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SPECIAL MODULES FOR $R(\mathrm{PSL}(2,q))$

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Abstract. Let R be a fusion ring and $R_{\mathbb{C}} := R \otimes_{\mathbb{Z}} \mathbb{C}$ be the corresponding fusion algebra. We first show that the algebra $R_{\mathbb{C}}$ has only one left (right, two-sided) cell and the corresponding left (right, two-sided) cell module. Then we prove that, up to isomorphism, $R_{\mathbb{C}}$ admits a unique special module, which is 1-dimensional and given by the Frobenius-Perron homomorphism FPdim. Moreover, as an example, we explicitly determine the special module of the interpolated fusion algebra $R(\mathrm{PSL}(2,q)) := r(\mathrm{PSL}(2,q)) \otimes_{\mathbb{Z}} \mathbb{C}$ up to isomorphism, where $r(\mathrm{PSL}(2,q))$ is the interpolated fusion ring with even $q \geqslant 2$.

Keywords: Frobenius-Perron theorem; special module; fusion ring

MSC 2020: 16G99

1. Introduction

Kazhdan and Lusztig in [6] defined the left, right and two-sided cells of Coxeter groups to study representations of Coxeter groups and Hecke algebras. Then in [13], Mazorchuk and Miemietz defined, constructed and described right cell 2-representations of finitary 2-categories. Kildetoft and Mazorchuk found that on the level of the Grothendieck group, a cell 2-representation is a based module over a finite dimensional positively based algebra, and so they gave the definitions of left, right and two-sided cells and the corresponding cell modules over arbitrary finite dimensional positively based algebras. Furthermore, by using Frobenius-Perron theorem, they defined the special modules over any finite dimensional positively based algebras. At last, in some sense, they classified the special modules for arbitrary finite dimensional positively based algebras up to isomorphism, see [7], Section 9.

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However, for a finite dimensional positively based algebra, the module structures of its special modules are not clear. For the works of cell modules of concrete positively based algebras, one can refer [1] or [9].

In [2], Chapter 3, Etingof et al. introduced the concepts of $\mathbb{Z}_+$-rings, unital $\mathbb{Z}_+$-rings and based rings, etc. Let R be a fusion ring with basis $X = \{X_i\colon 1 \leqslant i \leqslant n\}$, and $1 = X_1 \in X$. Then the complexified algebra $R_{\mathbb{C}} = R \otimes_{\mathbb{Z}} \mathbb{C}$ is a positively based algebra with positive basis $X = \{X_i\colon 1 \leqslant i \leqslant n\}$. For a matrix with nonnegative entries, Frobenius-Perron theorem shows that this matrix has a maximal nonnegative eigenvalue. In this sense, one can define a group homomorphism $\mathrm{FPdim}\colon R \to \mathbb{C}$ as follows. For any $X_i \in X$, let $\mathrm{FPdim}(X_i)$ be the maximal nonnegative eigenvalue of the matrix corresponding to the left multiplication by X_i. Etingof et al. showed that FPdim is also a ring homomorphism, and gave many properties of FPdim. This provides an accessible way to study special module of $R_{\mathbb{C}}$.

The family of finite simple groups of Lie type $\mathrm{PSL}(2,q)$, with q prime-power, admits a generic character table depending on whether q is even, $q \equiv 1 \pmod 4$ or $q \equiv -1 \pmod 4$, see [10], Section 3. In [10], by using the Schur orthogonality relations, Liu et al. computed the fusion rules of interpolated fusion ring $r(\mathrm{PSL}(2,q))$. In particular, if q is prime-power, the interpolated fusion ring $r(\mathrm{PSL}(2,q))$ is nothing but the Grothendieck ring of $\mathrm{Rep}(\mathrm{PSL}(2,q))$.

In this paper, we use the properties of FPdim to explicitly classify the cell, cell module and special module (up to isomorphism) of the fusion algebra $R_{\mathbb{C}}$. As an example, we study the special module of $R(\mathrm{PSL}(2,q))$, where $q \geqslant 2$ is even. This paper is organized as follows. In Section 2, we recall the definitions of fusion rings and Frobenius-Perron theorem. In Section 3, we review the concepts of positively based algebras, cell modules and special modules. In Section 4, we prove Lemma 4.1, which states that the fusion algebra $R_{\mathbb{C}}$ has only one left (right, two-sided) cell and the corresponding cell module. And Corollary 4.2 and Theorem 4.3 show that $R_{\mathbb{C}}$ only has a unique special module up to isomorphism, which is 1-dimensional, and $\mathrm{FPdim}\colon R \to \mathbb{C}$ defined in Definition 3.3.3 of [2] is exactly the special representation of $R_{\mathbb{C}}$. Moreover, we state the relations between the Casimir number and the Perron-Frobenius element for a fusion algebra, see Lemma 4.4 and Corollary 4.5. In Section 5, we first compute the Casimir number of $r(\mathrm{PSL}(2,q))$ for even $q \geqslant 2$, see Lemma 5.2 and Corollary 5.5. Then, we use Casimir number to explicitly determine the special module of $R(\mathrm{PSL}(2,q))$, see Theorems 5.7 and 5.8.

2. Fusion rings

In this section, we first recall the definitions of fusion rings, Frobenius-Perron theorem, Frobenius-Perron dimension (FPdim), related results and properties of FPdim and the concepts of Casimir element $c(1)$ of a fusion ring. Moreover, we introduce the interpolated fusion ring $r(\mathrm{PSL}(2,q))$, where $q \geqslant 2$ is even.

Let R be a ring with $\mathbb{Z}_+$-basis $X = \{X_i\colon 1 \leqslant i \leqslant n\}$, that is for any $1 \leqslant i, j \leqslant n$,

$$X_i X_j = \sum_{1 \leqslant k \leqslant n} N_{ij}^k X_k,$$

where $N_{ij}^k \in \mathbb{Z}_+$.

Definition 2.1. A ring R with $\mathbb{Z}_+$-basis $X = \{X_i\colon 1 \leqslant i \leqslant n\}$ is called a *fusion ring* (see [2]) if the followings hold:

(1) $X_1 = 1 \in X$.

(2) There exists a map $i \mapsto i^*$ of the index set $\{1, 2, \ldots, n\}$ such that the induced map $*\colon R \to R$,

$$x = \sum_{1 \leqslant i \leqslant n} k_i X_i \mapsto x^* = \sum_{1 \leqslant i \leqslant n} k_i X_{i^*}, \quad k_i \in \mathbb{Z},$$

is an anti-involution of the ring R.

(3) There exists a group homomorphism defined by

$$\tau(X_i) = \begin{cases} 1, & i = 1, \\ 0, & \text{otherwise}, \end{cases}$$

such that

$$\tau(X_i X_j) = \delta_{ij^*},$$

where δ_{ij^*} is the Kronecker symbol.

Lemma 2.2. *Each fusion ring R is transitive, that is, for any $W, Z \in X$ there exist $Y_1, Y_2 \in X$ such that WY_1 and Y_2W contain Z with nonzero coefficient.*

P r o o f. It follows from [2], Definition 3.3.1, Exercise 3.3.2. □

Now, we are ready to recall Frobenius-Perron theorem, see [2], [3], [4], [5], [14]. It is a crucial tool in the definitions of special modules, FPdim of a fusion ring and its basis elements.

Theorem 2.3 (Frobenius-Perron). *Let $M \in \mathrm{Mat}_{k\times k}(\mathbb{R}_{\geqslant 0})$. Then there is a non-negative real number λ such that:*

(1) *The number λ is an eigenvalue of M.*

(2) *Any other eigenvalue $\mu \in \mathbb{C}$ satisfies $|\mu| \leqslant \lambda$.*

Let $M \in \mathrm{Mat}_{k\times k}(\mathbb{R}_{>0})$. *Then there is a positive real number* λ *such that:*

(3) *The number* λ *is an eigenvalue of* M.

(4) *Any other eigenvalue* $\mu \in \mathbb{C}$ *of* M *satisfies* $|\mu| < \lambda$.

(5) *The eigenvalue* λ *has algebraic (and hence also geometric) multiplicity* 1.

(6) *There is a* $v \in \mathbb{R}^k_{>0}$ *such that* $Mv = \lambda v$. *There is also a* $\widehat{v} \in \mathbb{R}^k_{>0}$ *such that* $\widehat{v}M = \lambda\widehat{v}^T$ ($\widehat{v}^T$ *is the transpose of* $\widehat{v}$).

(7) *Any* $w \in \mathbb{R}^k_{\geqslant 0}$ *which is an eigenvector of* M *(with some eigenvalue) is a scalar multiple of* v, *and similarly for* $\widehat{v}$.

The eigenvalue λ is called the *Perron-Frobenius eigenvalue* of M, denoted by $\mathrm{FPdim}(M)$. In particular, if M has strictly positive entries, then the eigenvector v of the Perron-Frobenius eigenvalue $\mathrm{FPdim}(M)$ is called the *Perron-Frobenius eigenvector*. In this case, it follows from Theorem 2.3 (5) and (7) that $\mathrm{FPdim}(M)$ is unique (the algebraic multiplicity is 1), and the Perron-Frobenius eigenvector is unique up to a positive scalar.

By Frobenius-Perron theorem, one can define the FPdim of a fusion ring and its basis elements. Let R be a fusion ring with $\mathbb{Z}_+$-basis $X = \{X_i\colon 1 \leqslant i \leqslant n\}$. Define $\mathbf{N}_{X_i}$ to be the matrix given by the left multiplication of X_i with respect to a basis $X_1, X_2, \ldots, X_n$. That is,

$$X_i(X_1, X_2, \ldots, X_n) = (X_1, X_2, \ldots, X_n)\mathbf{N}_{X_i}. \tag{2.1}$$

Definition 2.4. The maximal real eigenvalue λ_i of $\mathbf{N}_{X_i}$ is called the FPdim of the basis element X_i, denoted by $\mathrm{FPdim}(X_i) = \lambda_i$, i.e., $\mathrm{FPdim}(X_i) = \mathrm{FPdim}(\mathbf{N}_{X_i})$ (the Perron-Frobenius eigenvalue of $\mathbf{N}_{X_i}$). The FPdim of the fusion ring R is defined by $\mathrm{FPdim}(R) = \sum\limits_{i=1}^{n} \mathrm{FPdim}(X_i)^2$.

In fact, $\mathrm{FPdim}\colon R \to \mathbb{C}$ is a ring homomorphism. Furthermore, we have the following properties and results of $R_{\mathbb{C}}$ and FPdim, see [2], Propositions 3.3.4, 3.3.6, 3.3.11.

Lemma 2.5. *Let* R *be a fusion ring with* $\mathbb{Z}_+$*-basis* $X = \{X_i\colon 1 \leqslant i \leqslant n\}$ *and* $\mathrm{FPdim}\colon R \to \mathbb{C}$ *be defined as above. Then the following statements hold.*

(1) *The number* $\mathrm{FPdim}(X_i)$ *is an algebraic integer and* $\mathrm{FPdim}(X_i) \geqslant 1$.

(2) *There exists a unique, up to scaling, nonzero element* $c \in R_{\mathbb{C}}$ *such that* $Wc = \mathrm{FPdim}(W)c$ *for all* $W \in R$. *Moreover,* $cY = \mathrm{FPdim}(Y)c$ *for all* $Y \in R$.

(3) *An element* $c \in R \otimes_{\mathbb{Z}} \mathbb{R}$ *as in* (2) *will be called a regular element of* R.

(4) *The element* $r = \sum\limits_{i=1}^{n} \mathrm{FPdim}(X_i)X_i$ *is a regular element of* R.

Throughout the following, we write X_i^* for X_{i^*} (see Definition 2.1) simply. Recall that the matrix $\mathbf{N}_{X_i}$ is determined by the left multiplication of X_i given in (2.1). We call a basis element X_i self-dual if $X_i^* = X_i$. Equivalently, X_i is self-dual if and only if the matrix $\mathbf{N}_{X_i}$ is symmetric, if and only if the unit element 1 is contained in the decomposition formulas of X_i^2, which corresponds to the fact that in a fusion category the unit object is a direct summand of the tensor product of an object with its dual.

The Casimir operator c (see [11], Section 3.1) of the fusion ring R is a map from R to its center $Z(R)$ defined by

$$c(x) = \sum_{i=1}^{n} X_i x X_i^* \quad \forall\, x \in R.$$

Definition 2.6. Let R be a fusion ring with pair of dual bases $\{X_i\colon 1 \leqslant i \leqslant n\}$ and $\{X_i^*\colon 1 \leqslant i \leqslant n\}$. Then the element $c(1) = \sum\limits_{i=1}^{n} X_i X_i^*$ is called the Casimir element of R.

Lemma 2.7. *Let R be a fusion ring with $\mathbb{Z}_+$-basis $\{X_i\colon 1 \leqslant i \leqslant n\}$. Then* $\mathrm{FPdim}(R)$ *is the Perron-Frobenius eigenvalue of the matrix* $[c(1)]$ *determined by the left multiplication of Casimir element* $c(1)$.

P r o o f. It follows from [2], Propositions 3.3.6 (1), 3.3.9 that $\mathrm{FPdim}\colon R \to \mathbb{C}$ is a ring homomorphism and $\mathrm{FPdim}(X) = \mathrm{FPdim}(X^*)$. Notice that $c(1) = \sum\limits_{i=1}^{n} X_i X_i^*$. Then

$$\begin{aligned}
\mathrm{FPdim}([c(1)]) = \mathrm{FPdim}(c(1)) &= \mathrm{FPdim}\Bigl(\sum_{i=1}^{n} X_i X_i^*\Bigr) \\
&= \sum_{i=1}^{n} \mathrm{FPdim}(X_i X_i^*) = \sum_{i=1}^{n} \mathrm{FPdim}(X_i)\mathrm{FPdim}(X_i^*) \\
&= \sum_{i=1}^{n} \mathrm{FPdim}(X_i)^2 = \mathrm{FPdim}(R).
\end{aligned}$$

This completes the proof. □

In the following, we introduce the interpolated fusion ring $r(\mathrm{PSL}(2,q))$, where $q \geqslant 2$ is even. When $q \geqslant 4$, the interpolated fusion ring $r(\mathrm{PSL}(2,q))$ has $q+1$ basis elements $\{x_{1,1}, x_{q-1,c}, x_{q,1}, x_{q+1,c'}\colon c \in \{1,2,\dots,q/2\}, c' \in \{1,2,\dots,q/2-1\}\}$, where $x_{d,c}$, $d \in \{1, q-1, q, q+1\}$, (or $x_{d,c'}$) is the cth (or c'th) basis element of FPdim d, and $x_{1,1}$ is the unit element. Moreover, $\mathrm{FPdim}(r(\mathrm{PSL}(2,q))) = q(q^2-1)$. In the following, for any even $q \geqslant 4$, we always denote the sets $\{1,2,\dots,q/2\}$ and $\{1,2,\dots,q/2-1\}$ by I_q and J_q, respectively.

Definition 2.8. The interpolated fusion ring $r(\mathrm{PSL}(2,q))$ has $q+1$ basis elements $\{x_{1,1}, x_{q-1,c}, x_{q,1}, x_{q+1,c'}\colon\ c \in I_q, c' \in J_q\}$, and the fusion rules of $r(\mathrm{PSL}(2,q))$ are given by (see [10], Section 4.1):

$$\begin{aligned}
x_{q-1,c_1}x_{q-1,c_2} &= \delta_{c_1,c_2}x_{1,1} + \sum_{\substack{c_3\in I_q\\ c_1+c_2+c_3\neq q+1\\ \text{and } 2\max(c_1,c_2,c_3)}} x_{q-1,c_3} + (1-\delta_{c_1,c_2})x_{q,1} + \sum_{c'\in J_q} x_{q+1,c'},\\
x_{q-1,c_1}x_{q,1} &= \sum_{c_2\in I_q}(1-\delta_{c_1,c_2})x_{q-1,c_2} + x_{q,1} + \sum_{c'\in J_q} x_{q+1,c'},\\
x_{q-1,c_1}x_{q+1,c_2} &= \sum_{c\in I_q} x_{q-1,c} + x_{q,1} + \sum_{c'\in J_q} x_{q+1,c'},\\
x_{q,1}x_{q,1} &= x_{1,1} + \sum_{c\in I_q} x_{q-1,c} + x_{q,1} + \sum_{c'\in J_q} x_{q+1,c'},\\
x_{q,1}x_{q+1,c_1} &= \sum_{c\in I_q} x_{q-1,c} + x_{q,1} + \sum_{c'\in J_q}(1+\delta_{c_1,c_2})x_{q+1,c'},\\
x_{q+1,c_1}x_{q+1,c_2} &= \delta_{c_1,c_2}x_{1,1} + \sum_{c\in I_q} x_{q-1,c} + (1+\delta_{c_1,c_2})x_{q,1}\\
&\quad + \sum_{\substack{c_3\in J_q\\ c_1+c_2+c_3\neq q-1\\ \text{and } 2\max(c_1,c_2,c_3)}} x_{q+1,c_3} + 2\sum_{\substack{c_4\in J_q\\ c_1+c_2+c_4=q-1\\ \text{or } 2\max(c_1,c_2,c_4)}} x_{q+1,c_4}.
\end{aligned}$$

In particular, when $q=2$, $r(\mathrm{PSL}(2,2))$ has 3 basis elements $x_{1,1}$, $x_{1,2}$ and $x_{2,1}$, and the fusion rules are determined by

$$x_{1,2}^2 = x_{1,1}, \quad x_{1,2}x_{2,1} = x_{2,1}, \quad x_{2,1}^2 = x_{1,1} + x_{1,2} + x_{2,1}.$$

By the fusion rules of $r(\mathrm{PSL}(2,q))$ (even $q \geqslant 2$) given above, we have the following.

Lemma 2.9. *The basis elements* $\{x_{1,1}, x_{q-1,c}, x_{q,1}, x_{q+1,c'}\colon\ c \in I_q, c' \in J_q\}$ *of* $r(\mathrm{PSL}(2,q))$ *are self-dual, where* $q \geqslant 4$ *is even. For* $q = 2$*, the basis elements* $\{x_{1,1}, x_{1,2}, x_{2,1}\}$ *of* $r(\mathrm{PSL}(2,2))$ *are self-dual.*

3. Cell and special structures of positively based algebras

In this section, we recall the definitions of positively based algebras, cell modules and special modules in the sense of Kildetoft-Mazorchuk, see [7].

Let A be an algebra over $\mathbb{C}$ with basis $B = \{b_i\colon\ i \in \mathcal{I}\}$. Here we always assume that $1 \in B$. The basis B is called positive if all structure constants of A with respect to B are nonnegative real numbers. That is, for all $i, j \in \mathcal{I}$,

$$b_i b_j = \sum_{k \in \mathcal{I}} r_{ij}^k b_k,$$

where $r_{ij}^k \in \mathbb{R}_{\geqslant 0}$ for all i, j, k. An algebra with fixed positive basis is called a *positively based algebra* (see [7], Subsection 2.1). It is obvious that for a fusion ring R with $\mathbb{Z}_+$-basis $X = \{X_i\colon 1 \leqslant i \leqslant n\}$, the fusion algebra $R_{\mathbb{C}}$ is a positively based algebra with positive basis $X = \{X_i\colon 1 \leqslant i \leqslant n\}$.

Let A be a positively based algebra with positive basis $B = \{b_i\colon i \in \mathcal{I}\}$. Define the multioperation $\star\colon \mathcal{I} \times \mathcal{I} \to 2^{\mathcal{I}}$ for any $i, j \in \mathcal{I}$,

$$i \star j := \{k\colon r_{ij}^k > 0\},$$

where $b_i b_j = \sum\limits_{k \in \mathcal{I}} r_{ij}^k b_k$. Thus, $(\mathcal{I}, \star)$ turns into a multisemigroup, see [8], Subsection 3.7. For $i, j \in \mathcal{I}$, we set $i \leqslant_L j$ if there exists an $s \in \mathcal{I}$ such that $j \in s \star i$. It is easy to see that $\leqslant_L$ is a partial preorder on $\mathcal{I}$. We write $i \sim_L j$ if $i \leqslant_L j$ and $j \leqslant_L i$. Then $\sim_L$ becomes an equivalence relation on $\mathcal{I}$. The equivalence classes for $\sim_L$ are called *left cells*. Furthermore, the preorder $\leqslant_L$ also induces a genuine partial order on the set of all left cells in $\mathcal{I}$, denoted also by $\leqslant_L$ without ambiguity. Similarly, using multiplication by s on the right, one can define the right preorder $\leqslant_R$, the corresponding equivalence relation $\sim_R$ and the right cells. Moreover, one can define the two-sided preorder $\leqslant_J$, the corresponding equivalence relation $\sim_J$ and the two-sided cells, using multiplication by s on the left and by t on the right. We write $i <_L j$ if $i \leqslant_L j$ and $i \nsim_L j$, and similarly for $i <_R j$ and $i <_J j$.

Definition 3.1. A two-sided cell $\mathcal{J}$ is called *idempotent* if there exist $i, j, k \in \mathcal{J}$ such that $k \in i \star j$.

Let A be a positively based algebra with positive basis $B = \{b_i\colon i \in \mathcal{I}\}$. Let $\mathcal{L}$ be a left cell in $\mathcal{I}$ and $\overline{\mathcal{L}}$ be the union of all left cells $\mathcal{L}'$ in $\mathcal{I}$ with $\mathcal{L} \leqslant_L \mathcal{L}'$. Define $\underline{\overline{\mathcal{L}}} := \overline{\mathcal{L}} \setminus \mathcal{L}$. Consider the $\mathbb{C}$-submodule $M_{\mathcal{L}}$ of A spanned by all b_j, $j \in \overline{\mathcal{L}}$, and the $\mathbb{C}$-submodule $N_{\mathcal{L}}$ of $M_{\mathcal{L}}$ spanned by all b_j, $j \in \underline{\overline{\mathcal{L}}}$. Kildetoft and Mazorchuk proved that both $M_{\mathcal{L}}$ and $N_{\mathcal{L}}$ are A-submodules of the regular left module ${}_AA$, see [7], Proposition 1. The left cell module of A associated to $\mathcal{L}$ is defined as the quotient module $C_{\mathcal{L}} = M_{\mathcal{L}}/N_{\mathcal{L}}$. When $\underline{\overline{\mathcal{L}}} = \emptyset$, we regard $N_{\mathcal{L}} = 0$ and $C_{\mathcal{L}} = M_{\mathcal{L}}$. Similarly, one can define right cell modules and two-sided cell modules.

Lemma 3.2. *Let A be a positively based algebra with positive basis $B = \{b_i\colon i \in \mathcal{I}\}$. Then the left cell module $C_{\mathcal{L}} = {}_AA$ if and only if $\mathcal{L} = \mathcal{I}$, see* [1], *Lemma* 2.1.

Let A be a finite dimensional $\mathbb{C}$-algebra and V be a finite dimensional A-module with fixed basis $\boldsymbol{v} = \{v_1, v_2, \ldots, v_n\}$. Then the pair $(V, \boldsymbol{v})$ is called a *based A-module.*

An element $a \in A$ is said to be a Perron-Frobenius element for a based A-module $(V, \boldsymbol{v})$ if all entries of the matrix of the action of a on V with respect to this basis $\boldsymbol{v}$ are positive real numbers.

Given a Perron-Frobenius element $a \in A$ for a based module $(V, \boldsymbol{v})$, let λ be the Perron-Frobenius eigenvalue of the linear operator a on V. Then we have the following.

Definition 3.3. A simple A-subquotient L of V is called a *special subquotient* with respect to a if λ is an eigenvalue of a acting on L.

Remark 3.4. Given a Perron-Frobenius element $a \in A$ for a based A-module $(V, \boldsymbol{v})$, there is a unique, up to isomorphism, special subquotient L of V with respect to a, see [7], Corollary 3.

Let A be a positively based $\mathbb{C}$-algebra with positive basis $B = \{b_i \colon i \in \mathcal{I}\}$. Given a left cell $\mathcal{L}$ in $\mathcal{I}$, consider the corresponding left cell module $C_{\mathcal{L}}$ defined as above. Denote by $B_{\mathcal{L}}$ the standard basis of $C_{\mathcal{L}}$ given by the image of the elements b_i, where $i \in \mathcal{L}$. Set $\boldsymbol{c} \in \mathbb{R}^{\mathcal{I}}_{>0} = \{c_i \in \mathbb{R}_{>0} \colon i \in \mathcal{I}\}$, and

$$a(\boldsymbol{c}) = \sum_{i \in \mathcal{I}} c_i b_i \in A. \tag{3.1}$$

Kildetoft and Mazorchuk showed that the element $a(\boldsymbol{c})$ is a Perron-Frobenius element for the based module $(C_{\mathcal{L}}, B_{\mathcal{L}})$, see [7], Lemma 4. Thus, for each left cell $\mathcal{L}$ and each $\boldsymbol{c} \in \mathbb{R}^{\mathcal{I}}_{>0}$, one can consider the corresponding special subquotient $L_{\mathcal{L},\boldsymbol{c}}$ of $C_{\mathcal{L}}$. It follows from [7], Theorem 5 that for a fixed left cell and any $\boldsymbol{c}, \boldsymbol{c}' \in \mathbb{R}^{\mathcal{I}}_{>0}$, $L_{\mathcal{L},\boldsymbol{c}} \cong L_{\mathcal{L},\boldsymbol{c}'}$. Since $L_{\mathcal{L},\boldsymbol{c}}$ is independent of $\boldsymbol{c}$, we write $L_{\mathcal{L}}$ for $L_{\mathcal{L},\boldsymbol{c}}$ simply. Furthermore, Theorem 6 of [7] implies that $L_{\mathcal{L}} \cong L_{\mathcal{L}'}$ if two left cells $\mathcal{L}$ and $\mathcal{L}'$ belong to the same two-sided cell.

Let (A, B) be a positively based algebra and $(V, \boldsymbol{v})$ a based A-module. We will say that $(V, \boldsymbol{v})$ is positively based if for any $b_i \in B$ and any $v_s \in \boldsymbol{v}$, the element $b_i \cdot v_s$ is a linear combination of elements in $\boldsymbol{v}$ with nonnegative real coefficients. For example, the left regular A-module ${}_AA$ is positively based with respect to the basis B. For $v_s, v_t \in \boldsymbol{v}$, we write $v_s \to v_t$ if there exists a $b_i \in B$ such that the coefficient at v_t in $b_i \cdot v_s$ is nonzero. The relation $\to$ is, clearly, reflexive and transitive. A based A-module $(V, \boldsymbol{v})$ will be called *transitive* if $\to$ is the full relation. For example, for each left cell $\mathcal{L}$, $(C_{\mathcal{L}}, B_{\mathcal{L}})$ is a transitive A-module. For a transitive A-module $(V, \boldsymbol{v})$, we have the following, see [7], Section 9.4, Corollary 23.

Definition 3.5. A simple A-module is called *special* if it is isomorphic to a special subquotient for a transitive A-module.

Lemma 3.6. *There is a one-to-one correspondence between the set of isomorphism classes of special A-modules and the set of idempotent two-sided cells for A.*

In particular, if A is semi-simple, then we have the following, see [7], Proposition 13.

Lemma 3.7. *Let A be a semi-simple positively based algebra. Then the following statements hold.*

(1) *Each two-sided cell for A is idempotent.*

(2) *Let $\mathcal{L}$ be a left cell and $\mathcal{J}$ a two-sided cell containing $\mathcal{L}$. Then the dimension of $L_{\mathcal{L}}$ equals the number of left cells in $\mathcal{J}$.*

4. Special module for fusion algebra

Let R be a fusion ring with $\mathbb{Z}_+$-basis $X = \{X_i\colon\ 1 \leqslant i \leqslant n\}$, where $X_1 = 1$. In this section, we first classify the special module up to isomorphism of the complexified fusion algebra $R_{\mathbb{C}}$. We then state the relations between the Casimir element $c(1)$ and Perron-Frobenius element for a fusion algebra.

Lemma 4.1. *The fusion algebra $R_{\mathbb{C}}$ has unique left, right and two-sided cell $\boldsymbol{n}$.*

Proof. We first show that $R_{\mathbb{C}}$ only has one left cell $\boldsymbol{n}$. On one hand, for any $i \in \boldsymbol{n}$, $X_i = X_i X_1$, i.e., $i \in i \star 1$, and so $1 \leqslant_L i$. On the other hand, $X_i^* X_i = X_1 + \ldots$, i.e., $1 \in i^* \star i$, and hence $i \leqslant_L 1$. Thus, we prove that $1 \sim_L i$ for any $i \in \boldsymbol{n}$, that is, the unique left cell $\mathcal{L} = \boldsymbol{n}$. Similarly, one can prove that $R_{\mathbb{C}}$ has a unique right, two-sided cell $\boldsymbol{n}$. □

Corollary 4.2. *The fusion algebra $R_{\mathbb{C}}$ has only one left cell module ${}_{R_{\mathbb{C}}}R_{\mathbb{C}}$ (the regular left module) and only one special module up to isomorphism. Furthermore, the special module is of dimension* 1.

Proof. By Lemmas 3.2 and 4.1, one can immediately know that $R_{\mathbb{C}}$ has only one left cell module ${}_{R_{\mathbb{C}}}R_{\mathbb{C}}$. It follows from [2], Corollary 3.3.7 or [12], 1.2 (a) that $R_{\mathbb{C}}$ is semi-simple. Hence, by Lemma 3.7 (1), the two-sided cell $\mathcal{J} = \boldsymbol{n}$ is idempotent, and so by Lemma 3.6, $R_{\mathbb{C}}$ has only one special cell module up to isomorphism. Moreover, Lemma 3.7 (2) shows that the dimension of this special module is 1. Thus, the proof is finished. □

Recall that for the based module $({}_{R_{\mathbb{C}}}R_{\mathbb{C}}, X_{\boldsymbol{n}})$, any $\boldsymbol{c} \in \mathbb{R}^{\boldsymbol{n}}_{>0}$, $a(\boldsymbol{c}) = \sum\limits_{i=1}^{n} c_i X_i$ is a Perron-Frobenius element for $({}_{R_{\mathbb{C}}}R_{\mathbb{C}}, X_{\boldsymbol{n}})$. Then we have the following.

Theorem 4.3. *$\mathbb{C}r$ is a special module of $R_{\mathbb{C}}$, where $r = \sum\limits_{i=1}^{n} \mathrm{FPdim}(X_i) X_i$ is the regular element given in Lemma* 2.5. *Moreover,* FPdim: $R \to \mathbb{C}$ *defined in Lemma* 2.5 *is exactly the unique special representation up to isomorphism.*

P r o o f. By (3.1), we know that $c_{\boldsymbol{n}} := \sum_{i=1}^{n} X_i$ is a Perron-Frobenius element for $({}_{R_{\mathbb{C}}}R_{\mathbb{C}}, X_{\boldsymbol{n}})$. Furthermore, it follows from Lemma 2.5 (2) that $c_{\boldsymbol{n}} \cdot r = \mathrm{FPdim}(c_{\boldsymbol{n}})r = \mathrm{FPdim}([c_{\boldsymbol{n}}])r$. Now, in order to prove that $\mathbb{C}r$ is a special module of $R_{\mathbb{C}}$, it remains to show that $\mathbb{C}r$ is a simple $R_{\mathbb{C}}$-module. By Lemma 2.5 (2) again, $X_i \cdot r = \mathrm{FPdim}(X_i)r$ for any $i \in \boldsymbol{n}$, which shows that $\mathbb{C}r$ is indeed a module over $R_{\mathbb{C}}$. Hence, $\mathbb{C}r$ is a special subquotient of $({}_{R_{\mathbb{C}}}R_{\mathbb{C}}, X_{\boldsymbol{n}})$, and $\mathrm{FPdim}\colon R \to \mathbb{C}$ is the unique special representation of $R_{\mathbb{C}}$ up to isomorphism. □

The following results state the relations between the Casimir element and the Perron-Frobenius element for a fusion algebra.

Lemma 4.4. *The Casimir element $c(1)$ of R is a Perron-Frobenius element for the left cell module $({}_{R_{\mathbb{C}}}R_{\mathbb{C}}, X_{\boldsymbol{n}})$ if and only if $[c(1)]$ has strictly positive entries.*

P r o o f. It follows from a straightforward verification. □

Corollary 4.5. *If the coefficient of each basis element in $c(1)$ is not 0, then $c(1)$ is a Perron-Frobenius element for $({}_{R_{\mathbb{C}}}R_{\mathbb{C}}, X_{\boldsymbol{n}})$.*

P r o o f. It follows from the transitivity of a fusion ring, see Lemma 2.2. □

5. Special module for $R(\mathrm{PSL}(2, q))$

In this section, for the case of interpolated fusion algebra $R(\mathrm{PSL}(2, q))$, we set the Casimir number $c(1)_q$ as the Perron-Frobenius element to explicitly determine the special module of $R(\mathrm{PSL}(2, q))$ up to isomorphism. For simplicity, we write $r_q := r(\mathrm{PSL}(2, q))$ and $R_q := R(\mathrm{PSL}(2, q))$.

From Section 2 we know that for each even $q \geqslant 4$, R_q is a positively based algebra with positive basis $U := \{x_{1,1}, x_{q-1,c}, x_{q,1}, x_{q+1,c'}\colon c \in I_q, c' \in J_q\}$. In this case, denote the set $\{(1,1), (q-1,c), (q,1), (q+1,c')\colon c \in I_q, c' \in J_q\}$ by $\boldsymbol{q}$. When $q = 2$, we write $\mathbf{2} = \{(1,1), (1,2), (2,1)\}$. Then by Lemma 4.1, we have the following result.

Corollary 5.1. *For each even $q \geqslant 2$, the fusion algebra R_q has only one left cell $\boldsymbol{q}$, and a unique left cell module ${}_{R_q}R_q$.*

Next, we will compute the Casimir element $c(1)_q$ of r_q and prove that $c(1)_q$ is a Perron-Frobenius element for $({}_{R_q}R_q, U_{\boldsymbol{q}})$ for each even $q \geqslant 2$.

Lemma 5.2. *When $q = 2$, $c(1)_2 = 3x_{1,1} + x_{1,2} + x_{2,1}$.*

P r o o f. It follows from a straightforward computation. □

Throughout the following, unless otherwise stated, assume that $q \geqslant 4$ is even. We first give two lemmas, which will be useful in the study of the Casimir element $c(1)_q$ and the special module of R_q.

Lemma 5.3. *The following statements hold:*

(1) $\sum\limits_{c_1\in I_q} x^2_{q-1,c_1} = \frac{1}{2}qx_{1,1} + (\frac{1}{2}q-1)\sum\limits_{c\in I_q} x_{q-1,c} + \frac{1}{2}q\sum\limits_{c'\in J_q} x_{q+1,c'}$;

(2) $x^2_{q,1} = x_{1,1} + \sum\limits_{c\in I_q} x_{q-1,c} + x_{q,1} + \sum\limits_{c'\in J_q} x_{q+1,c'}$;

(3) $\sum\limits_{c_2\in J_q} x^2_{q+1,c_2} = (\frac{1}{2}q-1)x_{1,1} + (\frac{1}{2}q-1)\sum\limits_{c\in I_q} x_{q-1,c} + (q-2)x_{q,1} + \frac{1}{2}q\sum\limits_{c'\in J_q} x_{q+1,c'}$.

P r o o f. The relation (2) follows from a straightforward computation. We prove (1) and (3). For any $c_1 \in I_q$ we have

$$x^2_{q-1,c_1} = x_{1,1} + \sum_{c\in I_q} x_{q-1,c} - \sum_{c_3\in\{q+1-2c_1,2c_1\}} x_{q-1,c_3} + \sum_{c'\in J_q} x_{q+1,c'}.$$

Hence,

$$\sum_{c_1\in I_q} x^2_{q-1,c_1} = \frac{q}{2}x_{1,1} + \frac{q}{2}\sum_{c\in I_q} x_{q-1,c} - \sum_{\substack{c_3\in\{q+1-2c_1,2c_1\}\\ c_1\in I_q}} x_{q-1,c_3} + \frac{q}{2}\sum_{c'\in J_q} x_{q+1,c'}. \tag{5.1}$$

It is easy to check that $\bigcup\limits_{c_1\in I_q}\{q+1-2c_1, 2c_1\} = \{1,2,\dots,q\}$. Since $c_3 \in I_q$,

$$\sum_{\substack{c_3\in\{q+1-2c_1,2c_1\}\\ c_1\in I_q}} x_{q-1,c_3} = \sum_{c\in I_q} x_{q-1,c}.$$

Thus, (5.1) becomes

$$\sum_{c_1\in I_q} x^2_{q-1,c_1} = \frac{q}{2}x_{1,1} + \Big(\frac{q}{2}-1\Big)\sum_{c\in I_q} x_{q-1,c} + \frac{q}{2}\sum_{c'\in J_q} x_{q+1,c'}.$$

This shows (1). For any $c_2 \in J_q$ we have

$$x^2_{q+1,c_2} = x_{1,1} + \sum_{c\in I_q} x_{q-1,c} + 2x_{q,1} + \sum_{c'\in J_q} x_{q+1,c'} + \sum_{c_4\in\{q-1-2c_2,2c_2\}} x_{q+1,c_4}.$$

Similarly to (5.1), we have

$$\begin{aligned}\sum_{c_2\in J_q} x^2_{q+1,c_2} = \Big(\frac{q}{2}-1\Big)x_{1,1} + \Big(\frac{q}{2}-1\Big)\sum_{c\in I_q} x_{q-1,c} + (q-2)x_{q,1}\\ + \Big(\frac{q}{2}-1\Big)\sum_{c'\in J_q} x_{q+1,c'} + \sum_{\substack{c_4\in\{q-1-2c_2,2c_2\}\\ c_2\in J_q}} x_{q+1,c_4}.\end{aligned} \tag{5.2}$$

It is not difficult to check that $\bigcup_{c_2\in J_q}\{q-1-2c_2, 2c_2\} = \{1,2,\ldots,q-2\}$. Since $c_4 \in J_q$,

$$\sum_{\substack{c_4\in\{q-1-2c_2,2c_2\}\\ c_2\in J_q}} x_{q+1,c_4} = \sum_{c'\in J_q} x_{q+1,c'}.$$

Hence, (5.2) becomes

$$\sum_{c_2\in J_q} x^2_{q+1,c_2} = \Big(\frac{q}{2}-1\Big)x_{1,1} + \Big(\frac{q}{2}-1\Big)\sum_{c\in I_q} x_{q-1,c} + (q-2)x_{q,1} + \frac{q}{2}\sum_{c'\in J_q} x_{q+1,c'}.$$

This shows (3). □

Lemma 5.4. *The following statements hold:*

(1) *For any $c_1 \in I_q$,*

$$x_{q-1,c_1}\sum_{c\in I_q} x_{q-1,c} = x_{1,1} + \Big(\frac{q}{2}-1\Big)x_{q-1,c_1} + \Big(\frac{q}{2}-2\Big)\sum_{c_2\in I_q\setminus\{c_1\}} x_{q-1,c_2} + \Big(\frac{q}{2}-1\Big)x_{q,1} + \frac{q}{2}\sum_{c'\in J_q} x_{q+1,c'}.$$

(2) *For any $c_3 \in J_q$,*

$$x_{q+1,c_3}\sum_{c'\in J_q} x_{q+1,c'} = x_{1,1} + \Big(\frac{q}{2}-1\Big)\sum_{c\in I_q} x_{q-1,c} + \frac{q}{2}x_{q,1} + \frac{q}{2}x_{q+1,c_3} + \Big(\frac{q}{2}+1\Big)\sum_{c_4\in J_q\setminus\{c_3\}} x_{q+1,c_4}.$$

P r o o f. For any $c_1, c_5 \in I_q$ we have

$$x_{q-1,c_1}x_{q-1,c_5} = \delta_{c_1,c_5}x_{1,1} + \sum_{c\in I_q} x_{q-1,c} - \sum_{\substack{c_6\in I_q\\ c_1+c_5+c_6=q+1\\ \text{or } 2\max(c_1,c_5,c_6)}} x_{q-1,c_6} + (1-\delta_{c_1,c_5})x_{q,1} + \sum_{c'\in J_q} x_{q+1,c'}.$$

Then

$$x_{q-1,c_1}\sum_{c_5\in I_q} x_{q-1,c_5} = x_{1,1} + \frac{q}{2}\sum_{c\in I_q} x_{q-1,c} - \sum_{c_5\in I_q}\sum_{\substack{c_6\in I_q\\ c_1+c_5+c_6=q+1\\ \text{or } 2\max(c_1,c_5,c_6)}} x_{q-1,c_6} + \Big(\frac{q}{2}-1\Big)x_{q,1} + \frac{q}{2}\sum_{c'\in J_q} x_{q+1,c'}.$$

We need to consider $c_1+c_5+c_6 = q+1$ or $c_1+c_5+c_6 = 2\max(c_1,c_5,c_6)$ when c_5 runs through I_q.

▷ $c_1 + c_5 + c_6 = q + 1 \Rightarrow c_6 = q + 1 - c_1 - c_5$: It is easy to see that when $c_5 = \frac{1}{2}q$, $\frac{1}{2}q - 1, \dots, \frac{1}{2}q - c_1 + 1$,

$$c_6 = \frac{q}{2} - c_1 + 1, \frac{q}{2} - c_1 + 2, \dots, \frac{q}{2}, \tag{5.3}$$

and in the case when $c_5 < \frac{1}{2}q - c_1 + 1$, we have $q + 1 - c_1 - c_5 > \frac{1}{2}q$.

▷ If $c_1 + c_5 + c_6 = 2\max(c_1, c_5, c_6) = 2\max(c_1, c_5)$, then $c_6 = |c_1 - c_5|$: When $c_5 = c_1 - 1$, $c_1 - 2, \dots, 1$,

$$c_6 = 1, 2, \dots, c_1 - 1. \tag{5.4}$$

In the case when $c_5 = c_1 + 1$, $c_1 + 2, \dots, \frac{1}{2}q$,

$$c_6 = 1, 2, \dots, \frac{q}{2} - c_1, \tag{5.5}$$

and if $c_5 = c_1$, $|c_1 - c_5| = 0$.

▷ If $c_1 + c_5 + c_6 = 2\max(c_1, c_5, c_6) = 2c_6$, then $c_6 = c_1 + c_5$: When $c_5 = 1, 2, \dots, \frac{1}{2}q - c_1$,

$$c_6 = c_1 + 1, c_1 + 2, \dots, \frac{q}{2}. \tag{5.6}$$

In the case when $\frac{1}{2}q - 1 < c_5 \leqslant \frac{1}{2}q$, we have $\frac{1}{2}q < c_1 + c_5 \leqslant \frac{1}{2}q + c_1$.

Thus, when c_5 runs over I_q, (5.3) and (5.5) show that

$$c_6 = 1, 2, \dots, \frac{q}{2},$$

and (5.4), (5.6) imply that

$$c_6 = 1, 2, \dots, c_1 - 1, c_1 + 1, \dots \frac{q}{2}.$$

Summarizing the discussion above, we have

$$\sum_{c_5 \in I_q} \sum_{\substack{c_6 \in I_q \\ c_1 + c_5 + c_6 = q+1 \\ \text{or } 2\max(c_1, c_5, c_6)}} x_{q-1, c_6} = x_{q-1, c_1} + 2 \sum_{c_2 \in I_q \setminus \{c_1\}} x_{q-1, c_2}.$$

Hence,

$$\begin{aligned} x_{q-1, c_1} \sum_{c \in I_q} x_{q-1, c} = x_{1,1} &+ \left(\frac{q}{2} - 1\right) x_{q-1, c_1} + \left(\frac{q}{2} - 2\right) \sum_{c_2 \in I_q \setminus \{c_1\}} x_{q-1, c_2} \\ &+ \left(\frac{q}{2} - 1\right) x_{q,1} + \frac{q}{2} \sum_{c' \in J_q} x_{q+1, c'}. \end{aligned}$$

Thus, (1) follows. Similarly, one can prove (2). □

By the discussion above, we have the following.

Corollary 5.5. *We have*

$$c(1)_q = (q+1)x_{1,1} + (q-1)\sum_{c\in I_q} x_{q-1,c} + (q-1)x_{q,1} + (q+1)\sum_{c'\in J_q} x_{q+1,c'}.$$

Proof. It follows from a straightforward verification and Lemma 5.4. □

Corollary 5.6. *For any even $q \geqslant 2$, the Casimir element $c(1)_q$ is a Perron-Frobenius element for the left cell module $({}_{R_q}R_q, U_{\boldsymbol{q}})$. Moreover, $q(q^2-1)$ (the* FPdim *of r_q) is the Perron-Frobenius eigenvalue of the matrix $[c(1)_q]$ determined by the left multiplication of Casimir element $c(1)_q$, and its algebraic multiplicity is* 1.

Proof. It follows from Lemma 2.7, Corollaries 4.5 and 5.5. □

By the definition of r_q, the regular element of R_q is given by

$$e_q = x_{1,1} + (q-1)\sum_{c\in I_q} x_{q-1,c} + qx_{q,1} + (q+1)\sum_{c'\in J_q} x_{q+1,c'}.$$

Moreover, FPdim: $R_q \to \mathbb{C}$ is given by

$$\begin{aligned}(5.7)\qquad &\mathrm{FPdim}(x_{1,1}) = 1, \quad \mathrm{FPdim}(x_{q-1,c}) = q-1, \quad c\in I_q,\\ &\mathrm{FPdim}(x_{q,1}) = q, \quad \mathrm{FPdim}(x_{q+1,c'}) = q+1, \quad c'\in J_q.\end{aligned}$$

When $q=2$, $e_2 = x_{1,1} + x_{1,2} + 2x_{2,1}$ and

$$(5.8)\qquad \mathrm{FPdim}(x_{1,1}) = \mathrm{FPdim}(x_{1,2}) = 1, \quad \mathrm{FPdim}(x_{2,1}) = 2.$$

Theorem 5.7. *When $q=2$, $\mathbb{C}e_2$ is a special module of R_2. Furthermore, the action of R_2 on $\mathbb{C}e_2$ ($\mathbb{C}e_2 = \mathbb{C}$, as $\mathbb{C}$-vector spaces) is given by*

$$x_{1,1}\cdot 1 = x_{1,2}\cdot 1 = 1, \quad x_{2,1}\cdot 1 = 2.$$

Proof. It follows from Lemma 2.7 and Corollary 5.6 that $c(1)_2$ is a Perron-Frobenius element of R_2, and $\mathrm{FPdim}(c(1)_2) = \mathrm{FPdim}([c(1)_2]) = 6$. Moreover, by the fusion rules of r_2, we have

$$x_{1,1}\cdot e_2 = x_{1,2}\cdot e_2 = e_2, \quad x_{2,1}\cdot e_2 = 2e_2.$$

Hence, $\mathbb{C}e_2$ is a module over R_2. It is left to show that $c(1)_2 \cdot e_2 = \mathrm{FPdim}(c(1)_2)e_2$. Note that the action of R_2 on $\mathbb{C}e_2$ can be seen as

$$x_{1,1} \cdot 1 = x_{1,2} \cdot 1 = 1, \quad x_{2,1} \cdot 1 = 2.$$

Thus,

$$c(1)_2 \cdot 1 = (x_{1,1} + x_{1,2} + 2x_{2,1}) \cdot 1 = 1 + 1 + 2 \times 2 = 6 = \mathrm{FPdim}(c(1)_2).$$

This completes the proof. □

Theorem 5.8. *Let $q \geqslant 4$ be even. Then $\mathbb{C}e_q$ is a special module of R_q. Moreover, the action of R_q on $\mathbb{C}e_q$ ($\mathbb{C}e_q = \mathbb{C}$, as $\mathbb{C}$-vector spaces) is determined by*

$$x_{1,1} \cdot 1 = 1, \quad x_{q-1,c} \cdot 1 = q - 1, \quad c \in I_q,$$
$$x_{q,1} \cdot 1 = q, \quad x_{q+1,c'} \cdot 1 = q + 1, \quad c' \in J_q.$$

P r o o f. It follows from Corollary 5.6 that $c(1)_q$ is the Perron-Frobenius element for $({}_{R_q}R_q, U_{\boldsymbol{q}})$, and $\mathrm{FPdim}([c(1)_q]) = \mathrm{FPdim}(c(1)_q) = q(q^2 - 1)$. Now, we show that $\mathbb{C}e_q$ is a module of R_q. It suffices to show that the action of each basis element $x_{1,1}$, x_{q-1,c_1} ($c_1 \in I_q$), $x_{q,1}$ and x_{q+1,c_2} ($c_2 \in J_q$) of R_q on e_q is equal to a scalar multiple of e_q. It is easy to see that $x_{1,1} \cdot e_q = e_q$. For each $c_1 \in I_q$, by Lemma 5.4 (1) and the fusion rules, we have

$$\begin{aligned}
x_{q-1,c_1} \cdot e_q &= x_{q-1,c_1} \cdot \Bigg(x_{1,1} + (q-1)\sum_{c \in I_q} x_{q-1,c} + q x_{q,1} + (q+1)\sum_{c' \in J_q} x_{q+1,c'}\Bigg) \\
&= x_{q-1,c_1} + (q-1)\Bigg(x_{1,1} + \Big(\frac{q}{2} - 1\Big)x_{q-1,c_1} \\
&\quad + \Big(\frac{q}{2} - 2\Big)\sum_{c_2 \in I_q \setminus \{c_1\}} x_{q-1,c_2} + \Big(\frac{q}{2} - 1\Big)x_{q,1} + \frac{q}{2}\sum_{c' \in J_q} x_{q+1,c'}\Bigg) \\
&\quad + q\Bigg(\sum_{c_2 \in I_q \setminus \{c_1\}} x_{q-1,c_2} + x_{q,1} + \sum_{c' \in J_q} x_{q+1,c'}\Bigg) \\
&\quad + (q+1)\Big(\frac{q}{2} - 1\Big)\Bigg(\sum_{c \in I_q} x_{q-1,c} + x_{q,1} + \sum_{c' \in J_q} x_{q+1,c'}\Bigg) \\
&= (q-1)x_{1,1} + (q-1)^2 x_{q-1,c_1} + (q-1)^2 \sum_{c_2 \in I_q \setminus \{c_1\}} x_{q-1,c_2} \\
&\quad + (q^2 - q)x_{q,1} + (q^2 - 1)\sum_{c' \in J_q} x_{q+1,c'} \\
&= (q-1)e_q.
\end{aligned}$$

Similarly, by Lemma 5.4 (2), one can check that for any $c_2 \in J_q$, $x_{q+1,c_2} \cdot e_q = (q+1)e_q$. At last, we show that $x_{q,1} \cdot e_q = qe_q$. In fact, by the fusion rules, we have

$$\begin{aligned} x_{q,1} \cdot e_q = x_{q,1} \cdot \Big(x_{1,1} + (q-1) \sum_{c \in I_q} x_{q-1,c} + q x_{q,1} + (q+1) \sum_{c' \in J_q} x_{q+1,c'} \Big) \\ = x_{q,1} + (q-1) \Big(\Big(\frac{q}{2} - 1 \Big) \sum_{c \in I_q} x_{q-1,c} + \frac{q}{2} x_{q,1} + \frac{q}{2} \sum_{c' \in J_q} x_{q+1,c'} \Big) \\ + q \Big(x_{1,1} + \sum_{c \in I_q} x_{q-1,c} + x_{q,1} + \sum_{c' \in J_q} x_{q+1,c'} \Big) \\ + (q+1) \Big(\Big(\frac{q}{2} - 1 \Big) \sum_{c \in I_q} x_{q-1,c} + \Big(\frac{q}{2} - 1 \Big) x_{q,1} + \frac{q}{2} \sum_{c' \in J_q} x_{q+1,c'} \Big) \\ = q x_{1,1} + (q^2 - q) \sum_{c \in I_q} x_{q-1,c} + q^2 x_{q,1} + (q^2 + q) \sum_{c' \in J_q} x_{q+1,c'} = q e_q. \end{aligned}$$

Summarizing the discussion above, $\mathbb{C}e_q$ is a module over R_q. Moreover, the action of $\mathbb{C}e_q$ can be regarded as follows:

$$x_{1,1} \cdot 1 = 1, \quad x_{q-1,c} \cdot 1 = q - 1, \ c \in I_q, \quad x_{q,1} \cdot 1 = q, \quad x_{q+1,c'} \cdot 1 = q + 1, \ c' \in J_q.$$

It is left to prove that $c(1)_q \cdot 1 = \mathrm{FPdim}(c(1)_q) = q(q^2 - 1)$. In fact, by Corollary 5.5,

$$\begin{aligned} c(1)_q \cdot 1 &= \Big((q+1) x_{1,1} + (q-1) \sum_{c_1 \in I_q} x_{q-1,c_1} + (q-1) x_{q,1} + (q+1) \sum_{c_2 \in J_q} x_{q+1,c_2} \Big) \cdot 1 \\ &= (q+1) + (q-1)(q-1) \frac{q}{2} + (q-1)q + (q+1)(q+1) \Big(\frac{q}{2} - 1 \Big) \\ &= q^3 - q = q(q^2 - 1). \end{aligned}$$

Thus, the proof is finished. □

Remark 5.9. The action of R_q on $\mathbb{C}e_q$ with even $q \geqslant 2$ is the same as (5.7) or (5.8). Furthermore, Theorems 5.7 and 5.8 illustrate Theorem 4.3.

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