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LINEAR PRESERVER OF $n \times 1$ FERRERS VECTORS

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Abstract. Let $A = [a_{ij}]_{m\times n}$ be an $m \times n$ matrix of zeros and ones. The matrix A is said to be a Ferrers matrix if it has decreasing row sums and it is row and column dense with nonzero $(1,1)$-entry. We characterize all linear maps perserving the set of $n \times 1$ Ferrers vectors over the binary Boolean semiring and over the Boolean ring $\mathbb{Z}_2$. Also, we have achieved the number of these linear maps in each case.

Keywords: Ferrers matrix; linear preserver; Boolean semiring

MSC 2020: 15A04, 05B20

1. Introduction

In the analysis of a biological lattice, $(0,1)$-matrices are an essential tool. Some samples are the hunt predictor patterns, the climate-growth patterns, the pollinator-plant patterns, see [1]. In this paper, we will characterize the linear maps that preserve the set of $n \times 1$ Ferrers vectors over the binary Boolean semiring or over the Boolean ring $\mathbb{Z}_2$. Some works on linear preserver problems can be found in [3] and [4]. The required definitions are listed below.

A semiring is a set S together with two binary operators $S(+,\cdot)$ satisfying the following conditions:

(1) additive associativity: for all $a,b,c \in S$, $a + (b + c) = (a + b) + c$,
(2) additive commutativity: for all $a,b \in S$, $a + b = b + a$,
(3) multiplicative associativity: for all $a,b,c \in S$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$,
(4) left and right distributivity: for all $a,b,c \in S$, $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$.

In other words, a semiring is an algebraic structure similar to a ring, but without the requirement that each element must have an additive inverse.

A ring R is called a *Boolean ring* if $x^2 = x$ for all $x \in R$.

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Example 1.1. Let X be a nonempty set. Define

$$A \oplus B = A \cup B, \quad A \cdot B = A \cap B \quad \forall\, A, B \in P(X).$$

Then $(P(X), \oplus, \cdot)$ is a Boolean semiring having $\emptyset$ and X as its zero and identity, respectively. The empty set $\emptyset$ is the only additively invertible element of $(P(X), \oplus, \cdot)$. So, $(P(X), \oplus, \cdot)$ is not a Boolean ring. Also, $A \oplus A = A$ for all $A \in P(X)$, see [5].

Example 1.2. Let

$$S = \{0\} \cup \left[\frac{1}{2}, 1\right]$$

and define

$$\begin{aligned} x \oplus 0 = 0 \oplus x &= x, && \forall\, x \in S, \\ x \oplus y &= \frac{1}{2}, && \forall\, x, y \in \left[\frac{1}{2}, 1\right], \\ x \cdot y &= \min\{x, y\}, && \forall\, x, y \in S. \end{aligned}$$

It is easy to show that $(S, \oplus, \cdot)$ is a Boolean semiring with zero 0 and identity 1. Moreover, 0 is the only additively invertible element of the semiring $(S, \oplus, \cdot)$, see [5].

On $\{0, 1\}$ there are two semiring structures. The first is the Boolean semiring that corresponds to $1 + 1 = 1$ and we will denote it by $\mathcal{B}$, see [2].

$+$	0	1
0	0	1
1	1	1

$\cdot$	0	1
0	0	0
1	0	1

The second is the Boolean ring $\mathbb{Z}_2$ that corresponds to $1 + 1 = 0$, see [2].

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

By $M_{m,n}(\mathcal{B})$ we mean the set of all $m \times n$ matrices with entries from $\mathcal{B}$. Let $\mathcal{B}^n = M_{n,1}(\mathcal{B})$ and let $e_j \in \mathcal{B}^n$ be the vector with exactly one nonzero entry 1 in the jth position. The vectors e_j are called *cells*. The zero vector of $M_{n,1}(\mathcal{B})$ is denoted by $O_{n,1}$.

A function $T\colon M_{m,n}(\mathcal{B}) \to M_{m,n}(\mathcal{B})$ is said to be a linear operator of $M_{m,n}(\mathcal{B})$ if for any $A, B \in M_{m,n}(\mathcal{B})$ and for any $\alpha \in \mathcal{B}$, $T(A + B) = T(A) + T(B)$ and $T(\alpha A) = \alpha T(A)$. Let $T\colon M_{m,n}(\mathcal{B}) \to M_{m,n}(\mathcal{B})$ be a linear operator. For a set $X \subseteq M_{m,n}(\mathcal{B})$, if $A \in X$ implies that $T(A) \in X$, we say that T preserves X. Furthermore, we say that T strongly preservers X if $A \in X$, if and only if $T(A) \in X$.

In [1], Beasley has found the structure of bijective linear maps preserving $m \times n$ Ferrers matrices over the Boolean semiring as follows.

Theorem 1.3 ([1]). *Let $T\colon M_{m,n}(\mathcal{B}) \to M_{m,n}(\mathcal{B})$ be a bijective linear operator that maps the set of all Ferrers matrices in $M_{m,n}(\mathcal{B})$ to itself. Then either:*
(1) *T is the identity, or*
(2) *$m = n$ and T is the transpose operator.*

This paper consists of two sections. In the first and second sections we will characterize the structure of linear maps preserving $n \times 1$ Ferrers vectors over the Boolean semiring $\mathcal{B}$ and over the ring $\mathbb{Z}_2$, respectively. In this paper we denote the ith row of a matrix A by a_i and the (i, j) entry of a matrix A by a_{ij}.

Definition 1.4. An $m \times n$ matrix of zeros and ones is said to be a Ferrers matrix if:
▷ it has decreasing row sums,
▷ it is row and column dense, i.e., there are no zeros between two nonzero entries for every row and every column,
▷ its $(1, 1)$-entry is 1.

In other words, $A = [a_{ij}]$ is a Ferrers matrix if and only if $a_{ij} = 1$ implies that $a_{kl} = 1$ for all $k \leqslant i$ and $l \leqslant j$. For example,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

is a 3×3 Ferrers matrix.

In this paper, we denote a Ferrers vector by v, also we denote the set of $n\times 1$ Ferrers vectors over a set S by $(FVS)_n$. It is observable that $(FVS)_n = \{v(1), \ldots, v(n)\}$, where $v(i) = e_1 + e_2 + \ldots + e_i$.

Definition 1.5. Let $1 \leqslant i \leqslant n$. A Ferrers vector $v = (v_1, v_2, \ldots, v_n)^\top$ is said to be an i-weight Ferrers vector if $v_i = 1$ and $v_{i+1} = 0$ (unless $i = n$).

For example, $v = (1, 1, 1, 0, \ldots, 0)^\top$ is a 3-weight Ferrers vector.

2. Linear preserver of $n \times 1$ Ferrers vectors over $\mathcal{B}$

Definition 2.1. A nonzero row vector $v = (v_1, v_2, \ldots, v_n)^\top$ is said to be a stair (of order i), if there exists i $(1 \leqslant i \leqslant n)$ such that $v_i = 1$ and $v_1 = \ldots = v_{i-1} = 0$ (unless $i = 1$). The order of a stair v is denoted by $s(v)$. Also the zero row vector is said to be a stair of order $n + 1$.

The following theorem characterizes the structure of linear maps preserving Ferrers vectors over $\mathcal{B}$.

Theorem 2.2. *Let $T\colon \mathcal{B}^n \to \mathcal{B}^n$ be a linear map and let A be the representation matrix of T with respect to $\{e_1, \ldots, e_n\}$. Then T is a linear preserver of $n\times 1$ Ferrers vectors if and only if for all $i = 1, \ldots, n$, a_i is a stair and $1 = s(a_1) \leqslant s(a_2) \leqslant \ldots \leqslant s(a_n) \leqslant n + 1$, where a_i is the ith row A. In other words, A has the following form:*

$$
\tag{2.1}
\begin{array}{r}
\\ \\ \\ \\ \\ \text{row } p_1 \\ \\ \\ \text{row } p_2 \\ \\ \\ \text{row } p_3 \\ \\ \\ \\ \text{row } p_t \\ \\
\end{array}
\begin{array}{c}
\quad\;\; \text{column } k_1 \quad \text{column } k_2 \qquad\quad \text{column } k_s \qquad\qquad \\
\left(\begin{array}{cccccc}
1 & & & & & \\
\vdots & * & * & & * & * \\
1 & & & & & \\
 & 1 & & & & \\
0 & \vdots & * & & * & * \\
 & 1 & & & & \\
 & & 1 & & & \\
0 & 0 & \vdots & & * & * \\
 & & 1 & & & \\
 & & & \ddots & & \\
 & & & & 1 & * \\
0 & 0 & 0 & 0 & \vdots & \\
 & & & & 1 & * \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right),
\end{array}
$$

where $1 \leqslant p_1 \leqslant \ldots \leqslant p_t \leqslant n$ and $1 \leqslant k_1 \leqslant \ldots \leqslant k_s \leqslant n$.

Proof. First, we show that any matrix of the above form is the representation matrix of a linear preserver of $n \times 1$ Ferrers vectors. Suppose that $v = (v_1, v_2, \ldots, v_n)^\top$ is any arbitrary Ferrers vector. Since $s(a_1) = 1$ and $v_1 = 1$, $(Av)_1 = 1$. If $(Av)_l = 1$ for some l $(1 \leqslant l \leqslant n)$, we show that $(Av)_i = 1$ for all $i < l$.

By the assumtion, a_l is a stair of order k $(1 \leqslant k \leqslant n)$. Since $s(a_l) = k$ and $(Av)_l = 1$, we have

$$1 = (Av)_l = \sum_{j=1}^{n} A_{lj} v_j = \sum_{j=k}^{n} A_{lj} v_j.$$

So, there exist some j $(k \leqslant j \leqslant n)$ such that $a_{lj} = 1$ and $v_j = 1$. Since $j \geqslant k$ and v is a Ferrers vector, $v_1 = v_2 = \ldots = v_k = 1$. Now, let $i \leqslant l$. By the assumption, $s(a_i) \leqslant s(a_l) = k$ and hence there exists some m $(1 \leqslant m \leqslant k)$ such that $a_{im} = 1$. Since $m \leqslant k$, $v_m = 1$ and these imply that

$$(Av)_i = a_{im} v_m + \sum_{\substack{j=1 \\ j \neq m}}^{n} a_{ij} v_j = 1.$$

So, $(Av)_1 = \ldots = (Av)_l = 1$ and hence Av is a Ferrers vector. Then T is a linear preserver of $n \times 1$ Ferrers vectors.

Conversely, let T be a linear preserver of $n \times 1$ Ferrers vectors. We first show that $s(a_1) = 1$. Since e_1 is a Ferrers vector and T is a linear preserver of $n \times 1$ Ferrers vectors, $T(e_1) = Ae_1$ is a Ferrers vector and hence $a_{11} \neq 0$. Now, we show that if $p < q$, then $s(a_p) \leqslant s(a_q)$. Assume if possible that $k = s(a_p) > m = s(a_q)$. So, $a_{pk} = a_{qm} = 1$. Consider the Ferrers vector $v = (v_1, v_2, \ldots, v_n)^\top = e_1 + e_2 + \ldots + e_m$. Since $m < k$, $v_k = \ldots = v_n = 0$ and hence

$$(Av)_p = \sum_{j=1}^{n} a_{pj} v_j = \sum_{j=k}^{n} a_{pj} v_j = 0.$$

On the other hand, $a_{qm} = v_m = 1$ and consequently,

$$(Av)_q = \sum_{j=1}^{n} a_{qj} v_j = a_{qm} v_m + \sum_{\substack{j=1 \\ j \neq m}}^{n} a_{qj} v_j = 1.$$

So, $(Av)_q = 1$ and $(Av)_p = 0$. Since $p < q$, Av is not a Ferrers vector, which is a contradiction. Therefore,

$$1 = s(a_1) \leqslant s(a_2) \leqslant \ldots \leqslant s(a_n),$$

and the proof is complete. □

We need the following lemma to find the number of linear preservers of $n \times 1$ Ferrers vectors over $\mathcal{B}$.

Lemma 2.3. *Let $\mathcal{F}_n = \{f \colon \{1, \ldots, n\} \to \{1, \ldots, n\} \colon f$ is nondecreasing$\}$. Then*

$$|\mathcal{F}_n| = |\{(i_1, \ldots, i_n) \colon 1 \leqslant i_1 \leqslant \ldots \leqslant i_n \leqslant n\}| = \sum_{i_1=1}^{n} \sum_{i_2=i_1}^{n} \ldots \sum_{i_n=i_{n-1}}^{n} 1.$$

Proof. For $1 \leqslant i_1 \leqslant i_2 \leqslant \ldots \leqslant i_k \leqslant n$, let

$$\mathcal{F}_n(i_1, \ldots, i_k) = \{f \in \mathcal{F}_n \colon f(1) = i_1, \ldots, f(k) = i_k\}.$$

We show that

$$|\mathcal{F}_n(i_1, \ldots, i_k)| = \sum_{i_{k+1}=i_k}^{n} |\mathcal{F}_n(i_1, \ldots, i_{k+1})|.$$

For $f \in \mathcal{F}_n(i_1, \ldots, i_k)$, let $f(k+1) = i_{k+1}$. This implies that

$$i_{k+1} \in \{i_k, i_k + 1, \ldots, n\},$$

and hence

$$\mathcal{F}_n(i_1, \ldots, i_k) = \bigcup_{i_{k+1}=i_k}^{n} \mathcal{F}_n(i_1, \ldots, i_k, i_{k+1}).$$

So, we have

$$|\mathcal{F}_n(i_1,\dots,i_k)| = \Big|\bigcup_{i_{k+1}=i_k}^{n} \mathcal{F}_n(i_1,\dots,i_k,i_{k+1})\Big|.$$

Since the sets $\mathcal{F}_n(i_1,\dots,i_k,i_k)$, $\mathcal{F}_n(i_1,\dots,i_k,i_k+1),\dots,\mathcal{F}_n(i_1,\dots,i_k,n)$ are mutually disjoint, we conclude that

$$|\mathcal{F}_n(i_1,\dots,i_k)| = \sum_{i_{k+1}=i_k}^{n} |\mathcal{F}_n(i_1,\dots,i_k,i_{k+1})|.$$

On the other hand, we have $\mathcal{F}_n = \bigcup\limits_{i_1=1}^{n} \mathcal{F}_n(i_1)$ and therefore,

$$|\mathcal{F}_n| = \sum_{i_1=1}^{n} |\mathcal{F}_n(i_1)| = \sum_{i_1=1}^{n}\sum_{i_2=i_1}^{n} |\mathcal{F}_n(i_1,i_2)| = \sum_{i_1=1}^{n}\sum_{i_2=i_1}^{n}\dots\sum_{i_n=i_{n-1}}^{n} |\mathcal{F}_n(i_1,\dots,i_n)|.$$

Since $|\mathcal{F}_n(i_1,\dots,i_n)| = 1$, we conclude that

$$|\mathcal{F}_n| = \sum_{i_1=1}^{n}\sum_{i_2=i_1}^{n}\dots\sum_{i_n=i_{n-1}}^{n} 1.$$

□

We denote the set of all linear preservers of $n \times 1$ Ferrers vectors over $\mathcal{B}$ by $(\mathrm{LFV}\mathcal{B})_n$.

Theorem 2.4. *Let* $|(\mathrm{LFV}\mathcal{B})_n|$ *be the cardinal number of* $(\mathrm{LFV}\mathcal{B})_n$. *Then*

$$|(\mathrm{LFV}\mathcal{B})_n| = \sum_{i_1=1}^{n}\sum_{i_2=i_1}^{n}\dots\sum_{i_n=i_{n-1}}^{n} 2^{i_1+\dots+i_{n-1}}.$$

Proof. Let T be an arbitrary linear preserver of $n \times 1$ Ferrers vectors over $\mathcal{B}$. Now consider all possible choices for operator T. We have $T((\mathrm{FV}\mathcal{B})_n) \subseteq (\mathrm{FV}\mathcal{B})_n$ and hence,

$$T(v(1)) = v(i_1), \quad T(v(2)) = v(i_2),\dots, \quad T(v(n)) = v(i_n)$$

for some $\{i_1,\dots,i_n\} \subseteq \{1,\dots,n\}$. We show that $i_1 \leqslant i_2 \leqslant \dots \leqslant i_n$. Let $j \in \{2,\dots,n\}$. Then

$$\text{(I)}\quad v(j) = v(j-1) + e_j \Rightarrow T(v(j)) = T(v(j-1)) + T(e_j)$$
$$\Rightarrow v(i_j) = v(i_{j-1}) + T(e_j)$$
$$\Rightarrow (v(i_j))_k = (v(i_{j-1}))_k + (T(e_j))_k \quad \forall\, k \in \{1,\dots,n\}.$$

Since $(v(i_{j-1}))_k = 1$ for all $1 \leqslant k \leqslant i_{j-1}$ and $(T(e_j))_k \in \{0,1\}$, we have $(v(i_j))_k = 1$ for all $1 \leqslant k \leqslant i_{j-1}$. This implies that $i_{j-1} \leqslant i_j \leqslant n$.

Now, let $\mathcal{N}(T(e_j))$ be the number of all possible choices of $T(e_j)$ such that $T(v(j)) = v(i_j)$. For all $1 \leqslant k \leqslant i_{j-1}$ and for all $2 \leqslant j \leqslant n$ we have $(v(i_j))_k = (v(i_{j-1}))_k = 1$. By the use of equation (I) and noting that $1+1 = 1+0 = 1$, we obtain that

$$(T(e_j))_k = 0 \quad \text{or} \quad (T(e_j))_k = 1.$$

Then

$$\mathcal{N}(T(e_j)) = 2^{i_{j-1}} \quad \forall\, j \in \{2, \ldots, n\}.$$

Let $\mathcal{N}(T)$ be the number of all possible choices for all $T(e_j)$ $(2 \leqslant j \leqslant n)$. Then

$$\mathcal{N}(T) = 2^{i_1} \times \ldots \times 2^{i_{n-1}} = 2^{i_1+\ldots+i_{n-1}}.$$

Since $i_1 \leqslant \ldots \leqslant i_n$, by the use of Lemma 2.3, we can conclude that

$$|(\mathrm{LFV}\mathcal{B})_n| = \sum_{i_1=1}^{n} \sum_{i_2=i_1}^{n} \cdots \sum_{i_n=i_{n-1}}^{n} 2^{i_1+\ldots+i_{n-1}}.$$

□

Example 2.5. The representation matrix of a linear preserver of 3×1 Ferrers vectors according to Theorem 2.2 has one of the following 172 forms:

$$T_1 = \begin{bmatrix} 1 & \alpha_{12} & \alpha_{13} \\ 0 & 1 & \alpha_{23} \\ 0 & \alpha_{32} & \alpha_{33} \end{bmatrix}, \quad \{\alpha_{12}, \alpha_{13}, \alpha_{23}, \alpha_{32}, \alpha_{33}\} \in \{0,1\}, \quad n(T_1) = 2^5 = 32,$$

$$T_2 = \begin{bmatrix} 1 & \alpha_{12} & \alpha_{13} \\ 0 & 0 & 1 \\ 0 & 0 & \alpha_{33} \end{bmatrix}, \quad \{\alpha_{12}, \alpha_{13}, \alpha_{33}\} \in \{0,1\}, \quad n(T_2) = 2^3 = 8,$$

$$T_3 = \begin{bmatrix} 1 & \alpha_{12} & \alpha_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \{\alpha_{12}, \alpha_{13}\} \in \{0,1\}, \quad n(T_3) = 2^2 = 4,$$

$$T_4 = \begin{bmatrix} 1 & \alpha_{12} & \alpha_{13} \\ 1 & \alpha_{22} & \alpha_{23} \\ 0 & \alpha_{32} & \alpha_{33} \end{bmatrix}, \quad \{\alpha_{12}, \alpha_{13}, \alpha_{22}, \alpha_{23}, \alpha_{32}, \alpha_{33}\} \in \{0,1\}, \quad n(T_4) = 2^6 = 64,$$

$$T_5 = \begin{bmatrix} 1 & \alpha_{12} & \alpha_{13} \\ 1 & \alpha_{22} & \alpha_{23} \\ 1 & \alpha_{32} & \alpha_{33} \end{bmatrix}, \quad \{\alpha_{12}, \alpha_{13}, \alpha_{22}, \alpha_{23}, \alpha_{32}, \alpha_{33}\} \in \{0,1\}, \quad n(T_5) = 2^6 = 64.$$

Also, by the use of Theorem 2.4 we observe that

$$
\begin{aligned}
|(\mathrm{LFV}\mathcal{B})_3| &= \sum_{i_1=1}^{3}\sum_{i_2=i_1}^{3}\sum_{i_3=i_2}^{3} 2^{i_1+i_2} = \sum_{i_2=1}^{3}\sum_{i_3=i_2}^{3} 2^{1+i_2} + \sum_{i_2=2}^{3}\sum_{i_3=i_2}^{3} 2^{2+i_2} + 2^{3+3} \\
&= \sum_{i_3=1}^{3} 2^{1+1} + \sum_{i_3=2}^{3} 2^{1+2} + 2^{1+3} + \sum_{i_3=2}^{3} 2^{2+2} + 2^{2+3} + 2^{3+3} \\
&= 3 \times 2^2 + 2 \times 2^3 + 2^4 + 2 \times 2^4 + 2^5 + 2^6 = 172.
\end{aligned}
$$

Example 2.6. The MATLAB code for calculating $|(\mathrm{LFV}\mathcal{B})_n|$ is as follows for every n:

```
function S=prj_gen(n)
h=fopen('prog.m','w');
fprintf(h,'function S=prog(n)\n');
fprintf(h,'S=0;\n');
for i=1:n
  if i==1
    ind='1';
  else
    ind=['i' num2str(i-1)];
  end
  fprintf(h,'for i%d=%s:n\n',i,ind);
end
fprintf(h,'S=S+2^(');
for i=1:n-1
  if i~=n-1
    pow=['i' num2str(i) '+'];
  else
    pow=['i' num2str(i)];
  end
  fprintf(h,pow);
end
fprintf(h,');\n');
for i=1:n
  fprintf(h,'end \n');
end
fclose(h);
S=prog(n);
```

```
function S=prog(n)
S=0;
  for i1=1:n
    for i2=i1:n
      for i3=i2:n
        for i4=i3:n
          for i5=i4:n
            for i6=i5:n
              for i7=i6:n
                for i8=i7:n
                  for i9=i8:n
                    for i10=i9:n
                      for i11=i10:n
                        for i12=i11:n
                          S=S+2^(i1+i2+i3+i4+i5+i6+i7+i8+i9+i10+i11);
                        end
                      end
                    end
                  end
                end
              end
            end
          end
        end
      end
    end
  end
clc
clear
  for n=2:10
  S=prj_gen(n);
  disp(' ')
  disp(['n = ' num2str(n)])
  disp(['|LFV' num2str(n) '(B)| = ' num2str(S)])
end
```

n	$\lvert(\mathrm{LFV}\mathcal{B})_n)\rvert$
2	8
3	172
4	12528
5	3412496
6	3604201088
7	14993777471936
8	2.475778594880996 e + 17
9	1.628864745018387 e + 22
10	4.27831114511251 e + 27

3. Linear preserver of $n \times 1$ Ferrers vectors over $\mathbb{Z}_2$

In this section, we consider the Boolean ring $\mathbb{Z}_2 = \{0, 1\}$ such that $1 + 1 = 0$. We find all linear preservers of $n \times 1$ Ferrers vectors over $\mathbb{Z}_2$.

For a matrix A, let $a_i^{(k)} = \sum\limits_{j=1}^{k} a_{ij}$. The following theorem characterizes the structure of linear operators preserving Ferrers vectors over $\mathbb{Z}_2$.

Theorem 3.1. *Let $T\colon \mathbb{Z}_2^n \to \mathbb{Z}_2^n$ be a linear operator. Let $A = [a_{ij}]$ be the representation matrix of T with respect to $\{e_1, \ldots, e_n\}$. Then T is a linear preserver of $n \times 1$ Ferrers vectors if and only if the following conditions hold:*

(1) $a_{11} = 1$ *and* $a_{12} = \ldots = a_{1n} = 0$,

(2) *for every* $1 \leqslant i \leqslant n - 1$ *and* $1 \leqslant k \leqslant n$, *if* $a_i^{(k)} = 0$, *then* $a_{i+1}^{(k)} = 0$.

Proof. Let T be a linear preserver of $n \times 1$ Ferrers vectors. We first show that $a_{11} = 1$ and $a_{1j} = 0$ for all $1 \leqslant j \leqslant n$. By the assumption, there exists $1 \leqslant j_1 \leqslant n$ such that $Av(1) = v(j_1)$, so $a_{11} = 1$. Also, there exists $1 \leqslant j_2 \leqslant n$ such that $Av(2) = v(j_2)$, so $(Av(2))_1 = 1$, which implies $1 + a_{12} = 1$ and hence $a_{12} = 0$.

By induction we show that $a_{12} = \ldots = a_{1n} = 0$. Let $a_{12} = \ldots = a_{1n-1} = 0$. For $Av(n)$, there exists $1 \leqslant j_n \leqslant n$ such that $Av(n) = v(j_n)$. Consequently, $1 + 0 + \ldots + 0 + a_{1n} = 1$ and so $a_{1n} = 0$.

Now, we prove that for all $1 \leqslant k \leqslant n$, $a_i^{(k)} = 0$ implies that $a_{i+1}^{(k)} = 0$. Assume if possible that there exists $1 \leqslant l \leqslant n$ such that $a_i^{(l)} = 0$ but $a_{i+1}^{(l)} \neq 0$. Consider the Ferrers vector $v(l) = (1, \ldots, 1, 0, \ldots, 0)^\top$. So $(Av(l))_i = 0$, but $(Av(l))_{i+1} = 1$ and this is a contradiction.

Conversely, suppose that $a_{11} = 1$ and $a_{12} = \ldots = a_{1n} = 0$ and $a_i^{(k)} = 0$ for all $1 \leqslant k \leqslant n$ implies that $a_{i+1}^{(k)} = 0$. We show that A is a linear preserver of $n \times 1$

Ferrers vectors on $\mathbb{Z}_2$. Let $1 \leqslant k \leqslant n$ be arbitrary and suppose that there exists $1 \leqslant i \leqslant n$ such that $(Av(k))_i = 0$. We show that $(Av(k))_{i+1} = 0$.

Since $(Av(k))_i = a_{i1} + a_{i2} + \ldots + a_{ik} = 0$, we have $a_i^k = \sum_{j=1}^{k} a_{ij} = 0$. Then by the assumption $a_i^{(k)} = 0$, so $a_{i+1}^{(k)} = 0$. Since $a_{i+1}^k = (Av(k))_{i+1}$, we have $(Av(k))_{i+1} = 0$. Therefore, A is a linear preserver of $n \times 1$ Ferrers vectors on $\mathbb{Z}_2$. □

Lemma 3.2. *Let* $|(\mathrm{LFV}\mathbb{Z}_2)_n|$ *be the cardinal number of* $(\mathrm{LFV}\mathbb{Z}_2)_n$*. Then*

$$|(\mathrm{LFV}\mathbb{Z}_2)_n| = n^n.$$

Proof. Let T be an arbitrary linear preserver of $n \times 1$ Ferrers vectors on $\mathbb{Z}_2$. Now, consider all possible choices for T. We have $T((\mathrm{FV}\mathbb{Z}_2)_n) \subseteq (\mathrm{FV}\mathbb{Z}_2)_n$ and hence

$$T(v(1)) = v(i_1), \quad T(v(2)) = v(i_2), \ldots, \quad T(v(n)) = v(i_n),$$

where $\{i_1, i_2, \ldots, i_n\} \subseteq \{1, 2, \ldots, n\}$. Since $e_j = v(j-1) + v(j)$, we have

$$T(e_j) = T(v(j-1)) + T(v(j)). \tag{3.1}$$

So, for every $1 \leqslant j \leqslant n$, $T(e_j)$ is completely determined by $T(v(1)), \ldots, T(v(n))$. Thus, for finding a linear preserver T over $\mathbb{Z}_2$ we need to know $T(v(1)), \ldots, T(v(n))$. If T is a linear preserver, then for $1 \leqslant j \leqslant n$ there exists $1 \leqslant i_j \leqslant n$ such that $T(v(j)) = v(i_j)$. Now, by the use of multiplication principle, we have

$$|(\mathrm{LFV}\mathbb{Z}_2)_n| = n^n.$$

□

Example 3.3. For every $1 \leqslant n \leqslant 10$ we compare $|(\mathrm{LFV}\mathbb{Z}_2)_n|$ and $|(\mathrm{LFV}\mathcal{B})_n|$ in the following table:

n	$\|(\mathrm{LFV}\mathcal{B})_n\|$	$\|(\mathrm{LFV}\mathbb{Z}_2)_n\|$
2	8	
2	8	4
3	172	27
4	12528	256
5	3412496	3125
6	3604201088	46656
7	14993777471936	823543
8	2.475778594880996 e + 17	16777216
9	1.628864745018387 e + 22	387420489
10	4.2731114511251 e + 27	1 e + 10

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