{-(1-1/p)/\beta}, \quad \|\partial_x K^{\beta}_{\gamma}(t,\cdot)\|_{L^p(\mathbb{R})} \lesssim t^{-(1-1/p)/\beta-1/\beta},
$$
$$
\||D|^{\theta}[K^{\beta}_{\gamma}(t,\cdot)]\|_{L^p(\mathbb{R})} \lesssim t^{-(1-1/p)/\beta-\theta/\beta},
$$
$$
\||D|^{\theta}[\partial_x K^{\beta}_{\gamma}(t,\cdot)]\|_{L^p(\mathbb{R})} \lesssim t^{-(1-1/p)/\beta-(1+\theta)/\beta}
$$

for a constant $C>0$.

P r o o f. The properties of (5.3), analogous to the ones for (2.8), are given in [2], Lemma 2.1, and combining the self-similarity, the mass conservation of (5.3) and its derivative, and the fact that these are bounded on $(0,T)\times\mathbb{R}$ for any $T>0$, we conclude the first and second estimates.

For the third estimate, we apply the self-similarity property of K_γ^β and rescale as follows:

(5.4)
$$||D|^\theta[K_\gamma^\beta(t,\cdot)](x)| = \frac{1}{t^{1/\beta}}\Big||D|^\theta\Big[K_\gamma^\beta\Big(1,\frac{\cdot}{t^{1/\beta}}\Big)\Big](x)\Big| = \frac{1}{t^{(1+\theta)/\beta}}\Big||D|^\theta[K_\gamma^\beta(1,\cdot)]\Big(\frac{x}{t^{1/\beta}}\Big)\Big|.$$

If the L^p-norm of (5.4) is finite, we get, applying the change of variable $X = x/t^{1/\beta}$, the desired estimate:

$$\begin{aligned}\||D|^\theta[K_\gamma^\beta(t,\cdot)](x)\|_{L^p(\mathbb{R})} &= \frac{1}{t^{(1+\theta)/\beta}}t^{1/(p\beta)}\Big(\int_\mathbb{R} ||D|^\theta[K_\gamma^\beta(1,\cdot)](X)|^p\,\mathrm{d}X\Big)^{1/p}\\ &\lesssim t^{-(1-1/p)/\beta-\theta/\beta}.\end{aligned}$$

Thus, it remains to show that the L^p-norm is finite. One gets the boundedness of the integrand using definition (2.4):

$$\begin{aligned}||D|^\theta[K_\gamma^\beta(1,\cdot)](X)| &= \frac{1}{\sqrt{2\pi}}\Big|\int_\mathbb{R}|\xi|^\theta \mathrm{e}^{-|\xi|^\beta \mathrm{e}^{\mathrm{i}\,\mathrm{sgn}(\xi)\gamma\pi/2}}\mathrm{e}^{\mathrm{i}X\xi}\,\mathrm{d}\xi\Big|\\ &\leqslant \frac{1}{\sqrt{2\pi}}\int_\mathbb{R}|\xi|^\theta\mathrm{e}^{-|\xi|^\beta\cos(\gamma\pi/2)}\,\mathrm{d}\xi<\infty,\end{aligned}\tag{5.5}$$

where $|\gamma|\leqslant 2-\beta<1$, which implies that $\cos(\frac12\gamma\pi)>0$. Hence, in order to conclude, it is sufficient to control the behaviour for large $|X|$. Starting from (5.5), we write

$$\begin{aligned}|D|^\theta[K_\gamma^\beta(1,\cdot)](X) &= \frac{1}{\sqrt{2\pi}}\int_\mathbb{R}|\xi|^\theta\mathrm{e}^{-|\xi|^\beta\mathrm{e}^{\mathrm{i}\,\mathrm{sgn}(\xi)\gamma\pi/2}}\mathrm{e}^{\mathrm{i}X\xi}\,\mathrm{d}\xi\\ &= \frac{1}{\sqrt{2\pi}}\int_\mathbb{R}|\xi|^\theta\mathrm{e}^{-|\xi|^\beta(\cos(\gamma\pi/2)+\mathrm{i}\,\mathrm{sgn}(\xi)\sin(\gamma\pi/2))}\mathrm{e}^{\mathrm{i}X\xi}\,\mathrm{d}\xi\\ &= \frac{1}{\sqrt{2\pi}}\int_0^\infty\xi^\theta\mathrm{e}^{-\xi^\beta(\cos(\gamma\pi/2)+\mathrm{i}\sin(\gamma\pi/2))}\mathrm{e}^{-\mathrm{i}(-X)\xi}\,\mathrm{d}\xi\\ &\quad+\frac{1}{\sqrt{2\pi}}\int_0^\infty\xi^\theta\mathrm{e}^{-\xi^\beta(\cos(\gamma\pi/2)-\mathrm{i}\sin(\gamma\pi/2))}\mathrm{e}^{-iX\xi}\,\mathrm{d}\xi.\end{aligned}$$

Now, let
$$\sigma=\cos\Big(\frac{\gamma\pi}{2}\Big)+\mathrm{i}\sin\Big(\frac{\gamma\pi}{2}\Big).$$

Now we apply [30], Lemma 2, which implies

$$||D|^\theta[K_\gamma^\beta(1,\cdot)](X)|\lesssim\frac{1}{|X|^{1+\theta}},\quad |X|\gg1.\tag{5.6}$$

Since $\theta > 0$, we can apply the lemma if the condition

$$\sigma, \overline{\sigma} \in \left\{a + \mathrm{i}b \in \mathbb{C}\colon -\cos\Big(\frac{\beta\pi}{2}\Big) \leqslant a \leqslant 1,\ |b| \leqslant -\tan\Big(\frac{\beta\pi}{2}\Big)\right\}$$

is satisfied. This holds since

$$|\gamma| \leqslant \min\{\beta, 2-\beta\} \Rightarrow \frac{\gamma\pi}{2} \in \Big(-\frac{(2-\beta)\pi}{2}, \frac{(2-\beta)\pi}{2}\Big) \subset \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big)$$

and this implies that if $\sigma = a + \mathrm{i}b$,

$$\cos\Big(\frac{(2-\beta)\pi}{2}\Big) = -\cos\Big(\frac{\beta\pi}{2}\Big) \leqslant a = \cos\Big(\frac{\gamma\pi}{2}\Big) \leqslant 1$$

and the imaginary part satisfies

$$|b| = \Big|\sin\Big(\frac{\gamma\pi}{2}\Big)\Big| \leqslant \sin\Big(\frac{(2-\beta)\pi}{2}\Big) \leqslant \tan\Big(\frac{(2-\beta)\pi}{2}\Big) = -\tan\Big(\frac{\beta\pi}{2}\Big).$$

As a result of (5.5) and the previous behaviour given in (5.6), we conclude that

$$|D|^{\theta}[K_{\gamma}^{\beta}(1,\cdot)](X) \in L^p(\mathbb{R}) \quad \text{for any } 1 \leqslant p \leqslant \infty.$$

Finally, the fourth estimate follows similarly. We leave the details to the reader. □

Now with Lemma 5.1, we can proceed as in the proof of [22], Proposition 3.1 to obtain Proposition 2.1 for (5.1).

In order to conclude the corresponding weak viscous entropy inequality, similarly to Theorem 2.2 and the Oleinik type of inequality and all other *a priori* estimates, similarly to Proposition 3.1 and Lemma 3.2, for positive solutions, we need the following lemma:

Lemma 5.2 (Partial integration by parts and energy estimate)**.** *Let $\beta \in (1,2)$ and $|\gamma| \leqslant \min\{\beta, 2-\beta\}$. Then:*

(i) *For functions g and h such that $D_{\gamma}^{\beta}[g], \mathcal{D}^{\theta_1}[g], h, \overline{\mathcal{D}^{\theta_2}}[h] \in L^2(\mathbb{R})$,*

$$\int_{\mathbb{R}} D_{\gamma}^{\beta}[g](x)h(x)\,\mathrm{d}x = -\frac{1}{d_{\beta+1}}(c_{\gamma}^1 + c_{\gamma}^2)\int_{\mathbb{R}} \mathcal{D}^{\theta_1}[g](x)\overline{\mathcal{D}^{\theta_2}}[h](x)\,\mathrm{d}x,$$

where $\frac{1}{2} < \theta_1$, $\theta_2 < 1$ and $\beta = \theta_1 + \theta_2$.

(ii) *Moreover, for $1 < \beta < 2$ and g, $D_{\gamma}^{\beta}[g] \in L^2(\mathbb{R}) \cap C_b^2(\mathbb{R})$, we have*

$$-\int_{\mathbb{R}} g(x)D_{\gamma}^{\beta}[g](x)\,\mathrm{d}x \geqslant 0.$$

P r o o f. We note that Lemma 2.3 has easy generalisation

$$\int_{\mathbb{R}} \partial_x \overline{\mathcal{D}^{\beta-1}}[g](x)h(x)\,\mathrm{d}x = -\int_{\mathbb{R}} \mathcal{D}^{\theta_1}[g](x)\overline{\mathcal{D}^{\theta_2}}[h](x)\,\mathrm{d}x$$

for $\frac{1}{2} < \theta_1,\, \theta_2 < 1$ with $\beta = \theta_1 + \theta_2$. This and Lemma 2.3 with $\alpha = \beta - 1$ implies (i), using representation (5.2).

In order to show (ii), we again use representation (5.2) and Lemma 2.1. This gives

$$\begin{aligned}\int_{\mathbb{R}} gD_\gamma^\beta[g](x)\,\mathrm{d}x &= \frac{1}{d_{\beta+1}}\Big(c_\gamma^1 \int_{\mathbb{R}} g\partial_x\mathcal{D}^{\beta-1}[g](x)\,\mathrm{d}x + c_\gamma^2 \int_{\mathbb{R}} g\partial_x\overline{\mathcal{D}^{\beta-1}}[g](x)\,\mathrm{d}x\Big)\\ &= \frac{1}{d_{\beta+1}}(c_\gamma^1 + c_\gamma^2)\int_{\mathbb{R}} g\partial_x\mathcal{D}^{\beta-1}[g](x)\,\mathrm{d}x \leqslant 0,\end{aligned}$$

where the last inequality is proved as in e.g., [12]. □

Part (i) of the above lemma allows to prove a weak entropy inequality, namely

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}} \Big(&|u(t,x)-k|\partial_t\varphi + \frac{1}{q}\operatorname{sgn}(u(t,x)-k)(|u(t,x)|^{q-1}u(t,x) - |k|^{q-1}k)\partial_x\varphi\\ &+|u(t,x)-k|\overline{D_\gamma^\beta}[\varphi(t,\cdot)](x)\Big)\,\mathrm{d}x\,\mathrm{d}t \geqslant 0,\end{aligned}$$

where, as we have defined also in [15],

$$\overline{D_\gamma^\beta}[g](x) = \frac{1}{d_{\beta+1}}(c_\gamma^1\partial_x\overline{\mathcal{D}^{\beta-1}}[g](x) + c_\gamma^2\partial_x\mathcal{D}^{\beta-1}[g](x)).$$

We observe that the above lemma is necessary to conclude the analogous of Lemma 3.2, in particular, property (vii). Indeed, we need an energy estimate similar to (3.3). Let us briefly indicate how this is obtained. First, we multiply the equation by u and integrate by parts:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} u^2\,\mathrm{d}x - \int_{\mathbb{R}} uD_\gamma^\beta[u](x)\,\mathrm{d}x = 0.$$

Now, using (i) above, we obtain the energy type of identity:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} u^2\,\mathrm{d}x + \frac{1}{d_{\beta+1}}(c_\gamma^1 + c_\gamma^2)\int_{\mathbb{R}} \mathcal{D}^{\beta/2}[u](x)\overline{\mathcal{D}^{\beta/2}}[u](x)\,\mathrm{d}x = 0.$$

The second term is positive by (ii), this means that, in fact,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} u^2\,\mathrm{d}x + \frac{1}{d_{\beta+1}}(c_\gamma^1 + c_\gamma^2)\int_{\mathbb{R}} |\mathcal{D}^{\beta/2}[u](x)|^2\,\mathrm{d}x = 0.$$

The rest of the argument follows unchanged, combining all the results that we have mentioned. Thus, we can generalise the large time asymptotic result Theorem 1.1 for equation (5.1) in the sub-critical case, $1 < q < \beta$, for nonnegative solutions, obtaining the same rate of convergence.

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