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Czechoslovak Mathematical Journal, Vol. 73 (2023), No. 4, 1201–1217

Persistent URL: http://dml.cz/dmlcz/151955

Czechoslovak Mathematical Journal, 73 (148) (2023), 1201–1217

SOBOLEV TYPE INEQUALITIES FOR FRACTIONAL MAXIMAL FUNCTIONS AND RIESZ POTENTIALS IN MORREY SPACES OF VARIABLE EXPONENT ON HALF SPACES

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Received October 4, 2022. Published online May 24, 2023.

Abstract. Our aim is to establish Sobolev type inequalities for fractional maximal functions $M_{\mathbb{H},\nu}f$ and Riesz potentials $I_{\mathbb{H},\alpha}f$ in weighted Morrey spaces of variable exponent on the half space $\mathbb{H}$. We also obtain Sobolev type inequalities for a C^1 function on $\mathbb{H}$. As an application, we obtain Sobolev type inequality for double phase functionals with variable exponents $\Phi(x,t)=t^{p(x)}+(b(x)t)^{q(x)}$, where $p(\cdot)$ and $q(\cdot)$ satisfy log-Hölder conditions, $p(x)<q(x)$ for $x\in\mathbb{H}$, and $b(\cdot)$ is nonnegative and Hölder continuous of order $\theta\in(0,1]$.

Keywords: variable exponent; fractional maximal function; Riesz potential; Sobolev's inequality; weighted Morrey space; double phase functional

MSC 2020: 46E30, 42B25, 31B15

1. Introduction

The well-known theorem by Hardy and Littlewood says that the maximal operator is bounded in $L^p(\mathbb{R}^n)$ when $p>1$. The central Hardy-Littlewood maximal function Mf on $\mathbb{R}^n$ is defined by

$$Mf(x)=\sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|\,\mathrm{d}y,$$

where $B(x,r)$ is the ball in $\mathbb{R}^n$ centered at x of radius $r>0$ and $|B(x,r)|$ denotes its Lebesgue measure. When we consider the maximal function on an open set $G\subset\mathbb{R}^n$ defined by

$$M_Gf(x)=\sup_{\{r>0\colon B(x,r)\subset G\}}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|\,\mathrm{d}y,$$

DOI: 10.21136/CMJ.2023.0442-22

the boundedness of the maximal operator $f \to M_G f$ is closely related to a class of functions f on G and the size of the boundary; M_G is sometimes referred to as the local maximal operator, see [16].

In this paper, we are concerned with weighted Lebesgue spaces consisting of functions f on the half space $\mathbb{H} = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}^1 \colon x_n > 0\}$ satisfying

$$\sup_{r>0, x\in\mathbb{H}} \frac{r^\sigma}{|B(x,r)|} \int_{\mathbb{H}\cap B(x,r)} (|f(y)| y_n^\beta)^p \, dy < \infty$$

for constants $\sigma \geqslant 0$, β and $p > 1$; in what follows, p is extended to a variable exponent of log-Hölder continuity. Since the weight $\omega(y) = |y_n|^{\beta p}$ is not the Muckenhoupt A_p weight when β is not in the interval $(-1/p, 1-1/p)$, we need to modify the maximal functions. In fact, it may occur that Mf is identically equal to ∞ in $\mathbb{H}$; see [27], Remarks 2.4 and 2.9 for details.

For this purpose, let us consider the fractional central Hardy-Littlewood maximal function $M_{\mathbb{H},\nu} f$ defined by

$$M_{\mathbb{H},\nu} f(x) = \sup_{\{r>0\colon B(x,r)\subset\mathbb{H}\}} \frac{r^\nu}{|B(x,r)|} \int_{B(x,r)} |f(y)| \, dy$$

for a locally integrable function f on $\mathbb{H}$ with $0 \leqslant \nu \leqslant n$. The mapping $f \mapsto M_{\mathbb{H},\nu} f$ is called the *fractional central maximal operator*. When $\nu = 0$, $M_{\mathbb{H},0} f = M_{\mathbb{H}} f$.

Variable exponent Lebesgue spaces and Sobolev spaces have been intensely studied to analyze nonlinear partial differential equations with a nonstandard growth condition by many mathematicians. For the progress in this field, see [10], [12]. Capone, Cruz-Uribe and Fiorenza in [7] studied the boundedness of the usual fractional maximal operator $M_\nu f$ in $L^{p(\cdot)}(\mathbb{R}^n)$. Cruz-Uribe, Fiorenza and Neugebauer in [11] proved the boundedness of the maximal operator Mf in variable weighted Lebesgue spaces when the weight is an $A_{p(\cdot)}$ weight. There are many related results; see [3], [13], [17], [18], [20], [25], [28], [30], [32] and so on.

Recently, we studied the boundedness of $M_{\mathbb{H},\nu} f$ in weighted Morrey spaces; see [27], Theorem 2.1:

Theorem A. *Let $1/p^* = 1/p - \nu/\sigma > 0$, $0 < \sigma < \frac{1}{2}(n+1)$. Suppose $\beta < (n+1)/(2p')$, where $1/p + 1/p' = 1$. Then there exists a constant $C > 0$ such that*

$$\sup_{\{r>0\colon B(x,r)\subset\mathbb{H}\}} \frac{r^\sigma}{|B(x,r)|} \int_{B(x,r)} (z_n^\beta M_{\mathbb{H},\nu} f(z))^{p^*} \, dz \leqslant C$$

when

$$\sup_{r>0, x\in\mathbb{H}} \frac{r^\sigma}{|B(x,r)|} \int_{\mathbb{H}\cap B(x,r)} (|f(y)| y_n^\beta)^p \, dy \leqslant 1.$$

Our first aim in this paper is to extend Theorem A to weighted Morrey spaces of variable exponent, see Theorem 3.1 below. Our result gives the Sobolev type inequality for $M_{\mathbb{H},\nu}f$.

In relation to $M_{\mathbb{H},\nu}f$, we consider the Riesz potential of order α on $\mathbb{H}$ defined by

$$I_{\mathbb{H},\alpha}f(x) = \int_{B(x,x_n)} |x-y|^{\alpha-n} f(y)\,\mathrm{d}y$$

for $0<\alpha<n$ and $f \in L^1_{\rm loc}(\mathbb{H})$. When studying a C^1 function u on $\mathbb{H}$, we can apply the representation formula

$$\begin{aligned}(1.1)\qquad \biggl| u(x) - \frac{1}{|B(x,r)|}\int_{B(x,r)} u(y)\,\mathrm{d}y \biggr| &\leqslant \frac{1}{n|B(0,1)|}\int_{B(x,r)} |z-x|^{1-n}|\nabla u(z)|\,\mathrm{d}z \\ &\leqslant \frac{1}{n|B(0,1)|} I_{\mathbb{H},1}|\nabla u|(x)\end{aligned}$$

for $B(x,r)\subset\mathbb{H}$; for its proof we refer to [14], (26).

In Theorem 3.4 of [29] we proved a Sobolev type inequality for $I_{\mathbb{H},\alpha}f$ in weighted Morrey spaces:

Theorem B. *Let $1/p^* = 1/p - \alpha/\sigma > 0$. Suppose $\beta < (n+1)/(2p')$ and $0<\sigma<\frac{1}{2}(n+1)$. Then there exists a constant $C>0$ such that*

$$\sup_{\{r>0\colon B(x,r)\subset\mathbb{H}\}} \frac{r^\sigma}{|B(x,r)|}\int_{B(x,r)} (z_n^\beta I_{\mathbb{H},\alpha}f(z))^{p^*}\,\mathrm{d}z \leqslant C$$

when $f\geqslant 0$ is such that

$$\sup_{r>0,x\in\mathbb{H}} \frac{r^\sigma}{|B(x,r)|}\int_{\mathbb{H}\cap B(x,r)} (|f(y)|y_n^\beta)^p\,\mathrm{d}y \leqslant 1.$$

Applying the technique by Hedberg (see [2]) and the arguments expanded in the proof of Theorem 3.1 below, we prove a Sobolev type inequality for $I_{\mathbb{H},\alpha}f$ in weighted Morrey spaces of variable exponent (see Theorem 4.3) as an extension of Theorem B, see also [3], [21], [23], [25]. Thanks to (1.1), we show a Sobolev type inequality for a C^1 function on $\mathbb{H}$, see Corollary 4.4.

Regarding regularity theory of differential equations, Baroni, Colombo and Mingione in [4], [5], [8], [9] studied a double phase functional $\widehat{\Phi}(x,t)=t^p+a(x)t^q$. In [29], we studied Sobolev type inequalities for $M_{\mathbb{H},\nu}f$ and $I_{\mathbb{H},\alpha}f$ of functions in weighted

Morrey spaces of the double phase functionals $\widehat{\Phi}(x,t)$ in the constant exponent case. In the present paper, relaxing the continuity of $a(\cdot)$, let us consider the double phase functional

$$\Phi(x,t) = t^{p(x)} + (b(x)t)^{q(x)},$$

where $p(\cdot)$ and $q(\cdot)$ satisfy log-Hölder conditions, $p(x) < q(x)$ for $x \in \mathbb{H}$ and $b(\cdot)$ is nonnegative and Hölder continuous of order $\theta \in (0,1]$ (see [19]); if we write $\Phi(x,t) = t^p + a(x)t^q$ with $a(x) = b(x)^q$ when $p(\cdot) = p$, $q(\cdot) = q$ are constants, then a is not always Hölder continuous of order θq when $\theta q > 1$. As an application, we give Sobolev type inequalities for $M_{\mathbb{H},\nu}f$ and $I_{\mathbb{H},\alpha}f$ of functions f in weighted Morrey spaces of the double phase functionals $\Phi(x,t)$ (see Theorems 5.1 and 5.2 given later) as an extension of [29], Theorems 2.1 and 4.1; for related results, see [19], [23], [24]. As a corollary, we obtain Sobolev type inequalities for a C^1 function on $\mathbb{H}$ in the framework of the double phase functionals, see Corollary 5.3. We also refer to [6], [15], [22], [26], [31] for other double phase problems.

Throughout this paper, let C denote various constants independent of the variables in question. The symbol $g \sim h$ means that $C^{-1}h \leqslant g \leqslant Ch$ for some constant $C > 0$.

2. Preliminaries

In this section we recall all the definitions and preliminary results which will be useful in the sequel.

We say that $p(\cdot)$ satisfies condition (P) if

(P1) $1 < p_- := \inf_{x\in\mathbb{R}^n} p(x) \leqslant \sup_{x\in\mathbb{R}^n} p(x) =: p_+ < \infty$;

(P2) there exists a constant $c_p \geqslant 0$ such that

$$|p(x) - p(y)| \leqslant \frac{c_p}{\log(\mathrm{e} + |x-y|^{-1})} \quad \text{for } x, y \in \mathbb{R}^n;$$

(P3) there exist $p(\infty) > 1$ and $c_{p(\infty)}$ such that

$$|p(x) - p(\infty)| \leqslant \frac{c_{p(\infty)}}{\log(\mathrm{e} + |x|)} \quad \text{for } x \in \mathbb{R}^n.$$

A typical example of $p(\cdot)$ is of the form

$$p(x) := p + \frac{c_1}{\log(\mathrm{e} + |x|^{-1})} + \frac{c_2}{\log(\mathrm{e} + |x|)}$$

for some constants c_1 and c_2.

Throughout this paper, we always assume that $p(\cdot)$ satisfies condition (P). Let $1/p(x)+1/p'(x)=1$ and $\mathbb{H}$ be the half space, that is,

$$\mathbb{H}=\{x=(x',x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}^1\colon\ x_n>0\}.$$

Let G be an open set in $\mathbb{R}^n$. For σ with $0<\sigma\leqslant n$, the Morrey space $L^{p(\cdot),\sigma}(G)$ of variable exponent is defined by

$$L^{p(\cdot),\sigma}(G)=\Big\{f\in L^1_{\rm loc}(G)\colon \sup_{B(x,r)\subset G}\frac{r^\sigma}{|B(x,r)|}\int_{B(x,r)}\Big(\frac{|f(y)|}{\lambda}\Big)^{p(y)}\,\mathrm{d}y<\infty \quad \text{for some } \lambda>0\Big\}.$$

It is a Banach space with respect to the norm

$$\|f\|_{L^{p(\cdot),\sigma}(G)}=\inf\Big\{\lambda>0\colon \sup_{B(x,r)\subset G}\frac{r^\sigma}{|B(x,r)|}\int_{B(x,r)}\Big(\frac{|f(y)|}{\lambda}\Big)^{p(y)}\,\mathrm{d}y\leqslant 1\Big\}.$$

It is convenient to see that

$$\|f\|_{L^{p(\cdot),\sigma}(G)}\leqslant 1\Leftrightarrow \sup_{B(x,r)\subset G}\frac{r^\sigma}{|B(x,r)|}\int_{B(x,r)}(|f(y)|)^{p(y)}\,\mathrm{d}y\leqslant 1. \tag{2.1}$$

Let us recall some lemmas, which will be useful in the sequel.

Lemma 2.1 ([27], Lemma 2.3). *For $\varepsilon>\frac12(n-1)$, set*

$$I(x)=\int_{B(x,x_n)}y_n^{\varepsilon-n}\,\mathrm{d}y.$$

Then there exists a constant $C>0$ such that

$$I(x)\leqslant Cx_n^\varepsilon.$$

Lemma 2.2 ([27], Lemma 2.5). *For $\varepsilon<\frac12(n-1)$, set*

$$J(y)=\int_{\{x\in\mathbb{H}\colon\ |x-y|<x_n\}}x_n^{\varepsilon-n}\,\mathrm{d}x.$$

Then there exists a constant $C>0$ such that

$$J(y)\leqslant Cy_n^\varepsilon.$$

We know the following result on the boundedness of the usual fractional maximal operator M_ν in variable exponent Morrey spaces.

Lemma 2.3 ([3], Corollary 2, [13], Lemma 4)**.** *Let $1/p^*(x) = 1/p(x) - \nu/\sigma \geqslant 1/p_+ - \nu/\sigma > 0$ on $\mathbb{R}^n$. Then there exists a constant $C > 0$ such that*

$$\sup_{r>0, x\in\mathbb{R}^n} \frac{r^\sigma}{|B(x,r)|} \int_{B(x,r)} (M_\nu g(x))^{p^*(x)} \,\mathrm{d}x \leqslant C$$

when

$$\sup_{r>0, x\in\mathbb{R}^n} \frac{r^\sigma}{|B(x,r)|} \int_{B(x,r)} |g(y)|^{p(y)} \,\mathrm{d}y \leqslant 1.$$

Moreover, we need the following technical lemmas.

Lemma 2.4. *Suppose $-\beta - \sigma/p_- + n > \frac{1}{2}(n-1)$. Then for $0 < x_n < 1$,*

$$\int_{B(x,x_n)} y_n^{-\beta-\sigma/p(y)} \,\mathrm{d}y \leqslant C x_n^{-\beta-\sigma/p(x)+n}.$$

P r o o f. Since $|y - x'|^2 < 2x_n y_n < 2y_n$ and $y_n < 2x_n < 2$ for $y \in B(x, x_n)$ with $0 < x_n < 1$, we have by (P2)

$$y_n^{-1/p(y)+1/p(x')} \leqslant C y_n^{-C/\log(\mathrm{e}+|y-x'|^{-1})} \leqslant C y_n^{-C/\log(\mathrm{e}+y_n^{-1})} \leqslant C.$$

Hence, we obtain by Lemma 2.1

$$\begin{aligned}\int_{B(x,x_n)} y_n^{-\beta-\sigma/p(y)} \,\mathrm{d}y &\leqslant C \int_{B(x,x_n)} y_n^{-\beta-\sigma/p(x')} \,\mathrm{d}y \\ &\leqslant C x_n^{-\beta-\sigma/p(x')+n} \leqslant C x_n^{-\beta-\sigma/p(x)+n}\end{aligned}$$

since $-\beta - \sigma/p_- + n > \frac{1}{2}(n-1)$ and

$$x_n^{-1/p(x')-1/p(x)} \leqslant x_n^{-C/\log(\mathrm{e}+|x'-x|^{-1})} = x_n^{-C/\log(\mathrm{e}+x_n^{-1})} \leqslant C$$

by (P2), which completes the proof. $\square$

Lemma 2.5. *Suppose $-\beta - \sigma/p_- + n > \frac{1}{2}(n-1)$. Then for $x_n \geqslant 1$,*

$$\int_{B(x,x_n)} y_n^{-\beta-\sigma/p(y)} \,\mathrm{d}y \leqslant C x_n^{-\beta-\sigma/p(\infty)+n} \leqslant C x_n^{-\beta-\sigma/p(x)+n}.$$

P r o o f. Since

$$y_n^{-1/p(y)} \leqslant y_n^{-1/p(\infty)+c/\log(\mathrm{e}+|y|)} \leqslant C y_n^{-1/p(\infty)}$$

for $y_n \geqslant 1$ by (P3) and $-\beta - \sigma/p(\infty) + n > \frac{1}{2}(n-1)$, we have

$$\begin{aligned}\int_{\{y\in B(x,x_n)\colon\, y_n>1\}} y_n^{-\beta-\sigma/p(y)}\,\mathrm{d}y &\leqslant C\int_{\{y\in B(x,x_n)\colon\, y_n>1\}} y_n^{-\beta-\sigma/p(\infty)}\,\mathrm{d}y\\ &\leqslant C x_n^{-\beta-\sigma/p(\infty)+n}\end{aligned}$$

by Lemma 2.1.

Moreover, taking γ such that $\gamma < \frac{1}{2}(n+1)$ and $-\beta - \sigma/p_- + \gamma > 0$, it is

$$\begin{aligned}\int_{\{y\in B(x,x_n)\colon\, y_n<1\}} y_n^{-\beta-\sigma/p(y)}\,\mathrm{d}y &\leqslant \int_{\{y\in B(x,x_n)\colon\, y_n<1\}} y_n^{-\gamma}\,\mathrm{d}y\\ &\leqslant C x_n^{(n-1)/2}\int_0^1 r^{-\gamma+(n-1)/2}\,\mathrm{d}r\\ &\leqslant C x_n^{(n-1)/2} \leqslant C x_n^{-\beta-\sigma/p(\infty)+n}.\end{aligned}$$

Consequently,

$$\int_{B(x,x_n)} y_n^{-\beta-\sigma/p(y)}\,\mathrm{d}y \leqslant C x_n^{-\beta-\sigma/p(\infty)+n} \leqslant C x_n^{-\beta-\sigma/p(x)+n}$$

by (P3), which completes the proof. □

3. Boundedness of fractional maximal operators on the half space

In this section, we study the boundedness of the fractional central maximal operator in the weighted Morrey spaces of variable exponent.

Let us state our boundedness result in the following theorem.

Theorem 3.1. *Suppose $1/p^*(x) = 1/p(x) - \nu/\sigma \geqslant 1/p_+ - \nu/\sigma > 0$ on $\mathbb{R}^n$ and $\beta \leqslant \sigma/(p_-)' < (n+1)/(2(p_-)')$. Then there exists a constant $C > 0$ such that*

$$\sup_{\{r>0\colon\, B(x,r)\subset\mathbb{H}\}} \frac{r^\sigma}{|B(x,r)|}\int_{B(x,r)} (z_n^\beta M_{\mathbb{H},\nu} f(z))^{p^*(z)}\,\mathrm{d}z \leqslant C$$

when

$$\sup_{r>0,x\in\mathbb{H}} \frac{r^\sigma}{|B(x,r)|}\int_{\mathbb{H}\cap B(x,r)} (|f(y)|y_n^\beta)^{p(y)}\,\mathrm{d}y \leqslant 1.$$

This implies that

$$\|\omega M_{\mathbb{H},\nu} f\|_{L^{p^*(\cdot),\sigma}(\mathbb{H})} \leqslant C\|\omega f \chi_{\mathbb{H}}\|_{L^{p(\cdot),\sigma}(\mathbb{R}^n)},$$

where $\omega(x) = |x_n|^\beta$ for $x \in \mathbb{R}^n$ and χ_E denotes the characteristic function of $E \subset \mathbb{R}^n$.

Remark 3.2. One sees that ω^p is the Muckenhoupt A_p weight if and only if $-1/p < \beta < 1-1/p$. Even though β is not in the interval $(-1/p, 1-1/p)$, Theorem 3.1 is applicable for β such that $\beta p/(p-1) \leqslant \sigma < \frac{1}{2}(n+1)$.

To prove Theorem 3.1, first note that

$$\begin{aligned} K_1(x) &= \sup_{0<r<x_n/2} \frac{r^\nu}{|B(x,r)|} \int_{B(x,r)} |f(y)| \,\mathrm{d}y \\ &\leqslant \sup_{0<r<x_n/2} C x_n^{-\beta} \frac{r^\nu}{|B(x,r)|} \int_{B(x,r)} |f(y)| y_n^\beta \,\mathrm{d}y \leqslant C x_n^{-\beta} M_\nu g(x), \end{aligned}$$

where $g(y) = |f(y)||y_n|^\beta \chi_{\mathbb{H}}(y)$.

Next note that

$$\begin{aligned} K_2(x) &= \sup_{x_n/2 \leqslant r < x_n} \frac{r^\nu}{|B(x,r)|} \int_{B(x,r)} |f(y)| \,\mathrm{d}y \\ &\leqslant C \frac{x_n^\nu}{|B(x,x_n)|} \int_{B(x,x_n)} |f(y)| \,\mathrm{d}y = C K_3(x). \end{aligned}$$

Then

$$M_{\mathbb{H},\nu} f(x) \leqslant C x_n^{-\beta} M_\nu g(x) + C K_3(x). \tag{3.1}$$

With the aid of Lemmas 2.4 and 2.5, we can obtain the following estimate for K_3.

Lemma 3.3. *Suppose* $1/p^*(x) = 1/p(x) - \nu/\sigma \geqslant 1/p_+ - \nu/\sigma > 0$ *on* $\mathbb{R}^n$, $-\beta - \sigma/p_- + n > \frac{1}{2}(n-1)$ *and* $-\beta - \sigma/p_- + \sigma \geqslant 0$. *Then there exists a constant* $C > 0$ *such that*

$$x_n^\beta K_3(x) \leqslant C x_n^{-\sigma/p^*(x)}$$

when

$$\sup_{x\in\mathbb{H}} \frac{x_n^\sigma}{|B(x,x_n)|} \int_{B(x,x_n)} (|f(y)| y_n^\beta)^{p(y)} \,\mathrm{d}y \leqslant 1.$$

P r o o f. Set $K(y)=y_n^{-\beta-\sigma/p(y)}$. Since $-\beta-\sigma/p_-+n>\frac{1}{2}(n-1)$, we have by (P1), Lemma 2.4 and Lemma 2.5

$$
\begin{aligned}
(3.2)\qquad K_3(x) &= \frac{x_n^{\nu}}{|B(x,x_n)|}\int_{\{y\in B(x,x_n)\colon f(y)\leqslant K(y)\}}|f(y)|\,\mathrm{d}y \\
&\quad+\frac{x_n^{\nu}}{|B(x,x_n)|}\int_{\{y\in B(x,x_n)\colon f(y)>K(y)\}}|f(y)|\,\mathrm{d}y \\
&\leqslant \frac{x_n^{\nu}}{|B(x,x_n)|}\int_{B(x,x_n)}K(y)\,\mathrm{d}y \\
&\quad+\frac{x_n^{\nu}}{|B(x,x_n)|}\int_{B(x,x_n)}|f(y)|\Big(\frac{|f(y)|}{K(y)}\Big)^{p(y)-1}\,\mathrm{d}y \\
&\leqslant Cx_n^{\nu-\beta-\sigma/p(x)} \\
&\quad+C\frac{x_n^{\nu}}{|B(x,x_n)|}\int_{B(x,x_n)}y_n^{(p(y)-1)(\beta+\sigma/p(y))}|f(y)|^{p(y)}\,\mathrm{d}y.
\end{aligned}
$$

If $0<x_n<1$ and $y\in B(x,x_n)$, then by (P2)

$$y_n^{-\beta+\sigma(p(y)-1)/p(y)}\leqslant Cx_n^{-\beta+\sigma(p(y)-1)/p(y)}\leqslant Cx_n^{-\beta+\sigma(p(x)-1)/p(x)}$$

when $-\beta+\sigma(p(y)-1)/p(y)\geqslant 0$.

If $x_n\geqslant 1$, $y_n\geqslant 1$ and $y\in B(x,x_n)$, then by (P3)

$$
\begin{aligned}
y_n^{-\beta+\sigma(p(y)-1)/p(y)} &\leqslant Cy_n^{-\beta+\sigma(p(\infty)-1)/p(\infty)}\leqslant Cx_n^{-\beta+\sigma(p(\infty)-1)/p(\infty)} \\
&\leqslant Cx_n^{-\beta+\sigma(p(x)-1)/p(x)}
\end{aligned}
$$

when $-\beta+\sigma(p(\infty)-1)/p(\infty)\geqslant 0$.

If $0<y_n<1\leqslant x_n$ and $y\in B(x,x_n)$, then

$$y_n^{-\beta+\sigma(p(y)-1)/p(y)}\leqslant 1\leqslant x_n^{-\beta+\sigma(p(x)-1)/p(x)}$$

when $-\beta+\sigma(p(x)-1)/p(x)\geqslant 0$.

Hence, it follows from (3.2) that

$$
\begin{aligned}
K_3(x) &\leqslant Cx_n^{\nu-\beta-\sigma/p(x)}+Cx_n^{-\beta+\sigma(p(x)-1)/p(x)}\frac{x_n^{\nu}}{|B(x,x_n)|}\int_{B(x,x_n)}(|f(y)|y_n^{\beta})^{p(y)}\,\mathrm{d}y \\
&\leqslant Cx_n^{-\beta+\nu-\sigma/p(x)},
\end{aligned}
$$

which gives the required result. □

Now we are ready to prove Theorem 3.1.

P r o o f of Theorem 3.1. Let f be a measurable function on $\mathbb{R}^n$ such that

$$\sup_{r>0,x\in\mathbb{H}} \frac{r^\sigma}{|B(x,r)|}\int_{\mathbb{H}\cap B(x,r)} (|f(y)|y_n^\beta)^{p(y)}\,\mathrm{d}y \leqslant 1.$$

In view of (3.1), we recall

$$z_n^\beta M_{\mathbb{H},\nu} f(z) \leqslant CM_\nu g(z) + CK_3(z)z_n^\beta,$$

where $g(y) = |f(y)|y_n^\beta \chi_{\mathbb{H}}(y)$. By Lemmas 2.3, 3.3 and 2.1, we obtain for $r > 0$ such that $B(x,r) \subset \mathbb{H}$ that

$$\begin{aligned}
&\frac{r^\sigma}{|B(x,r)|}\int_{B(x,r)} (z_n^\beta M_{\mathbb{H},\nu} f(z))^{p^*(z)}\,\mathrm{d}z\\
&\quad\leqslant C\frac{r^\sigma}{|B(x,r)|}\int_{B(x,r)} (M_\nu g(z))^{p^*(z)}\,\mathrm{d}z + C\frac{x_n^\sigma}{|B(x,x_n)|}\int_{B(x,x_n)} (K_3(z)z_n^\beta)^{p^*(z)}\,\mathrm{d}z\\
&\quad\leqslant C + C\frac{x_n^\sigma}{|B(x,x_n)|}\int_{B(x,x_n)} z_n^{-\sigma}\,\mathrm{d}z \leqslant C
\end{aligned}$$

when $\sigma < \frac{1}{2}(n+1)$. Thus, the proof is completed. □

Corollary 3.4. *Suppose $1/p^*(x) = 1/p(x) - \nu/\sigma \geqslant 1/p_+ - \nu/\sigma > 0$ on $\mathbb{R}^n$, $-\beta-\sigma/p_- + n > \frac{1}{2}(n-1)$ and $-\beta-\sigma/p_- + \sigma \geqslant 0$. Then for $q \geqslant 1$ with $\sigma/q < \frac{1}{2}(n+1)$ there exists a constant $C > 0$ such that*

$$\sup_{\{r>0\colon B(x,r)\subset\mathbb{H}\}} \frac{r^{\sigma/q}}{|B(x,r)|}\int_{B(x,r)} (z_n^\beta M_{\mathbb{H},\nu} f(z))^{p^*(z)/q}\,\mathrm{d}z \leqslant C$$

when

$$\sup_{r>0,x\in\mathbb{H}} \frac{r^\sigma}{|B(x,r)|}\int_{\mathbb{H}\cap B(x,r)} (|f(y)|y_n^\beta)^{p(y)}\,\mathrm{d}y \leqslant 1.$$

For this, note from Hölder's inequality and Lemma 2.3 that

$$\frac{r^{\sigma/q}}{|B(x,r)|}\int_{B(x,r)} (M_\nu g(z))^{p^*(z)/q}\,\mathrm{d}z \leqslant \biggl(\frac{r^\sigma}{|B(x,r)|}\int_{B(x,r)} (M_\nu g(z))^{p^*(z)}\,\mathrm{d}z\biggr)^{1/q} \leqslant C$$

for $r > 0$ such that $B(x,r) \subset \mathbb{H}$. Moreover, by Lemmas 3.3 and 2.1, we find

$$\frac{x_n^{\sigma/q}}{|B(x,x_n)|}\int_{B(x,x_n)} (K_3(z)z_n^\beta)^{p^*(z)/q}\,\mathrm{d}z \leqslant C\frac{x_n^{\sigma/q}}{|B(x,x_n)|}\int_{B(x,x_n)} z_n^{-\sigma/q}\,\mathrm{d}z \leqslant C$$

when $\sigma/q < \frac{1}{2}(n+1)$.

Example 3.5. Let

$$\Gamma = \{x = (x_1, x_2)\colon\ 1 < 2x_1 < x_2 < 3x_1\}$$

and

$$p(x) = p(x_1, x_2) = \begin{cases} p_0 + \Big(\dfrac{\min\{x_1, 1\}}{\log(\mathrm{e} + x_1)}\Big)^{\theta} & \text{when } x_1 > 0; \\ p_0 & \text{when } x_1 \leqslant 0 \end{cases}$$

for $p_0 > 1$ and $0 < \theta < 1$.

Consider

$$f(y) = \begin{cases} |y|^{-2/p_0} & \text{when } y \in \Gamma; \\ 0 & \text{otherwise.} \end{cases}$$

(1) Since $(\log|y|)/(\log(\mathrm{e} + y_1))^{\theta} \geqslant C(\log|y|)^{1-\theta}$ for $y \in \Gamma$,

$$\int_{\mathbb{R}^2} f(y)^{p(y)}\,\mathrm{d}y = \int_{\Gamma} |y|^{-2} \exp\Big(-\frac{2}{p_0}(\log|y|)\Big(\frac{\min\{y_1, 1\}}{\log(\mathrm{e} + y_1)}\Big)^{\theta}\Big)\,\mathrm{d}y < \infty.$$

(2) For $x = (x_1, x_2)$ with $x_1 < 0$, $x_2 > 2$ and $-2x_1 < x_2$,

$$M_{\mathbb{H}} f(x) \geqslant C x_2^{-2} \int_{B(x, x_2)} f(y)\,\mathrm{d}y \geqslant C|x|^{-2/p_0},$$

so that

$$\int_{\mathbb{H}} \{M_{\mathbb{H}} f(x)\}^{p(x)}\,\mathrm{d}x \geqslant C \int_{\{(x_1, x_2) \in \mathbb{R}^2\colon\ x_1 < 0, x_2 > 2, -2x_1 < x_2\}} |x|^{-2}\,\mathrm{d}x = \infty.$$

This implies that (P3) is sharp in Theorem 3.1.

4. Riesz potentials

In this section, we will establish Sobolev's inequality for Riesz potentials $I_{\mathbb{H},\alpha} f$. Write

$$\begin{aligned} I_{\mathbb{H},\alpha} f(x) &= \int_{B(x, x_n/2)} |x - y|^{\alpha - n} f(y)\,\mathrm{d}y \\ &\quad + \int_{B(x, x_n) \setminus B(x, x_n/2)} |x - y|^{\alpha - n} f(y)\,\mathrm{d}y \\ &= T_1(x) + T_2(x). \end{aligned}$$

For $0 < \alpha < n$ and a locally integrable function g on $\mathbb{R}^n$, we define the usual Riesz potential $I_\alpha g$ of order α by

$$I_\alpha g(x) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n} g(y)\,\mathrm{d}y.$$

Let us recall the following result due to [21], Theorem 4.1, cf. [1].

Lemma 4.1 (Sobolev inequality for Morrey spaces). *Let $1/p^*(x) = 1/p(x) - \alpha/\sigma \geqslant 1/p_+ - \nu/\sigma > 0$ on $\mathbb{R}^n$. Then there exists a constant $C > 0$ such that*

$$\frac{r^\sigma}{|B(x,r)|} \int_{B(x,r)} |I_\alpha g(z)|^{p^*(z)}\,\mathrm{d}z \leqslant C$$

for $x \in \mathbb{R}^n$, $r > 0$ and measurable functions g on $\mathbb{R}^n$ with

$$\sup_{x\in\mathbb{R}^n, r>0} \frac{r^\sigma}{|B(x,r)|} \int_{B(x,r)} |g(y)|^{p(y)}\,\mathrm{d}y \leqslant 1.$$

Finally, in order to establish our main result, we need to prove the following lemma.

Lemma 4.2. *Suppose*

$$\sup_{x\in\mathbb{H}, r>0} \frac{r^\sigma}{|B(x,r)|} \int_{\mathbb{H}\cap B(x,r)} (|f(y)|y_n^\beta)^{p(y)}\,\mathrm{d}y \leqslant 1.$$

If $1/p^*(x) = 1/p(x) - \alpha/\sigma \geqslant 1/p_+ - \nu/\sigma > 0$ on $\mathbb{R}^n$, then there exists a constant $C > 0$ such that

$$\sup_{x\in\mathbb{H}, r>0} \frac{r^\sigma}{|B(x,r)|} \int_{\mathbb{H}\cap B(x,r)} (z_n^\beta |T_1(z)|)^{p^*(z)}\,\mathrm{d}z \leqslant C.$$

Proof. For $x \in \mathbb{H}$ we have

$$z_n^\beta |T_1(z)| \leqslant C \int_{B(z,z_n/2)} |z-y|^{\alpha-n} g(y)\,\mathrm{d}y \leqslant C I_\alpha g(z),$$

where $g(y) = |f(y)||y_n|^\beta \chi_{\mathbb{H}}(y)$. By Lemma 4.1 for $x \in \mathbb{H}$ and $r > 0$,

$$\frac{r^\sigma}{|B(x,r)|} \int_{\mathbb{H}\cap B(x,r)} (z_n^\beta |T_1(z)|)^{p^*(z)}\,\mathrm{d}z \leqslant C \frac{r^\sigma}{|B(x,r)|} \int_{B(x,r)} |I_\alpha g(z)|^{p^*(z)}\,\mathrm{d}z \leqslant C,$$

as required. $\square$

For T_2, note that

$$|T_2(x)| \leqslant \Big(\frac{x_n}{2}\Big)^{\alpha-n} \int_{B(x,x_n)} |f(y)|\,\mathrm{d}y$$

for $x \in \mathbb{H}$.

Now we are in position to prove a Sobolev type inequality for $I_{\mathbb{H},\alpha}f$.

Theorem 4.3. *Let $1/p^*(x) = 1/p(x) - \alpha/\sigma \geqslant 1/p_+ - \alpha/\sigma > 0$ on $\mathbb{R}^n$ and $\beta \leqslant \sigma/(p_-)' < (n+1)/(2(p_-)')$. Then there exists a constant $C > 0$ such that*

$$\sup_{\{r>0\colon B(x,r)\subset\mathbb{H}\}} \frac{r^\sigma}{|B(x,r)|} \int_{B(x,r)} (z_n^\beta |I_{\mathbb{H},\alpha}f(z)|)^{p^*(z)}\,\mathrm{d}z \leqslant C$$

when

$$\sup_{x\in\mathbb{H},r>0} \frac{r^\sigma}{|B(x,r)|} \int_{\mathbb{H}\cap B(x,r)} (|f(y)|y_n^\beta)^{p(y)}\,\mathrm{d}y \leqslant 1.$$

Proof. Let f be a measurable function on $\mathbb{H}$ such that

$$\sup_{x\in\mathbb{H},r>0} \frac{r^\sigma}{|B(x,r)|} \int_{\mathbb{H}\cap B(x,r)} (|f(y)|y_n^\beta)^{p(y)}\,\mathrm{d}y \leqslant 1.$$

By Lemmas 4.2, 3.3 with $\nu = \alpha$, and Lemma 2.1, we obtain for $r > 0$ such that $B(x,r) \subset \mathbb{H}$,

$$\begin{aligned}
&\frac{r^\sigma}{|B(x,r)|} \int_{B(x,r)} (z_n^\beta |I_{\mathbb{H},\alpha}f(z)|)^{p^*(z)}\,\mathrm{d}z \\
&\quad\leqslant C\frac{r^\sigma}{|B(x,r)|} \int_{B(x,r)} (z_n^\beta T_1(z))^{p^*(z)}\,\mathrm{d}z + C\frac{x_n^\sigma}{|B(x,x_n)|} \int_{B(x,x_n)} (z_n^\beta T_2(z))^{p^*(z)}\,\mathrm{d}z \\
&\quad\leqslant C + C\frac{x_n^\sigma}{|B(x,x_n)|} \int_{B(x,x_n)} z_n^{-\sigma}\,\mathrm{d}z \leqslant C
\end{aligned}$$

when $\sigma < \frac{1}{2}(n+1)$, as required. □

In view of (1.1), we find the following statement

Corollary 4.4. *Let $1/p^*(x) = 1/p(x) - 1/\sigma \geqslant 1/p_+ - 1/\sigma > 0$ on $\mathbb{R}^n$ and $\beta \leqslant \sigma/(p_-)' < (n+1)/(2(p_-)')$. Then there exists a constant $C > 0$ such that*

$$\sup_{\{r>0\colon B(x,r)\subset\mathbb{H}\}} \frac{r^\sigma}{|B(x,r)|} \int_{B(x,r)} \bigg(z_n^\beta \bigg|u(z) - \frac{1}{|B(z,z_n)|}\int_{B(z,z_n)} u(y)\,\mathrm{d}y\bigg|\bigg)^{p^*(z)}\,\mathrm{d}z \leqslant C$$

when u is a C^1 function on $\mathbb{H}$ such that

$$\sup_{x\in\mathbb{H},r>0} \frac{r^\sigma}{|B(x,r)|} \int_{\mathbb{H}\cap B(x,r)} (|\nabla u(y)|y_n^\beta)^{p(y)}\,\mathrm{d}y \leqslant 1.$$

5.1. Boundedness of fractional maximal operators. In this section, we consider the double phase functional

$$\Phi(x,t) = t^{p(x)} + (b(x)t)^{q(x)},$$

where $p(x) < q(x)$ for $x \in \mathbb{R}^n$, and $b(\cdot)$ is nonnegative and Hölder continuous of order $\theta \in (0,1]$, see [19].

By Theorem 3.1, we can obtain the following result.

Theorem 5.1. *Let $p_- > 1$, $1/q(x) = 1/p(x) - \theta/\sigma$, $1/p^*(x) = 1/p(x) - \nu/\sigma > 0$ and $1/q^*(x) = 1/q(x) - \nu/\sigma \geqslant 1/q_+ - \nu/\sigma > 0$ on $\mathbb{R}^n$. Set*

$$\Phi^*(x,t) = \Phi_{p^*,q^*}(x,t) = t^{p^*(x)} + (b(x)t)^{q^*(x)}.$$

Suppose $\beta \leqslant \sigma/(p_-)' < (n+1)/(2(p_-)')$. Then there exists a constant $C > 0$ such that

$$\sup_{\{r>0\colon B(x,r)\subset\mathbb{H}\}} \frac{r^\sigma}{|B(x,r)|}\int_{B(x,r)} \Phi^*(z, z_n^\beta M_{\mathbb{H},\nu} f(z))\,\mathrm{d}z \leqslant C \tag{5.1}$$

when

$$\sup_{x\in\mathbb{H},r>0} \frac{r^\sigma}{|B(x,r)|}\int_{\mathbb{H}\cap B(x,r)} \Phi(y, |f(y)|y_n^\beta)\,\mathrm{d}y \leqslant 1.$$

P r o o f. Let f be a measurable function on $\mathbb{H}$ such that

$$\sup_{x\in\mathbb{H},r>0} \frac{r^\sigma}{|B(x,r)|}\int_{\mathbb{H}\cap B(x,r)} \Phi(y, |f(y)|y_n^\beta)\,\mathrm{d}y \leqslant 1.$$

First we see from Theorem 3.1 that

$$\sup_{\{r>0\colon B(x,r)\subset\mathbb{H}\}} \frac{r^\sigma}{|B(x,r)|}\int_{B(x,r)} (z_n^\beta M_{\mathbb{H},\nu} f(z))^{p^*(z)}\,\mathrm{d}z \leqslant C.$$

Note that

$$\begin{aligned}
b(x)&\frac{r^\nu}{|B(x,r)|}\int_{B(x,r)} |f(y)|\,\mathrm{d}y \\
&= \frac{r^\nu}{|B(x,r)|}\int_{B(x,r)} \{b(x)-b(y)\}|f(y)|\,\mathrm{d}y + \frac{r^\nu}{|B(x,r)|}\int_{B(x,r)} b(y)|f(y)|\,\mathrm{d}y \\
&\leqslant C\frac{r^{\nu+\theta}}{|B(x,r)|}\int_{B(x,r)} |f(y)|\,\mathrm{d}y + \frac{r^\nu}{|B(x,r)|}\int_{B(x,r)} b(y)|f(y)|\,\mathrm{d}y \\
&\leqslant CM_{\mathbb{H},\nu+\theta} f(x) + M_{\mathbb{H},\nu}[bf](x)
\end{aligned}$$

when $B(x,r) \subset \mathbb{H}$. Hence, by Theorem 3.1, we establish

$$\sup_{\{r>0:\, B(x,r)\subset\mathbb{H}\}} \frac{r^\sigma}{|B(x,r)|} \int_{B(x,r)} (z_n^\beta b(z) M_{\mathbb{H},\nu} f(z))^{q^*(z)}\, \mathrm{d}z \leqslant C$$

since $1/q^*(x) = 1/q(x) - \nu/\sigma = 1/p(x) - (\nu+\theta)/\sigma$ and $1/q_- = 1/p_- - \theta/\sigma$, which completes the proof. □

5.2. Sobolev's inequality. As in the proof of Theorem 5.1, we can obtain the following theorem using Theorem 4.3. Its proof is omitted.

Theorem 5.2. *Let $1/q(x) = 1/p(x) - \theta/\sigma$, $1/p^*(x) = 1/p(x) - \alpha/\sigma > 0$ and $1/q^*(x) = 1/q(x) - \alpha/\sigma \geqslant 1/q_+ - \alpha/\sigma > 0$ on $\mathbb{R}^n$. Suppose $\beta \leqslant \sigma/(p_-)' < (n+1)/(2(p_-)')$. Then there exists a constant $C > 0$ such that*

$$\sup_{\{r>0:\, B(x,r)\subset\mathbb{H}\}} \frac{r^\sigma}{|B(x,r)|} \int_{B(x,r)} \Phi^*(z, z_n^\beta I_{\mathbb{H},\alpha} f(z))\, \mathrm{d}z \leqslant C$$

when

$$\sup_{x\in\mathbb{H},r>0} \frac{r^\sigma}{|B(x,r)|} \int_{\mathbb{H}\cap B(x,r)} \Phi(y, |f(y)|y_n^\beta)\, \mathrm{d}y \leqslant 1.$$

Corollary 5.3. *Let $1/q(x) = 1/p(x) - \theta/\sigma$, $1/p^*(x) = 1/p(x) - 1/\sigma > 0$ and $1/q^*(x) = 1/q(x) - 1/\sigma \geqslant 1/q_+ - 1/\sigma > 0$ on $\mathbb{R}^n$. Suppose $\beta \leqslant \sigma/(p_-)' < (n+1)/(2(p_-)')$. Then there exists a constant $C > 0$ such that*

$$\sup_{\{r>0:\, B(x,r)\subset\mathbb{H}\}} \frac{r^\sigma}{|B(x,r)|} \int_{B(x,r)} \Phi^*\left(z, z_n^\beta \left| u(z) - \frac{1}{|B(z,z_n)|} \int_{B(z,z_n)} u(y)\, \mathrm{d}y \right|\right) \mathrm{d}z \leqslant C$$

when

$$\sup_{x\in\mathbb{H},r>0} \frac{r^\sigma}{|B(x,r)|} \int_{\mathbb{H}\cap B(x,r)} \Phi(y, |\nabla u(y)|y_n^\beta)\, \mathrm{d}y \leqslant 1.$$

Acknowledgments. We would like to express our thanks to the referee for his/her suggestions.

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