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ON LIMIT CYCLES OF PIECEWISE DIFFERENTIAL SYSTEMS FORMED BY ARBITRARY LINEAR SYSTEMS AND A CLASS OF QUADRATIC SYSTEMS

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Abstract. We study the continuous and discontinuous planar piecewise differential systems separated by a straight line and formed by an arbitrary linear system and a class of quadratic center. We show that when these piecewise differential systems are continuous, they can have at most one limit cycle. However, when the piecewise differential systems are discontinuous, we show that they can have at most two limit cycles, and that there exist such systems with two limit cycles. Therefore, in particular, we have solved the extension of the 16th Hilbert problem to this class of differential systems.

Keywords: discontinuous piecewise differential system; continuous piecewise differential system; first integral; non-algebraic limit cycle; linear system; quadratic center

MSC 2020: 34C05, 34C07, 37G15

1. Introduction and statement of the main results

We know that in the qualitative theory of planar dynamical systems, one of the most important topics is related to the second part of the unsolved Hilbert 16th problem; for more details we refer to [10]. In the piecewise discontinuous and continuous context this problem has been studied recently by many authors and numerous applications can be cited, see for instance [2], [5], [13], [12], [16], among other papers.

In 1991, Lum and Chua (see [17], [18]) conjectured that planar continuous piecewise linear differential systems with two zones separated by a straight line have at most one limit cycle. In 1998, this conjecture was proved by Freire et al. in [8]. Limit cycles of discontinuous piecewise linear differential systems separated by a straight line have been studied by many authors; see for instance [1], [9], [12], [11], [15], [3] and the references therein. There are examples of such systems exhibiting three limit

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cycles (see [5], [13], [16]), but it is an open question to know if 3 is the maximum number of crossing limit cycles that such discontinuous differential systems can have.

For piecewise smooth quadratic systems, the authors in [4] showed that there are piecewise quadratic systems with 9 small amplitude limit cycles. In [14], the authors showed that at least 5 limit cycles can bifurcate from quadratic isochronous centers under piecewise smooth quadratic perturbations. Recently, the author in [19] showed that the piecewise smooth quadratic isochronous systems can have at least 6 limit cycles. On the other hand, it seems intuitively clear that "most" limit cycles of continuous and discontinuous piecewise differential systems have to be non-algebraic. Nevertheless, in all these papers devoted to the study of the crossing limit cycles of piecewise differential systems, explicit non-algebraic limit cycles do not appear.

The goal of this paper is to study the limit cycles of a class planar piecewise differential systems separated by a straight line $\Sigma = \{(x,y) \in \mathbb{R}^2 \colon x = 0\}$ and formed by arbitrary linear differential system

$$\dot{x} = \alpha x + \beta y, \quad \dot{y} = \eta x + \delta y \tag{1}$$

and a quadratic polynomial differential system with a center

$$\dot{x} = n(x^2 + bx + hy + l) + h, \quad \dot{y} = -m(x^2 + bx + hy + l) - 2x - b, \tag{2}$$

where b, n, m, l, h, β, α, δ and η are real constants.

The following lemma gives sufficient conditions for the existence of a center in differential system (2) and its first integrals.

Lemma 1. *If $hn > 0$, the point (P_1, P_2), where*

$$P_1 = \frac{hm - bn}{2n}, \quad P_2 = \frac{b^2n^2 - h^2m^2 - 4hn - 4ln^2}{4hn^2},$$

is a singularity of center type of differential system (2). Moreover, the differential system (2) has the first integral

$$H_2(x,y) = (x^2 + bx + hy + l)\mathrm{e}^{ny+mx}.$$

P r o o f. Through the linear transformations

$$x = X + P_1, \quad y = -\frac{1}{hn}(Y + hmX) + P_2,$$

we still denote X, Y by x, y for convenience.

The point (P_1, P_2) will be moved into $(0,0)$ of the new system, and the system (2) will be changed into the following system:

$$\dot{x} = nx^2 - y, \quad \dot{y} = 2hnx. \tag{3}$$

Notice that this system is symmetric with respect to the y-axis, i.e., it is invariant under the transformation $(x, y, t) \to (-x, y, -t)$. Since $hn > 0$, the eigenvalues of the Jacobian matrix at $(0,0)$ are $\pm \mathrm{i}\sqrt{2hn}$; then this is either a center or a focus for system (3), but since the system is symmetric with respect to the y-axis, the origin is a center.

It is easy to check that $\frac{\partial H_2}{\partial x}\dot{x} + \frac{\partial H_2}{\partial y}\dot{y} \equiv 0$ for the functions H_2 given in Lemma 1, and hence it is first integral of this system. □

In order to state precisely our results, we introduce first some notation and definitions. Consider the piecewise differential system

$$X_{\pm}\colon\ (\dot{x}, \dot{y}) = (f_{\pm}(x,y), g_{\pm}(x,y)),$$

defined in $\Sigma_{\pm} = \{(x,y) \in \mathbb{R}^2\colon\ \pm x > 0\}$. We use the techniques and approaches presented by Filippov in [7] and by di Bernardo et al. in [6] to establish these notations. In order to extend the definition of a trajectory to Σ, we split Σ into two parts depending on whether or not the vector field points towards it:

1. Crossing region: $\Sigma_{\mathrm{c}} = \{(0,y) \in \Sigma\colon\ f_+(0,y)f_-(0,y) > 0\}$.
2. Sliding region: $\Sigma_{\mathrm{s}} = \{(0,y) \in \Sigma\colon\ f_+(0,y)f_-(0,y) \leqslant 0\}$.

Periodic orbits that have neither a sliding part nor tangent points are called crossing periodic orbits, otherwise they are called sliding periodic orbits. We say that an isolated periodic orbit Γ is an algebraic limit cycle if all its points are contained in an algebraic curve of the plane; otherwise such a limit cycle is called non-algebraic. As we have already said, in this paper we deal with the non-algebraic limit cycles of piecewise differential systems

$$\begin{aligned} &\dot{x} = \alpha x + \beta y, && \dot{y} = \eta x + \delta y && \text{in } \Sigma_+,\\ &\dot{x} = n(x^2 + bx + hy + l) + h, && \dot{y} = -m(x^2 + bx + hy + l) - 2x - b && \text{in } \Sigma_-, \end{aligned} \tag{4}$$

where b, n, m, l, h, β, α, η and δ are real constants, such that $hn > 0$. We prove that when these systems are continuous, they have at most one limit cycle. However, when the piecewise differential systems are discontinuous, we show that they can have at most two limit cycles. Moreover, these limit cycles, if they exist, are not algebraic and are explicitly given. Concrete examples exhibiting the applicability of our result are introduced.

Our first main result is contained in the following theorem.

Theorem 2. *The discontinuous piecewise differential system* (4)*, can have at most two crossing non-algebraic limit cycles. Moreover, inside this class of discontinuous piecewise differential systems, there are systems with exactly either* 1 *or* 2 *such non-algebraic limit cycles.*

Theorem 2 is proved in Section 2. Here we shall prove that Theorem 2 cannot be extended to continuous piecewise linear differential system. Thus our second main result is:

Theorem 3. *The continuous piecewise differential system* (4) *can have at most one crossing non-algebraic limit cycle. Moreover, there are systems in this class having one limit cycle.*

Theorem 3 is proved in Section 3.

2. Proof of Theorem 2

The origin of a planar differential system (4) in the right zone is either a saddle or a diagonal node or a non-diagonal node, or a focus, or a center. It is easy to check that if the right subsystem of (4) is a saddle or a diagonal node or a non-diagonal node, then the crossing limit cycles cannot exist. Therefore, in order to be able to have a limit cycle, such an equilibrium point must be either a focus or center for the right subsystem of (4).

The following normal form for the linear differential systems in $\mathbb{R}^2$, having a focus or a center at the origin, and its first integrals will help us prove our main result.

Lemma 4 ([3])**.** *The linear system* (1)*, having a focus (or a center) at the origin can be written as*

$$\dot{x} = (2\lambda - \delta)x + \beta y, \quad \dot{y} = -\frac{1}{\beta}((\lambda - \delta)^2 + \omega^2)x + \delta y,$$

with $\omega > 0$ and $\lambda \neq 0$ (or $\omega > 0$ and $\lambda = 0$). Moreover, the first integral of this system is given by

$$H_1(x,y) = \begin{cases} (((\lambda-\delta)^2+\omega^2)x^2 + 2\beta(\lambda-\delta)xy + \beta^2y^2) \\ \qquad\qquad \times e^{(-2\lambda/\omega)\arctan \omega x/((\lambda-\delta)x+y\beta)} & \text{if } \lambda \neq 0, \\ (\delta^2+\omega^2)x^2 - 2\beta\delta xy + \beta^2 y^2 & \text{if } \lambda = 0, \end{cases} \tag{5}$$

P r o o f o f Theorem 2. The crossing limit cycles of the piecewise differential systems (4) intersect the line of discontinuity Σ in two different points $(0, y_0)$ and $(0, y_1)$. Clearly, these two points must satisfy the system of equations

$$H_1(0, y_0) - H_1(0, y_1) = 0, \quad H_2(0, y_0) - H_2(0, y_1) = 0, \tag{6}$$

since the origin is a focus type (or a center) for right subsystem of (4). This means that starting at any point $(0, y_0)$ with $y_0 < 0$, the orbits of the right subsystem of (4) will go into the zone Σ_- in a counterclockwise direction until they reach Σ at a point $(0, y_1)$ with $y_1 > 0$ after a time t_+. Now we can use the parametric form $(x_+(t), y_+(t))$ of the curve $H_1(x, y) = H_1(0, y_0)$ in the half-plane Σ_+ where

$$x_+(t) = \frac{\beta}{\omega} y_0 \mathrm{e}^{\lambda t} \sin \omega t, \quad y_+(t) = \frac{1}{\omega} y_0 \mathrm{e}^{\lambda t}(\omega \cos \omega t + (\delta - \lambda) \sin \omega t).$$

In the case where the point $(0, y_1)$ exists, the following must be satisfy: $y_1 = y_+(t_+)$ and $x_+(t_+) = 0$. By solving this last equation when $\beta y_0/\omega \neq 0$, the minimum positive time t_+ is $t_+ = \pi/\omega$. Substituting the previous time in $y_1 = y_+(t_+)$, we obtain

$$y_1 = y(t_+) = \begin{cases} -y_0 \mathrm{e}^{\lambda\pi/\omega} & \text{if } \lambda \neq 0, \\ -y_0 & \text{if } \lambda = 0. \end{cases} \tag{7}$$

This proves that $H_1(0, y_0) - H_1(0, y_1) = 0$. Therefore, the discontinuous piecewise differential system (4) has limit cycles if the equation

$$H_2(0, y_0) - H_2(0, y_1) = 0, \tag{8}$$

has isolated solutions.

If $\lambda \neq 0$, the equation (8) can be written in

$$h y_0 + l + (h \mathrm{e}^{\lambda\pi/\omega} y_0 - l) \mathrm{e}^{-n(\mathrm{e}^{\lambda\pi/\omega}+1) y_0} = 0. \tag{9}$$

Then, the existence and the number of a crossing limit cycles of (4) is equivalent to the existence and the number of a isolated negative values y_0 satisfying (9); in other words, the existence and the number of zeros for the function

$$f(y) = h y + l + (h \mathrm{e}^{\lambda\pi/\omega} y - l) \mathrm{e}^{-n(\mathrm{e}^{\lambda\pi/\omega}+1) y},$$

with respect to variable y. It is easy to see that f is an infinitely continuously differentiable with respect to y. A direct calculation shows that

$$\begin{aligned} f'(y) &= h + ((h + ln)\mathrm{e}^{\lambda\pi/\omega} + ln - hny\mathrm{e}^{\lambda\pi/\omega} - hny\mathrm{e}^{2\pi\lambda/\omega})\mathrm{e}^{-n(\mathrm{e}^{\lambda\pi/\omega}+1)y}, \\ f''(y) &= (\mathrm{e}^{\lambda\pi/\omega} + 1)(ln + (2h + ln)\mathrm{e}^{\lambda\pi/\omega} - hn(1 + \mathrm{e}^{\lambda\pi/\omega})\mathrm{e}^{\lambda\pi/\omega} y)\mathrm{e}^{-n(\mathrm{e}^{\lambda\pi/\omega}+1)y}. \end{aligned}$$

We can easily notice that $f''(y) = 0$ has at most one root y_0. Thus the equation $f(y) = 0$ has at most three roots y_{0i}, $i = 1, 2, 3$. In view of the expression of the function $f(y)$, it follows immediately that $y = 0$ is a solution of $f(y) = 0$, which cannot contribute a limit cycle. So, $f(y) = 0$ can have at most 2 solutions that can provide at most 2 limit cycles for the piecewise differential system (4). Using the first integrals of both differential systems of (4) and knowing that the non-algebraic crossing periodic orbits passes through the points $(0, y_{0i})$, $i = 1, 2$ when $t = 0$ and through the point $(0, y_{1i})$ when $t = \pi/\omega$ where y_{0i} are the zeros of $f(y)$ and $y_{1i} = -y_{0i}e^{\lambda\pi/\omega}$. Thus these expressions are:

$$\begin{aligned}\Gamma_i = \{(x, y) \in \Sigma_+ \colon\ &(((\lambda-\delta)^2 + \omega^2)x^2 + 2\beta(\lambda-\delta)xy + \beta^2y^2)\\ &\times e^{-(2\lambda/\omega)\arctan(\omega x/((\lambda-\delta)x+\beta y))} = \beta^2y_{0i}^2\}\\ \cup\, \{(x, y) \in \Sigma_- \colon\ &(x^2 + bx + hy + l)e^{ny+mx} = (hy_{0i} + l)e^{ny_{0i}}\}.\end{aligned}$$

and the Theorem 2 is proved if $\lambda \neq 0$.

If $\lambda = 0$, the equation (8) can be written in $g(y) = 0$ where

$$g(y) = (hy + l)e^{ny} + (hy - l)e^{-ny}.$$

It is easy to see that the function g is odd (i.e., $g(-y) = -g(y)$), Therefore, if y is a solution of $g(y) = 0$, then $-y$ is also a solution of $g(y) = 0$. Now, solving $g(y) = 0$ is equivalent to find the solutions of the equation $G(y) = 0$, where

$$G(y) = hy + l + (hy - l)e^{-2ny}.$$

In order to investigate the number of solutions of $G(y) = 0$, and since G is infinitely continuously differentiable with respect to y in $\mathbb{R}$, we shall use the first two derivatives of the function G. Simple calculations yield

$$G'(y) = h + (h + 2ln - 2hny)e^{-2ny}, \quad G''(y) = -4ne^{-2ny}(h + ln - hny).$$

It is obvious that $G''(y) = 0$ has at most one root, and it follows that $G(y) = 0$ has at most three roots y_{0i}, $i = 1, 2, 3$.

It is easy to see that $y = 0$ is a solution of the equation $G(y) = 0$. According to the previous remark, we know that if y_0 is a solution of $G(y) = 0$, then $-y_0$ is also a solution of $G(y) = 0$. By (7) the two values y_0 and $y_1 = -y_0$ (if there exists y_0) contribute to the same limit cycle. Thus when $\lambda = 0$, the discontinuous piecewise differential system (4) has at most one limit cycle. Moreover, we can choose the appropriate parameters h, l, and n in such a way that $G(y) = 0$ has exactly two real roots y_0, and a $y_1 = -y_0$ that can provide 1 limit cycle for the discontinuous piecewise differential system (4). Using the first integrals of both differential systems of (4) with $\lambda = 0$, and knowing that the non-algebraic crossing periodic orbits passes through the point $(0, y_0)$ when $t = 0$ and through the point $(0, -y_0)$ when $t = \pi/\omega$ where y_0 is a solution of $G(y) = 0$. Thus this expression is:

$$\begin{aligned}\Gamma = \{(x,y) \in \Sigma_+ \colon\ &(\delta^2+\omega^2)x^2 - 2\beta\delta xy + \beta^2 y^2 = \beta^2 y_0^2\}\\ \cup \{(x,y) \in \Sigma_- \colon\ &(x^2+bx+hy+l)\mathrm{e}^{ny+mx} = (hy_0+l)\mathrm{e}^{ny_0}\}.\end{aligned}$$

This completes the proof of Theorem 2. □

The next examples show that the upper bound for the maximum number of crossing limit cycles provided in Theorem 2 is reached.

Case 1: The right subsystem of (4) is a focus type.

Example 5. Consider a discontinuous piecewise differential system (4) with $b = -6$, $l = -13$, $n = \frac{1}{10}$, $m = -\frac{1}{2}$, $h = 1$, $\alpha = -1$, $\beta = 1$, $\eta = -4$ and $\delta = -1$:

$$\begin{aligned}&\dot{x} = -x + y, && \dot{y} = -4x - y && \text{in } \Sigma_+,\\ &\dot{x} = -6x + y + x^2 - 3, && \dot{y} = 5x^2 - 50x + 5y - 5 && \text{in } \Sigma_-.\end{aligned} \tag{10}$$

This system has exactly one explicit non-algebraic limit cycle Γ enclosing a sliding set $\Sigma_{\mathrm{s}} = \{(0,y) \in \Sigma \colon\ 0 \leqslant y \leqslant 3\}$ and is given by

$$\begin{aligned}\Gamma = \{(x,y) \in \Sigma_+ \colon\ &(4x^2+y^2)\mathrm{e}^{-\arctan(-2x/y)} = 1.7421\}\\ \cup \{(x,y) \in \Sigma_- \colon\ &(x^2-6x+y-13)\mathrm{e}^{y/10-x/2} = -12.549\},\end{aligned}$$

see the Figure 1.

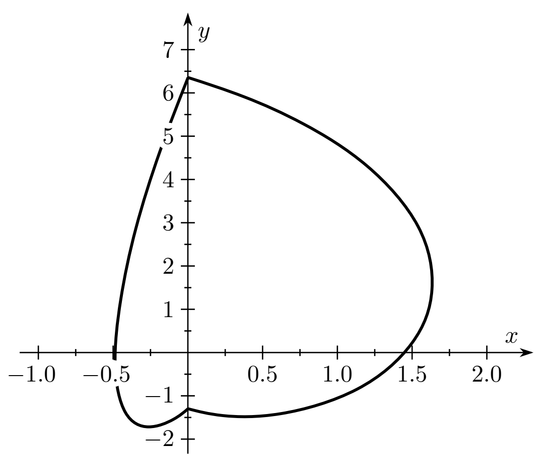

Figure 1. The crossing non-algebraic limit cycle of the discontinuous piecewise linear differential systems (10).

E x a m p l e 6. Consider a discontinuous piecewise differential system (4) with $b = -4, n = \frac{1}{3}, m = \beta = -1, h = 1, l = -2, \alpha = -3, \eta = 8$ and $\delta = 1$:

$$\begin{array}{llll} (11) & \dot{x} = -3x - y, & \dot{y} = 8x + y & \text{in } \Sigma_+, \\ & \dot{x} = -x^2 + 4x - y - 1, & \dot{y} = -3x^2 + 18x - 3y - 6 & \text{in } \Sigma_-. \end{array}$$

This system has exactly two explicit non-algebraic limit cycles Γ_i, $i = 1, 2$ enclosing a sliding set $\Sigma_{\rm s} = \{(0, y) \in \Sigma\colon -1 \leqslant y \leqslant 0\}$. These cycles are given by

$$\begin{aligned} \Gamma_1 = \{&(x, y) \in \Sigma_+\colon (8x^2 + 4xy + y^2)\mathrm{e}^{-\arctan(2x/(2x+y))} = 19.825\} \\ &\cup \{(x, y) \in \Sigma_-\colon (x^2 - 4x + y - 2)\mathrm{e}^{y/3-x} = -1.4627\}, \\ \Gamma_2 = \{&(x, y) \in \Sigma_+\colon (8x^2 + 4xy + y^2)\mathrm{e}^{-\arctan(2x/(2x+y))} = 50.971\} \\ &\cup \{(x, y) \in \Sigma_-\colon (x^2 - 4x + y - 2)\mathrm{e}^{y/3-x} = -0.84603\}, \end{aligned}$$

see Figure 2.

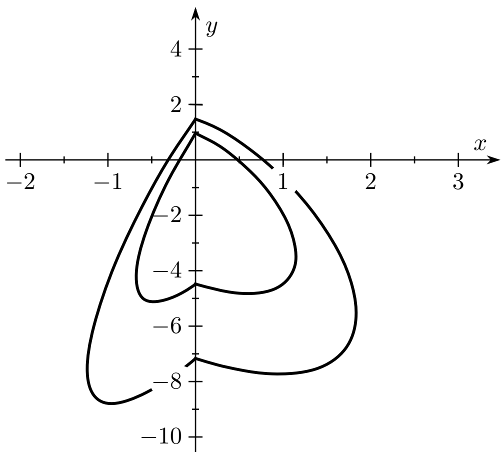

Figure 2. The two crossing non-algebraic limit cycles of the discontinuous piecewise linear differential systems (11).

Case 2: The right subsystem of (4) is of a center type

E x a m p l e 7. Consider a discontinuous piecewise differential system (4) with $b = -6$, $l = -3$, $n = -1$, $m = -\frac{1}{2}$, $h = -1$, $\alpha = 1$, $\beta = -1$, $\eta = 4$ and $\delta = -1$:

$$\begin{array}{llll} (12) & \dot{x} = x - y, & \dot{y} = 2x - y & \text{in } \Sigma_+, \\ & \dot{x} = 4x^2 - 24x - 4y - 6, & \dot{y} = 10x + y - x^2 - 10 & \text{in } \Sigma_-. \end{array}$$

This system has exactly one explicit non-algebraic limit cycle Γ that encloses a sliding set $\Sigma_s = \{(0, y) \in \Sigma\colon\ -\frac{3}{2} \leqslant y \leqslant 0\}$ and is given by

$$\Gamma = \{(x, y) \in \Sigma_+\colon\ 2x^2 - 2xy + y^2 = 3.9948\}$$
$$\cup \{(x, y) \in \Sigma_-\colon\ (x^2 - 6x - y - 2)\mathrm{e}^{-2y-x/2} = -7.3429 \times 10^{-2}\},$$

see Figure 3.

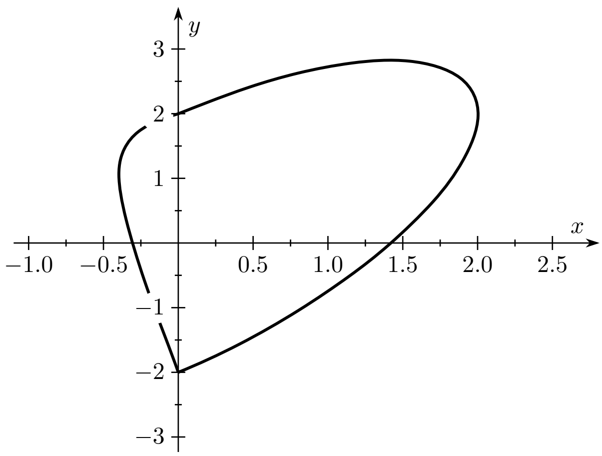

Figure 3. The crossing non-algebraic limit cycles of the discontinuous piecewise linear differential systems (12).

3. Proof of Theorem 3

Since we must have a continuous piecewise differential system for both systems of (4), in Σ_+ and in Σ_+ must coincide on $x = 0$. Therefore,

$$b = h\frac{m}{n}, \quad l = -\frac{h}{n}, \quad \beta = hn \quad \text{and} \quad \delta = -hm.$$

With the same techniques introduced in the proof of Theorem 2, it can be proved that the continuous piecewise differential system (4) can have at most one crossing non-algebraic limit cycle.

In order to have a crossing limit cycle that intersects Σ at the points $(0, y_0)$ with $y_0 < 0$ and $(0, y_1)$ with $y_1 > 0$, these points must satisfy system (6). The analytic form $(x_+(t), y_+(t))$ in the half-plane Σ_+ of the curve $H_1(0, y) = H_1(0, y_0)$ is given by

$$x_+(t) = \frac{hn}{\omega} y_0 \mathrm{e}^{\lambda t} \sin \omega t, \quad y_+(t) = \frac{1}{\omega} y_0 \mathrm{e}^{\lambda t}(\omega \cos \omega t - (hm + \lambda) \sin \omega t).$$

Since this orbit can reach Σ again at some point $(0, y_1)$, it must satisfy $y_1 = y_+(t_+)$ and $e^{\lambda t_+}(\sin \omega t_+) y_0 \beta/\omega = 0$. Solving this last equation when $y_0\beta/\omega \neq 0$, the minimum time t_+ is $t_+ = \pi/\omega$. Substituting the previous time in $y_1 = y_+(t_+)$, we obtain

$$y_1 = y(t_+) = \begin{cases} -y_0 e^{\lambda\pi/\omega} & \text{if } \lambda \neq 0, \\ -y_0 & \text{if } \lambda = 0. \end{cases}$$

In order for the continuous piecewise differential system (4) to have limit cycles, the equation

$$H_2(0, y_0) - H_2(0, y_1) = 0 \tag{13}$$

must be have negative isolated values y_0.

If $\lambda \neq 0$, the equation (13) can be written in

$$n y_0 - 1 + (1 + n y_0 e^{\pi\lambda/\omega}) e^{-n(e^{\lambda\pi/\omega}+1) y_0} = 0.$$

For convenience we put

$$f(y) = n y - 1 + (1 + n y e^{\pi\lambda/\omega}) e^{-n(e^{\lambda\pi/\omega}+1) y}.$$

It is not difficult to see that f is infinitely continuously differentiable with respect to y. With a simple calculation we get

$$\begin{aligned} f'(y) &= n - n(n y e^{\pi\lambda/\omega} + n y e^{2\pi\lambda/\omega} + 1) e^{-n(e^{\lambda\pi/\omega}+1) y}, \\ f''(y) &= n^2 (e^{\pi\lambda/\omega} + 1)(n y e^{\pi\lambda/\omega} - e^{\pi\lambda/\omega} + n y e^{2(\pi\lambda/\omega)} + 1) e^{-n(e^{\lambda\pi/\omega}+1) y}. \end{aligned}$$

Clearly $f''(y) = 0$ has at most one root y_0, and thus the equation $f(y) = 0$ has at most three solutions y_{0i}, $i = 1, 2, 3$. From the expressions of the functions $f(y)$ and $f'(y)$, we can easily see that $f(0) = f'(0) = 0$. Hence 0 is a strict local maximum or minimum of the function $f(y)$, i.e., $f(y) > f(0)$ or $f(y) < f(0)$ for all points y in a deleted. So, in this case, $f(y) = 0$ can have at most one solution $y_0 \neq 0$ that can provide at most one limit cycle for the discontinuous piecewise differential system (4). We use the first integrals of both differential systems of (4) with $b = hm/n$, $l = -h/n$, $\beta = hn$ and $\delta = -hm$ and know that the non-algebraic crossing periodic orbit passes through the points $(0, y_0)$ when $t = 0$ and through the point $(0, -y_0 e^{\lambda\pi/\omega})$ when $t = \pi/\omega$ where y_0 are the zeros of $f(y)$. Thus these expressions are:

$$\begin{aligned} \Gamma = \{(x, y) \in \Sigma_+ \colon\ &(((\lambda + hm)^2 + \omega^2) x^2 + 2hn(\lambda + hm) x y + (hn)^2 y^2) \\ &\times e^{(-2\lambda/\omega)\arctan(\omega x/((\lambda+hm)x+hny))} = (hny_0)^2\} \\ \cup \{(x, y) \in \Sigma_- \colon\ &(x^2 + (mh/n)x + hy - h/n) e^{ny+mx} = (hy_0 - h/n) e^{ny_0}\} \end{aligned}$$

and the theorem is proved if $\lambda \neq 0$.

If $\lambda = 0$, the equation (13) can be written in $g(y_0) = 0$ where

$$g(y_0) = \Big(hy_0 - \frac{h}{n}\Big)\mathrm{e}^{ny_0} + \Big(hy_0 + \frac{h}{n}\Big)\mathrm{e}^{-ny_0}.$$

It is easy to see that the function g is odd (i.e., $g(-y) = -g(y)$). Therefore, if y is a solution of $g(y) = 0$, then $-y$ is also a solution of $g(y) = 0$. Now, solving $g(y) = 0$ is equivalent to finding the solutions y_0 of the equation $G(y) = 0$, where

$$G(y) = hy - \frac{h}{n} + \Big(hy + \frac{h}{n}\Big)\mathrm{e}^{-2ny}.$$

In order to investigate the number of solutions of $G(y) = 0$, and since G is infinitely continuously differentiable with respect to y in $\mathbb{R}$, we shall use the first two derivatives of the function G. Simple calculations yield

$$G'(y) = h - (h + 2hny)\mathrm{e}^{-2ny}, \quad G''(y) = 4hn^2\mathrm{e}^{-2ny}y.$$

Clearly $G''(y) = 0$ has at most one root y_0, and so $G(y) = 0$ has at most three solutions y_{0i}, $i = 1, 2, 3$. On the other hand, it is obvious that $G(0) = G'(0) = G''(0) = 0$, and thus there is a horizontal tangent to the graph of G at the origin. Thus $G(y) = 0$ has at most two roots. According to the previous remark, we know that if y_0 is a solution of $G(y) = 0$, then $-y_0$ is also a solution of $G(y) = 0$. Since $G(0) = 0$ then $y_0 = 0$ is the unique solution of $G(y) = 0$, then the continuous piecewise differential system (4) has no limit cycle. This completes the proof of Theorem 3. □

The next example shows that the upper bound for the maximum number of crossing limit cycles provided in Theorem 3 is reached.

E x a m p l e 8. Consider a continuous piecewise differential system (4) with $n = -1$, $b = 2$, $l = -2$, $\beta = -2$, $h = -2$, $m = 1$, $\eta = \frac{5}{2}$, $\alpha = 0$ and $\delta = 2$:

$$\begin{aligned} &\dot{x} = -2y, && \dot{y} = \frac{5}{2}x + 2y && \text{in } \Sigma_+, \\ &\dot{x} = -x^2 - 2x + 2y, && \dot{y} = -x^2 - 4x + 2y && \text{in } \Sigma_-. \end{aligned} \tag{14}$$

This system has exactly one explicit non-algebraic limit cycle Γ given by

$$\begin{aligned} \Gamma = \{(x,y) \in \Sigma_+ \colon\ &(5x^2 + 4xy + 4y^2)\mathrm{e}^{\arctan(2x/(x+2y))} = 3.8483\} \\ \cup \{(x,y) \in \Sigma_- \colon\ &(x^2 + 2x - 2y - 2)\mathrm{e}^{-y+x} = -0.10214\}, \end{aligned}$$

see Figure 4.

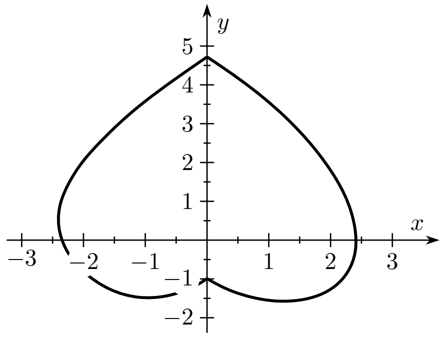

Figure 4. The crossing non-algebraic limit cycles of the continuous piecewise linear differential systems (14).

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