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SYMMETRIES IN CONNECTED GRADED ALGEBRAS AND THEIR PBW-DEFORMATIONS

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Abstract. We focus on connected graded algebras and their PBW-deformations endowed with additional symmetric structures. Many well-known algebras such as negative parts of Drinfeld-Jimbo's quantum groups, cubic Artin-Schelter algebras and three-dimensional Sklyanin algebras appear in our research framework. As an application, we investigate a $\mathcal{K}_2$ algebra $\mathcal{A}$ which was introduced to compute the cohomology ring of the Fomin-Kirillov algebra $\mathcal{FK}_3$, and explicitly construct all the (self-)symmetric and sign-(self-)symmetric PBW-deformations of $\mathcal{A}$.

Keywords: connected graded algebra; PBW-deformation; self-symmetry; sign-symmetry; $\mathcal{K}_2$ algebra

MSC 2020: 16S80

1. Introduction

A filtered algebra $\mathcal{U}$ over a field $\mathbb{K}$ is called a *Poincaré-Birkhoff-Witt-deformation* (for short PBW-deformation) of a graded $\mathbb{K}$-algebra $\mathcal{A}$ if its associated graded algebra $\mathrm{gr}(\mathcal{U})$ is isomorphic to $\mathcal{A}$. The PBW-deformation theory of connected graded algebras including Koszul algebras and d-Koszul algebras is extensively investigated.

Koszul algebras are a class of connected graded algebras with good homological properties. The polynomial algebra $\mathbb{K}[x_1, x_2, \ldots, x_n]$ may be the most important example of Koszul algebra. For an n-dimensional Lie algbra $\mathfrak{g}$, the symmetric algebra $S(\mathfrak{g})$ is isomorphic to the polynomial algebra $\mathbb{K}[x_1, x_2, \ldots, x_n]$. The theory of Lie algebras shows that the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ is just

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a PBW-deformation of the algebra $S(\mathfrak{g})$, which plays an important role in establishing the PBW theorem of $U(\mathfrak{g})$, see [12]. For an arbitrary Koszul algebra $\mathcal{A}$, in [4] Braverman and Gaitsgory gave a Jacobi condition which decides when a deformation $\mathcal{U}$ of $\mathcal{A}$ is a PBW-deformation, see also [20]. For an arbitrary d-Koszul algebra $\mathcal{A}$, a generalized Jacobi type condition was given for the determination of PBW-deformations, see [2], [8].

For an arbitrary connected graded algebra $\mathcal{A}$ over a field $\mathbb{K}$, using the notion of regular central extension of algebras, Cassidy and Shelton in [5] established the most general Jacobi condition for deciding when certain deformations of $\mathcal{A}$ obtained by altering its defining relations are PBW-ones. The Jacobi condition transforms the determination of PBW-deformations into a finite linear algebra problem, where a homological constant called complexity (see Definition 2.2) controls the degree of difficulty in a sense.

For a finite dimensional complex semisimple Lie algebra $\mathfrak{g}$, the negative part $U_q^-(\mathfrak{g})$ of Drinfeld-Jimbo's quantum group $U_q(\mathfrak{g})$ is a connected graded algebra with defining relations of mixed degrees except the cases $U_q^-(\mathfrak{sl}_2)$ and $U_q^-(\mathfrak{sl}_3)$. Some concrete PBW-deformations of $U_q^-(\mathfrak{g})$, which we call PBW-deformations of quantum group, have appeared in many researches about coideal subalgebras of $U_q(\mathfrak{g})$ (cf. [15], [16], [17], [18], [19]) and nonstandard quantum deformations of $U(\mathfrak{g})$ (cf. [10], [13], [14]). In [25] Yang and the first author focused on a class of PBW-deformations $\mathfrak{B}_q(\mathfrak{g})$ of quantum groups, see also [23], [24]. They proved a PBW theorem for $\mathfrak{B}_q(\mathfrak{g})$, where the role of $\mathfrak{B}_q(\mathfrak{g})$ as a PBW-deformation is also very crucial. Using the Jacobi condition given in [5], they also proposed an algorithm for the judgement of PBW-deformations of $U_q^-(\mathfrak{g})$ and applied it to compute the PBW-deformations of quantum groups for $\mathfrak{g}$ of Dynkin type $\mathbb{A}_2$, $\mathbb{B}_2$, see Theorem 3.5 in [25]. The above practical computations show that constructing all the PBW-deformations via direct calculations is not easy in general, even though the Jacobi condition comes down to a linear algebra problem.

Motivated by the above researches, we aim to construct some special PBW-deformations of connected graded algebras instead of all. For this purpose, in this paper we introduce the notions such as τ-(self-)symmetric connected graded algebra, sign-(self-)symmetric PBW-deformation and (self-)symmetric PBW-deformation, see Definition 3.5. In fact, the τ-commutative homogenous algebras defined by Berger in [1] are a special class of τ-symmetric connected graded algebras. Another reason why we pay close attention to connected graded algebras and their PBW-deformations with certain symmetries arises from a wealth of existed examples, see Section 3. As it is stated above, the connected graded algebras with some good homological properties such as Koszul algebras and d-Koszul algebras often emerge in the PBW-deformation theory. As a generalization of a (d-)Koszul algebra, Cassidy and Shelton introduced the notion of a $\mathcal{K}_2$ algebra in [5], [6]. To compute the cohomology of the 12-dimensional Fomin-Kirillov algebra $\mathcal{FK}_3$, Stefan and Vay defined a $\mathcal{K}_2$ algebra $\mathcal{A}$ in [21]. In virtue of the Jacobi condition in [5], we construct all the (self-)symmetric and sign-(self-)symmetric PBW-deformations of the $\mathcal{K}_2$ algebra $\mathcal{A}$, see Theorem 4.2.

This paper is organized as follows. In Section 2, we fix some notations and collect some background materials that will be necessary in the sequel. In Section 3, we state the definitions and some properties of connected graded algebras and their PBW-deformations with certain symmetries, then present some examples. In Section 4, using Cassidy-Shelton's PBW-deformation theory, we compute the complexity of a $\mathcal{K}_2$ algebra $\mathcal{A}$ and explicitly describe its four kinds of PBW-deformations.

Throughout, we, respectively, denote by $\mathbb{K}$ and $\mathbb{Z}$ an algebraically closed field with suitable characteristic and the set of integers. All linear spaces, algebras and modules are over the field $\mathbb{K}$ unless otherwise stated.

2. Preliminaries

In this section, we recall some necessary preliminaries about PBW-deformation theory of connected graded algebras formed by Cassidy and Shelton in [5].

Fix $X = \{x_1, x_2, \dots, x_n\}$ and $V = \mathrm{Span}_{\mathbb{K}} X$. Let $\mathcal{T}$ be the free algebra $\mathbb{K}\langle X\rangle = \mathbb{K}\langle x_1, x_2, \dots, x_n\rangle$ with a standard grading, that is, $\mathrm{Deg}(x_i) = 1$ for $1 \leqslant i \leqslant n$. Denote by

$$\mathcal{A} = \mathbb{K}\langle X\rangle / \langle r_1, r_2, \dots, r_{m_0}\rangle$$

the quotient algebra of $\mathcal{T}$ with m_0 homogeneous relations $r_1, r_2, \dots, r_{m_0}$. Throughout this paper, we assume that $R = \{r_1, r_2, \dots, r_{m_0}\}$ is a minimal set of relations for $\mathcal{A}$ and none of the relations is linear. By a deformation of $\mathcal{A}$ we mean a $\mathbb{K}$-algebra

$$\mathcal{U} = \mathbb{K}\langle X\rangle / \langle r_1 + l_1, r_2 + l_2, \dots, r_{m_0} + l_{m_0}\rangle$$

with the set of relations $P = \{r_1 + l_1, r_2 + l_2, \dots, r_{m_0} + l_{m_0}\}$, where $l_1, l_2, \dots, l_{m_0}$ are (not necessarily homogeneous) elements of $\mathcal{T}$ such that $\mathrm{Deg}(l_i) < \mathrm{Deg}(r_i)$ for all i. The algebra $\mathcal{A}$ is graded and the algebra $\mathcal{U}$ is filtered. We denote by $\mathcal{F}^k(\mathcal{U})$ $(k \in \mathbb{Z})$ the filtration of $\mathcal{U}$ and define $\mathrm{gr}(\mathcal{U}) = \bigoplus_{k\in\mathbb{Z}} \mathcal{F}^k(\mathcal{U})/\mathcal{F}^{k-1}(\mathcal{U})$ to be the graded algebra associated with $\mathcal{U}$.

Definition 2.1 ([5]). The nongraded deformation $\mathcal{U}$ of the graded $\mathbb{K}$-algebra $\mathcal{A}$ is said to be a PBW-deformation if its associated graded algebra $\mathrm{gr}(\mathcal{U})$ is isomorphic to $\mathcal{A}$.

Let $\mathcal{A}$-$\underline{\text{Mod}}$ be the category of $\mathbb{Z}$-graded $\mathcal{A}$-modules. For $d \in \mathbb{Z}$ let $\mathcal{A}\{d\}$ denote the graded left $\mathcal{A}$-module ${}_{\mathcal{A}}\mathcal{A}$ with grading shifted by d, that is, $\mathcal{A}\{d\}_k = \mathcal{A}_{d+k}$ for $k \in \mathbb{Z}$. For $\overrightarrow{d} = (d'_1, d'_2, \ldots, d'_{r_0}) \in \mathbb{Z}^{r_0}$ we denote

$$\mathcal{A}\{\overrightarrow{d}\} := \mathcal{A}\{d'_1\} \oplus \mathcal{A}\{d'_2\} \ldots \oplus \mathcal{A}\{d'_{r_0}\}.$$

Similarly, for each object M in $\mathcal{A}$-$\underline{\text{Mod}}$, symbol $M\{d\}$ denotes the graded left $\mathcal{A}$-module M with grading shifted by d, i.e., $M\{d\}_k = M_{d+k}$ for $k \in \mathbb{Z}$. For M, N in $\mathcal{A}$-$\underline{\text{Mod}}$, we define

$$\begin{aligned}\text{Hom}^j_{\mathcal{A}}(M,N) &:= \{\phi \in \text{Hom}_{\mathcal{A}}(M,N)\colon\ \phi(M_i) \subseteq N_{i-j}\},\\ \underline{\text{Hom}}_{\mathcal{A}}(M,N) &:= \bigoplus_{j\in\mathbb{Z}} \text{Hom}^j_{\mathcal{A}}(M,N).\end{aligned}$$

Obviously, $\underline{\text{Hom}}_{\mathcal{A}}(M,N)$ is a $\mathbb{Z}$-graded vector space. Fix a minimal projective resolution

$$\ldots \to C_2 \xrightarrow{\varphi_2} C_1 \xrightarrow{\varphi_1} C_0 \xrightarrow{\varepsilon} \mathbb{K} \to 0 \tag{2.1}$$

for the trivial $\mathcal{A}$-module $\mathbb{K}$ in $\mathcal{A}$-$\underline{\text{Mod}}$. Each module C_i is graded-free and can be expressed in the form $C_i = \mathcal{A}\{\overrightarrow{d_i}\}$, where $\overrightarrow{d_i} = (d_{1,i}, d_{2,i}, \ldots, d_{t_i,i})$ and t_i may be infinite for $i > 2$. If we apply the functor $\underline{\text{Hom}}_{\mathcal{A}}(\cdot, \mathbb{K})$ to the truncated complex $P_\bullet$ of the above resolution (2.1), then the cohomology of the resulting cochain complex $\underline{\text{Hom}}_{\mathcal{A}}(P_\bullet, \mathbb{K})$ of abelian groups equals $E(\mathcal{A}) = \oplus \text{Ext}^{r,s}_{\mathcal{A}}(\mathbb{K}, \mathbb{K})$, which is the associated bigraded Yoneda algebra of $\mathcal{A}$ with r the cohomology degree and $-s$ the internal degree inherited from the grading on $\mathcal{A}$.

Definition 2.2 ([5])**.** The complexity of the graded $\mathbb{K}$-algebra $\mathcal{A}$ is defined by

$$c(\mathcal{A}) = \sup\{s\colon\ \text{Ext}^{3,s}_{\mathcal{A}}(\mathbb{K}, \mathbb{K}) \neq 0\} - 1$$

if the global dimension of $\mathcal{A}$ is at least 3. For global dimension less than 3 we set $c(\mathcal{A}) = 0$.

Take $\mathcal{F}^k(\mathcal{T}) = \bigoplus_{i\leqslant k} V^{\otimes i}$. Let

$$\mathcal{P}_1 = \text{Span}_{\mathbb{K}}(P \cap \mathcal{F}^1(\mathcal{T})), \tag{2.2}$$

$$\mathcal{P}_k = V\mathcal{P}_{k-1} + \mathcal{P}_{k-1}V + \text{Span}_{\mathbb{K}}(P \cap \mathcal{F}^k(\mathcal{T})) \quad \text{for } k > 1. \tag{2.3}$$

In [5], a Jacobi type condition was given for determining PBW-deformations of $\mathcal{A}$.

Theorem 2.3 ([5]). *Let $\mathcal{A}$ be a graded $\mathbb{K}$-algebra of finite complexity $c(\mathcal{A})$ and let $\mathcal{U}$ be a deformation of $\mathcal{A}$. Then $\mathcal{U}$ is a PBW-deformation of $\mathcal{A}$ if and only if $\mathcal{P}_1 = 0$ and the following Jacobi condition is satisfied:*

$$\mathcal{P}_{k+1} \cap \mathcal{F}^k(\mathcal{T}) \subset \mathcal{P}_k \quad \forall\, 1 \leqslant k \leqslant c(\mathcal{A}). \tag{2.4}$$

For more details about unexplained concepts, we refer the readers to [5], [6].

3. Connected graded algebras and PBW-deformations with symmetries

In this section, we introduce some notions around connected graded algebras and their PBW-deformations with certain symmetries, list some basic properties about them and present some concrete examples.

For convenience, we firstly fix some notations and conventions. Let $\mathbf{S}_X$ be the set consisting of all permutations of X and $X^- = \{-x_1, -x_2, \ldots, -x_n\}$. Set $\widehat{X} = X \cup X^-$. For any integer $m \geqslant 2$, denote by Γ^m the Cartesian product $\underbrace{\Gamma \times \Gamma \times \ldots \times \Gamma}_{m}$ with $\Gamma = \{1, -1\}$.

Definition 3.1. Assume that $\sigma_X \in \mathbf{S}_X$ and $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \Gamma^n$. For a given pair (σ_X, ε) define a map $\tau\colon X \to \widehat{X}$ by $\tau(x_i) = \varepsilon_i \sigma_X(x_i)$ for $x_i \in X$. We call τ a sign-permutation of X and denote by $\mathbf{SP}_X$ the set of all sign-permutations of X.

Remark 3.2. We can naturally identify $\mathbf{S}_X$ with the symmetric group $\mathbb{S}_n$. Furthermore, there exists a bijection $\varphi\colon \mathbf{SP}_X \to \mathbb{S}_n \times \Gamma^n$ defined as $\varphi(\tau) = (\sigma_X, \varepsilon)$. So we will identify τ with its image $\varphi(\tau)$.

Each element f in $\mathbb{K}\langle X\rangle$ can be uniquely expressed as

$$f = f_0 + f_1 + f_2 + \ldots + f_q,$$

where $q \in \mathbb{Z}^{\geqslant 0}$ and $f_s = \sum_{1 \leqslant i_1, i_2, \ldots, i_s \leqslant n} a_{i_1 i_2 \ldots i_s} x_{i_1} x_{i_2} \ldots x_{i_s}$ is the s-homogeneous part of f with $0 \leqslant s \leqslant q$ and $a_{i_1 i_2 \ldots i_s} \in \mathbb{K}$. For $\tau \in \mathbf{SP}_X$ and $f \in \mathbb{K}\langle X\rangle$ we define

$$\begin{aligned}
\tau_0(f_s) &= \sum_{1 \leqslant i_1, i_2, \ldots, i_s \leqslant n} a_{i_1 i_2 \ldots i_s} \tau(x_{i_1}) \tau(x_{i_2}) \ldots \tau(x_{i_s}), \\
\tau_1(f_s) &= \sum_{1 \leqslant i_1, i_2, \ldots, i_s \leqslant n} a_{i_1 i_2 \ldots i_s} \tau(x_{i_s}) \tau(x_{i_{s-1}}) \ldots \tau(x_{i_1}), \\
\tau_0(f) &= f_0 + \tau_0(f_1) + \tau_0(f_2) + \ldots + \tau_0(f_q), \\
\tau_1(f) &= f_0 + \tau_1(f_1) + \tau_1(f_2) + \ldots + \tau_1(f_q).
\end{aligned}$$

Definition 3.3. Assume that $\mathcal{A} = \mathbb{K}\langle X\rangle/\langle R\rangle$ with R not necessarily homogenous and $\tau \in \mathbf{SP}_X$.

(1) If there exist a permutation σ_R of R and $\delta = (\delta_1, \delta_2, \ldots, \delta_{m_0}) \in \Gamma^{m_0}$ such that $\tau_0(r_i) = \delta_i \sigma_R(r_i)$ (or $\tau_1(r_i) = \delta_i \sigma_R(r_i)$) for all $1 \leqslant i \leqslant m_0$, then we call τ a sign-permutation (or an anti-sign-permutation) of $\mathcal{A}$. In particular, when $\tau \in \mathbf{S}_X$, we call a sign-permutation (or an anti-sign-permutation) τ of $\mathcal{A}$ just a permutation (or an anti-permutation) of $\mathcal{A}$. Respectively, denote by $\mathbf{SP}_0(\mathcal{A})$, $\mathbf{SP}_1(\mathcal{A})$, $\mathbf{S}_0(\mathcal{A})$ and $\mathbf{S}_1(\mathcal{A})$ the set of all sign-permutations, all anti-sign-permutations, all permutations and all anti-permutations of $\mathcal{A}$.

(2) Let $\overline{\tau}$ be an automorphism of $\mathcal{A}$. If the restriction of $\overline{\tau}$ on X belongs to $\mathbf{SP}_0(\mathcal{A})$ (or $\mathbf{S}_0(\mathcal{A})$), then we call $\overline{\tau}$ a sign-permutation (or permutation) automorphism of $\mathcal{A}$. Denote by $\mathrm{Aut}_{\mathrm{sp}}(\mathcal{A})$ (or $\mathrm{Aut}_{\mathrm{p}}(\mathcal{A})$) the set of all sign-permutation (or permutation) automorphisms of $\mathcal{A}$.

(3) Let $\overline{\tau}$ be an anti-automorphism of $\mathcal{A}$. If the restriction of $\overline{\tau}$ on X belongs to $\mathbf{SP}_1(\mathcal{A})$ (or $\mathbf{S}_1(\mathcal{A})$), then we call $\overline{\tau}$ a sign-permutation (or permutation) anti-automorphism of $\mathcal{A}$. Denote by $\mathrm{Aut}^{\mathrm{op}}_{\mathrm{sp}}(\mathcal{A})$ (or $\mathrm{Aut}^{\mathrm{op}}_{\mathrm{p}}(\mathcal{A})$) the set of all sign-permutation (or permutation) anti-automorphisms of $\mathcal{A}$.

Remark 3.4. In this paper, almost all concepts and results depend on the choice of presentation of an algebra. For example, take $\mathcal{A}_1 = \mathbb{K}\langle x, y\rangle/\langle x^3, y^3\rangle$ and $\mathcal{A}_2 = \mathbb{K}\langle x, y\rangle/\langle x^3 + 2y^3, x^3 + 3y^3\rangle$. It is easy to see that $\mathcal{A}_1 \cong \mathcal{A}_2$ but $\mathbf{S}_k(\mathcal{A}_1) \neq \mathbf{S}_k(\mathcal{A}_2)$, $\mathbf{SP}_k(\mathcal{A}_1) \neq \mathbf{SP}_k(\mathcal{A}_2)$ for $0 \leqslant k \leqslant 1$.

Now we introduce some concepts concerning connected graded algebras and their PBW-deformations with certain symmetries.

Definition 3.5. Retain the notations $\mathcal{A} = \mathbb{K}\langle X\rangle/\langle R\rangle$ and $\mathcal{U} = \mathbb{K}\langle X\rangle/\langle P\rangle$ in Section 3.

(1) For a given $\tau \in \mathbf{SP}_X$, we call $\mathcal{A}$ a τ-self-symmetric (or τ-symmetric) connected graded algebra if $\tau \in \mathbf{SP}_0(\mathcal{A})$ (or $\tau \in \mathbf{SP}_1(\mathcal{A})$).

(2) When $\mathbf{SP}_X = \mathbf{SP}_0(\mathcal{A})$ (or $\mathbf{SP}_X = \mathbf{SP}_1(\mathcal{A})$), we call $\mathcal{A}$ a completely sign-self-symmetric (or sign-symmetric) connected graded algebra.

(3) When $\mathbf{S}_X = \mathbf{S}_0(\mathcal{A})$ (or $\mathbf{S}_X = \mathbf{S}_1(\mathcal{A})$), we call $\mathcal{A}$ a completely self-symmetric (or symmetric) connected graded algebra.

(4) Assume that $\mathcal{U}$ is a PBW-deformation of $\mathcal{A}$.

   (i) For a given $\tau \in \mathbf{SP}_X$, we call $\mathcal{U}$ a τ-self-symmetric (or τ-symmetric) PBW-deformation of $\mathcal{A}$ if $\tau \in \mathbf{SP}_0(\mathcal{A}) \cap \mathbf{SP}_0(\mathcal{U})$ (or $\tau \in \mathbf{SP}_1(\mathcal{A}) \cap \mathbf{SP}_1(\mathcal{U})$).

   (ii) If $\mathbf{SP}_0(\mathcal{A}) = \mathbf{SP}_0(\mathcal{U})$ (or $\mathbf{SP}_1(\mathcal{A}) = \mathbf{SP}_1(\mathcal{U})$, $\mathbf{S}_0(\mathcal{A}) = \mathbf{S}_0(\mathcal{U})$, $\mathbf{S}_1(\mathcal{A}) = \mathbf{S}_1(\mathcal{U})$), then $\mathcal{U}$ is said to be a sign-self-symmetric (or sign-symmetric, self-symmetric, symmetric) PBW-deformation of $\mathcal{A}$.

Remark 3.6. Suppose that $\mathcal{A} = \mathbb{K}\langle X\rangle/\langle R\rangle$ is a homogeneous algebra. If we choose τ to be a sign-permutation corresponding to the pair (σ_X, ε) with $\sigma_X = \mathrm{id}$, then $\mathcal{A}$ is a τ-symmetric connected graded algebra if and only if $\mathcal{A}$ is a τ-commutative algebra in the sense of Berger, cf. [1].

Around the notions in Definitions 3.3 and 3.5, we have the following statements.

Proposition 3.7.

(1) *The algebra $\mathcal{A}$ is a τ-self-symmetric (or τ-symmetric) connected graded algebra if and only if there exists a sign-permutation automorphism (or anti-automorphism) $\overline{\tau}\colon \mathcal{A} \to \mathcal{A}$ such that $\overline{\tau}|_X = \tau$.*

(2) *For $\tau, \tau' \in \mathbf{SP}_0(\mathcal{A})$, define $X \overset{\tau\cdot\tau'}{\longrightarrow} \widehat{X}$ by*

$$(\tau\cdot\tau')(x_i) = \begin{cases} \tau[\tau'(x_i)] & \text{if } \tau'(x_i) \in X, \\ -\tau[-\tau'(x_i)] & \text{if } \tau'(x_i) \in X^-. \end{cases}$$

Then $\mathbf{SP}_0(\mathcal{A})$ is a group under the above multiplication.

(3) *As groups $\mathbf{SP}_0(\mathcal{A}) \cong \mathrm{Aut}_{\mathrm{sp}}(\mathcal{A})$ and $\mathbf{S}_0(\mathcal{A}) \cong \mathrm{Aut}_{\mathrm{p}}(\mathcal{A})$.*

(4) *A completely sign-self-symmetric (or sign-symmetric) connected graded algebra is completely self-symmetric (or symmetric).*

(5) *For $0 \leqslant k \leqslant 1$, one has $\mathbf{SP}_k(\mathcal{A}) \supseteq \mathbf{SP}_k(\mathcal{U})$ and $\mathbf{S}_k(\mathcal{A}) \supseteq \mathbf{S}_k(\mathcal{U})$.*

(6) *A sign-self-symmetric (or sign-symmetric) PBW-deformation $\mathcal{U}$ of the connected graded algebra $\mathcal{A}$ is a self-symmetric (or symmetric) PBW-deformation.*

P r o o f. We can easily obtain (1), (2) and (3) if we note that there exists a natural bijection from $\mathbf{SP}_0(\mathcal{A})$ (or $\mathbf{SP}_1(\mathcal{A})$, $\mathbf{S}_0(\mathcal{A})$) to $\mathrm{Aut}_{\mathrm{sp}}(\mathcal{A})$ (or $\mathrm{Aut}^{\mathrm{op}}_{\mathrm{sp}}(\mathcal{A})$, $\mathrm{Aut}_{\mathrm{p}}(\mathcal{A})$). It follows from Definition 3.5 (2) and (3) that the statement in (4) holds. For a given $\tau \in \mathbf{SP}_X$ (or $\tau \in \mathbf{S}_X$), it follows from Definition 3.3 that $\tau \in \mathbf{SP}_k(\mathcal{U})$ (or $\tau \in \mathbf{S}_k(\mathcal{U})$) if and only if there exist a permutation σ_P of P and $\delta = (\delta_1, \delta_2, \ldots, \delta_{m_0}) \in \Gamma^{m_0}$ such that $\tau_k(r_i + l_i) = \delta_i\sigma_P(r_i + l_i)$ for all $1 \leqslant i \leqslant m_0$, where the latter condition is equivalent to saying that $\tau_k(r_i) = \delta_i\sigma_P(r_i)$ and $\tau_k(l_i) = \delta_i\sigma_P(l_i)$. Hence, (5) holds. Since $\mathbf{S}_k(\mathcal{A}) \subseteq \mathbf{SP}_k(\mathcal{A}) = \mathbf{SP}_k(\mathcal{U})$ implies $\mathbf{S}_k(\mathcal{A}) \subseteq \mathbf{S}_k(\mathcal{U})$, by (5) we can obtain (6). □

Remark 3.8. It follows from Definition 3.5 (4) (ii) and Proposition 3.7 (3) that self-symmetric (or sign-self-symmetric) PBW-deformations of $\mathcal{A}$ are just the PBW-deformations of $\mathcal{A}$ in the representation category of the group $\mathbf{S}_0(\mathcal{A})$ ($\mathbf{SP}_0(\mathcal{A})$).

Next we present some examples about τ-symmetric connected graded algebras and their PBW-deformations with certain symmetries.

Example 3.9. Let $\mathcal{A} = \mathbb{K}\langle X\rangle/\langle R\rangle$, where

$$R = \left\{ \sum_{\sigma\in S_d} (-1)^{\mathrm{sgn}(\sigma)} x_{i_{\sigma(1)}} x_{i_{\sigma(2)}} \dots x_{i_{\sigma(d)}} \colon 1 \leqslant i_1 < i_2 < \dots < i_d \leqslant n \right\}.$$

Then $\mathcal{A}$ is not only completely sign-self-symmetric but also completely sign-symmetric. In fact, $\mathcal{A}$ is a d-Koszul algebra and $\mathcal{A} = \mathbb{K}[x_1, x_2, \dots, x_n]$ when $d = 2$. The PBW-deformations of $\mathcal{A}$ were explicitly described in Theorems 4.1 and 4.2 in [8] for $n \geqslant d + 2$.

Example 3.10. Let $\mathcal{A} = \mathbb{K}\langle X\rangle/\langle R\rangle$, where

$$R = \left\{ \sum_{\sigma\in S_d} x_{i_{\sigma(1)}} x_{i_{\sigma(2)}} \dots x_{i_{\sigma(d)}} \colon 1 \leqslant i_1 \leqslant i_2 \leqslant \dots \leqslant i_d \leqslant n \right\}.$$

Then $\mathcal{A}$ is completely sign-self-symmetric and completely sign-symmetric. It is a d-Koszul algebra and its PBW-deformation is characterized in Theorem 4.6 in [8]. Especially when $d = 2$, the above algebra $\mathcal{A}$ is generated by $x_1, x_2, \dots, x_n$ with relations

$$x_i^2 = 0 \quad (1 \leqslant i \leqslant n), \quad x_i x_j + x_j x_i = 0 \quad (1 \leqslant i \neq j \leqslant n).$$

The Clifford algebra $\mathrm{CL}(V)$ can be presented by generators $x_1, x_2, \dots, x_n$ and relations

$$x_i^2 = Q(x_i) \quad (1 \leqslant i \leqslant n), \quad x_i x_j + x_j x_i = 0 \quad (1 \leqslant i \neq j \leqslant n),$$

where $V = \mathrm{Span}_{\mathbb{K}} X$ and $Q\colon V \to \mathbb{K}$ is a quadratic form on V. It is easy to see that $\mathrm{CL}(V)$ is a sign-self-symmetric, sign-symmetric, self-symmetric or symmetric PBW-deformation of $\mathcal{A}$ if and only if $Q(x_i) = Q(x_j)$ for all $1 \leqslant i, j \leqslant n$.

Example 3.11. Assume that $\mathfrak{g}$ is a complex simple Lie algebra of rank n ($\geqslant 2$) and $A = (a_{ij})_{n\times n}$ is the Cartan matrix of $\mathfrak{g}$. Denote by $U_q^-(\mathfrak{g})$ the negative part of Drinfeld-Jimbo quantum group $U_q(\mathfrak{g})$. For $\mathfrak{g}$ of different Dynkin type, $\mathbf{SP}_0(U_q^-(\mathfrak{g}))$ and $\mathbf{SP}_1(U_q^-(\mathfrak{g}))$ can be described in the following table:

Type of $\mathfrak{g}$	$\mathbb{A}_n$, $\mathbb{D}_n(n \geqslant 5)$, $\mathbb{E}_6$	$\mathbb{D}_4$	$\mathbb{B}_n(n \geqslant 2)$, $\mathbb{C}_n(n \geqslant 3)$, $\mathbb{E}_7$, $\mathbb{E}_8$, $\mathbb{F}_4$, $\mathbb{G}_2$
$\mathbf{SP}_0(U_q^-(\mathfrak{g}))$	$\mathbb{Z}_2$	$\mathbb{S}_3$	$\{\mathrm{id}\}$
$\mathbf{SP}_1(U_q^-(\mathfrak{g}))$	$\mathbb{Z}_2 \times \Gamma^n$	$\mathbb{S}_3 \times \Gamma^4$	Γ^n

In fact, the group $\mathbf{SP}_0(U_q^-(\mathfrak{g}))$ is just the group of all automorphisms of Dynkin diagram corresponding to $\mathfrak{g}$, cf. [9].

The deformation $\mathfrak{B}_q(\mathfrak{g})$ of $U_q^-(\mathfrak{g})$ is the $\mathbb{K}$-algebra with generators B_i ($1 \leqslant i \leqslant n$) and relations:

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_i B_i^{1-a_{ij}-k} B_j B_i^k = h(B_i, B_j) \tag{3.1}$$

for all $i \neq j$, where

$$h(B_i, B_j) = \begin{cases} 0 & \text{if } a_{ij} = 0, \\ -q_i^{-1} B_j & \text{if } a_{ij} = -1, \\ -q^{-1}[2]^2(B_iB_j - B_jB_i) & \text{if } a_{ij} = -2, \\ -q^{-1}([3]^2+1)(B_i^2B_j + B_jB_i^2) & \\ +q^{-1}[2]([2][4]+q^2+q^{-2})B_iB_jB_i - q^{-2}[3]^2B_j & \text{if } a_{ij} = -3. \end{cases}$$

The algebra $\mathfrak{B}_q(\mathfrak{g})$ is a class of coideal subalgebras of $U_q(\mathfrak{g})$ and Letzter's work in [18] showed that $\mathfrak{B}_q(\mathfrak{g})$ is a PBW-deformation of $U_q^-(\mathfrak{g})$, cf. [15], [24], [25]. It is routine to check that $\mathfrak{B}_q(\mathfrak{g})$ is a sign-self-symmetric and sign-symmetric PBW-deformation of $U_q^-(\mathfrak{g})$ for each $\mathfrak{g}$.

Example 3.12.

(1) The cubic Artin-Schelter algebra $\mathcal{C}_{\mathrm{AS}}(\mathbb{A})$ of type $\mathbb{A}$ is presented by generators x, y and relations

$$ay^2x + byxy + axy^2 + x^3 = 0,$$
$$y^3 + ayx^2 + bxyx + ax^2y = 0,$$

where $a, b \in \mathbb{K}$. Then $\mathcal{C}_{\mathrm{AS}}(\mathbb{A})$ is completely sign-self-symmetric and completely sign-symmetric. It follows from [8] that all the sign-(self-)symmetric PBW-deformations of $\mathcal{C}_{\mathrm{AS}}(\mathbb{A})$ are the algebras $\mathcal{D}_{\mathrm{AS}}(\mathbb{A})$ with generators x, y and relations

$$ay^2x + byxy + axy^2 + x^3 = cx,$$
$$y^3 + ayx^2 + bxyx + ax^2y = cy,$$

where $c \in \mathbb{K}$.

(2) The cubic Artin-Schelter algebra $\mathcal{C}_{\mathrm{AS}}(\mathbb{S}_2')$ of type $\mathbb{S}_2'$ is generated by x, y and subject to relations

$$y^2x + xy^2 + x^3 = 0,$$
$$yx^2 - x^2y = 0,$$

where $a, b \in \mathbb{K}$. In this case,

$$\mathbf{SP}_0(\mathcal{C}_{\mathrm{AS}}(\mathbb{S}_2')) = \mathbf{SP}_1(\mathcal{C}_{\mathrm{AS}}(\mathbb{S}_2')) = \{\tau\colon\ \tau(x) = \pm x, \tau(y) = \pm y\},$$

while $\mathbf{S}_0(\mathcal{C}_{\mathrm{AS}}(\mathbb{S}'_2)) = \mathbf{S}_1(\mathcal{C}_{\mathrm{AS}}(\mathbb{S}'_2)) = \{\mathrm{id}\}$. It follows from [8] that all the PBW-deformations of $\mathcal{C}_{\mathrm{AS}}(\mathbb{S}'_2)$ are the following algebras $\mathcal{D}_{\mathrm{AS}}(\mathbb{S}'_2)$ with generators x, y and relations

$$y^2x + xy^2 + x^3 = a_{11}x^2 + a_{14}y^2 + a_{21}x + a_3, \tag{3.2}$$
$$yx^2 - x^2y = -a_{14}xy + a_{14}yx, \tag{3.3}$$

which are exactly all the (self-)symmetric PBW-deformations. Moreover, it is easy to check that $\mathcal{D}_{\mathrm{AS}}(\mathbb{S}'_2)$ is a sign-(self-)symmetric PBW-deformation if and only if $a_{11} = a_{14} = a_3 = 0$.

Example 3.13. Denote by $\mathbb{P}^2_{\mathbb{K}}$ the projective plane. For every $i = 1, 2, 3$, let $a, b, c, d_i, e_i \in \mathbb{K}$ with

$$[a : b : c] \in \mathbb{P}^2_{\mathbb{K}} \setminus \{[a : b : c] \colon\ abc = 0 \text{ or } a^3 = b^3 = c^3 = 1\}.$$

The three-dimensional Sklyanin algebra $\mathcal{S}(a, b, c)$ is generated by x, y, z and subject to relations

$$ayz + bzy + cx^2 = 0,$$
$$azx + bxz + cy^2 = 0,$$
$$axy + byx + cz^2 = 0.$$

Then up to isomorphism or bijection one has

	$\mathbf{S}_0(\mathcal{S}(a,b,c))$	$\mathbf{SP}_0(\mathcal{S}(a,b,c))$	$\mathbf{S}_1(\mathcal{S}(a,b,c))$	$\mathbf{SP}_1(\mathcal{S}(a,b,c))$
$a = b$	$\mathbb{S}_3$	$\mathbb{S}_3 \oplus \mathbb{Z}_2$	$\mathbb{S}_3$	$\mathbb{S}_3 \times \mathbb{Z}_2$
$a \neq b$	$\mathbb{Z}_3$	$\mathbb{Z}_3 \oplus \mathbb{Z}_2$	$\{(12), (13), (23)\}$	$\{(12), (13), (23)\} \times \mathbb{Z}_2$

Hence, the algebra $\mathcal{S}(a, b, c)$ is completely (self-)symmetric but not completely sign-(self)-symmetric.

Let $\mathcal{S}_{\mathrm{d}}(a, b, c)$ be the deformation of $\mathcal{S}(a, b, c)$ given by generators x, y, z and three relations

$$ayz + bzy + cx^2 + d_1x + e_1 = 0,$$
$$azx + bxz + cy^2 + d_2y + e_2 = 0,$$
$$axy + byx + cz^2 + d_3z + e_3 = 0.$$

It can be seen from [3], [7], [22] that $\mathcal{S}_{\mathrm{d}}(a, b, c)$ is a PBW-deformation of $\mathcal{S}(a, b, c)$. In our terms, $\mathcal{S}_{\mathrm{d}}(a, b, c)$ is a (self-)symmetric PBW-deformation of $\mathcal{S}(a, b, c)$, while $\mathcal{S}_{\mathrm{d}}(a, b, c)$ is a sign-(self-)symmetric PBW-deformation of $\mathcal{S}(a, b, c)$ only if $d_1 = d_2 = d_3 = 0$ and $e_1 = e_2 = e_3$.

Example 3.14. The Fomin-Kirillov algebra $\mathcal{FK}_3$ is defined by generators a, b, c and relations

$$a^2 = b^2 = c^2 = 0,$$
$$ab + bc + ca = ba + cb + ac = 0.$$

One has $\mathbf{S}_0(\mathcal{FK}_3) = \mathbf{S}_1(\mathcal{FK}_3) = \mathbb{S}_3$ and $\mathbf{SP}_0(\mathcal{FK}_3) = \mathbf{SP}_1(\mathcal{FK}_3) = \mathbb{S}_3 \times \mathbb{Z}_2$. It is easy to check that $\mathcal{FK}_3$ is completely (self-)symmetric but not completely sign-(self-)symmetric. For $\alpha_1, \alpha_2 \in \mathbb{K}$, let $\mathcal{D}_3(\alpha_1, \alpha_2)$ be the deformation of $\mathcal{FK}_3$ given by generators a, b, c and relations

$$a^2 = b^2 = c^2 = \alpha_1,$$
$$ab + bc + ca = ba + cb + ac = \alpha_2.$$

Then $\mathcal{D}_3(\alpha_1, \alpha_2)$ is a (self-)symmetric and sign-(self-)symmetric PBW-deformation of $\mathcal{FK}_3$, cf. [11].

4. PBW-deformations of a $\mathcal{K}_2$ algebra

In this section, we focus on a $\mathcal{K}_2$ algebra $\mathcal{A}$ which is a subalgebra of the Fomin-Kirillov algebra $\mathcal{FK}_3$, and aim to explicitly constructing its four kinds of PBW-deformations in Definition 3.5 (4) (ii).

Let

$$\mathcal{A} = \mathbb{K}\langle a, b\rangle / \langle a^2, b^2, aba - bab\rangle, \tag{4.1}$$
$$\mathcal{R} = \mathbb{K}\langle c\rangle / \langle c^2\rangle. \tag{4.2}$$

The Fomin-Kirillov algebra $\mathcal{FK}_3$ can be realized by a twisted tensor product $\mathcal{A} \otimes_\sigma \mathcal{R}$, where $\sigma\colon \mathcal{R} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{R}$ is a twisting map, cf. [21]. The identification of $\mathcal{FK}_3$ and $\mathcal{A} \otimes_\sigma \mathcal{R}$ plays a fundamental role in computing the cohomology of $\mathcal{FK}_3$, see [21].

Now we can obtain the accurate value of the complexity $c(\mathcal{A})$ of $\mathcal{A}$ as follows.

Proposition 4.1.

(1) *The complexity $c(\mathcal{A})$ of the $\mathcal{K}_2$ algebra $\mathcal{A}$ is* 3.

(2) $\mathbf{S}_0(\mathcal{A}) = \mathbf{S}_1(\mathcal{A}) = \{\sigma_i \in \mathbf{S}_X\colon 1 \leqslant i \leqslant 2\}$ *and* $\mathbf{SP}_0(\mathcal{A}) = \mathbf{SP}_1(\mathcal{A}) = \{\sigma_i \in \mathbf{SP}_X\colon 1 \leqslant i \leqslant 4\}$, *where* $X = \{a, b\}$ *and* σ_i $(1 \leqslant i \leqslant 4)$ *are, respectively, determined by*

$$\begin{cases} \sigma_1(a) = a, \\ \sigma_1(b) = b, \end{cases} \quad \begin{cases} \sigma_2(a) = b, \\ \sigma_2(b) = a, \end{cases} \quad \begin{cases} \sigma_3(a) = -a, \\ \sigma_3(b) = -b, \end{cases} \quad \begin{cases} \sigma_4(a) = -b, \\ \sigma_4(b) = -a. \end{cases} \tag{4.3}$$

P r o o f. (1) In [21], a first quadrant double complex of left $\mathcal{A}$-modules is obtained as follows:

(4.4)
$$
\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
{\scriptstyle \varrho_b}\downarrow & & \downarrow{\scriptstyle \varrho_{-a}} & & \downarrow{\scriptstyle \varrho_b} & & \downarrow{\scriptstyle \varrho_{-a}} & & \downarrow{\scriptstyle \varrho_{ba}} & & \\
\mathcal{A} & \xleftarrow[\varrho_{ba}]{} & \mathcal{A} & \xleftarrow[\varrho_{ab}]{} & \mathcal{A} & \xleftarrow[\varrho_{ba}]{} & \mathcal{A} & \xleftarrow[\varrho_{b}]{} & \mathcal{A} & \xleftarrow[\varrho_{b}]{} & \cdots \\
{\scriptstyle \varrho_b}\downarrow & & \downarrow{\scriptstyle \varrho_{-a}} & & \downarrow{\scriptstyle \varrho_b} & & \downarrow{\scriptstyle \varrho_{ab}} & & \downarrow{\scriptstyle \varrho_{ab}} & & \\
\mathcal{A} & \xleftarrow[\varrho_{ba}]{} & \mathcal{A} & \xleftarrow[\varrho_{ab}]{} & \mathcal{A} & \xleftarrow[\varrho_{-a}]{} & \mathcal{A} & \xleftarrow[\varrho_{-a}]{} & \mathcal{A} & \xleftarrow[\varrho_{-a}]{} & \cdots \\
{\scriptstyle \varrho_b}\downarrow & & \downarrow{\scriptstyle \varrho_{-a}} & & \downarrow{\scriptstyle \varrho_{ba}} & & \downarrow{\scriptstyle \varrho_{ba}} & & \downarrow{\scriptstyle \varrho_{ba}} & & \\
\mathcal{A} & \xleftarrow[\varrho_{ba}]{} & \mathcal{A} & \xleftarrow[\varrho_{b}]{} & \mathcal{A} & \xleftarrow[\varrho_{b}]{} & \mathcal{A} & \xleftarrow[\varrho_{b}]{} & \mathcal{A} & \xleftarrow[\varrho_{b}]{} & \cdots \\
{\scriptstyle \varrho_b}\downarrow & & \downarrow{\scriptstyle \varrho_{ab}} & & \downarrow{\scriptstyle \varrho_{ab}} & & \downarrow{\scriptstyle \varrho_{ab}} & & \downarrow{\scriptstyle \varrho_{ab}} & & \\
\mathcal{A} & \xleftarrow[\varrho_{-a}]{} & \mathcal{A} & \xleftarrow[\varrho_{-a}]{} & \mathcal{A} & \xleftarrow[\varrho_{-a}]{} & \mathcal{A} & \xleftarrow[\varrho_{-a}]{} & \mathcal{A} & \xleftarrow[\varrho_{-a}]{} & \cdots
\end{array}
$$

where ϱ_x denotes the right multiplication by $x \in \mathcal{A}$. Then a minimal projective resolution for the trivial $\mathcal{A}$-module $\mathbb{K}$ in $\mathcal{A}$-Mod can be obtained as the following total complex associated to the double complex (4.4)

$$\ldots \to \mathcal{A}\{-3,-4,-4,-3\} \xrightarrow{\varphi_3} \mathcal{A}\{-2,-3,-2\} \xrightarrow{\varphi_2} \mathcal{A}\{-1,-1\} \xrightarrow{\varphi_1} \mathcal{A} \xrightarrow{\varepsilon} \mathbb{K} \to 0.$$

Hence, by Definition 2.2 one knows that $c(\mathcal{A}) = 3$.

(2) We can directly check (2) according to Definition 3.3 (1). $\square$

Every deformation $\mathcal{U}$ of $\mathcal{A}$ can be presented with generators a, b and relations

(4.5) $$a^2 + \lambda_{11}a + \lambda_{12}b + \lambda_{13} = 0,$$

(4.6) $$b^2 + \lambda_{21}a + \lambda_{22}b + \lambda_{23} = 0,$$

(4.7) $$aba - bab + \lambda_{31}a^2 + \lambda_{32}ab + \lambda_{33}ba + \lambda_{34}b^2 + \lambda_{35}a + \lambda_{36}b + \lambda_{37} = 0,$$

where $\lambda_{ij} \in \mathbb{K}$ are called the *structure coefficients* of $\mathcal{U}$. For convenience, we set

(4.8) $$r_1 = a^2, \quad r_2 = b^2, \quad r_3 = aba - bab,$$

(4.9) $$l_1 = \lambda_{11}a + \lambda_{12}b + \lambda_{13}, \quad l_2 = \lambda_{21}a + \lambda_{22}b + \lambda_{23},$$

(4.10) $$l_3 = \lambda_{31}a^2 + \lambda_{32}ab + \lambda_{33}ba + \lambda_{34}b^2 + \lambda_{35}a + \lambda_{36}b + \lambda_{37}.$$

We are ready to determine all the self-symmetric, sign-self-symmetric, symmetric and sign-symmetric PBW-deformations of the $\mathcal{K}_2$ algebra $\mathcal{A}$, where the main tool is the Jacobi condition (2.4) in Theorem 2.3.

Theorem 4.2.

(1) *A deformation $\mathcal{U}$ is a PBW-deformation of $\mathcal{A}$ if and only if its structure coefficients λ_{ij} satisfy the conditions*

$$\tag{4.11} \begin{cases} \lambda_{12} = \lambda_{21} = 0, \\ \lambda_{11} - \lambda = \lambda_{22} + \lambda, \\ (\lambda_{11} - \lambda)\lambda = \lambda_{13} - \lambda_{23}, \\ \lambda_{11}\lambda_{31} + \lambda_{11}\mu + \lambda_{22}\lambda_{34} - \lambda\mu - \lambda_{35} - \lambda_{36} = 0, \\ \lambda_{11}\lambda_{31}\lambda - \lambda_{13}\lambda_{31} - \lambda_{23}\mu - \lambda_{23}\lambda_{34} - \lambda\lambda_{35} + \lambda_{37} = 0, \\ \lambda_{22}\lambda\lambda_{34} + \lambda_{13}\lambda_{31} + \lambda_{13}\mu + \lambda_{23}\lambda_{34} - \lambda\lambda_{36} - \lambda_{37} = 0, \\ \lambda_{13}\lambda_{31}\lambda - \lambda_{23}\lambda\lambda_{34} + \lambda_{11}\lambda_{37} - \lambda_{13}\lambda_{35} - \lambda_{23}\lambda_{36} - \lambda\lambda_{37} = 0, \end{cases}$$

where $(\lambda, \mu) = (\lambda_{32}, \lambda_{33})$ and $(\lambda_{33}, \lambda_{32})$.

(2) *$\mathcal{U}$ is a (self-)symmetric PBW-deformation of $\mathcal{A}$ if and only if $\mathcal{U}$ satisfies the relations*

$$\tag{4.12} \begin{cases} a^2 + \lambda_1 a + \lambda_2 = 0, \\ b^2 + \lambda_1 b + \lambda_2 = 0, \\ aba - bab + \lambda_3(a^2 - b^2) + \lambda_4(a - b) = 0. \end{cases}$$

(3) *$\mathcal{U}$ is a sign-(self-)symmetric PBW-deformation of $\mathcal{A}$ if and only if $\mathcal{U}$ satisfies the relations*

$$\tag{4.13} \begin{cases} a^2 + \lambda_1 = 0, \\ b^2 + \lambda_1 = 0, \\ aba - bab + \lambda_2(a - b) = 0. \end{cases}$$

P r o o f. In the following, we will firstly prove (1) for determining all the PBW-deformations of $\mathcal{A}$, then, respectively, choose from them all the self-symmetric, sign-self-symmetric, symmetric and sign-symmetric PBW-deformations. We will omit the proof of (3) because it is similar as (2).

(1) For the case $\mathcal{A}$, $P_1 = 0$ in (2.2) and for $2 \leqslant k \leqslant 4$, in (2.3) one has

$$\tag{4.14} P_k = \operatorname{Span}_{\mathbb{K}}\{x_1x_2\ldots x_s(r_i + l_i)x_{s+1}x_{s+2}\ldots x_t\colon \\ 1 \leqslant i \leqslant 3,\ t + \operatorname{Deg}(r_i) \leqslant k,\ x_j = a, b,\ 1 \leqslant j \leqslant t\}.$$

Denote

$$P'_k = \operatorname{Span}_{\mathbb{K}}\{x_1x_2\ldots x_s(r_i + l_i)x_{s+1}x_{s+2}\ldots x_t\colon \\ 1 \leqslant i \leqslant 3,\ t + \operatorname{Deg}(r_i) = k + 1,\ x_j = a, b,\ 1 \leqslant j \leqslant t\}.$$

Note that $P_{k+1} = P_k \cup P'_k$, so the Jacobi condition (2.4) in Theorem 2.3 is equivalent to

$$P'_k \cap \mathcal{F}^k(\mathcal{T}) \subseteq P_k \tag{4.15}$$

for all $1 \leqslant k \leqslant c(\mathcal{A}) = 3$. Since

$$P_2 = \mathrm{Span}_{\mathbb{K}}\{r_1 + l_1, r_2 + l_2\} \tag{4.16}$$

for any element $\alpha_1(r_1 + l_1) + \alpha_2(r_2 + l_2) \in P'_1 \cap \mathcal{F}^1(\mathcal{T})$ $(\alpha_1, \alpha_2 \in \mathbb{K})$, one has

$$\alpha_1 r_1 + \alpha_2 r_2 = 0. \tag{4.17}$$

It can be obtained from (4.8) and (4.17) that $\alpha_1 = \alpha_2 = 0$. Hence, $P'_1 \cap \mathcal{F}^1(\mathcal{T}) = 0 \subseteq P_1$ always holds. Because

(4.18) $P_3 = \mathrm{Span}_{\mathbb{K}}\{r_i + l_i, x(r_j + l_j), (r_j + l_j)x\colon\ 1 \leqslant i \leqslant 3,\ 1 \leqslant j \leqslant 2,\ x = a, b\}$,

(4.19) $P'_2 = \mathrm{Span}_{\mathbb{K}}\{r_3 + l_3, x(r_j + l_j), (r_j + l_j)x\colon\ 1 \leqslant j \leqslant 2,\ x = a, b\}$,

any element in $P'_2 \cap \mathcal{F}^2(\mathcal{T})$ can be expressed as

$$\alpha_3(r_3 + l_3) + \sum_{\substack{x=a,b,\\ 1\leqslant j\leqslant 2}} \alpha_{xj} x(r_j + l_j) + \sum_{\substack{1\leqslant j\leqslant 2,\\ x=a,b}} \alpha_{jx}(r_j + l_j)x \tag{4.20}$$

with $\alpha_3, \alpha_{xj}, \alpha_{jx} \in \mathbb{K}$ and

$$\alpha_3 r_3 + \sum_{\substack{x=a,b,\\ 1\leqslant j\leqslant 2}} \alpha_{xj} x r_j + \sum_{\substack{1\leqslant j\leqslant 2,\\ x=a,b}} \alpha_{jx} r_j x = 0. \tag{4.21}$$

Putting (4.8) into (4.21), one can obtain

$$\alpha_3(aba - bab) + (\alpha_{a1} + \alpha_{1a})a^3 + \alpha_{a2}ab^2 + \alpha_{b1}ba^2 + \alpha_{2a}b^2a + \alpha_{1b}a^2b + (\alpha_{b2} + \alpha_{2b})b^3 = 0.$$

By linear independence, the coefficients in (4.20) should satisfy the following linear equations:

$$\begin{cases} \alpha_3 = \alpha_{a2} = \alpha_{b1} = \alpha_{2a} = \alpha_{1b} = 0, \\ \alpha_{a1} + \alpha_{1a} = 0, \\ \alpha_{b2} + \alpha_{2b} = 0. \end{cases} \tag{4.22}$$

Now the condition $P_2' \cap \mathcal{F}^2(\mathcal{T}) \subseteq P_2$ holds if and only if there exist $\beta_i \in \mathbb{K}$ $(1 \leqslant i \leqslant 4)$ such that

$$\tag{4.23} \begin{cases} -al_1 + l_1 a = \beta_1(r_1 + l_1) + \beta_2(r_2 + l_2), \\ -bl_2 + l_2 b = \beta_3(r_1 + l_1) + \beta_4(r_2 + l_2), \end{cases}$$

which is equivalent to $\lambda_{12} = \lambda_{21} = 0$ after substituting (4.8), (4.9) into (4.23) and using linear independence. Because

$$\tag{4.24} \begin{aligned} P_4 = \mathrm{Span}_{\mathbb{K}}\{& r_i + l_i, x(r_i + l_i), (r_i + l_i)x, xy(r_j + l_j), \\ & x(r_j + l_j)y, (r_j + l_j)xy \colon\ 1 \leqslant i \leqslant 3,\ 1 \leqslant j \leqslant 2,\ x, y = a, b\}, \end{aligned}$$

$$\tag{4.25} \begin{aligned} P_3' = \mathrm{Span}_{\mathbb{K}}\{& x(r_3 + l_3), (r_3 + l_3)x, xy(r_j + l_j), \\ & x(r_j + l_j)y, (r_j + l_j)xy \colon\ 1 \leqslant j \leqslant 2,\ x, y = a, b\}, \end{aligned}$$

any element in $P_3' \cap \mathcal{F}^3(\mathcal{T})$ can be expressed as

$$\tag{4.26} \begin{aligned} & \sum_{x=a,b} [\alpha_{x3} x(r_3 + l_3) + \alpha_{3x}(r_3 + l_3)x] \\ & \qquad + \sum_{\substack{1 \leqslant j \leqslant 2, \\ x,y=a,b}} [\alpha_{xyj} xy(r_j + l_j) + \alpha_{xjy} x(r_j + l_j)y + \alpha_{jxy}(r_j + l_j)xy] \end{aligned}$$

with $\alpha_{3x}, \alpha_{x3}, \alpha_{xyj}, \alpha_{xjy}, \alpha_{jxy} \in \mathbb{K}$ and

$$\tag{4.27} \sum_{x=a,b} (\alpha_{x3} x r_3 + \alpha_{3x} r_3 x) + \sum_{\substack{1 \leqslant j \leqslant 2, \\ x,y=a,b}} (\alpha_{xyj} xy r_j + \alpha_{xjy} x r_j y + \alpha_{jxy} r_j xy) = 0.$$

Put (4.8) into (4.27), then the linear independence implies that the coefficients in (4.26) should satisfy the set of linear equations:

$$\tag{4.28} \begin{cases} \alpha_{aa1} + \alpha_{a1a} + \alpha_{1aa} = 0, & \alpha_{a3} + \alpha_{1ba} = 0, & \alpha_{bb2} + \alpha_{b2b} + \alpha_{2bb} = 0, \\ \alpha_{aa2} + \alpha_{1bb} = 0, & \alpha_{a3} - \alpha_{3b} = 0, & \alpha_{b1b} = 0, \\ \alpha_{ab1} + \alpha_{3a} = 0, & \alpha_{ba1} + \alpha_{b1a} = 0, & \alpha_{b2a} + \alpha_{2ba} = 0, \\ \alpha_{ab2} + \alpha_{a2b} = 0, & \alpha_{ba2} - \alpha_{3b} = 0, & \alpha_{b3} - \alpha_{2ab} = 0, \\ \alpha_{a1b} + \alpha_{1ab} = 0, & \alpha_{bb1} + \alpha_{2aa} = 0, & \alpha_{b3} - \alpha_{3a} = 0, \\ \alpha_{a2a} = 0, & & \end{cases}$$

where α's are arranged in the lexicographic order of a, b, 1, 2, 3 with

$$\tag{4.29} a < b < 1 < 2 < 3.$$

The condition $P_3' \cap \mathcal{F}^3(\mathcal{T}) \subseteq P_3$ exactly means that to each basic solution of (4.28) we assign the linear combination

$$(4.30)\qquad \sum_{x=a,b}(\alpha_{x3}xl_3+\alpha_{3x}l_3x)+\sum_{\substack{1\leqslant j\leqslant 2,\\ x,y=a,b}}(\alpha_{xyj}xyl_j+\alpha_{xjy}xl_jy+\alpha_{jxy}l_jxy)\in P_3,$$

that is to say,

$$(4.31)\qquad a^2l_1-al_1a,\quad bal_1-bl_1a,\quad a^2l_1-l_1a^2,\quad al_1b-l_1ab\in P_3,$$

$$(4.32)\qquad abl_2-al_2b,\quad b^2l_2-bl_2b,\quad bl_2a-l_2ba,\quad b^2l_2-l_2b^2\in P_3$$

$$(4.33)\qquad a^2l_2-l_1b^2,\quad b^2l_1-l_2a^2\in P_3,$$

$$(4.34)\qquad abl_1-bl_3-l_2ab-l_3a,\quad al_3+bal_2-l_1ba+l_3b\in P_3.$$

The linear combinations in (4.31) and (4.32) are zero since $\lambda_{12}=\lambda_{21}=0$, while the ones in (4.33) can be expressed as follows:
(4.35)

$$\begin{cases} a^2l_2-l_1b^2=\lambda_{23}(r_1+l_1)-\lambda_{13}(r_2+l_2)-\lambda_{11}a(r_2+l_2)+\lambda_{22}(r_1+l_1)b\in P_3,\\ b^2l_1-l_2a^2=-\lambda_{23}(r_1+l_1)+\lambda_{13}(r_2+l_2)+\lambda_{11}(r_2+l_2)a-\lambda_{22}b(r_1+l_1)\in P_3.\end{cases}$$

By substituting (4.8)–(4.10) into (4.34) and using linear independence, we can check that (4.34) is equivalent to the equalities in (4.11).

(2) By Definition 3.5 (4) (ii), a PBW-deformation $\mathcal{U}$ is a self-symmetric (or symmetric) PBW-deformation of $\mathcal{A}$ if $\mathbf{S}_0(\mathcal{A})=\mathbf{S}_0(\mathcal{U})$ (or $\mathbf{S}_1(\mathcal{A})=\mathbf{S}_1(\mathcal{U})$). Moreover, $\mathbf{S}_0(\mathcal{A})=\mathbf{S}_0(\mathcal{U})$ if and only if the structure coefficients of $\mathcal{U}$ satisfy

$$(4.36)\qquad \begin{cases}\lambda_{11}=\lambda_{22},\quad \lambda_{12}=\lambda_{21},\quad \lambda_{13}=\lambda_{23},\\ \lambda_{31}+\lambda_{34}=0,\quad \lambda_{32}+\lambda_{33}=0,\quad \lambda_{35}+\lambda_{36}=0,\quad \lambda_{37}=0,\end{cases}$$

while $\mathbf{S}_1(\mathcal{A})=\mathbf{S}_1(\mathcal{U})$ if and only if the structure coefficients of $\mathcal{U}$ satisfy

$$(4.37)\qquad \begin{cases}\lambda_{11}=\lambda_{22},\quad \lambda_{12}=\lambda_{21},\quad \lambda_{13}=\lambda_{23},\\ \lambda_{31}+\lambda_{34}=0,\quad \lambda_{32}=\lambda_{33}=0,\quad \lambda_{35}+\lambda_{36}=0,\quad \lambda_{37}=0.\end{cases}$$

Now we can obtain (2) by combining (4.11), (4.36) and (4.37). □

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