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ON PAIRS OF GOLDBACH-LINNIK EQUATIONS WITH UNEQUAL POWERS OF PRIMES

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Abstract. It is proved that every pair of sufficiently large odd integers can be represented by a pair of equations, each containing two squares of primes, two cubes of primes, two fourth powers of primes and 105 powers of 2.

Keywords: Goldbach-Waring-Linnik problem; circle method; powers of 2

MSC 2020: 11P32, 11P05, 11P55

1. Introduction and main result

The Goldbach conjecture is one of the most famous problems and there are many variations derived from the conjecture. In 1951, Linnik in [6] proved under the assumption of the Generalized Riemann Hypothesis (GRH) that every large even integer N can be written as the sum of two primes and finite number of powers of 2, and later in 1953 he proved this conjecture unconditionally (see [7]), that is

$$N = p_1 + p_2 + 2^{v_1} + 2^{v_2} + \ldots + 2^{v_k}. \tag{1.1}$$

The explicit value of the number k was first obtained by Liu, Liu and Wang (see [10]), in which $k = 54\,000$ is acceptable. Afterwards, several researchers improved the value of k and in 2002 Heath-Brown and Puchta in [1] showed that $k = 13$ and under the GRH $k = 7$. Pintz and Ruzsa in [12] established the unconditional result which states that $k = 8$. In 2017, Liu in [9] firstly considered that every sufficiently large even integer can be written as a sum of two squares of primes, two cubes of primes, two fourth powers of primes and a bounded number of powers of 2,

$$N = p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^4 + p_6^4 + 2^{v_1} + \ldots + 2^{v_k}. \tag{1.2}$$

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He obtained $k = 41$. In 2019, Lü in [9] improved the value 41 to 24, and then Zhao in [14] improved the result $k = 24$ of Lü to 22. In 2012, Kong in [2] stated a simultaneous version of the Goldbach-Linnik problem. Kong in [2] proved that the simultaneous equations

$$N_1 = p_1 + p_2 + 2^{v_1} + 2^{v_2} + \ldots + 2^{v_k}, \quad N_2 = p_3 + p_4 + 2^{v_1} + 2^{v_2} + \ldots + 2^{v_k} \tag{1.3}$$

are solvable for every pair of sufficiently large positive even integers N_1, N_2 satisfying $N_2 \gg N_1 \geqslant N_2$ for $k = 63$ in general and $k = 31$ under the GRH. Then the result was improved by Kong and Liu (see [3]) in 2017, who showed that the simultaneous equations (1.2) can be solvable for $k = 34$ unconditionally and for $k = 18$ under the GRH. In this paper, we will consider the simultaneous representation of pairs of positive integers $N_2 \gg N_1 > N_2$ in the form

$$\begin{aligned} N_1 &= p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^4 + p_6^4 + 2^{v_1} + 2^{v_2} + \ldots + 2^{v_k}, \\ N_2 &= p_7^2 + p_8^2 + p_9^3 + p_{10}^3 + p_{11}^4 + p_{12}^4 + 2^{v_1} + 2^{v_2} + \ldots + 2^{v_k}, \end{aligned} \tag{1.4}$$

where k is a positive integer. Our result is stated as follows.

Theorem 1.1. *For $k = 105$, the equations* (1.4) *are solvable for every pair of sufficiently large positive even integers N_1 and N_2 satisfying $N_2 \gg N_1 > N_2$.*

The proof of our result employs the Hardy-Littlewood circle method in combination with some new methods of Lü, see [11].

2. Notations and some preliminary lemmas

We adopt the notations of [9]. Assume that N is a sufficiently large even integer and set

$$U_i = \sqrt{(1-\eta)N_i}, \quad V_i = (\eta N_i/2)^{1/3}, \quad W_i = (\eta N_i/2)^{1/4}, \tag{2.1}$$

and

$$L = \frac{\log(N_i/\log N_i)}{\log 2},$$

where η is a positive constant $< 10^{-10}$. Let

$$R(N_1, N_2) = \sum \log p_1 \log p_2 \ldots \log p_{12}$$

be the weighted number of solutions of (1.1) in $(p_1, p_2, \ldots p_{12}, v_1, v_2, \ldots, v_k)$ with

$$\begin{gathered} p_1 \sim U_1, \quad p_2 \sim U_1, \quad p_3 \sim V_1, \quad p_4 \sim V_1, \quad p_5 \sim W_1, \quad p_6 \sim W_1, \\ p_7 \sim U_2, \quad p_8 \sim U_2, \quad p_9 \sim V_2, \quad p_{10} \sim V_2, \quad p_{11} \sim W_2, \quad p_{12} \sim W_2, \end{gathered}$$

$4 \leqslant v_j \leqslant L$ for $j = 1, 2, \ldots, k$. For this purpose, we introduce

$$f(\alpha_i, U_i) = \sum_{p \sim U_i} (\log p) e(p^2 \alpha_i), \quad g(\alpha_i, V_i) = \sum_{p \sim V_i} (\log p) e(p^3 \alpha_i), \tag{2.2}$$

$$h(\alpha_i, W_i) = \sum_{p \sim W_i} (\log p) e(p^4 \alpha_i), \quad G(\alpha_i) = \sum_{4 \leqslant v \leqslant L} e(2^v \alpha_i), \tag{2.3}$$

$$\mathcal{E}_\lambda := \{\alpha_i \in [0,1] \colon \ |G(\alpha_i)| \geqslant \lambda L\}, \tag{2.4}$$

where $e(x) := \exp(2\pi i x)$ and $i = 1, 2$. By orthogonality, we have

$$\begin{aligned} R(N_1, N_2) = \int_0^1 \int_0^1 & f^2(\alpha_1, U_1) g^2(\alpha_1, V_1) h^2(\alpha_1, W_1) f^2(\alpha_2, U_2) g^2(\alpha_2, V_2) \\ & \times h^2(\alpha_2, W_2) G^k(\alpha_1 + \alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) \, d\alpha_1 \, d\alpha_2. \end{aligned} \tag{2.5}$$

Let

$$P_i = N_i^{3/20 - 2\varepsilon}, \quad Q_i = N_i^{17/20 + \varepsilon}$$

and define the major arcs M_i and minor arcs $C(M_i)$ as

$$M_i := \bigcup_{1 \leqslant q \leqslant P_i} \bigcup_{\substack{1 \leqslant a_i \leqslant q_i \\ (a_i, q_i) = 1}} M(a_i, q_i), \quad C(M_i) = \Big[\frac{1}{Q_i}, 1 + \frac{1}{Q_i}\Big] \setminus M_i, \tag{2.6}$$

$$M_i(a_i, q_i) = \Big\{\alpha_i \in [0,1] \colon \ \Big|\alpha_i - \frac{a_i}{q_i}\Big| \leqslant \frac{1}{q_i Q_i}\Big\}.$$

It follows from $2P_i \leqslant Q_i$ that the major arcs $M_1(a_1, q_1)$ and $M_2(a_2, q_2)$ are mutually disjoint, respectively. Then $R(N_1, N_2)$ can be written as

$$\begin{aligned} R(N_1, N_2) &= \int_0^1 \int_0^1 f^2(\alpha_1, U_1) g^2(\alpha_1, V_1) h^2(\alpha_1, W_1) f^2(\alpha_2, U_2) g^2(\alpha_2, V_2) \\ &\quad \times h^2(\alpha_2, W_2) G^k(\alpha_1 + \alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) \, d\alpha_1 \, d\alpha_2 \\ &= \Big\{\int_{M_1} + \int_{C(M_1) \cap \mathcal{E}_\lambda} + \int_{C(M_1) \setminus \mathcal{E}_\lambda}\Big\} \Big\{\int_{M_2} + \int_{C(M_2) \cap \mathcal{E}_\lambda} + \int_{C(M_2) \setminus \mathcal{E}_\lambda}\Big\} \\ &\quad \times f^2(\alpha_1, U_1) g^2(\alpha_1, V_1) h^2(\alpha_1, W_1) f^2(\alpha_2, U_2) g^2(\alpha_2, V_2) \\ &\quad \times h^2(\alpha_2, W_2) G^k(\alpha_1 + \alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) \, d\alpha_1 \, d\alpha_2 \\ &:= \sum_{s=1}^{3} \sum_{s=1}^{3} R_{st}(N_1, N_2), \end{aligned} \tag{2.7}$$

where $R_{st}(N_1, N_2)$ denotes the combination of the sth term in the first bracket and the tth term in the second bracket. We will establish Theorem 1.1 by estimating the term $R_{st}(N_1, N_2)$ for all $1 \leqslant s, t \leqslant 3$. We need to show that $R(N_1, N_2) > 0$ for every pair of sufficiently large even positive integers $N_2 \geqslant N_1 > N_2$.

We need the following lemmas to prove Theorem 1.1. For the Dirichlet character χ and q, let

$$C_k(\chi,a)=\sum_{h=1}^{q}\overline{\chi}(h)e\Big(\frac{ah^k}{q}\Big),\quad C_k(q,a)=\sum_{\substack{a=1\\(m,q)=1}}^{q}e\Big(\frac{am^k}{q}\Big),$$

where $k=2,3,4$. If χ_1, χ_2, χ_3, χ_4, χ_5 and χ_6 are characters $\bmod q$, then we write

$$\begin{aligned}(2.8)\quad & B(n,q;\chi_1,\chi_2,\chi_3,\chi_4,\chi_5,\chi_6)\\ &:=\sum_{\substack{a=1\\(a,q)=1}}^{q}C_2(\chi_1,a)C_2(\chi_2,a)C_3(\chi_3,a)C_3(\chi_4,a)C_4(\chi_5,a)C_4(\chi_6,a)e\Big(-\frac{an}{q}\Big),\end{aligned}$$

$$B(n,q)=B(n,q;\chi^0,\chi^0,\chi^0,\chi^0,\chi^0,\chi^0),$$

$$A(n,q)=\frac{B(n,q)}{\varphi^6(q)},\quad \mathfrak{S}(n)=\sum_{q=1}^{\infty}A(n,q).$$

Lemma 2.1. *We have*

$$\operatorname{meas}(\mathcal{E}_\lambda)\ll N_i^{-E(\lambda)}$$

with $E(\lambda)>17/24+10^{-10}$, *where* $\lambda=0.862\,654$.

P r o o f. We follow the procedure of Heath-Brown and Puchta and take $\xi=1.15$ and $h=23$ in it, see [1] for more detail. □

Remark 2.1. The algorithm by Pintz and Ruzsa is more efficient than the argument by Heath-Brown and Puchta. This algorithm has been implemented in full generality by Languasco and Zaccagnini, hower it only improves the remainder term and there is no influence on the result of the article, we omit the specific calculations and one can see Lemma 5 in [5] for further details.

Lemma 2.2. *Let* M_i *be as in* (2.6). *Then for* $2\leqslant n\leqslant N_i$, *we have*

$$\int_{M_i}f^2(\alpha_i,U_i)g^2(\alpha_i,V_i)h^2(\alpha_i,W_i)e(-\alpha_i n)\,\mathrm{d}\alpha=\frac{1}{2^2\cdot3^2\cdot4^2}\mathfrak{S}(n)J(n)+O(N_i^{7/6}L^{-1}).$$

Here $\mathfrak{S}(n)$ *is the singular series, which is defined as* (2.8) *and satisfies* $\mathfrak{S}(n)\gg1$ *for* $n\equiv0\pmod 2$. *Further* $J(n)$ *is defined as*

$$J(n):=\sum_{\substack{m_1+\ldots+m_6=n\\(U_i/2)^2<m_1,m_2\leqslant U_i\\(V_i/2)^2<m_3,m_4\leqslant V_i\\(W_i/2)^2<m_5,m_6\leqslant W_i}}(m_1m_2)^{-1/2}(m_3m_4)^{-2/3}(m_5m_6)^{-3/4}$$

and satisfies $V_i^2W_i^2\ll J(n)\ll V_i^2W_i^2$.

P r o o f. This is Lemma 2.1 in [9]. □

Lemma 2.3. *For all integers $n \equiv 0 \pmod 2$, one has $\mathfrak{S}(n) > 1.072\,808$.*

Proof. For details one can see Section 3 in Liu [9]. $\square$

Lemma 2.4. *Let $f(\alpha_i, U_i), g(\alpha_i, V_i), h(\alpha_i, W_i)$ be defined by (2.2) and (2.3), $C(M_i)$ by (2.6). Then*

$$\max_{\alpha \in C(M_i)} |f(\alpha_i, N_i)| \ll N_i^{1/2-1/16+\varepsilon}, \qquad \max_{\alpha \in C(M_i)} |g(\alpha_i, N_i)| \ll N_i^{1/3-1/42+\varepsilon}.$$

Proof. The proof of the lemma can be found in [9]. Thanks to the enlarged major arcs in [8], we can get the results of Lemma 2.4 by using Theorem 2 in [4]. $\square$

Lemma 2.5. *Let $f(\alpha_i, U_i)$, $g(\alpha_i, V_i)$, $h(\alpha_i, W_i)$ be defined by (2.2) and (2.3). Then we obtain that*

$$\int_0^1 |f^2(\alpha_i, U_i) h^2(\alpha_i, W_i)| \, \mathrm{d}\alpha_i \leqslant W_i^4 L^C, \tag{2.9}$$

$$\int_0^1 |f^2(\alpha_i, U_i) g^2(\alpha_i, V_i) h^2(\alpha_i, W_i)| \, \mathrm{d}\alpha_i \leqslant 0.58814 V_i^2 W_i^2. \tag{2.10}$$

Proof. For (2.9), it is in Lemma 4.3 in [13] and the proof of (2.10) can be found in [11]. $\square$

Lemma 2.6. *Let $B(N_i, k) = \{n_i \geqslant 2\colon\ n_i = N_i - 2^{v_1} - 2^{v_2} - \ldots - 2^{v_k}\}$ with $k \geqslant 2$. Then for $N_1 \equiv N_2 \equiv 0 \pmod 2$, we have*

$$\sum_{\substack{n_1 \in B(N_1,k) \\ n_2 \in B(N_2,k) \\ n_1 \equiv n_2 \equiv 0 \ (\mathrm{mod}\ 2)}} J(n_1) J(n_2) \geqslant (3\pi - 180\eta)^2 V_1^2 V_2^2 W_1^2 W_2^2 L^k.$$

Proof. Using Lemma 3.1 in [11], we have

$$\sum_{\substack{n_1 \in B(N_1,k) \\ n_2 \in B(N_2,k) \\ n_1 \equiv n_2 \equiv 1 \ (\mathrm{mod}\ 2)}} J(n_1) J(n_2) \geqslant (3\pi - 180\eta)^2 V_1^2 V_2^2 W_1^2 W_2^2 \sum_{((v))} 1,$$

where $((v))$ means that $v_1, v_2, \ldots, v_k$ satisfy

$$1 \leqslant v_1, \ldots, v_k \leqslant \log_2(N_1/(kL)), 2^{v_1} + 2^{v_2} + \ldots + 2^{v_k} \equiv N_1 \pmod 2.$$

Then following the argument of Lemma 4.1 in [9], we have

$$\sum_{((v))} 1 \geqslant (1 - \varepsilon) L^k.$$

Thus, we get the proof of this lemma. $\square$

Lemma 2.7. *Let $f(\alpha_i, U_i)$, $g(\alpha_i, V_i)$, $h(\alpha_i, W_i)$ be defined by* (2.2) *and* (2.3). *Then we obtain that*

$$\int_{C(M_i)} |f(\alpha_i, U_i)|^2 |g(\alpha_i, V_i) h(\alpha_i, W_i)|^{5/2} \, \mathrm{d}\alpha_i \leqslant N_i^{17/12+\varepsilon}. \tag{2.11}$$

Proof. The proof of the lemma can be found in [11]. It is based on the Cauchy-Schwarz inequality and Lemmas 2.4 and 2.5. □

3. Proof of Theorem 1.1

In this section, we shall give the proof of Theorem 1.1. We begin with the estimate for $R_{11}(N_1, N_2)$. Applying Lemmas 2.2, 2.3 and 2.6, and introducing the notation $B(N_i, k)$, we get

$$\begin{aligned} R_{11}(N_1, N_2) &= \int_{M_1} f^2(\alpha_1, U_1) g^2(\alpha_1, V_1) h^2(\alpha_1, W_1) G^k(\alpha_1) e(-\alpha_1 N_1) \, \mathrm{d}\alpha_1 \\ &\quad \times \int_{M_2} f^2(\alpha_2, U_2) g^2(\alpha_2, V_2) h^2(\alpha_2, W_2) G^k(\alpha_2) e(-\alpha_2 N_2) \, \mathrm{d}\alpha_2 \\ &= \sum_{\substack{n_1 \in B(N_1, k) \\ n_2 \in B(N_2, k)}} \int_{M_1} f^2(\alpha_1, U_1) g^2(\alpha_1, V_1) h^2(\alpha_1, W_1) e(-\alpha_1 n_1) \, \mathrm{d}\alpha_1 \\ &\quad \times \int_{M_2} f^2(\alpha_2, U_2) g^2(\alpha_2, V_2) h^2(\alpha_2, W_2) e(-\alpha_2 n_2) \, \mathrm{d}\alpha_2 \\ &\geqslant \Big(\frac{1}{2^2 \cdot 3^2 \cdot 4^2}\Big)^2 \sum_{\substack{n_1 \in B(N_1, k) \\ n_2 \in B(N_2, k)}} \mathfrak{S}(n_1) \mathfrak{S}(n_2) J(n_1) J(n_2) \\ &\quad + O(N_1^{7/6} N_2^{7/6} L^{k-1}) \\ &\geqslant \Big(\frac{1}{2^2 \cdot 3^2 \cdot 4^2}\Big)^2 \cdot (1.072\,808)^2 \cdot (3\pi - 180\eta)^2 \cdot V_1^2 V_2^2 W_1^2 W_2^2 L^k, \end{aligned} \tag{3.1}$$

where we used $n_i/N_i = 1 + O(L^{-1})$ for $n_i \in B(N_i, k)$.

Now we turn to give an upper bound for $R_{12}(N_1, N_2)$. The estimate for $R_{21}(N_1, N_2)$ is similar. By Cauchy's inequality, we get

$$|G(\alpha_1 + \alpha_2)| \leqslant \sqrt{G(2\alpha_1) G(2\alpha_2)}.$$

For $\alpha \in C(M_2) \setminus \mathcal{E}_\lambda$ and sufficiently large N_1, we have

$$|G(2\alpha_i)| \leqslant |G(\alpha_i)| + 2 \leqslant \lambda L + 2 \leqslant (1 + o(1)) \lambda L.$$

Then using the definition of $\mathcal{E}_\lambda$, the trivial bound of $G(\alpha_2)$, Lemmas 2.1, 2.4, 2.5 and 2.7, we have

(3.2)

$$\begin{aligned}
&R_{12}(N_1, N_2)\\
&\quad = \int_{M_1}\int_{C(M_2)\cap\mathcal{E}_\lambda} f^2(\alpha_1, U_1)g^2(\alpha_1, V_1)h^2(\alpha_1, W_1)f^2(\alpha_2, U_2)g^2(\alpha_2, V_2)\\
&\qquad \times h^2(\alpha_2, W_2)G^k(\alpha_1+\alpha_2)e(-\alpha_1 N_1 - \alpha_2 N_2)\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\\
&\quad \ll \int_{M_1} |f^2(\alpha_1, U_1)g^2(\alpha_1, V_1)h^2(\alpha_1, W_1)G^{k/2}(2\alpha_1)|\,\mathrm{d}\alpha_1\\
&\qquad \times \int_{C(M_2)\cap\mathcal{E}_\lambda} |f^2(\alpha_2, U_2)g^2(\alpha_2, V_2)h^2(\alpha_2, W_2)G^{k/2}(2\alpha_2)|\,\mathrm{d}\alpha_2\\
&\quad \ll L^{k/2}\int_{M_1} |f^2(\alpha_1, U_1)g^2(\alpha_1, V_1)h^2(\alpha_1, W_1)|\,\mathrm{d}\alpha_1 L^{k/2}\max_{\alpha_2\in C(M_2)}|f(\alpha_2, U_2)|\\
&\qquad \times \Big(\int_{C(M_2)} |f(\alpha_2, U_2)|^2|g(\alpha_2, V_2)h(\alpha_2, W_2)|^{5/2}\,\mathrm{d}\alpha_2\Big)^{4/5}\Big(\int_{\mathcal{E}_\lambda} 1\,\mathrm{d}\alpha_2\Big)^{1/5}\\
&\quad \ll L^k\cdot 0.588\,14\cdot V_1^2W_1^2\cdot(N_2^{1/2-1/16+\varepsilon})^{2/5}\cdot(N_2^{17/12+\varepsilon})^{4/5}\cdot(N_2^{-17/24-10^{-10}})^{1/5}\\
&\quad \leqslant V_1^2V_2^2W_1^2W_2^2L^{k-1}.
\end{aligned}$$

Similarly, we get

$$R_{21}(N_1, N_2) \leqslant V_1^2V_2^2W_1^2W_2^2L^{k-1}. \tag{3.3}$$

Next we turn to give an upper bound for $R_{13}(N_1, N_2)$. By Lemma 2.5, using the trivial bound $|G(2\alpha)| \leqslant L$ when $\alpha \in M_1$ and the bound $|G(2\alpha)| \leqslant (1+o(1))\lambda L$ when $\alpha \in C(M_2)\setminus\mathcal{E}_\lambda$, we have

(3.4)

$$\begin{aligned}
|R_{13}(N_1, N_2)| &= \int_{M_1}\int_{C(M_2)\setminus\mathcal{E}_\lambda} f^2(\alpha_1, U_1)g^2(\alpha_1, V_1)h^2(\alpha_1, W_1)f^2(\alpha_2, U_2)g^2(\alpha_2, V_2)\\
&\quad \times h^2(\alpha_2, W_2)G^k(\alpha_1+\alpha_2)e(-\alpha_1 N_1 - \alpha_2 N_2)\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\\
&\leqslant \int_{M_1} |f^2(\alpha_1, U_1)g^2(\alpha_1, V_1)h^2(\alpha_1, W_1)G^{k/2}(2\alpha_1)|\,\mathrm{d}\alpha_1\\
&\quad \times \int_{C(M_2)\setminus\mathcal{E}_\lambda} |f^2(\alpha_2, U_2)g^2(\alpha_2, V_2)h^2(\alpha_2, W_2)G^{k/2}(2\alpha_2)|\,\mathrm{d}\alpha_2\\
&\leqslant L^{k/2}\int_{M_1} |f^2(\alpha_1, U_1)g^2(\alpha_1, V_1)h^2(\alpha_1, W_1)|\,\mathrm{d}\alpha_1\\
&\quad \times (\lambda L)^{k/2}\int_0^1 |f^2(\alpha_2, U_2)g^2(\alpha_2, V_2)h^2(\alpha_2, W_2)|\,\mathrm{d}\alpha_2\\
&\leqslant (0.588\,14)^2\lambda^{k/2}V_1^2V_2^2W_1^2W_2^2L^k.
\end{aligned}$$

We can obtain the estimate for $R_{31}(N_1, N_2)$ analogously,

$$|R_{31}(N_1, N_2)| \leqslant (0.588\,14)^2 \lambda^{k/2} V_1^2 V_2^2 W_1^2 W_2^2 L^k. \tag{3.5}$$

We give the estimate for $R_{22}(N_1, N_2)$ by the trivial bound for $G(\alpha)$, Lemmas 2.1, 2.4 and 2.7, and the definition of $\mathcal{E}_\lambda$,

$$\begin{aligned}
(3.6)\quad & R_{22}(N_1, N_2) \\
&= \int_{C(M_1)\cap\mathcal{E}_\lambda} \int_{C(M_2)\cap\mathcal{E}_\lambda} f^2(\alpha_1, U_1) g^2(\alpha_1, V_1) h^2(\alpha_1, W_1) f^2(\alpha_2, U_2) g^2(\alpha_2, V_2) \\
&\quad \times h^2(\alpha_2, W_2) G^k(\alpha_1 + \alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2)\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 \\
&\ll \int_{C(M_1)\cap\mathcal{E}_\lambda} |f^2(\alpha_1, U_1) g^2(\alpha_1, V_1) h^2(\alpha_1, W_1) G^{k/2}(2\alpha_1)|\,\mathrm{d}\alpha_1 \\
&\quad \times \int_{C(M_2)\cap\mathcal{E}_\lambda} |f^2(\alpha_2, U_2) g^2(\alpha_2, V_2) h^2(\alpha_2, W_2) G^{k/2}(2\alpha_2)|\,\mathrm{d}\alpha_1 \\
&\ll L^{k/2} \max_{\alpha_1\in C(M_1)} |f(\alpha_1, U_1)| \\
&\quad \times \left(\int_{C(M_1)} |f(\alpha_1, U_1)|^2 |g(\alpha_1, V_1) h(\alpha_1, W_1)|^{5/2}\,\mathrm{d}\alpha_1\right)^{4/5} \left(\int_{\mathcal{E}_\lambda} 1\,\mathrm{d}\alpha_2\right)^{1/5} \\
&\quad \times L^{k/2} \max_{\alpha_2\in C(M_2)} |f(\alpha_2, U_2)| \\
&\quad \times \left(\int_{C(M_2)} |f(\alpha_2, U_2)|^2 |g(\alpha_2, V_2) h(\alpha_2, W_2)|^{5/2}\,\mathrm{d}\alpha_2\right)^{4/5} \left(\int_{\mathcal{E}_\lambda} 1\,\mathrm{d}\alpha_2\right)^{1/5} \\
&\ll L^k \cdot (N_1^{1/2-1/16+\varepsilon})^{2/5} \cdot (N_1^{17/12+\varepsilon})^{4/5} \cdot (N_1^{-17/24-10^{-10}})^{1/5} \\
&\quad \times (N_2^{1/2-1/16+\varepsilon})^{2/5} \cdot (N_2^{17/12+\varepsilon})^{4/5} \cdot (N_2^{-17/24-10^{-10}})^{1/5} \\
&\leqslant V_1^2 V_2^2 W_1^2 W_2^2 L^{k-1}.
\end{aligned}$$

For $R_{23}(N_1, N_2)$, we can similarly get

$$\begin{aligned}
(3.7)\quad & R_{23}(N_1, N_2) \\
&= \int_{C(M_1)\cap\mathcal{E}_\lambda} \int_{C(M_2)\backslash\mathcal{E}_\lambda} f^2(\alpha_1, U_1) g^2(\alpha_1, V_1) h^2(\alpha_1, W_1) f^2(\alpha_2, U_2) g^2(\alpha_2, V_2) \\
&\quad \times h^2(\alpha_2, W_2) G^k(\alpha_1 + \alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2)\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 \\
&\ll \int_{C(M_1)\cap\mathcal{E}_\lambda} |f^2(\alpha_1, U_1) g^2(\alpha_1, V_1) h^2(\alpha_1, W_1) G^{k/2}(2\alpha_1)|\,\mathrm{d}\alpha_1 \\
&\quad \times \int_{C(M_2)\backslash\mathcal{E}_\lambda} |f^2(\alpha_2, U_2) g^2(\alpha_2, V_2) h^2(\alpha_2, W_2) G^{k/2}(2\alpha_2)|\,\mathrm{d}\alpha_2
\end{aligned}$$

$$
\begin{aligned}
&\ll L^{k/2} \max_{\alpha_1 \in C(M_1)} |f(\alpha_1, U_1)| \\
&\quad \times \Big(\int_{C(M_1)} |f(\alpha_1, U_1)|^2 |g(\alpha_1, V_1) h(\alpha_1, W_1)|^{5/2} \, \mathrm{d}\alpha_1 \Big)^{4/5} \Big(\int_{\mathcal{E}_\lambda} 1 \, \mathrm{d}\alpha_2 \Big)^{1/5} \\
&\quad \times (\lambda L)^{k/2} \int_0^1 |f^2(\alpha_2, U_2) g^2(\alpha_2, V_2) h^2(\alpha_2, W_2)| \, \mathrm{d}\alpha_2 \\
&\leqslant \lambda^{k/2} L^k (N_1^{1/2-1/16+\varepsilon})^{2/5} \cdot (N_1^{17/12+\varepsilon})^{4/5} \cdot (N_1^{-17/24-10^{-10}})^{1/5} \\
&\quad \times 0.588\,14 \cdot V_2^2 \cdot W_2^2 \\
&\leqslant V_1^2 V_2^2 W_1^2 W_2^2 L^{k-1}.
\end{aligned}
$$

We can obtain the estimate for $R_{3,2}(N_1, N_2)$ analogously,

$$
R_{32}(N_1, N_2) \leqslant V_1^2 V_2^2 W_1^2 W_2^2 L^{k-1}. \tag{3.8}
$$

In the end, we provide the upper bound for $R_{33}(N_1, N_2)$.

$$
\begin{aligned}
R_{33}(N_1, N_2) &= \int_{C(M_1)\setminus\mathcal{E}_\lambda} \int_{C(M_2)\setminus\mathcal{E}_\lambda} f^2(\alpha_1, U_1) g^2(\alpha_1, V_1) h^2(\alpha_1, W_1) f^2(\alpha_2, U_2) \\
&\quad \times g^2(\alpha_2, V_2) h^2(\alpha_2, W_2) G^k(\alpha_1 + \alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) \, \mathrm{d}\alpha_1 \, \mathrm{d}\alpha_2 \\
&\ll \int_{C(M_1)\setminus\mathcal{E}_\lambda} |f^2(\alpha_1, U_1) g^2(\alpha_1, V_1) h^2(\alpha_1, W_1) G^{k/2}(2\alpha_1)| \, \mathrm{d}\alpha_1 \\
&\quad \times \int_{C(M_2)\setminus\mathcal{E}_\lambda} |f^2(\alpha_2, U_2) g^2(\alpha_2, V_2) h^2(\alpha_2, W_2) G^{k/2}(2\alpha_2)| \, \mathrm{d}\alpha_2 \\
&\ll (\lambda L)^{k/2} \int_0^1 |f^2(\alpha_1, U_1) g^2(\alpha_1, V_1) h^2(\alpha_1, W_1)| \, \mathrm{d}\alpha_1 \\
&\quad \times (\lambda L)^{k/2} \int_0^1 |f^2(\alpha_2, U_2) g^2(\alpha_2, V_2) h^2(\alpha_2, W_2)| \, \mathrm{d}\alpha_2 \\
&\leqslant 0.588\,14^2 \cdot \lambda^k V_1^2 V_2^2 W_1^2 W_2^2 L^k.
\end{aligned} \tag{3.9}
$$

Combining the above formulas (3.1)–(3.9), we can obtain

$$
\begin{aligned}
R(N_1, N_2) &> R_{11}(N_1, N_2) - R_{13}(N_1, N_2) - R_{31}(N_1, N_2) \\
&\quad - R_{33}(N_1, N_2) + O(V_1^2 V_2^2 W_1^2 W_2^2 L^{k-1}) \\
&> \bigg(\Big(\frac{1}{2^2 \cdot 3^2 \cdot 4^2} \Big)^2 \cdot (1.072\,808)^2 \cdot (3\pi - 180\eta)^2 \\
&\quad - 2 \cdot 0.588\,14^2 \cdot \lambda^{k/2} - 0.588\,14^2 \cdot \lambda^k \bigg) \\
&\quad \times (1 - \varepsilon) V_1^2 V_2^2 W_1^2 W_2^2 L^k.
\end{aligned}
$$

When $k \geqslant 105$, $\varepsilon = 10^{-20}$ and $\lambda = 0.862\,654$, we have

$$R(N_1, N_2) > 0$$

for all sufficiently large even integers N_1 and N_2 satisfying $N_2 \gg N_1 \geqslant N_2$ can written in the form of (1.4). Thus, Theorem 1.1 follows.

Remark 3.1. One may slightly improve the value 105 to 96 by Theorem 1.3 in [14].

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