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A CLASS OF QUANTUM DOUBLES OF POINTED HOPF ALGEBRAS OF RANK ONE

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Abstract. We construct a class of quantum doubles $D(H_{D_n})$ of pointed Hopf algebras of rank one $H_{\mathcal{D}}$. We describe the algebra structures of $D(H_{D_n})$ by generators with relations. Moreover, we give the comultiplication Δ_D, counit ε_D and the antipode S_D, respectively.

Keywords: pointed Hopf algebra; quantum double; rank one

MSC 2020: 16T05, 16G30

1. Introduction and preliminaries

During the past years, great progress has been made in the study of quantum groups and their representation theories. For example, the representation theory of the small quantum group $\mathbf{u_q}(\mathfrak{sl}_2)$ and the restricted quantum universal enveloping Hopf algebra $\mathfrak{U}_q(\mathfrak{sl}_2)$ associated to $\mathfrak{sl}_2$ have been studied in [6], [8], [12], [13]. Quantum groups are mathematical objects which arose from the study of the quantum inverse scattering method, especially the quantum Yang-Baxter equation. They are non-commutative and non-cocommutative Hopf algebras. However, how to construct more non-commutative and non-cocommutative Hopf algebras has always been a problem of great interest to us.

The most common way to construct a new Hopf algebra is to generate a new Hopf algebra from a known Hopf algebra through its deformation. Roche, Dijkgraaf and Pasquier gave a method to construct a quasi Hopf algebra, see [3]. Given a finite group G and a 3-cocycle ω, they can construct a quasi Hopf algebra $D^{\omega}(G)$. If ω is trivial, then $D^{\omega}(G)$ is isomorphic to the quantum double $D(\Bbbk G)$ of the

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group algebra $\Bbbk G$. If ω is a boundary, then $D^\omega(G)$ is a twisted deformation of $D(\Bbbk G)$. If ω is not a boundary, then $D^\omega(G)$ is a genuine quasi Hopf algebra, that is, $D^\omega(G)$ is not isomorphic to any 2-cocycle deformation of a Hopf algebra. Panov studied Ore extensions in the class of Hopf algebras, which enables one to describe the Hopf-Ore extensions for group algebras in [10]. Krop and Radford defined the rank as a measure of complexity for Hopf algebras, see [7]. They classified all finite dimensional pointed Hopf algebras of rank one over an algebraically closed field $\Bbbk$ of characteristic 0. It was shown in [7], [11] that a finite dimensional pointed Hopf algebra of rank one over an algebraically closed field is isomorphic to a quotient of a Hopf-Ore extension of its coradical. Doi studied the 2-cocycle σ of Hopf algebra H, and introduced a class of 2-cocycle deformed Hopf algebras H^σ, see [4]. Chen studied the general properties of 2-cocycle and double crossproducts. He proved that all lazy 2-cocycles form a group with respect to convolution, see [2].

In 1986, Drinfeld introduced quasitriangular Hopf algebra, and gave a method to construct quasitriangular Hopf algebra $D(H)$ from a finite dimensional Hopf algebra H. The finite-dimensional Hopf algebra $D(H)$ is called a *quantum double* (or *Drinfeld double*) of the Hopf algebra H. The representation category of a quasitriangular Hopf algebra is a braided tensor category, and its braided structure can provide solutions for the quantum Yang-Baxter equation. In particular, the representation of the quantum double of a finite dimensional Hopf algebra can provide solutions for the quantum Yang-Baxter equation, so the quantum doubles of finite dimensional Hopf algebras and their representation theories have attracted much attention. Chen constructed a class of noncomutative and noncocomutative Hopf algebras (see [1]); these Hopf algebras are isomorphic to the quantum doubles of Taft algebras. In this paper, we construct a class of quantum doubles from pointed Hopf algebras of rank one $H_{\mathcal{D}}$, where $\mathcal{D} = \{G, \chi, g, \mu\}$ is a group datum and G is a dihedral group.

Throughout, we work over an algebraically closed field $\Bbbk$ with char($\Bbbk$)=0. Unless otherwise stated, all algebras and Hopf algebras are defined over $\Bbbk$. Our references for basic concepts and notations about Hopf algebras are [5], [9]. In particular, for a Hopf algebra, we will use ε, Δ and S to denote the counit, comultiplication and antipode, respectively.

2. Pointed Hopf algebras of rank one

In this section, we first recall the construction of a finite dimensional pointed Hopf algebra of rank one from a group datum $\mathcal{D}$. A quadruple $\mathcal{D}$ is called a *group datum* if G is a finite group, g is an element in the center of G, χ is a $\Bbbk$-linear character of G, and $\mu \in \Bbbk$ subject to $\chi^n = 1$ or $\mu(g^n - 1) = 0$, where n is the order of $\chi(g)$. If $\mu(g^n - 1) = 0$, then the group datum $\mathcal{D}$ is said to be of nilpotent type. Otherwise, it is of non-nilpotent type, see [7].

Given a group datum $\mathcal{D} = \{G, \chi, g, \mu\}$, we let $H_{\mathcal{D}}$ be an associative algebra generated by y and h in G such that $\Bbbk G$ is a subalgebra of $H_{\mathcal{D}}$ and

$$y^n = \mu(g^n - 1), \quad yh = \chi(h)hy \quad \text{for } h \in G.$$

The algebra $H_{\mathcal{D}}$ is finite dimensional with a canonical $\Bbbk$-basis $\{y^i h\colon\ h \in G,\ 0 \leqslant i \leqslant n-1\}$. Thus, $\dim H_{\mathcal{D}} = n|G|$, where G is the order of G. In fact, $H_{\mathcal{D}}$ is endowed with a Hopf algebra structure. The comultiplication Δ, the counit ε, and the antipode S are given respectively by

$$\Delta(y) = y \otimes g + 1 \otimes y, \quad \varepsilon(y) = 0, \quad S(y) = -yg^{-1},$$
$$\Delta(h) = h \otimes h, \quad \varepsilon(h) = 1, \quad S(h) = h^{-1}$$

for all $h \in G$. It is easy to see that G is the group of group-like elements of $H_{\mathcal{D}}$ and $H_{\mathcal{D}}$ is a pointed Hopf algebra of rank one, see [7]. If the group datum $\mathcal{D}$ is of nilpotent type, $H_{\mathcal{D}}$ is said to be a pointed rank one Hopf algebra of type. Otherwise, it is of non-nilpotent type.

Throughout this paper, let $G = D_n = \langle a, b\colon\ a^n = b^2 = (ba)^2 = 1\rangle$ be the dihedral group, where $n = 2m$ and $m > 1$ is an odd integer. Assume $\mu = 0$, and char($\Bbbk$)=0. Let ω be a root of unity with $|\omega| = 2n$. In this case, $\Bbbk D_n$ is semisimple and $a^m \in Z(D_n)$. Let $\chi \in \widehat{D}_n$ be given by $\chi(a) = -1$ and $\chi(b) = 1$. Then we can construct a pointed Hopf algebra of rank one with group datum $\mathcal{D} = \{D_n, \chi, a^m, 0\}$. Denote these Hopf algebras by H_{D_n}. Then H_{D_n} is generated as an algebra by a, b, y such that

$$a^n = 1, \quad b^2 = 1, \quad (ba)^2 = 1, \quad y^2 = 0, \quad ya = -ay, \quad by = by.$$

The comultiplication Δ, the counit ε, and the antipode S are given respectively by

$$\Delta(y) = y \otimes a^m + 1 \otimes y, \quad \Delta(a) = a \otimes a, \quad \Delta(b) = b \otimes b,$$
$$\varepsilon(a) = 1, \quad \varepsilon(b) = 1, \quad \varepsilon(y) = 0,$$
$$S(a) = a^{-1}, \quad S(b) = b^{-1}, \quad S(y) = -ya^m.$$

Then H_{D_n} is of dimension $4n$ with a $\Bbbk$-basis $\{b^i a^j y^k\colon\ 0 \leqslant i, k \leqslant 1,\ 0 \leqslant j \leqslant n-1\}$.

3. Drinfeld doubles of the Hopf algebras H_{D_n}

In this section, we describe the structures of the Drinfeld doubles (or the quantum doubles) of the Hopf algebras H_{D_n}.

Definition 3.1 ([5], Definition IX 2.2)**.** A pair (X, A) of bialgebras is matched if there exist linear maps $\alpha\colon A\otimes X \to X$ and $\beta\colon A\otimes X\to A$ turning X into a module coalgebra over A, and turning A into a right module-coalgebra over X, respectively, such that if we set

$$\alpha(a\otimes x) = a\cdot x \quad \text{and} \quad \beta(a\otimes x) = a^x,$$

the following conditions are satisfied:

$$a\cdot(xy) = \sum_{(a)(x)} (a_1\cdot x_1)({a_2}^{x_2}\cdot y), \quad a\cdot 1 = \varepsilon(a)1,$$

$$(ab)^x = \sum_{(b)(x)} a^{b_1\cdot x_1}{b_2}^{x_2}, \quad 1^x = \varepsilon(x)1,$$

$$\sum_{(a)(x)} {a_1}^{x_1}\otimes a_2\cdot x_2 = \sum_{(a)(x)} {a_2}^{x_2}\otimes a_1\cdot x_1$$

for all $a, b\in A$ and $x, y\in X$.

Lemma 3.2 ([5], Theorem IX.2.3)**.** *Let (X, A) be a matched pair of Hopf algebras. There exists a unique Hopf algebra structure on the vector space $X\otimes A$, with unit equal to $1\otimes 1$, such that its product is given by*

$$(x\otimes a)(y\otimes b) = \sum_{(a)(y)} x(a_1\cdot y_1)\otimes {a_2}^{y_2}b,$$

its coproduct by

$$\triangle(x\otimes a) = \sum_{(a)(x)} (x_1\otimes a_1)\otimes(x_2\otimes a_2),$$

its counit by

$$\varepsilon(x\otimes a) = \varepsilon(x)\varepsilon(a)$$

and its antipode S given by

$$S(x\otimes a) = \sum_{(x)(a)} S_A(a_2)\cdot S_X(x_2)\otimes S_A(a_1)^{S_X(x_1)},$$

where S_X and S_A are antipode of X and A, respectively. Further $X\otimes A$ is called bicrossed product of X and A and denoted $X\bowtie A$. Obviously, X and A are subalgebras of $X\bowtie A$ under the injective $i_X(x) = x\otimes 1$ and $i_A(a) = 1\otimes a$, respectively.

Theorem 3.3 ([5], Theorem IX.3.5)**.** *Let* $(H, \mu, \eta, \Delta, \varepsilon, S, S^{-1})$ *be a finite-dimensional Hopf algebra with invertible antipode. Consider the Hopf algebra*

$$X = (H^{\mathrm{op}})^* = (H^*)^{\mathrm{cop}} = (H^*, \Delta^*, \varepsilon^*, (\mu^{\mathrm{op}})^*, \eta^*, (S^{-1})^*, S^*).$$

Let $\alpha\colon H \otimes X \to X$ *and* $\beta\colon H \otimes X \to H$ *be the linear maps given by*

$$\alpha(a \otimes f) = a \cdot f = \sum_{(a)} f(S^{-1}(a_2)?a_1) \quad \text{and} \quad \beta(a \otimes f) = a^f = \sum_{(a)} f(S^{-1}(a_3)a_1)a_2,$$

where $a \in H$ *and* $f \in X$*. Then the pair* (H, X) *of Hopf algebras is matched in the sense of Definition* 3.1*.*

Proposition 3.4 ([5], Definition IX.4.1)**.** *The quantum double* $D(H_{D_n})$ *of the Hopf algebra* H_{D_n} *is the bicrossed product of* H_{D_n} *and of* $(H^*_{D_n})^{\mathrm{cop}}$*:*

$$D(H_{D_n}) = (H^*_{D_n})^{\mathrm{cop}} \bowtie H_{D_n}.$$

Proposition 3.5. *The multiplication, comultiplication and counit in* $D(H_{D_n})$ *are given by*

$$(f \otimes a)(g \otimes b) = \sum_{(a)} fg(S^{-1}(a_3)?a_1) \otimes a_2 b, \quad \varepsilon_D(f \otimes a) = \varepsilon(a)f(1)$$

$$\Delta_D(f \otimes a) = \sum_{(a)(f)} (f_1 \otimes a_1) \otimes (f_2 \otimes a_2),$$

where $f, g \in (H^*_{D_n})^{\mathrm{cop}}$ *and* $a, b \in H_{D_n}$*,* $\sum f_1 \otimes f_2$ *is determined by* $\sum f_1(x)f_2(y) = f(yx)$ *for all* $x, y \in H_{D_n}$*.*

Note that H_{D_n} and $(H^*_{D_n})^{\mathrm{cop}}$ are Hopf subalgebras of $D(H_{D_n})$ via the identifications $y = \varepsilon \bowtie y$, $y \in H_{D_n}$ and $f = f \bowtie 1$, $f \in (H^*_{D_n})^{\mathrm{cop}}$, respectively, where ε is the unit element of $(H^*_{D_n})^{\mathrm{cop}}$. Let $\{\overline{b^i a^j y^k}\colon 0 \leqslant i, k \leqslant 1, 0 \leqslant j \leqslant n-1\}$ be the basis in $H^*_{D_n}$ dual to the basis $\{b^i a^j y^k\colon 0 \leqslant i, k \leqslant 1, 0 \leqslant j \leqslant n-1\}$ of H_{D_n}. That is, $\overline{b^i a^j y^k}(b^i a^j y^k) = 1$, and $\overline{b^i a^j y^k}(b^{i'} a^{j'} y^{k'}) = 0$ if $(i', j', k') \neq (i, j, k)$, where $0 \leqslant i, i'$, $k, k' \leqslant 1$, $0 \leqslant j, j' \leqslant n-1$.

Lemma 3.6. *The multiplication of* $H^*_{D_n}$ *is determined by*

$$\overline{b^i a^j y^s}\,\overline{b^k a^l y^t} = \begin{cases} \overline{b^i a^j y^t} & \text{if } s = 0,\ i = k \text{ and } j = l, \\ \overline{b^i a^j y} & \text{if } s = 1,\ t = 0,\ k = i \text{ and } l \equiv m + j \pmod{n}, \\ 0, & \text{otherwise,} \end{cases}$$

where $0 \leqslant i, k, s, t \leqslant 1$ *and* $0 \leqslant j, l \leqslant n-1$*.*

Proof. By the coalgebra structure of H_{D_n}, we have $\Delta(b^t a^h) = b^t a^h \otimes b^t a^h$, and $\Delta(b^t a^h y) = b^t a^h y \otimes b^t a^{h+m} + b^t a^h \otimes b^t a^h y$ for $0 \leqslant t \leqslant 1$ and $0 \leqslant h \leqslant n-1$. We prove the lemma for $s = t = 0$; $s = 0$, $t = 1$; $s = 1$, $t = 0$ and $s = t = 1$, respectively.

Case 1: If $s = t = 0$, then $\overline{b^i a^j}\,\overline{b^k a^l}(b^t a^h) = \overline{b^i a^j}(b^t a^h)\overline{b^k a^l}(b^t a^h)$, where $0 \leqslant i,k,t \leqslant 1$, $0 \leqslant j,l,h \leqslant n-1$. Hence, $\overline{b^i a^j}\,\overline{b^k a^l}(b^t a^h) = 1$ if and only if $t = i = k$ and $j = l = h$. Obviously, $\overline{b^i a^j}\,\overline{b^k a^l}(b^t a^h y) = 0$, and so we have

$$\overline{b^i a^j}\,\overline{b^k a^l} = \begin{cases} \overline{b^i a^j} & \text{if } i = k \text{ and } j = l, \\ 0, & \text{otherwise.} \end{cases}$$

Case 2: If $s = 0$, $t = 1$, then $\overline{b^i a^j}\,\overline{b^k a^l y}(b^t a^h y) = \overline{b^i a^j}(b^t a^h)\,\overline{b^k a^l y}(b^t a^h y)$, where $0 \leqslant i,k,t \leqslant 1$, $0 \leqslant j,l,h \leqslant n-1$. Hence, $\overline{b^i a^j}\,\overline{b^k a^l y}(b^t a^h y) = 1$ if and only if $t = i = k$ and $j = l = h$. Obviously, $\overline{b^i a^j}\,\overline{b^k a^l y}(b^t a^h) = 0$, and so we have

$$\overline{b^i a^j}\,\overline{b^k a^l y} = \begin{cases} \overline{b^i a^j y} & \text{if } i = k \text{ and } j = l, \\ 0, & \text{otherwise.} \end{cases}$$

Case 3: If $s = 1$, $t = 0$, then $\overline{b^i a^j y}\,\overline{b^k a^l}(b^t a^h y) = \overline{b^i a^j y}(b^t a^h y)\overline{b^k a^l}(b^t a^{h+m})$, where $0 \leqslant i,k,t \leqslant 1$, $0 \leqslant j,l,h \leqslant n-1$. Hence, $\overline{b^i a^j y}\,\overline{b^k a^l}(b^t a^h y) = 1$ if and only if $t = i = k$, $j = h$ and $l \equiv h + m \pmod{n}$. Obviously, $\overline{b^i a^j y}\,\overline{b^k a^l}(b^t a^h) = 0$, and so we have

$$\overline{b^i a^j y}\,\overline{b^k a^l} = \begin{cases} \overline{b^i a^j y} & \text{if } i = k \text{ and } j + m = l, \\ 0, & \text{otherwise.} \end{cases}$$

Case 4: If $s = t = 1$, then one can easily check that $\overline{b^i a^j y}\,\overline{b^k a^l y}(b^t a^h) = 0$, and $\overline{b^i a^j y}\,\overline{b^k a^l y}(b^t a^h y) = 0$ for $0 \leqslant i,k,t \leqslant 1$, $0 \leqslant j,l,h \leqslant n-1$. This completes the proof. □

Obviously, $\sum\limits_{i=0}^{1}\sum\limits_{j=0}^{n-1}\overline{b^i a^j} = \varepsilon$ is the identity of the algebra $H_{D_n}^*$. Put $\alpha = \sum\limits_{i=0}^{1}\sum\limits_{j=0}^{n-1}\omega^{i+2j}\overline{b^i a^j}$, $\beta = \sum\limits_{i=0}^{1}\sum\limits_{j=0}^{n-1}\overline{b^i a^j y}$.

Lemma 3.7. *The algebra $H_{D_n}^*$ is generated, as an algebra, by α, β.*

Proof. Let A be the subalgebra of $H_{D_n}^*$ generated by α, β. By Lemma 3.6, we have

$$\alpha^0 = \sum_{i=0}^{1}\sum_{j=0}^{n-1}\overline{b^i a^j} = \varepsilon, \quad \alpha^1 = \sum_{i=0}^{1}\sum_{j=0}^{n-1}\omega^{i+2j}\overline{b^i a^j}, \quad \alpha^2 = \sum_{i=0}^{1}\sum_{j=0}^{n-1}\omega^{2(i+2j)}\overline{b^i a^j},$$
$$\alpha^3 = \sum_{i=0}^{1}\sum_{j=0}^{n-1}\omega^{3(i+2j)}\overline{b^i a^j}, \ldots, \quad \alpha^{2n-1} = \sum_{i=0}^{1}\sum_{j=0}^{n-1}\omega^{(2n-1)(i+2j)}\overline{b^i a^j}.$$

For any $0 \leqslant i, i' \leqslant 1$ and $0 \leqslant j, j' \leqslant n-1$, we have $\omega^{i+2j} = \omega^{i'+2j'}$ if and only if $i = i'$ and $j = j'$. Hence, $\overline{b^i a^j} \in A$ for any $0 \leqslant i \leqslant 1, 0 \leqslant j \leqslant n-1$. Moreover, by $\overline{b^i a^j}\beta = \overline{b^i a^j y}$, $\overline{b^i a^j y} \in A$, $0 \leqslant i \leqslant 1$, $0 \leqslant j \leqslant n-1$. Consequently, $H^*_{D_n} = A$. □

Corollary 3.8. *The following holds in* $H^*_{D_n}$.

$$\alpha^{2n} = \varepsilon, \quad \beta^2 = 0, \quad \alpha\beta = -\beta\alpha.$$

P r o o f. It follows from Lemma 3.6 and the proof of Lemma 3.7. □

Corollary 3.9. *The algebra* $H^*_{D_n}$ *has a* $\Bbbk$*-basis* $\{\alpha^i\beta^j \colon 0 \leqslant i \leqslant 2n-1, 0 \leqslant j \leqslant 1\}$.

P r o o f. It follows from Lemma 3.7 and Corollary 3.8. □

Proposition 3.10. *The comultiplication* Δ^{op}*, the counit* ε *and the antipode* S *of* $(H^*_{D_n})^{\mathrm{cop}}$ *are given by*

$$\begin{aligned}
\Delta^{\mathrm{op}}(\alpha) &= \frac{1}{2}\alpha \otimes (\alpha + \alpha^{-1}) + \frac{1}{2}\alpha^{n+1} \otimes (\alpha - \alpha^{-1}),\\
\Delta^{\mathrm{op}}(\beta) &= \beta \otimes \varepsilon + \frac{1}{2}[(1+\omega^{-m})\alpha^m + (1-\omega^{-m})\alpha^{-m}] \otimes \beta,\\
\varepsilon(\alpha) &= 1, \quad \varepsilon(\beta) = 0,\\
S(\alpha) &= \frac{1}{2}(\alpha + \alpha^{-1} + \alpha^{n-1} - \alpha^{n+1}),\\
S(\beta) &= \frac{1}{2}[(1+\omega^{-m})\alpha^m + (1-\omega^{-m})\alpha^{-m}]\beta.
\end{aligned}$$

P r o o f. In $H^*_{D_n}$, for any $0 \leqslant i \leqslant n-1$, we claim

$$\begin{aligned}
\Delta(\overline{a^i}) = &\sum_{0 \leqslant k \leqslant i} \overline{a^k} \otimes \overline{a^{i-k}} + \sum_{i+1 \leqslant k \leqslant n-1} \overline{a^k} \otimes \overline{a^{n+i-k}}\\
&+ \sum_{0 \leqslant k \leqslant n-i-1} \overline{ba^k} \otimes \overline{ba^{i+k}} + \sum_{n-i \leqslant k \leqslant n-1} \overline{ba^k} \otimes \overline{ba^{i+k-n}}.
\end{aligned}$$

In fact, assume

$$\Delta(\overline{a^i}) = \sum_{j,j',l,l'=0}^{1} \sum_{k,k'=0}^{n-1} \theta^{j,k,l}_{j',k',l'} \overline{b^j a^k y^l} \otimes \overline{b^{j'} a^{k'} y^{l'}},$$

where $\theta^{j,k,l}_{j',k',l'} = \overline{a^i}(b^j a^k y^l b^{j'} a^{k'} y^{l'})$. By a straightforward computation, one gets that $\theta^{j,k,l}_{j',k',l'} = 1$ if and only if j, k, l, j', k', l' satisfy one of the following cases:

Case 1: $l = l' = j = j' = 0$, and $k + k' \equiv i \pmod{n}$.

Case 2: $l = l' = 0$, $j = j' = 1$, and $k' = k + i$, $0 \leqslant k \leqslant n-i-1$.

Case 3: $l = l' = 0$, $j = j' = 1$, and $k = k' + n - i$, $n - i \leqslant k \leqslant n-1$.

Moreover, we have $\theta^{j,k,l}_{j',k',l'}=0$ for other cases. Similarly, one can prove

$$\begin{aligned}\Delta(\overline{ba^i}) = &\sum_{0\leqslant k\leqslant n-i-1}\overline{a^k}\otimes\overline{ba^{i+k}}+\sum_{n-i\leqslant k\leqslant n-1}\overline{a^k}\otimes\overline{ba^{i+k-n}}\\ &+\sum_{0\leqslant k\leqslant i}\overline{ba^k}\otimes\overline{a^{i-k}}+\sum_{i+1\leqslant k\leqslant n-1}\overline{ba^k}\otimes\overline{a^{i-k+n}},\\ \Delta(\overline{a^iy}) = &\sum_{0\leqslant k\leqslant i}\overline{a^k}\otimes\overline{a^{i-k}y}+\sum_{i+1\leqslant k\leqslant n-1}\overline{a^k}\otimes\overline{a^{n+i-k}y}\\ &+\sum_{0\leqslant k\leqslant n-i-1}\overline{ba^k}\otimes\overline{ba^{i+k}y}+\sum_{n-i\leqslant k\leqslant n-1}\overline{ba^k}\otimes\overline{ba^{i+k-n}y}\\ &+\sum_{0\leqslant k\leqslant i}(-1)^{i-k}\overline{a^ky}\otimes\overline{a^{i-k}}+\sum_{i+1\leqslant k\leqslant n-1}(-1)^{i-k}\overline{a^ky}\otimes\overline{a^{n+i-k}}\\ &+\sum_{0\leqslant k\leqslant n-i-1}(-1)^{i+k}\overline{ba^ky}\otimes\overline{ba^{i+k}}\\ &+\sum_{n-i\leqslant k\leqslant n-1}(-1)^{i+k}\overline{ba^ky}\otimes\overline{ba^{i+k-n}}\end{aligned}$$

and

$$\begin{aligned}\Delta(\overline{ba^iy}) = &\sum_{0\leqslant k\leqslant n-i-1}\overline{a^k}\otimes\overline{ba^{i+k}y}+\sum_{n-i\leqslant k\leqslant n-1}\overline{a^k}\otimes\overline{ba^{i+k-n}y}\\ &+\sum_{0\leqslant k\leqslant i}\overline{ba^k}\otimes\overline{a^{i-k}y}+\sum_{i+1\leqslant k\leqslant n-1}\overline{ba^k}\otimes\overline{a^{i-k+n}y}\\ &+\sum_{0\leqslant k\leqslant i}(-1)^{i-k}\overline{ba^ky}\otimes\overline{a^{i-k}}+\sum_{i+1\leqslant k\leqslant n-1}(-1)^{i-k}\overline{ba^ky}\otimes\overline{a^{i-k+n}}\\ &+\sum_{0\leqslant k\leqslant n-i-1}(-1)^{i+k}\overline{a^ky}\otimes\overline{ba^{i+k}}\\ &+\sum_{n-i\leqslant k\leqslant n-1}(-1)^{i+k}\overline{a^ky}\otimes\overline{ba^{i+k-n}}.\end{aligned}$$

Notice that $\Delta(\alpha)=\sum\limits_{i=0}^{1}\sum\limits_{j=0}^{n-1}\omega^{i+2j}\Delta(\overline{b^ia^j})$, and $\Delta(\beta)=\sum\limits_{i=0}^{1}\sum\limits_{j=0}^{n-1}\Delta(\overline{b^ia^jy})$. Let $A=\sum\limits_{j=0}^{n-1}\omega^{2j}\overline{a^j}$, $B=\sum\limits_{j=0}^{n-1}\omega^{2j+1}\overline{ba^j}$. Then $\alpha=A+B$. By the identities above, one gets

$$\begin{aligned}\Delta(\alpha) &= \overline{1}\otimes(A+B)+\overline{a}\otimes(\omega^2A+\omega^{-2}B)+\overline{a^2}\otimes(\omega^4A+\omega^{-4}B)+\dots\\ &\quad+\overline{a^{n-1}}\otimes(\omega^{2(n-1)}A+\omega^{-2(n-1)}B)+\overline{b}\otimes(\omega A+\omega^{-1}B)\\ &\quad+\overline{ba}\otimes(\omega^3A+\omega^{-3}B)+\overline{ba^2}\otimes(\omega^5A+\omega^{-5}B)+\dots\\ &\quad+\overline{ba^{n-1}}\otimes(\omega^{2n-1}A+\omega^{-2(n-1)}B)\\ &=\alpha\otimes A+\alpha^{-1}\otimes B.\end{aligned}$$

By a straightforward computation, we have $\alpha^{n+1} = A - B$. Consequently, $A = \frac{1}{2}(\alpha + \alpha^{n+1})$ and $B = \frac{1}{2}(\alpha - \alpha^{n+1})$, and hence we have

$$\Delta(\alpha) = \frac{1}{2}(\alpha + \alpha^{-1}) \otimes \alpha + \frac{1}{2}(\alpha - \alpha^{-1}) \otimes \alpha^{n+1}.$$

Similarly, let $A' = \sum_{j=0}^{n-1} (-1)^j \overline{a^j}$ and $B' = \sum_{j=0}^{n-1} (-1)^j \overline{ba^j}$. One can check that $A' + \omega^m B' = \alpha^m$ and $A' - \omega^m B' = \alpha^{-m}$. Consequently, $A' = \frac{1}{2}(\alpha^m + \alpha^{-m})$ and $B' = \frac{1}{2}\omega^{-m}(\alpha^m - \alpha^{-m})$. Moreover, by a straightforward computation, one has $\Delta(\beta) = \varepsilon \otimes \beta + \beta \otimes (A' + B')$. Hence, we have

$$\Delta(\beta) = \varepsilon \otimes \beta + \frac{1}{2}\beta \otimes [(1 + \omega^{-m})\alpha^m + (1 - \omega^{-m})\alpha^{-m}].$$

It is easy to see that

$$\varepsilon(\overline{b^i a^j y^k}) = \begin{cases} 1 & \text{if } i = j = k = 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $0 \leqslant i,\ k \leqslant 1$, and $0 \leqslant j \leqslant n-1$. Consequently, $\varepsilon(\alpha) = 1$, $\varepsilon(\beta) = 0$. Moreover, a straightforward verification shows that $S(\overline{a^i}) = \overline{a^{n-i}}$, $S(\overline{ba^i}) = \overline{ba^i}$, $S(\overline{a^i y}) = (-1)^{i+1}\overline{a^{m-i}y}$, and $S(\overline{ba^i y}) = (-1)^{i+1}\overline{ba^{i-m}y}$, where $0 \leqslant i \leqslant n-1$. Hence,

$$S(\alpha) = \overline{1} + \omega^{-2}\overline{a} + \ldots + \omega^{-2(n-1)}\overline{a^{n-1}} + \omega\overline{b} + \omega^3\overline{ba} + \ldots + \omega^{2n-1}\overline{ba^{n-1}}.$$

Let $\overline{A} = \sum_{j=0}^{n-1} \omega^{-2j}\overline{a^j}$, $\overline{B} = \sum_{j=0}^{n-1} \omega^{-2j-1}\overline{ba^j}$. Then $S(\alpha) = \overline{A} + B$, where B is given as before. Moreover, one can check that $\alpha^{-1} = \overline{A} + \overline{B}$ and $\alpha^{n-1} = \overline{A} - \overline{B}$, and so we have $\overline{A} = \frac{1}{2}(\alpha^{-1} + \alpha^{n-1})$, and $\overline{B} = \frac{1}{2}(\alpha^{-1} - \alpha^{n-1})$. It follows that $S(\alpha) = \frac{1}{2}(\alpha + \alpha^{-1} + \alpha^{n-1} - \alpha^{n+1})$. Similarly, one can get $S(\beta) = \frac{1}{2}[(1 + \omega^{-m})\alpha^m + (1 - \omega^{-m})\alpha^{-m}]\beta$. This completes the proof. $\square$

Corollary 3.11. *In $(H^*_{D_n})^{\mathrm{cop}}$, we have*

(1) $\Delta^{\mathrm{op}}(\alpha^{-1}) = \frac{1}{2}(\alpha^{-1} + \alpha^{n-1}) \otimes \alpha^{-1} + \frac{1}{2}(\alpha^{-1} - \alpha^{n-1}) \otimes \alpha$,
(2) $\Delta^{\mathrm{op}}(\alpha^{n+1}) = \frac{1}{2}(\alpha + \alpha^{n+1}) \otimes \alpha^{n+1} - \frac{1}{2}(\alpha - \alpha^{n+1}) \otimes \alpha^{n-1}$,
(3) $\Delta^{\mathrm{op}}(\alpha^m) = \frac{1}{2}(\alpha^m + \alpha^{-m}) \otimes \alpha^m + \frac{1}{2}(\alpha^m - \alpha^{-m}) \otimes \alpha^{-m}$,
(4) $\Delta^{\mathrm{op}}(\alpha^{-m}) = \frac{1}{2}(\alpha^{-m} - \alpha^m) \otimes \alpha^m + \frac{1}{2}(\alpha^{-m} + \alpha^m) \otimes \alpha^{-m}$,
(5) $S(\alpha^{-1}) = \frac{1}{2}(\alpha + \alpha^{-1} + \alpha^{n+1} - \alpha^{n-1})$,
(6) $S(\alpha^{n+1}) = \frac{1}{2}(\alpha^{-1} - \alpha + \alpha^{n+1} + \alpha^{n-1})$,
(7) $S(\alpha^m) = \alpha^m$, $S(\alpha^{-m}) = \alpha^{-m}$.

Proof. It is similar to the proof of Proposition 3.10. $\square$

Proposition 3.12. *Let $\triangle_D$, ε_D and S_D be the comultiplication, counit and the antipode of $D(H_{D_n})$. Then we have*

$$\triangle_D(\varepsilon \bowtie a) = (\varepsilon \bowtie a) \otimes (\varepsilon \bowtie a), \quad \triangle_D(\varepsilon \bowtie b) = (\varepsilon \bowtie b) \otimes (\varepsilon \bowtie b),$$
$$\triangle_D(\varepsilon \bowtie y) = (\varepsilon \bowtie y) \otimes (\varepsilon \bowtie a)^m + (\varepsilon \bowtie 1) \otimes (\varepsilon \bowtie y),$$
$$\triangle_D(\alpha \bowtie 1) = \frac{1}{2}(\alpha \bowtie 1) \otimes (\alpha \bowtie 1) + \frac{1}{2}(\alpha \bowtie 1) \otimes (\alpha \bowtie 1)^{-1} + \frac{1}{2}(\alpha \bowtie 1)^{n+1} \otimes (\alpha \bowtie 1) - \frac{1}{2}(\alpha \bowtie 1)^{n+1} \otimes (\alpha \bowtie 1)^{-1},$$
$$\triangle_D(\beta \bowtie 1) = (\beta \bowtie 1) \otimes (\varepsilon \bowtie 1) + \frac{1}{2}(1+\omega^{-m})(\alpha \bowtie 1)^m \otimes (\beta \bowtie 1) + \frac{1}{2}(1-w^{-m})(\alpha \bowtie 1)^{-m} \otimes (\beta \bowtie 1),$$
$$\varepsilon_D(\varepsilon \bowtie a) = 1, \quad \varepsilon_D(\varepsilon \bowtie b) = 1, \quad \varepsilon_D(\varepsilon \bowtie y) = 0, \quad \varepsilon_D(\alpha \bowtie 1) = 1, \quad \varepsilon_D(\beta \bowtie 1) = 0,$$
$$S_D(\varepsilon \bowtie a) = (\varepsilon \bowtie a)^{-1}, \quad S_D(\varepsilon \bowtie b) = \varepsilon \bowtie b, \quad S_D(\varepsilon \bowtie y) = -(\varepsilon \bowtie a)^m(\varepsilon \bowtie y),$$
$$S_D(\alpha \bowtie 1) = \frac{1}{2}(\alpha \bowtie 1) + \frac{1}{2}(\alpha \bowtie 1)^{-1} + \frac{1}{2}(\alpha \bowtie 1)^{n-1} - \frac{1}{2}(\alpha \bowtie 1)^{n+1},$$
$$S_D(\beta \bowtie 1) = \frac{1}{2}(1+\omega^{-m})(\alpha \bowtie 1)^m(\beta \bowtie 1) + \frac{1}{2}(1-\omega^{-m})(\alpha \bowtie 1)^{-m}(\beta \bowtie 1),$$

where $\varepsilon \bowtie 1$ is the unit element of $D(H_{D_n})$.

Proof. We only check the rules of the antipode since other rules can be easily obtained. By the definition of the antipode given above, we have $S_D(\varepsilon \bowtie \alpha) = a^{-1} \cdot \varepsilon \bowtie (a^{-1})^\varepsilon = \varepsilon \bowtie a^{-1} = (\varepsilon \bowtie a)^{-1}$. Similarly, we can get $S_D(\varepsilon \bowtie b) = \varepsilon \bowtie b$.

$$S_D(\varepsilon \bowtie y) = a^{-m} \cdot \varepsilon \bowtie (-a^m y)^\varepsilon + (-a^m y) \cdot \varepsilon \bowtie 1^\varepsilon = -\varepsilon \bowtie a^m y = -(\varepsilon \bowtie a)^m(\varepsilon \bowtie y).$$

By Lemma 3.11, we have

$$\begin{aligned} S_D(\alpha \bowtie 1) &= 1 \cdot S(\alpha + \alpha^{-1}) \bowtie 1^{S(\alpha)/2} + 1 \cdot S(\alpha - \alpha^{-1}) \bowtie 1^{S(\alpha^{n+1})/2} \\ &= 1 \cdot S(\alpha) \bowtie 1^{S(\alpha)/2} + 1 \cdot S(\alpha^{-1}) \bowtie 1^{S(\alpha)/2} \\ &\quad + 1 \cdot S(\alpha) \bowtie 1^{S(\alpha^{n+1})/2} - 1 \cdot S(\alpha^{-1}) \bowtie 1^{S(\alpha^{n+1})/2} \\ &= S(\alpha) \bowtie 1 \\ &= \frac{1}{2}(\alpha \bowtie 1) + \frac{1}{2}(\alpha \bowtie 1)^{-1} + \frac{1}{2}(\alpha \bowtie 1)^{n-1} - \frac{1}{2}(\alpha \bowtie 1)^{n+1}, \end{aligned}$$

and

$$\begin{aligned} S_D(\beta \bowtie 1) &= 1 \cdot S(\varepsilon) \bowtie 1^{S(\beta)} + 1 \cdot S(\beta) \bowtie 1^{S((1+\omega^{-m})\alpha^m + (1-\omega^{-m})\alpha^{-m})/2} \\ &= S(\beta) \bowtie 1 \\ &= \frac{1}{2}(1+\omega^{-m})(\alpha \bowtie 1)^m(\beta \bowtie 1) + \frac{1}{2}(1-\omega^{-m})(\alpha \bowtie 1)^{-m}(\beta \bowtie 1). \end{aligned}$$

□

Let $H_n(\omega)$ be an algebra generated by x_1, x_2, x_3, y_1 and y_2 subject to the following relations:

$$x_1^n = 1, \quad x_2^2 = 1, \quad (x_2x_1)^2 = 1, \quad x_3^2 = 0, \quad x_3x_1 = -x_1x_3, \quad x_3x_2 = x_2x_3,$$
$$y_1^{2n} = 1, \quad y_1^2 = 0, \quad y_1y_2 = -y_2y_1,$$
$$x_1y_2 = -y_2x_1, \quad x_2y_2 = y_2x_2, \quad x_3y_1 = -y_1x_3,$$
$$x_1y_1 = \frac{1}{2}(1+\omega^4)y_1x_1 + \frac{1}{2}(1-\omega^4)y_1^{n+1}x_1,$$
$$x_2y_1 = \frac{1}{2}(1+\omega^2)y_1^{-1}x_2 + \frac{1}{2}(1-\omega^2)y_1^{n-1}x_2,$$
$$x_3y_2 = y_2x_3 + x_1^m - \frac{1}{2}(1+\omega^{-m})y_1^m - \frac{1}{2}(1-\omega^{-m})y_1^{-m}.$$

One can easily check that $H_n(\omega)$ is spanned as a vector space by $\{y_1^iy_2^jx_2^px_1^qx_3^l\colon 0 \leqslant i \leqslant 2n-1,\ 0 \leqslant j,p,l \leqslant 1,\ 0 \leqslant q \leqslant n-1\}$, and so $\dim(H_n(\omega)) \leqslant 16n^2$.

Theorem 3.13. *There is an algebra isomorphism* φ *from* $H_n(\omega)$ *to* $D(H_{D_n})$ *given by*

$$\varphi(x_1) = \varepsilon \bowtie a, \quad \varphi(x_2) = \varepsilon \bowtie b, \quad \varphi(x_3) = \varepsilon \bowtie y, \quad \varphi(y_1) = \alpha \bowtie 1, \quad \varphi(y_2) = \beta \bowtie 1.$$

Moreover, $H_n(\omega)$ *has a* $\Bbbk$*-basis* $\{y_1^iy_2^jx_2^px_1^qx_3^l\colon 0 \leqslant i \leqslant 2n-1,\ 0 \leqslant j,p,l \leqslant 1,\ 0 \leqslant q \leqslant n-1\}$ *and* $\dim(H_n(\omega)) = 16n^2$.

Proof. Let $X = \varepsilon \bowtie a$, $Y = \varepsilon \bowtie b$, $Z = \varepsilon \bowtie y$, $D = \alpha \bowtie 1$, and $E = \beta \bowtie 1$. Since H_{D_n} and $(H_{D_n}^*)^{\mathrm{cop}}$ are Hopf subalgebras of $D(H_{D_n})$ as stated before, we have $X^n = 1$, $Y^2 = 1$, $(YX)^2 = 1$, $Z^2 = 0$, $ZX = -XZ$, $YZ = ZY$, $D^{2n} = 1$, $E^2 = 0$, and $DE = -ED$. By Proposition 3.5, one gets that

$$XD = (A+\omega^4B) \bowtie a, \qquad DX = \alpha \bowtie a, \qquad D^{n+1}X = \alpha^{n+1} \bowtie a,$$
$$YD = (\overline{A}+\omega^2\overline{B}) \bowtie b, \qquad DY = \alpha \bowtie b, \qquad D^{-1}Y = \alpha^{-1} \bowtie b,$$
$$ZD = -(A+B) \bowtie y, \quad DZ = \alpha \bowtie y,$$
$$XE = -\beta \bowtie a, \quad EX = \beta \bowtie a,$$
$$YE = \beta \bowtie b, \quad EY = \beta \bowtie b,$$
$$ZE = \varepsilon \bowtie a^m + \beta \bowtie y - T \bowtie 1, \quad EZ = \beta \bowtie y,$$

where $T = A' + B'$, A, B, $A'B'$, $\overline{A}$ and $\overline{B}$ are given as before. Moreover, one has

$$DX - XD = (1-\omega^4)B \bowtie a = \frac{1}{2}(1-\omega^4)(\alpha - \alpha^{n+1}) \bowtie a = \frac{1}{2}(1-\omega^4)(DX - D^{n+1}X).$$

Consequently, we have $XD = \frac{1}{2}(1+\omega^4)DX + \frac{1}{2}(1-\omega^4)D^{n+1}X$. Furthermore,

$$\begin{aligned} D^{-1}Y - YD &= (1-\omega^2)\overline{B} \bowtie b = \frac{1}{2}(1-\omega^2)(\alpha^{-1} - \alpha^{n-1}) \bowtie b \\ &= \frac{1}{2}(1-\omega^2)(D^{-1}Y - D^{n-1}Y). \end{aligned}$$

Consequently, we have $YD = \frac{1}{2}(1+\omega^2)D^{-1}Y + \frac{1}{2}(1-\omega^2)D^{n-1}Y$. Obviously, we have $ZD = -DZ$, $XE = -EX$, and $YE = EY$. Similarly, we can get $ZE = EZ + X^m - \frac{1}{2}(1+\omega^{-m})D^m - \frac{1}{2}(1-\omega^{-m})D^{-m}$. It follows that there exists a unique algebra map $\psi\colon H_n(\omega) \to D(H_{D_n})$ such that $\psi(x_1) = X$, $\psi(x_2) = Y$, $\psi(x_3) = Z$, $\psi(y_1) = D$ and $\psi(y_2) = E$. By Lemma 3.7 and the definition of $D(H_{D_n})$, $D(H_{D_n})$ is generated as an algebra by X, Y, Z, D and E. Hence, ψ is surjective, and so $16n^2 \geqslant \dim(H_n(\omega)) \geqslant \dim(D(H_{D_n}) = 16n^2$. Thus, $\dim(H_n(\omega)) = 16n^2$. It follows that ψ is an algebra isomorphism. □

Proposition 3.14. *Let $H_n(\omega)$ be a Hopf algebra with the comultiplication, the counit and the antipode determined by*

$$\begin{gathered} \triangle(x_1) = x_1 \otimes x_1, \quad \triangle(x_2) = x_2 \otimes x_2, \quad \triangle(x_3) = x_3 \otimes x_1^m + 1 \otimes x_3, \\ \triangle(y_1) = \frac{1}{2} y_1 \otimes (y_1 + y_1^{-1}) + \frac{1}{2} y_1^{n+1} \otimes (y_1 - y_1^{-1}), \\ \triangle(y_2) = y_2 \otimes 1 + \frac{1}{2}[(1+\omega^{-m})y_1^m + (1-w^{-m})y_1^{-m}] \otimes y_2, \\ \varepsilon(x_1) = 1, \quad \varepsilon(x_2) = 1, \quad \varepsilon(x_3) = 0, \quad \varepsilon(y_1) = 1, \quad \varepsilon(y_2) = 0, \\ S(x_1) = x_1^{-1}, \quad S(x_2) = x_2, \quad S(x_3) = -x_1^m x_3, \\ S(y_1) = \frac{1}{2}(y_1 + y_1^{-1} + y_1^{n-1} - y_1^{n+1}), \\ S(y_2) = \frac{1}{2}(1+\omega^{-m})y_1^m y_2 + \frac{1}{2}(1-\omega^{-m})y_1^{-m} y_2. \end{gathered}$$

Moreover, φ is an Hopf algebra isomorphism.

P r o o f. It follows from Proposition 3.12 and Theorem 3.13. □

If $n = 2$, then the R-matrix of $H_2(\omega)$ can be described by the following equation.

$$\begin{aligned} R = 1 \otimes 1 &+ \frac{1}{4} x_3 \otimes (y_2 + y_1 y_2 + y_1^2 y_2 + y_1^3 y_2) + \frac{1}{4} x_1 \otimes (1 + y_1^2 - y_1 - y_1^3) \\ &+ \frac{1}{4} x_1 x_3 \otimes (y_2 + y_1^2 y_2 - y_1 y_2 - y_1^3 y_2) + \frac{1}{4} x_2 \otimes (1 - y_1^2 - \omega y_1 + \omega y_1^3) \\ &+ \frac{1}{4} x_2 x_3 \otimes (y_2 - y_1^2 y_2 - \omega y_1 y_2 + \omega y_1^3 y_2) + \frac{1}{4} x_1 x_2 \otimes (1 - y_1^2 + \omega y_1 - \omega y_1^3) \\ &+ \frac{1}{4} x_2 x_1 x_3 \otimes (y_2 - y_1^2 y_2 + \omega y_1 y_2 - \omega y_1^3 y_2). \end{aligned}$$

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