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ON Π-PROPERTY OF SOME MAXIMAL SUBGROUPS OF SYLOW SUBGROUPS OF FINITE GROUPS

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Abstract. Let H be a subgroup of a finite group G. We say that H satisfies the Π-property in G if for any chief factor L/K of G, $|G/K : N_{G/K}(HK/K \cap L/K)|$ is a $\pi(HK/K \cap L/K)$-number. We study the influence of some p-subgroups of G satisfying the Π-property on the structure of G, and generalize some known results.

Keywords: finite group; p-supersoluble group, p-nilpotent group, Π-property

MSC 2020: 20D10, 20D20

1. Introduction

All groups considered in this paper are finite. We use conventional notions as in [8]. Throughout the paper, G always denotes a finite group, p denotes a fixed prime, π denotes a set of primes and $\pi(G)$ denotes the set of all primes dividing $|G|$. An integer n is called a π-*number* if all prime divisors of n belong to π. In particular, an integer n is called a p-*number* if it is a power of p.

Suppose that P is a p-group. Let $\mathcal{M}(P)$ be the set of all maximal subgroups of P. Let d be the smallest generator number of P, i.e., $p^d = |P/\Phi(P)|$, where $\Phi(P)$ is the Frattini subgroup of P. Following [13], $\mathcal{M}_d(P) = \{P_1, \ldots, P_d\}$ is a subset of $\mathcal{M}(P)$ such that $\bigcap_{i=1}^{d} P_i = \Phi(P)$. Notice that the subset $\mathcal{M}_d(P)$ is not unique for a fixed

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p-group P in general. We know that $|\mathcal{M}(P)| = (p^d - 1)/(p-1)$, $|\mathcal{M}_d(P)| = d$ and $\lim\limits_{d\to\infty} (p^d-1)/((p-1)d) = \infty$, so $|\mathcal{M}(P)| \gg |\mathcal{M}_d(P)|$.

In [12], Li introduced the concept of the Π-property of subgroups of finite groups. Let H be a subgroup of G. We say that H satisfies the Π-property in G if for any chief factor L/K of G, $|G/K : N_{G/K}(HK/K \cap L/K)|$ is a $\pi(HK/K \cap L/K)$-number. This embedding property of subgroups has a strong structural impact and generalises many other known properties, see Section 4.

In this note, we study the influence of some maximal subgroups of Sylow subgroups satisfying the Π-property on the structure of finite groups. Our first result is as follows:

Theorem 1.1. *Let P be a Sylow p-subgroup of G for a prime $p \in \pi(G)$. Suppose that every member of a fixed $\mathcal{M}_d(P)$ satisfies the Π-property in G. Then either $|P| = p$ or G is p-supersoluble.*

Based on Theorem 1.1, we can prove the following results.

Theorem 1.2. *Suppose that P is a Sylow p-subgroup of G and $N_G(P)$ is p-nilpotent for a prime $p \in \pi(G)$. Then G is p-nilpotent if and only if every member of a fixed $\mathcal{M}_d(P)$ satisfies the Π-property in G.*

Theorem 1.3. *Let p be a prime dividing the order of G with $(|G|, p-1) = 1$ and $P \in \mathrm{Syl}_p(G)$. Then G is p-nilpotent if and only if every member of a fixed $\mathcal{M}_d(P)$ satisfies the Π-property in G.*

Theorem 1.4. *Let G be a group. Suppose that every member of a fixed $\mathcal{M}_d(P)$ satisfies the Π-property in G for every non-cyclic Sylow subgroup P of G. Then G is supersoluble.*

2. Preliminaries

In this section, we give some lemmas that will be used in our proofs.

Lemma 2.1 ([12], Proposition 2.1 (1))**.** *Let H be a subgroup of G and N a normal subgroup of G. If H satisfies the Π-property in G, then HN/N satisfies the Π-property in G/N.*

Lemma 2.2 ([3], Chaper A, Lemma 1.2)**.** *Let U, V and W be subgroups of G. Then the following statements are equivalent:*
(1) $U \cap VW = (U \cap V)(U \cap W)$.
(2) $UV \cap UW = U(V \cap W)$.

Lemma 2.3 ([4], Lemma 2.4)**.** *Suppose that H is a non-abelian simple group. If the Sylow p-subgroups H_p of H are of order p, then the outer automorphism group* $\mathrm{Out}(H)$ *of H is a p'-group.*

Lemma 2.4 ([8], Kapitel I, Hauptsatz 17.4)**.** *Suppose that N is an abelian normal subgroup of G and $N \leqslant M \leqslant G$ such that $(|N|, |G : M|) = 1$. If N is complemented in M, then N is complemented in G.*

Lemma 2.5. *Let H be a p-subgroup of G for some prime $p \in \pi(G)$ and $K \trianglelefteq G$. If H satisfies the* Π*-property in G and $H \leqslant K$, then H satisfies the* Π*-property in K.*

P r o o f. Let A/B be an arbitrary chief factor of K. Let C/D be a chief factor of G below K such that $D \leqslant B \leqslant A \leqslant C$. Then $|G : N_G(HD \cap C)|$ is a p-number. Thus, $|K : N_K(HD \cap C)|$ is a p-number. Clearly, $N_K(HD \cap C) \leqslant N_K((HD \cap A)B) = N_K(HB \cap A)$ and hence $|K : N_K(HB \cap A)|$ is a p-number. This shows that H satisfies the Π-property in K. □

Lemma 2.6. *Let H be a p-subgroup of G for a prime $p \in \pi(G)$. If G is p-supersoluble, then H satisfies the* Π*-property in G.*

P r o o f. Let M/N be an arbitrary chief factor of G. Then $|M/N|$ is a p'-number or $|M/N| = p$. If $|M/N|$ is a p'-number, then $(H \cap M)N/N = 1$. If $|M/N| = p$, then $(H \cap M)N/N = M/N$ or 1. In any case, we have that $|G/N : N_{G/N}((H \cap M)N/N)| = 1$. Therefore, H satisfies the Π-property in G, as wanted. □

3. Proofs

P r o o f of Theorem 1.1. Suppose that the theorem is not true and G is a counterexample of minimal order. Then $|P| \geqslant p^2$ and G is not p-supersoluble. Set $\mathcal{M}_d(P) = \{P_1, \ldots, P_d\}$. We divide the proof into the following steps.

Step 1: $O_{p'}(G) = 1$. Set $\overline{G} = G/O_{p'}(G)$. It is clear that $\overline{P}$ is a Sylow p-subgroup of $\overline{G}$. Moreover, $\overline{P}$ has the same smallest generator number as P. So $\mathcal{M}_d(\overline{P}) = \{\overline{P_1}, \ldots, \overline{P_d}\}$ and $\bigcap_{i=1}^{d} \overline{P_i} = \Phi(\overline{P_i})$. By Lemma 2.1, $\overline{P_i}$ satisfies the Π-property in $\overline{G}$ for any $\overline{P_i} \in \mathcal{M}_d(\overline{P})$. Thus, $\overline{G}$ satisfies the hypotheses of the theorem. If $O_{p'}(G) > 1$, then either $|\overline{P}| = p$ or $\overline{G}$ is p-supersoluble by the minimal choice of G. It follows that either $|P| = p$ or G is p-supersoluble, a contradiction. Therefore, $O_{p'}(G) = 1$.

Step 2: P is non-cyclic. Assume that P is cyclic. Then P has a unique maximal subgroup P_1. Let K be a minimal normal subgroup of G. Since $O_{p'}(G) = 1$, it follows that $P \cap K > 1$. Note that $|P| \geqslant p^2$. Since P is a cyclic p-group, we have that $P_1 \cap K =$

$P_1 \cap (P \cap K) > 1$. By hypothesis, P_1 satisfies the Π-property in G. As a consequence, $|G : N_G(P_1 \cap K)|$ is a p-number. It follows that $P_1 \cap K \trianglelefteq G$. By Theorem 2.1 of [19], we deduce that G is p-soluble. Furthermore, G is p-supersoluble, a contradiction.

Step 3: Let $\Phi(P)_G$ be the core of $\Phi(P)$ in G, then $\Phi(P)_G = 1$. Assume that $\Phi(P)_G > 1$. Then we can pick a minimal normal subgroup K of G contained in $\Phi(P)_G$. Since $\Phi(P) \leqslant P_i$ for any $P_i \in \mathcal{M}_d(P)$, we have that $\bigcap_{i=1}^{d} P_i/K = \Phi(P/K)$. Obviously, P/K has the same smallest generator number as P. By Lemma 2.1, G/K satisfies the hypotheses of the theorem. The minimal choice of G implies that $|P/K| = p$ or G/K is p-supersoluble. If $|P/K| = p$, then P is cyclic, contrary to Step 2. If G/K is p-supersoluble, then G is p-supersoluble, also a contradiction.

Step 4: If N is a minimal normal subgroup of G contained in P, then $|N| = p$. If $N \leqslant P_i$ for every $P_i \in \mathcal{M}_d(P)$, then $N \leqslant \bigcap_{i=1}^{d} P_i = \Phi(P)$, which is contrary to Step 3. Hence, there exists $\widehat{P} \in \mathcal{M}_d(P)$ such that $N \nleqslant \widehat{P}$. By hypothesis, $\widehat{P}$ satisfies the Π-property in G. Then $|G : N_G(\widehat{P} \cap N)|$ is a p-number. Since $\widehat{P} \cap N \trianglelefteq P$, it follows that $\widehat{P} \cap N \trianglelefteq G$. Hence, $\widehat{P} \cap N = 1$. Note that $N \leqslant P$, we have $|N| = p$.

Step 5: All minimal normal subgroups of G are contained in $O_p(G)$. Assume that T is a minimal normal subgroup of G which is not a p-subgroup. By Step 1, we have that $p \mid |T|$ and $T = T_1 \times \ldots \times T_s$, where T_i $(i = 1, \ldots, s)$ is a non-abelian simple subgroup of T.

Substep 5.1: $P_i \cap T = 1$ for any P_i in $\mathcal{M}_d(P)$ and $|P \cap T| = p$. In addition, T is a non-abelian simple group. For any $P_i \in \mathcal{M}_d(P)$, P_i satisfies the Π-property in G. Then $|G : N_G(P_i \cap T)|$ is a p-number. Since $P_i \cap T \trianglelefteq P$, it follows that $P_i \cap T \trianglelefteq G$. Observe that T is not a p-group, we have $P_i \cap T = 1$, and thus $|P \cap T| = p$ by Step 1. Hence, $T = T_1$ is a non-abelian simple group.

Substep 5.2: Under the assumption, $O_p(G) = 1$. If $O_p(G) > 1$, we can pick a minimal normal subgroup N of G contained in $O_p(G)$. By Step 4, we know that N is of order p. Hence, $NT = N \times T$. By hypothesis, P_i satisfies the Π-property in G for any $P_i \in \mathcal{M}_d(P)$. Consider the chief factor TN/N. Then $|G : N_G(P_iN \cap TN)|$ is a p-number. Note that $P_iN \cap TN \trianglelefteq P$, and so $P_iN \cap TN \trianglelefteq G$. This yields that $P_iN \cap TN = N$ or $P_iN \cap TN = TN$. Since T is not a p-group, it follows that $TN \nleqslant P_iN$. Therefore, $P_iN \cap TN = N$. If $N \nleqslant P_i$, then $P_iN = P$. This implies that $P_iN \cap TN = P \cap TN = N(P \cap T) > N$, a contradiction. Hence, $N \leqslant P_i$. It follows that $N \leqslant \bigcap_{i=1}^{d} P_i = \Phi(P)$, which is contrary to Step 3.

Substep 5.3: $C_G(T) = 1$. Assume that $C_G(T) > 1$. Let L be a minimal normal subgroup of G contained in $C_G(T)$. By (5.1), we get that $T \cap L = 1$. Consider the chief factor LT/T. By hypothesis, P_i satisfies the Π-property in G for any $P_i \in \mathcal{M}_d(P)$.

Then $|G : N_G(P_iT \cap LT)|$ is a p-number, and thus $P_iT \cap LT \trianglelefteq G$. This forces that $LT \leqslant P_iT$ or $P_iT \cap LT = T$. If $LT \leqslant P_iT$, then $P_iLT/T = P_iT/T \cong P_i/(P_i \cap T)$, and thus P_iLT/T is a p-group. This shows that LT/T is a p-group. Since $L \cap T = 1$, we conclude that L is a non-identity p-group, which is contrary to Substep 5.2. Therefore, $P_iT \cap LT = T$. This forces that $P_i \cap L = 1$. By Lemma 2.2, we have that $T \cap P_iL = (T \cap P_i)(T \cap L) = T \cap P_i$. By Substep 5.1, there exists a subgroup $P_j \in \mathcal{M}_d(P)$ such that $T \cap P_j = 1$. Thus, $|P_jTL|_p > |P|$, a contradiction.

Substep 5.4: Finishing the proof of Step 5. Since $C_G(T) = 1$ by Substep 5.3, we see that G is isomorphic to a subgroup of $\mathrm{Aut}(T)$. Note that $Z(T) \leqslant C_G(T) = 1$, and so $|G/T|$ divides $|\mathrm{Aut}(T)/\mathrm{Inn}(T)|$. In view of Substep 5.1, we conclude that p divides $|\mathrm{Out}(T)|$. By Lemma 2.3, this is impossible.

Step 6: $O_p(G)$ is a direct product of some normal subgroups of G of order p and $G = O_p(G) \rtimes R$, the semi-direct product of $O_p(G)$ with a subgroup R of G. Let K_1 be a minimal normal subgroup of G contained in $O_p(G)$. Then $|K_1| = p$ by Step 4 and $K_1 \cap \Phi(P) = 1$ by Step 3. Hence, there exists a maximal subgroup M_1 of P such that $K_1 \cap M_1 = 1$. By Lemma 2.4, K_1 has a complement U in G, i.e., $G = K_1U$ and $K_1 \cap U = 1$. Hence, $P = K_1(P \cap U)$. Then $O_p(G) = K_1(O_p(G) \cap U)$. If $O_p(G) \cap U = 1$, then Step 6 holds. Now assume that $O_p(G) \cap U > 1$. Hence, we can pick a minimal normal subgroup K_2 of G contained in $O_p(G) \cap U$. Then $|K_2| = p$ by Step 4 and $K_2 \cap \Phi(P) = 1$ by Step 3. Hence, there exists a maximal subgroup M_2 of P such that $K_2 \cap M_2 = 1$. Then $P = K_2M_2 = (O_p(G) \cap U)M_2 = (P \cap U)M_2$. It is clear that $|(P \cap U) : (M_2 \cap U)| = |M_2(P \cap U) : M_2| = |P : M_2| = p$. Thus, $M_2 \cap U$ is a complement of K_2 in $P \cap U$. Therefore, K_2 has a complement V in U by Lemma 2.4. Then $G = K_1U = (K_1 \times K_2) \rtimes V$. Continuing this process, we finally have $G = O_p(G) \rtimes R$ and $O_p(G) = K_1 \times K_2 \times \cdots \times K_t$, where K_i $(i = 1, \ldots, t)$ is a normal subgroup of G of order p.

Step 7: The final contradiction. By Step 6, we know that $O_p(G)$ is a direct product of some normal subgroups of G of order p. Hence, $P \leqslant C_G(O_p(G))$. Notice that $C_G(O_p(G)) \cap R \trianglelefteq O_p(G)R = G$. By Step 5, we have $C_G(O_p(G)) \cap R = 1$. Then $P \cap R = 1$. This yields that $P = P \cap O_p(G)R = O_p(G)(P \cap R) = O_p(G)$. By Step 6, G is p-supersoluble, the final contradiction. Our proof is now complete. □

Proof of Theorem 1.2. By Lemma 2.6, we only need to prove the sufficiency. Applying Theorem 1.1, we know that either $|P| = p$ or G is p-supersoluble. If $|P| = p$, then $P \leqslant Z(N_G(P))$ because $N_G(P)$ is p-nilpotent. By Burnside's theorem (see [9], Theorem 5.13), G is p-nilpotent, as wanted. Hence, we may suppose that G is p-supersoluble. By [8], Kapitel VI, Hauptsatz 6.6 we know that the p-length of a p-supersoluble group is at most 1. Thus, $PO_{p'}(G)$ is normal in G. Write $\overline{G} = G/O_{p'}(G)$. Then $\overline{G} = N_{\overline{G}}(\overline{P}) = N_G(P)O_{p'}(G)/O_{p'}(G)$ is p-nilpotent. Hence, G is p-nilpotent, as wanted. □

Proof of Theorem 1.3. We only need to prove the sufficiency. By Theorem 1.1, we know that either $|P| = p$ or G is p-supersoluble. If $|P| = p$, then G is p-nilpotent by [6], Chapter 1, Lemma 3.39 as desired. If G is p-supersoluble, then G is p-nilpotent by [6], Chapter 2, Lemma 5.25 and we are done. □

Corollary 3.1. *Let N be a normal subgroup of G such that G/N is p-nilpotent and P is a Sylow p-subgroup of N, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Suppose that every member of some fixed $\mathcal{M}_d(P)$ satisfies the Π-property in G. Then G is p-nilpotent.*

Proof. By Theorem 1.3, we know that N is p-nilpotent. Let M be the normal p-complement of N. Then $M \trianglelefteq G$. By Lemma 2.1, G/M satisfies the hypotheses of the corollary. If $M > 1$, then G/M is p-nilpotent by induction. Thus, G is p-nilpotent, as desired. Now assume that $M = 1$, then $N = P$. Let K/P be the normal p-complement of G/P. Then $K \trianglelefteq G$ and P is the Sylow p-subgroup of K. By Lemma 2.5, we see that every member of $\mathcal{M}_d(P)$ satisfies the Π-property in K. Then K is p-nilpotent by Theorem 1.3. Let K_1 be the normal p-complement of K. Clearly, K_1 is also a normal p-complement of G. Hence, G is p-nilpotent and the proof is complete. □

Proof of Theorem 1.4. Let q be the smallest prime of $|G|$ and $Q \in \mathrm{Syl}_q(G)$. If Q is cyclic, then G is q-nilpotent by [9], Corollary 5.14. If Q is non-cyclic, then by Theorem 1.3, G is q-nilpotent. By the same arguments and induction, we see that G is a Sylow tower group. Applying Theorem 1.1, we conclude that G is supersoluble. □

Remark 3.2. There exists a saturated formation $\mathcal{F}$ containing $\mathcal{U}$, the class of all supersoluble groups, and a soluble group G with a normal subgroup N such that $G/N \in \mathcal{F}$, and for every non-cyclic Sylow subgroup P of N, every member of a fixed $\mathcal{M}_d(P)$ satisfies the Π-property in G. But $G \notin \mathcal{F}$.

For example, let f be a formation function defined by $f(p)$, the class of p'-groups for any prime p, and let $\mathcal{F}$ be the formation locally defined by $f(p)$. If M is a supersoluble group, then any p-chief factor L/K of M is cyclic of order p, and so $M/C_M(L/K)$ is cyclic of order dividing $p-1$. Hence, $M/C_M(L/K) \in f(p)$. Therefore, $M \in \mathcal{F}$ and so $\mathcal{F}$ contains $\mathcal{U}$. It is not difficult to see that $A_4 \in \mathcal{F}$.

Let $P = \langle a, b, c \rangle$ be an elementary abelian group of order 3^3, and let α, β be two automorphisms of P defined respectively by

$$\alpha = \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix}, \quad \beta = \begin{pmatrix} a & b & c \\ b & c^{-1} & a^{-1} \end{pmatrix}.$$

Then $\alpha^3 = \beta^3 = (\alpha\beta)^2 = 1$ and $H = \langle \alpha, \beta \rangle \cong A_4$. Thus, H acts on P by automorphisms. Let $G = P \rtimes H$ be the corresponding semidirect product. Then P is an irreducible and faithful A_4-module on $GF(3)$, and so it is a minimal normal subgroup of G with $C_H(P) = 1$. Since $A_4 \in \mathcal{F}$ and $G/P \cong H \cong A_4$, we have $G/P \in \mathcal{F}$. Let $R = PS$, where S is a Sylow 2-subgroup of G. We have $O^3(G) \leqslant R \trianglelefteq G$. Since S is elementary abelian of order 4, it follows that a minimal normal subgroup of R contained in P is of order 3. By Maschke's theorem (see [5], Chapter 3, Theorem 3.1), P is a completely reducible S-module. Hence, $P = \langle a_1 \rangle \times \langle a_2 \rangle \times \langle a_3 \rangle$, where $\langle a_i \rangle$ $(i = 1, 2, 3)$ is S-invariant. Let $P_i = \langle a_j \colon j \neq i \rangle$. Then every P_i is normalized by $O^3(G)$, and so P_i satisfies the Π-property in G. Set $\mathcal{M}_d(P) = \{P_1, P_2, P_3\}$. On the other hand, $P/1$ is a 3-chief factor of G and $G/C_G(P) = G/P \cong A_4$, which is not a $3'$-group. Hence, $G \notin \mathcal{F}$.

4. Final remarks and applications

In this section, we will show that the concept of the Π-property can be viewed as a generalization of many known embedding properties.

Recall that two subgroups H and K of G are said to be permutable if $HK = KH$. From [11], a subgroup H of G is said to be S-permutable (or π-quasinormal, S-quasinormal) in G if H permutes with all Sylow subgroups of G. According to [7], a subgroup H of G is said to be X-permutable with a subgroup T of G if there is an element $x \in X$ such that $HT^x = T^xH$, where X is a nonempty subset of G. Following [1], the $\mathcal{U}$-hypercenter $Z_{\mathcal{U}}(G)$ of G is the product of all normal subgroups H of G, such that all G-chief factors under H have prime orders, where $\mathcal{U}$ denotes the class of all supersoluble groups. A subgroup H of G is called a CAP-subgroup of G if H either covers or avoids every chief factor L/K of G, that is, $HL = HK$ or $H \cap L = H \cap K$, see [3], Chapter A, Definition 10.8.

Proposition 4.1. *Let H be a subgroup of G. Then H satisfies the Π-property in G if one of the following holds:*
(1) *H is normal in G;*
(2) *H is permutable in G;*
(3) *H is S-permutable in G;*
(4) *H is X-permutable with all Sylow subgroups of G, where X is a soluble normal subgroup of G;*
(5) *H is a CAP-subgroup of G;*
(6) *$H/H_G \leqslant Z_{\mathcal{U}}(G/H_G)$.*

Proof. Statements (1)–(6) were proved in Propositions 2.2–2.3 of [12]. □

A subgroup H of G is said to be S-semipermutable (see [2]) in G if $HG_p = G_pH$ for any Sylow p-subgroup G_p of G with $(p, |H|) = 1$. A subgroup H of G is said to be SS-quasinormal (see [14]) in G if there is a subgroup B of G such that $G = H$ and H permutes with every Sylow subgroup of B.

Proposition 4.2. *Let H be a p-subgroup of G for a prime $p \in \pi(G)$. Then H satisfies* Π-*property in G if one of the following holds:*

(1) *H is S-semipermutable in G;*

(2) *H is SS-quasinormal in G.*

P r o o f. (1) Let L/K be an arbitrary chief factor of G. Write $\overline{G} = G/K$. At first, we argue that $|\overline{G} : N_{\overline{G}}(\overline{H \cap L})|$ is a p-number. It is no loss of generality to assume that $\overline{H \cap L} > 1$. By Lemma 2.2 (4) of [16], $H \cap L$ is S-semipermutable in G. It follows from Lemma 2.2 (2) of [16] that $\overline{H} \cap \overline{L} = \overline{H \cap L}$ is S-semipermutable in $\overline{G}$. Then the normal closure $(\overline{H \cap L})^{\overline{G}}$ of $\overline{H \cap L}$ in $\overline{G}$ is soluble by Theorem A of [10]. Since $\overline{L}$ is a minimal normal subgroup of $\overline{G}$ and $\overline{H \cap L} > 1$, we have that $(\overline{H \cap L})^{\overline{G}} = \overline{L}$ is a normal p-subgroup of $\overline{G}$. Applying Lemmas 2.2 (3) and 2.1 (6) of [12], we get that $O^p(\overline{G}) \leqslant N_{\overline{G}}(\overline{H \cap L})$, and thus $|\overline{G} : N_{\overline{G}}(\overline{H \cap L})|$ is a p-number, as claimed. Therefore, H satisfies the Π-property in G.

(2) Applying Lemma 2.5 of [14], we know that H is S-semipermutable in G. By (1), the conclusion follows. □

By Propositions 4.1 and 4.2, we can obtain the following corollaries.

Corollary 4.3 ([14], Theorem 1.1)**.** *Let p be the smallest prime dividing the order of G and P a Sylow p-subgroup of G. If every member of a fixed $\mathcal{M}_d(P)$ is SS-quasinormal in G, then G is p-nilpotent.*

Corollary 4.4 ([14], Theorem 1.2)**.** *Let p be a prime dividing the order of G and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every member of a fixed $\mathcal{M}_d(P)$ is SS-quasinormal in G, then G is p-nilpotent.*

Corollary 4.5 ([14], Theorem 1.3)**.** *Let G be a p-solvable group for a prime p and P a Sylow p-subgroup of G. Suppose that every member of a fixed $\mathcal{M}_d(P)$ is SS-quasinormal in G. Then G is p-supersoluble.*

Corollary 4.6 ([17], Theorem 3.1)**.** *Let G be a p-soluble group and let P be a Sylow p-subgroup of G, where p is a fixed prime. Then G is p-supersoluble if and only if every member of a fixed $\mathcal{M}_d(P)$ is a* CAP-*subgroup of G.*

Corollary 4.7 ([17], Theorem 3.3)**.** *Let p be the smallest prime dividing the order of G and let P be a Sylow p-subgroup of G. Then G is p-nilpotent if and only if every member of a fixed $\mathcal{M}_d(P)$ is a* CAP*-subgroup of G.*

Corollary 4.8 ([17], Theorem 3.4)**.** *Suppose that P is a Sylow p-subgroup of G and $N_G(P)$ is p-nilpotent for a prime $p \in \pi(G)$. Then G is p-nilpotent if and only if every member of a fixed $\mathcal{M}_d(P)$ is a* CAP*-subgroup of G.*

Corollary 4.9 ([15], Theorem 3.1)**.** *Let G be a group and P be a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If all maximal subgroups of P are S-semipermutable in G, then G is p-nilpotent.*

Corollary 4.10 ([4], Main result)**.** *Let P be a Sylow p-subgroup of G for a prime $p \in \pi(G)$. Suppose that every member of some fixed $\mathcal{M}_d(P)$ is a* CAP*-subgroup of G. Then either $|P| = p$ or G is p-supersoluble.*

Corollary 4.11 ([18], Theorem 3.8)**.** *Let p be a prime dividing the order of a p-soluble group G and let P be a Sylow p-subgroup of G. If every member of a fixed $\mathcal{M}_d(P)$ is S-semipermutable in G, then G is p-supersoluble.*

Corollary 4.12 ([18], Theorem 3.9)**.** *Let p be an odd prime dividing the order of G and let P be a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every member of a fixed $\mathcal{M}_d(P)$ is S-semipermutable in G, then G is p-nilpotent.*

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