Tatiana Danielsson; Liselott Flodén; Pernilla Johnsen; Marianne Olsson Lindberg

Homogenization of monotone parabolic problems with an arbitrary number of spatial and temporal scales

Applications of Mathematics, Vol. 69 (2024), No. 1, 1–24

Persistent URL: `http://dml.cz/dmlcz/152248`

HOMOGENIZATION OF MONOTONE PARABOLIC PROBLEMS WITH AN ARBITRARY NUMBER OF SPATIAL AND TEMPORAL SCALES

Tatiana Danielsson, Liselott Flodén, Pernilla Johnsen, Marianne Olsson Lindberg, Östersund

Received November 14, 2022. Published online December 18, 2023.

In memory of Anders Holmbom (1958–2022)

Abstract. We prove a general homogenization result for monotone parabolic problems with an arbitrary number of microscopic scales in space as well as in time, where the scale functions are not necessarily powers of the scale parameter ε. The main tools for the homogenization procedure are multiscale convergence and very weak multiscale convergence, both adapted to evolution problems.

Keywords: homogenization; parabolic; monotone; two-scale convergence; multiscale convergence; very weak multiscale convergence

MSC 2020: 35B27

1. Introduction

In this paper we present a homogenization result for general monotone parabolic problems of the type

$$\begin{aligned}(1.1)\quad \partial_t u^\varepsilon(x,t) - \nabla\cdot a\Big(\frac{x}{\hat\varepsilon_1},\ldots,\frac{x}{\hat\varepsilon_n},\frac{t}{\check\varepsilon_1},\ldots,\frac{t}{\check\varepsilon_m},\nabla u^\varepsilon(x,t)\Big) &= f(x,t) \quad \text{in } \Omega_T,\\ u^\varepsilon(x,t) &= 0 \quad \text{on } \partial\Omega\times(0,T),\\ u^\varepsilon(x,0) &= u^0(x) \quad \text{in } \Omega\end{aligned}$$

with multiple spatial and temporal scales, where $f\in L^2(\Omega_T)$ and $u^0\in L^2(\Omega)$. Here Ω is an open bounded set in $\mathbb{R}^N$ with smooth boundary and $\Omega_T=\Omega\times(0,T)$. We let $Y=(0,1)^N$ and $S=(0,1)$ and we assume that a is Y-periodic in the n first variables and S-periodic in the following m variables. Finally, we let $\hat\varepsilon_k$ for $k=1,\ldots,n$ and $\check\varepsilon_j$

DOI: 10.21136/AM.2023.0269-22

for $j = 1, \ldots, m$ be scale functions depending on ε and tending to zero as ε does, where the scales are assumed to fulfill certain conditions of separatedness.

The mathematical theory of nonlinear partial differential equations plays an important role in, e.g., applied mathematics and physics. Homogenization theory concerns finding the effective properties, on a macroscopic level, of media with a strongly heterogeneous microstructure. Mathematically, the homogenization of (1.1) means studying the asymptotic behavior of the corresponding sequence of solutions u^ε as ε tends to zero and finding the limit problem

$$\begin{aligned} \partial_t u(x,t) - \nabla \cdot b(x,t,\nabla u(x,t)) &= f(x,t) && \text{in } \Omega_T, \\ u(x,t) &= 0 && \text{on } \partial\Omega \times (0,T), \\ u(x,0) &= u^0(x) && \text{in } \Omega, \end{aligned}$$

which admits the function u, the limit of $\{u^\varepsilon\}$, as its unique solution. Here b is characterized by local problems, one for each microscopic spatial scale. For more informative texts on homogenization theory we suggest, e.g., [1], [6] and [17].

The main tools used to carry out the homogenization process for (1.1) are multiscale convergence and very weak multiscale convergence in the evolution setting. Here very weak multiscale convergence, see, e.g., [10] and [12], is the key to handling the difficulties that appear when rapid time oscillations are present. The nonlinearity of the problem is treated by applying the perturbed test functions method.

Homogenization results for linear parabolic equations with oscillations in one spatial scale and one temporal scale were obtained by using asymptotic expansions in [5]. In [15] parabolic problems containing fast oscillations in space as well as in time were treated for the first time applying two-scale convergence methods. Using the same kind of methods, linear parabolic problems have also been investigated in, e.g., [14], where more than one fast temporal scale was considered for the first time and in [12], where an arbitrary number of scales in both space and time was treated. Furthermore, linear parabolic problems with one fast scale in space as well as in time have been studied using the periodic unfolding method in [3] and [4]. Homogenization results for not necessarily linear parabolic problems have been presented in, e.g., [13] and [26] using multiscale convergence methods and in [20] and [27] by the method of Σ-convergence, all of them for different combinations of moderate numbers of fixed scales, the number of fast scales being limited to at most one in either space or time or both, and in [21] for one microscopic spatial scale and arbitrarily many temporal scales.

The present paper contributes by collecting, combining and extending previous homogenization results of periodic parabolic multiscale problems. It serves as an overview of such problems in the sense that several other works can be obtained as special cases of the results in this paper. Novelties are that the problem studied is not necessarily linear while also exhibiting an arbitrary number of both spatial and temporal scales and that the scale functions do not have to be power functions of ε, as is the case in several other previous works (e.g., all mentioned above except [21] and [3]). The choice of arbitrary scale functions reconnects to the original setting of [2] by Allaire and Briane, where multiscale convergence (for more than two spatial scales) was first introduced and applied to the homogenization of elliptic multiscale problems.

Notation 1.1. *We let $F_\sharp(Y)$ be the space of all functions in $F_{\mathrm{loc}}(\mathbb{R}^N)$ which are the periodic repetition of a function in $F(Y)$. We also let $Y_k = Y$ for $k = 1, \ldots, n$, $Y^n = Y_1 \times \ldots \times Y_n$ (the nN-dimensional open unit cell), $y^n = y_1, \ldots, y_n$ (corresponding local spatial multivariable), $\mathrm{d}y^n = \mathrm{d}y_1 \ldots \mathrm{d}y_n$, $S_j = S$ for $j = 1, \ldots, m$, $S^m = S_1 \times \ldots \times S_m$ (the m-dimensional open unit cell), $s^m = s_1, \ldots, s_m$ (corresponding local temporal multivariable), $\mathrm{d}s^m = \mathrm{d}s_1 \ldots \mathrm{d}s_m$ and $\mathcal{Y}_{n,m} = Y^n \times S^m$, where we interpret $\mathcal{Y}_{0,m}$ as S^m. We let $\hat{\varepsilon}_k(\varepsilon)$, for $k = 1, \ldots, n$, and $\check{\varepsilon}_j(\varepsilon)$, $j = 1, \ldots, m$, be strictly positive functions such that $\hat{\varepsilon}_k(\varepsilon)$ and $\check{\varepsilon}_j(\varepsilon)$ go to zero when ε does. We also use the notations $\hat{\varepsilon}^n = \hat{\varepsilon}_1, \ldots, \hat{\varepsilon}_n$ and $\check{\varepsilon}^m = \check{\varepsilon}_1, \ldots, \check{\varepsilon}_m$ and furthermore, $x/\hat{\varepsilon}^n$ denotes $x/\hat{\varepsilon}_1, \ldots, x/\hat{\varepsilon}_n$ and similarly, by $t/\check{\varepsilon}^m$ we mean $t/\check{\varepsilon}_1, \ldots, t/\check{\varepsilon}_m$.*

2. Multiscale and very weak multiscale convergence

In [18] Nguetseng presented a new homogenization technique based on a certain type of convergence which has become known as two-scale convergence. Allaire has further developed the concept and in [1] he presented, e.g., compactness results for some alternative classes of test functions. See also, e.g., [29] and [17]. This concept of convergence was extended in [2] by Allaire and Briane to so-called multiscale convergence, which allows a use of multiple scales and makes it possible to capture numerous types of spatial microscopic oscillations. Below we define evolution multiscale convergence, which is a further development of multiscale convergence to include rapid temporal oscillations, see also [12].

Definition 2.1. A sequence $\{u^\varepsilon\}$ in $L^2(\Omega_T)$ is said to $(n+1, m+1)$-scale converge to $u_0 \in L^2(\Omega_T \times \mathcal{Y}_{n,m})$ if

$$\int_{\Omega_T} u^\varepsilon(x,t) v\Big(x, t, \frac{x}{\hat{\varepsilon}^n}, \frac{t}{\check{\varepsilon}^m}\Big) \,\mathrm{d}x \,\mathrm{d}t \to \int_{\Omega_T} \int_{\mathcal{Y}_{n,m}} u_0(x, t, y^n, s^m) \times v(x, t, y^n, s^m) \,\mathrm{d}y^n \,\mathrm{d}s^m \,\mathrm{d}x \,\mathrm{d}t$$

for any $v \in L^2(\Omega_T; C_\sharp(\mathcal{Y}_{n,m}))$. We write

$$u^\varepsilon(x,t) \overset{n+1,m+1}{\rightharpoonup} u_0(x, t, y^n, s^m).$$

The next definition concerns concepts regarding relations between scale functions.

Definition 2.2. We say that the scales in a list $\{\varepsilon_1, \ldots, \varepsilon_n\}$ are separated if

$$\lim_{\varepsilon \to 0} \frac{\varepsilon_{k+1}}{\varepsilon_k} = 0$$

for $k = 1, \ldots, n-1$ and that the scales are well-separated if there exists a positive integer l such that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon_k} \left(\frac{\varepsilon_{k+1}}{\varepsilon_k} \right)^l = 0$$

for $k = 1, \ldots, n-1$. For the special case when a list consists of only one element it is defined to be well-separated.

Worth noting is that separatedness is implied by well-separatedness. In the results with respect to evolution multiscale convergence (e.g., Theorem 2.6), there are assumptions on how the spatial and temporal microscales shall be related to each other. The notion of jointly (well-)separated lists of scales was first introduced by Persson, see, e.g., [22]. Note that it will not be enough that the lists of spatial and temporal scales are individually well-separated for the lists to be jointly well-separated.

Definition 2.3. Let $\{\hat{\varepsilon}^n\}$ and $\{\check{\varepsilon}^m\}$ be lists of (well-)separated scales. Collect all elements from both lists into one common list. If all possible pairs $\{\hat{\varepsilon}_k, \check{\varepsilon}_l\}$ such that for some $0 < C_{k,l} < \infty$,

$$\lim_{\varepsilon \to 0} \frac{\hat{\varepsilon}_k}{\check{\varepsilon}_l} = C_{k,l},$$

are removed and the list of all the remaining elements, sorted by order of magnitude, is (well-)separated, then the lists $\{\hat{\varepsilon}^n\}$ and $\{\check{\varepsilon}^m\}$ are said to be jointly (well-)separated. For the special case when the list of remaining elements is empty, the lists $\{\hat{\varepsilon}^n\}$ and $\{\check{\varepsilon}^m\}$ are defined to be jointly (well-)separated.

Note that the reason why pairs of asymptotically equal scale functions are removed is that it is already known that they are (well-)separated from all other scales and we do not have to take them into further consideration when continuing investigating the separatedness. To concretize the definition, we give the following example inspired by Example 2.60 in [22].

Example 2.4. The spatial list $\{\hat{\varepsilon}^2\} = \{2\sqrt{\varepsilon}, \varepsilon^2\}$ and the temporal list $\{\check{\varepsilon}^3\} = \{\mathrm{e}^{\varepsilon} - 1, \ln(1+\varepsilon^2), \varepsilon^3 \ln(1 + 1/\varepsilon)\}$ are both well-separated. Following Definition 2.3 we collect the scales in one common list $\{2\sqrt{\varepsilon}, \mathrm{e}^{\varepsilon} - 1, \varepsilon^2, \ln(1+\varepsilon^2), \varepsilon^3 \ln(1 + 1/\varepsilon)\}$ and since

$$\lim_{\varepsilon \to 0} \frac{\ln(1+\varepsilon^2)}{\varepsilon^2} = 1,$$

the pair $\{\varepsilon^2, \ln(1+\varepsilon^2)\}$ is removed and we have the joint list $\{2\sqrt{\varepsilon}, \mathrm{e}^{\varepsilon} - 1, \varepsilon^3 \ln(1 + 1/\varepsilon)\}$, which is well-separated. For an illustration, see Figure 1, where the well-separatedness is indicated by the space between the elements.

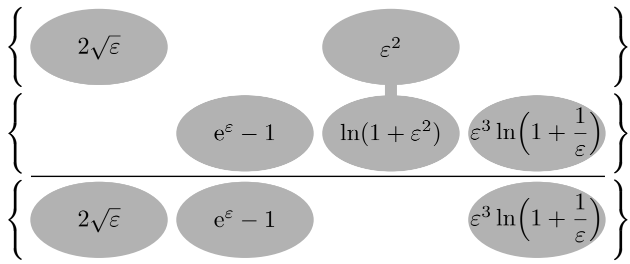

Figure 1. The two lists $\{\hat{\varepsilon}^2\} = \{2\sqrt{\varepsilon}, \varepsilon^2\}$ and $\{\breve{\varepsilon}^3\} = \{\mathrm{e}^{\varepsilon} - 1, \ln(1+\varepsilon^2), \varepsilon^3 \ln(1+1/\varepsilon)\}$ and the resulting well-separated joint list $\{2\sqrt{\varepsilon}, \mathrm{e}^{\varepsilon} - 1, \varepsilon^3 \ln(1+1/\varepsilon)\}$.

The lists $\{\hat{\varepsilon}'^3\} = \{2\sqrt{\varepsilon}, \varepsilon^2, \varepsilon^3\}$ and $\{\breve{\varepsilon}'^2\} = \{\varepsilon^2, \varepsilon^3 \ln(1+1/\varepsilon)\}$ are also both well-separated and they give the common list $\{2\sqrt{\varepsilon}, \varepsilon^2, \varepsilon^2, \varepsilon^3 \ln(1+1/\varepsilon), \varepsilon^3\}$. Obviously, the pair $\{\varepsilon^2, \varepsilon^2\}$ should be removed. Since

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^3 \ln(1+1/\varepsilon)} \left(\frac{\varepsilon^3}{\varepsilon^3 \ln(1+1/\varepsilon)} \right)^l = \infty$$

independently of the choice of the positive integer l while

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^3}{\varepsilon^3 \ln(1+1/\varepsilon)} = 0,$$

the pair $\{\varepsilon^3 \ln(1+1/\varepsilon), \varepsilon^3\}$ is merely separated. Hence, so is the joint list $\{2\sqrt{\varepsilon}, \varepsilon^3 \ln(1+1/\varepsilon), \varepsilon^3\}$, see Figure 2, where the mere separateness is indicated by the two elements in question being tangent to each other.

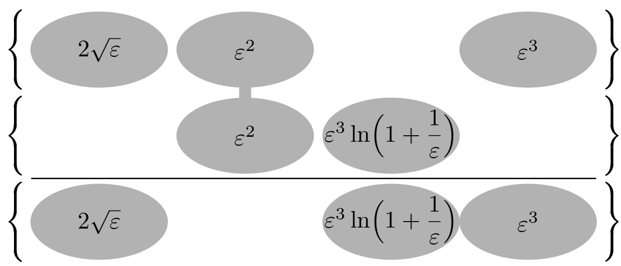

Figure 2. The two lists $\{\hat{\varepsilon}'^3\} = \{2\sqrt{\varepsilon}, \varepsilon^2, \varepsilon^3\}$ and $\{\breve{\varepsilon}'^2\} = \{\varepsilon^2, \varepsilon^3 \ln(1+1/\varepsilon)\}$ and the resulting separated joint list $\{2\sqrt{\varepsilon}, \varepsilon^3 \ln(1+1/\varepsilon), \varepsilon^3\}$.

To give some kind of intuitive understanding of (joint) separation of scales we present an elementary case with one spatial and one temporal microscale which together constitute a jointly well-separated setting. The pictures in Figure 3 illustrate, for three different values of ε, a sequence of functions $f(x/\varepsilon, t/\varepsilon^2)$, which could, e.g., represent a heat conductivity coefficient on $\Omega \times (0,T) = (0,1) \times (0,1)$. Even though both scale functions tend to zero, they can be distinguished from each another. One can easily see that the frequency increases more rapidly along the time axis than along the spatial axis as the value of ε decreases.

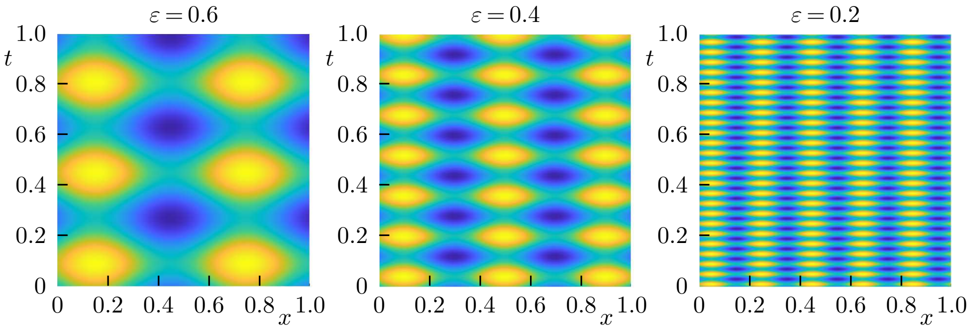

Figure 3. An illustration of joint scale separation.

The following two theorems state a compactness result for $(n+1, m+1)$-scale convergence and a characterization of multiscale limits for gradients, respectively.

Theorem 2.5. *Let $\{u^\varepsilon\}$ be a bounded sequence in $L^2(\Omega_T)$ and suppose that the lists $\{\hat{\varepsilon}^n\}$ and $\{\check{\varepsilon}^m\}$ are jointly separated. Then there exists u_0 in $L^2(\Omega_T \times \mathcal{Y}_{n,m})$ such that, up to a subsequence,*

$$u^\varepsilon(x,t) \overset{n+1,m+1}{\rightharpoonup} u_0(x,t,y^n,s^m).$$

P r o o f. See Theorem 2.66 in [22] or Theorem A.1 in [12]. □

The space $W^{1,2}(0,T;H_0^1(\Omega),L^2(\Omega))$ that appears in the theorem below is the space of all functions in $L^2(0,T;H_0^1(\Omega))$ such that the time derivative belongs to $L^2(0,T;H^{-1}(\Omega))$.

Theorem 2.6. *Let $\{u^\varepsilon\}$ be a bounded sequence in $W^{1,2}(0,T;H_0^1(\Omega),L^2(\Omega))$ and suppose that the lists $\{\hat{\varepsilon}^n\}$ and $\{\check{\varepsilon}^m\}$ are jointly well-separated. Then, up to a subsequence,*

$$\begin{aligned} u^\varepsilon(x,t) &\to u(x,t) \quad \text{in } \ L^2(\Omega_T),\\ u^\varepsilon(x,t) &\rightharpoonup u(x,t) \quad \text{in } L^2(0,T;H_0^1(\Omega)) \end{aligned}$$

and

$$\nabla u^\varepsilon(x,t) \overset{n+1,m+1}{\rightharpoonup} \nabla u(x,t) + \sum_{j=1}^{n} \nabla_{y_j} u_j(x,t,y^j,s^m),$$

where $u \in W^{1,2}(0,T;H_0^1(\Omega),L^2(\Omega))$ and $u_j \in L^2(\Omega_T \times \mathcal{Y}_{j-1,m};H_\sharp^1(Y_j)/\mathbb{R})$ for $j = 1,\dots,n$.

P r o o f. See Theorem 2.74 in [22] or Theorem 4 in [12]. □

Multiscale convergence is very useful for homogenization of problems involving rapid oscillations on several micro levels. Unfortunately, we can only use this for sequences bounded in the L^2-norm but when rapid time oscillations are present, we encounter sequences that do not possess this boundedness. Multiscale convergence has a large class of test functions and the limit captures both the global trend and the microscopic oscillations. If we downsize this class to only capture the microscopic fluctuations, it becomes possible to handle certain sequences that are not required to be bounded in any Lebesgue space. This is the idea behind so-called very weak multiscale convergence. A first compactness result of very weak multiscale convergence type was given in [15], see also [20], [10] and [11].

Definition 2.7. A sequence $\{w^\varepsilon\}$ in $L^1(\Omega_T)$ is said to $(n+1, m+1)$-scale converge very weakly to $w_0 \in L^1(\Omega_T \times \mathcal{Y}_{n,m})$ if

$$\int_{\Omega_T} w^\varepsilon(x,t) v_1\Big(x, \frac{x}{\hat{\varepsilon}^{n-1}}\Big) c\Big(t, \frac{t}{\check{\varepsilon}^m}\Big) v_2\Big(\frac{x}{\hat{\varepsilon}_n}\Big)\, \mathrm{d}x\, \mathrm{d}t$$

$$\to \int_{\Omega_T} \int_{\mathcal{Y}_{n,m}} w_0(x,t,y^n,s^m) v_1(x, y^{n-1}) c(t, s^m) v_2(y_n)\, \mathrm{d}y^n\, \mathrm{d}s^m\, \mathrm{d}x\, \mathrm{d}t$$

for any $v_1 \in D(\Omega; C_\sharp^\infty(Y^{n-1}))$, $v_2 \in C_\sharp^\infty(Y_n)/\mathbb{R}$ and $c \in D(0,T; C_\sharp^\infty(S^m))$, where

$$\int_{Y_n} w_0(x,t,y^n,s^m)\, \mathrm{d}y_n = 0.$$

We write

$$w^\varepsilon(x,t) \overset{n+1,m+1}{\underset{vw}{\rightharpoonup}} w_0(x,t,y^n,s^m).$$

The following theorem is essential for the homogenization of (1.1).

Theorem 2.8. *Let $\{u^\varepsilon\}$ be a bounded sequence in $W^{1,2}(0,T; H_0^1(\Omega), L^2(\Omega))$ and assume that the lists $\{\hat{\varepsilon}^n\}$ and $\{\check{\varepsilon}^m\}$ are jointly well-separated. Then there exists a subsequence such that*

$$\frac{u^\varepsilon(x,t)}{\hat{\varepsilon}_n} \overset{n+1,m+1}{\underset{vw}{\rightharpoonup}} u_n(x,t,y^n,s^m),$$

where $u_n \in L^2(\Omega_T \times \mathcal{Y}_{n-1,m}; H_\sharp^1(Y_n)/\mathbb{R})$ is the same as in Theorem 2.6 for $j = n$.

P r o o f. See Theorem 2.78 in [22] or Theorem 7 in [12]. □

3. The homogenization result

We study the homogenization of problem (1.1) given in the introduction, i.e.,

$$
\begin{aligned}
(3.1)\qquad \partial_t u^\varepsilon(x,t) - \nabla\cdot a\Big(\frac{x}{\hat{\varepsilon}^n},\frac{t}{\check{\varepsilon}^m},\nabla u^\varepsilon(x,t)\Big) &= f(x,t) && \text{in } \Omega_T,\\
u^\varepsilon(x,t) &= 0 && \text{on } \partial\Omega\times(0,T),\\
u^\varepsilon(x,0) &= u^0(x) && \text{in } \Omega,
\end{aligned}
$$

where $f\in L^2(\Omega_T)$ and $u^0\in L^2(\Omega)$. Here we assume that the flux function

$$a\colon\ \mathbb{R}^{nN}\times\mathbb{R}^m\times\mathbb{R}^N\to\mathbb{R}^N$$

satisfies the following structure conditions, where C_0 and C_1 are positive constants and $0<\alpha\leqslant 1$:

(i) $a(y^n,s^m,0)=0$ for all $(y^n,s^m)\in\mathbb{R}^{nN}\times\mathbb{R}^m$,
(ii) $a(\cdot,\cdot,\xi)$ is $\mathcal{Y}_{n,m}$-periodic and continuous for all $\xi\in\mathbb{R}^N$,
(iii) $a(y^n,s^m,\cdot)$ is continuous for all $(y^n,s^m)\in\mathbb{R}^{nN}\times\mathbb{R}^m$,
(iv) $(a(y^n,s^m,\xi)-a(y^n,s^m,\xi'))\cdot(\xi-\xi')\geqslant C_0|\xi-\xi'|^2$ for all $(y^n,s^m)\in\mathbb{R}^{nN}\times\mathbb{R}^m$ and all $\xi,\xi'\in\mathbb{R}^N$,
(v) $|a(y^n,s^m,\xi)-a(y^n,s^m,\xi')|\leqslant C_1(1+|\xi|+|\xi'|)^{1-\alpha}|\xi-\xi'|^\alpha$ for all $(y^n,s^m)\in\mathbb{R}^{nN}\times\mathbb{R}^m$ and all $\xi,\xi'\in\mathbb{R}^N$.

Under these conditions, problem (3.1) possesses a unique solution, see Theorem 30.A (a) in [28], and the a priori estimate

$$\|u^\varepsilon\|_{W^{1,2}(0,T;H_0^1(\Omega),L^2(\Omega))}<C$$

holds true for some $C>0$, see Proposition 3.16 in [22]. Finally, we assume that the lists $\{\hat{\varepsilon}^n\}$ and $\{\check{\varepsilon}^m\}$ in (3.1) are jointly well-separated.

In order to formulate the homogenization result (Theorem 3.1 below) in a neat way, we define some numbers determined by how the scale functions present are related to each other. Consider the spatial scale $\hat{\varepsilon}_i$. We define consecutively d_i and ϱ_i, $i=1,\dots,n$, as follows:

▷ If

$$\lim_{\varepsilon\to0}\frac{\check{\varepsilon}_1}{(\hat{\varepsilon}_i)^2}=0,\quad \text{then } d_i=m.$$

If, for some $j=1,\dots,m-1$,

$$\lim_{\varepsilon\to0}\frac{\check{\varepsilon}_j}{(\hat{\varepsilon}_i)^2}>0\quad\text{and}\quad\lim_{\varepsilon\to0}\frac{\check{\varepsilon}_{j+1}}{(\hat{\varepsilon}_i)^2}=0,\quad\text{then } d_i=m-j.$$

If

$$\lim_{\varepsilon\to 0}\frac{\check{\varepsilon}_m}{(\hat{\varepsilon}_i)^2} > 0, \quad \text{then } d_i = 0.$$

▷ If, for some $j = 1, \dots, m$,

$$\lim_{\varepsilon\to 0}\frac{(\hat{\varepsilon}_i)^2}{\check{\varepsilon}_j} = D_i, \quad 0 < D_i < \infty,$$

we say that we have resonance and we let $\varrho_i = D_i$, otherwise $\varrho_i = 0$.

We are now prepared to give and prove the main theorem of the paper. Here $W^{1,2}_\sharp(S; H^1_\sharp(Y)/\mathbb{R}, L^2_\sharp(Y)/\mathbb{R})$ denotes the space of all functions u such that $u \in L^2_\sharp(S; H^1_\sharp(Y)/\mathbb{R})$ and $\partial_s u \in L^2_\sharp(S; (H^1_\sharp(Y)/\mathbb{R})')$.

Theorem 3.1. *Let $\{u^\varepsilon\}$ be a sequence of solutions in $W^{1,2}(0,T; H^1_0(\Omega), L^2(\Omega))$ to (3.1). Then it holds that*

$$\begin{aligned} u^\varepsilon(x,t) &\to u(x,t) \quad \text{in } L^2(\Omega_T),\\ u^\varepsilon(x,t) &\rightharpoonup u(x,t) \quad \text{in } L^2(0,T; H^1_0(\Omega))\end{aligned}$$

and

$$\nabla u^\varepsilon(x,t) \overset{n+1,m+1}{\rightharpoonup} \nabla u(x,t) + \sum_{j=1}^n \nabla_{y_j} u_j(x,t,y^j,s^{m-d_j}),$$

where $u \in W^{1,2}(0,T; H^1_0(\Omega), L^2(\Omega))$ is the unique solution to the homogenized problem

$$\begin{aligned} \partial_t u(x,t) - \nabla\cdot b(x,t,\nabla u(x,t)) &= f(x,t) \quad \text{in } \Omega_T,\\ u(x,t) &= 0 \quad \text{on } \partial\Omega\times(0,T),\\ u(x,0) &= u^0(x) \quad \text{in } \Omega \end{aligned} \tag{3.2}$$

with

$$b(x,t,\nabla u(x,t)) = \int_{\mathcal{Y}_{n,m}} a\Big(y^n, s^m, \nabla u(x,t) + \sum_{j=1}^n \nabla_{y_j} u_j(x,t,y^j,s^{m-d_j})\Big)\,\mathrm{d}y^n\,\mathrm{d}s^m.$$

Here, for $i = 1, \dots, n$, u_i are the unique solutions to the system of local problems

$$\begin{aligned} \varrho_i \partial_{s_{m-d_i}} u_i(x,t,y^i,s^{m-d_i}) - \nabla_{y_i}\cdot \int_{S_{m-d_i+1}} \cdots \int_{S_m}\int_{Y_{i+1}} \\ \cdots \int_{Y_n} a\Big(y^n, s^m, \nabla u(x,t) + \sum_{j=1}^n \nabla_{y_j} u_j(x,t,y^j,s^{m-d_j})\Big) \\ \times\,\mathrm{d}y_n \dots \mathrm{d}y_{i+1}\,\mathrm{d}s_m \dots \mathrm{d}s_{m-d_i+1} = 0, \end{aligned} \tag{3.3}$$

where, when $\varrho_i = 0$, $u_i \in L^2(\Omega_T \times \mathcal{Y}_{i-1,m-d_i}; H^1_\sharp(Y_i)/\mathbb{R})$ and when $\varrho_i \neq 0$, $u_i \in L^2(\Omega_T \times \mathcal{Y}_{i-1,m-d_i-1}; W^{1,2}_\sharp(S_{m-d_i}; H^1_\sharp(Y_i)/\mathbb{R}, L^2_\sharp(Y_i)/\mathbb{R}))$.

Before giving the proof we provide some remarks concerning the theorem.

Remark 3.2. In words, the number d_i for $i = 1, \dots, n$ in the theorem specifies the number of temporal scales faster than the square of the spatial scale in question. The impact of this is that the corrector in question, u_i, will be independent of the local time variables corresponding to these rapid temporal scales. The number ϱ_i found in (3.3) indicates whether we have a parabolic or an elliptic local problem, i.e., the resonant case means a parabolic local problem whereas nonresonance an elliptic one.

Remark 3.3. In the theorem above, (3.2) and (3.3) constitute a coupled system of equations. In the special case when the equations are linear, it would be possible to decouple them using separation of variables. This was done already in [2] (Corollary 2.12) for elliptic problems with several spatial scales. See also Remark 3.2 in [7], where this is illustrated for a parabolic problem with three rapid scales in both space and time. In the case of nonlinearity, it is more complicated. One way to handle this case is to resort to numerical methods, see, e.g., [24], which presents an algorithm for a nonlinear parabolic problem with one rapid scale in both space and time.

For the convenience of the reader we also provide an outline for and some comments on the proof. The proof consists mainly of four steps, successively more extensive. In the first, rather brief, step, convergence results for the sequence of solutions are concluded up to a subsequence. In the second step, we derive the homogenized problem at a preliminary stage, here called pre-homogenized problem, in the sense that the evolution multiscale limit a_0 of the sequence of flux functions is not fully characterized. The third step consists of finding the local problems, still with the limit a_0 uncharacterized and hence called pre-local problems, including possible independencies of the local time variables in the correctors. This step is divided into two cases: nonresonance and resonance. These first three steps follow mainly the steps in the proof of Theorem 9 in [12], which treats the corresponding linear case. However, here we also need to handle the general scale functions. The proof is also abbreviated in the sense that the cases with and without independencies of the local time variables, respectively, have been merged, resulting in two cases instead of four. Due to the monotonicity there remains a vital part of the proof. This last main step consists of the characterization of a_0, giving the homogenized problem and the local problems their final form. Here the main tool is the method of perturbed test functions.

Proof of Theorem 3.1. *Main step 1:* Convergences of the solutions. The lists $\{\hat{\varepsilon}^n\}$ and $\{\check{\varepsilon}^m\}$ of scales are jointly well-separated and since $\{u^\varepsilon\}$ is bounded in $W^{1,2}(0,T;H_0^1(\Omega),L^2(\Omega))$, Theorem 2.6 is applicable and hence, up to a subsequence

$$
\begin{aligned}
u^\varepsilon(x,t) &\to u(x,t) \quad \text{in } L^2(\Omega_T),\\
u^\varepsilon(x,t) &\rightharpoonup u(x,t) \quad \text{in } L^2(0,T;H_0^1(\Omega))
\end{aligned}
$$

and

$$\nabla u^\varepsilon(x,t) \overset{n+1,m+1}{\rightharpoonup} \nabla u(x,t) + \sum_{j=1}^{n} \nabla_{y_j} u_j(x,t,y^j,s^m),$$

where $u \in W^{1,2}(0,T;H_0^1(\Omega),L^2(\Omega))$ and $u_j \in L^2(\Omega_T \times \mathcal{Y}_{j-1,m}; H_\sharp^1(Y_j)/\mathbb{R})$ for $j = 1,\dots,n$.

Main step 2: Pre-homogenized problem. The weak form of (3.1) reads: find $u^\varepsilon \in W^{1,2}(0,T;H_0^1(\Omega),L^2(\Omega))$ such that

$$\int_{\Omega_T} \Big(-u^\varepsilon(x,t)v(x)\partial_t c(t) + a\Big(\frac{x}{\hat{\varepsilon}^n}, \frac{t}{\check{\varepsilon}^m}, \nabla u^\varepsilon(x,t)\Big) \cdot \nabla v(x)c(t)\Big)\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} f(x,t)v(x)c(t)\,\mathrm{d}x\,\mathrm{d}t \tag{3.4}$$

for all $v \in H_0^1(\Omega)$ and $c \in D(0,T)$. By choosing $\xi' = 0$ in structure condition (v) we have

$$|a(y^n,s^m,\xi)| \leqslant C_1(1+|\xi|)^{1-\alpha}|\xi|^\alpha$$

and since

$$C_1(1+|\xi|)^{1-\alpha}|\xi|^\alpha < C_1(1+|\xi|)^{1-\alpha}(1+|\xi|)^\alpha,$$

we obtain

$$|a(y^n,s^m,\xi)| < C_1(1+|\xi|). \tag{3.5}$$

The boundedness of $\{u^\varepsilon\}$ in $L^2(0,T;H_0^1(\Omega))$ together with (3.5) gives, up to a subsequence, that

$$a\Big(\frac{x}{\hat{\varepsilon}^n}, \frac{t}{\check{\varepsilon}^m}, \nabla u^\varepsilon(x,t)\Big) \overset{n+1,m+1}{\rightharpoonup} a_0(x,t,y^n,s^m)$$

for some $a_0 \in L^2(\Omega_T \times \mathcal{Y}_{n,m})^N$ due to Theorem 2.5. We let ε tend to zero in (3.4) and obtain

$$\int_{\Omega_T} \Big(-u(x,t)v(x)\partial_t c(t) + \Big(\int_{\mathcal{Y}_{n,m}} a_0(x,t,y^n,s^m)\,\mathrm{d}y^n\,\mathrm{d}s^m\Big) \cdot \nabla v(x)c(t)\Big)\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} f(x,t)v(x)c(t)\,\mathrm{d}x\,\mathrm{d}t, \tag{3.6}$$

which is the homogenized problem if we can prove that

$$a_0(x,t,y^n,s^m) = a\Big(y^n,s^m,\nabla u(x,t) + \sum_{j=1}^{n} \nabla_{y_j} u_j(x,t,y^j,s^{m-d_j})\Big)$$

with u and u_j as given in the theorem. To characterize a_0 we will use the system of local problems (3.3), and deriving this will be our next aim.

Main step 3: Pre-local problems. In (3.4) we will use test functions defined according to the following. Let $\{r_\varepsilon\}$ be a sequence of positive numbers tending to zero as ε does. Fix $i = 1, \ldots, n$ and choose

$$v^\varepsilon(x) = r_\varepsilon v_1(x) v_2\Big(\frac{x}{\hat{\varepsilon}_1}\Big) \ldots v_{i+1}\Big(\frac{x}{\hat{\varepsilon}_i}\Big)$$

and

$$c^\varepsilon(t) = c_1(t) c_2\Big(\frac{t}{\check{\varepsilon}_1}\Big) \ldots c_{\lambda+1}\Big(\frac{t}{\check{\varepsilon}_\lambda}\Big), \quad \lambda = 1, \ldots, m$$

with $v_1 \in D(\Omega)$, $v_j \in C_\sharp^\infty(Y_{j-1})$ for $j = 2, \ldots, i$, $v_{i+1} \in C_\sharp^\infty(Y_i)/\mathbb{R}$, $c_1 \in D(0,T)$ and $c_l \in C_\sharp^\infty(S_{l-1})$ for $l = 2, \ldots, \lambda+1$. We get, after carrying out the differentiations,

$$\begin{aligned}
&\int_{\Omega_T} \Big(- u^\varepsilon(x,t) v_1(x) v_2\Big(\frac{x}{\hat{\varepsilon}_1}\Big) \ldots v_{i+1}\Big(\frac{x}{\hat{\varepsilon}_i}\Big) \Big(r_\varepsilon \partial_t c_1(t) c_2\Big(\frac{t}{\check{\varepsilon}_1}\Big) \ldots c_{\lambda+1}\Big(\frac{t}{\check{\varepsilon}_\lambda}\Big) \\
&\qquad + \sum_{l=2}^{\lambda+1} \frac{r_\varepsilon}{\check{\varepsilon}_{l-1}} c_1(t) c_2\Big(\frac{t}{\check{\varepsilon}_1}\Big) \ldots \partial_{s_{l-1}} c_l\Big(\frac{t}{\check{\varepsilon}_{l-1}}\Big) \ldots c_{\lambda+1}\Big(\frac{t}{\check{\varepsilon}_\lambda}\Big)\Big) \\
&\qquad + a\Big(\frac{x}{\hat{\varepsilon}^n}, \frac{t}{\check{\varepsilon}^m}, \nabla u^\varepsilon(x,t)\Big) \cdot \Big(r_\varepsilon \nabla v_1(x) v_2\Big(\frac{x}{\hat{\varepsilon}_1}\Big) \ldots v_{i+1}\Big(\frac{x}{\hat{\varepsilon}_i}\Big) \\
&\qquad + \sum_{j=2}^{i+1} \frac{r_\varepsilon}{\hat{\varepsilon}_{j-1}} v_1(x) v_2\Big(\frac{x}{\hat{\varepsilon}_1}\Big) \ldots \nabla_{y_{j-1}} v_j\Big(\frac{x}{\hat{\varepsilon}_{j-1}}\Big) \ldots v_{i+1}\Big(\frac{x}{\hat{\varepsilon}_i}\Big)\Big) \\
&\qquad \times c_1(t) c_2\Big(\frac{t}{\check{\varepsilon}_1}\Big) \ldots c_{\lambda+1}\Big(\frac{t}{\check{\varepsilon}_\lambda}\Big)\Big) \, \mathrm{d}x \, \mathrm{d}t \\
&= \int_{\Omega_T} f(x,t) r_\varepsilon v_1(x) v_2\Big(\frac{x}{\hat{\varepsilon}_1}\Big) \ldots v_{i+1}\Big(\frac{x}{\hat{\varepsilon}_i}\Big) c_1(t) c_2\Big(\frac{t}{\check{\varepsilon}_1}\Big) \ldots c_{\lambda+1}\Big(\frac{t}{\check{\varepsilon}_\lambda}\Big) \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned}$$

The next step will be to let ε tend to zero. The terms that do not include any inner derivative, e.g., the first term, immediately go to zero from the definition of r_ε and Theorem 2.6 while the remaining parts, i.e.,

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \int_{\Omega_T} \Big(- u^\varepsilon(x,t) v_1(x) v_2\Big(\frac{x}{\hat{\varepsilon}_1}\Big) \ldots v_{i+1}\Big(\frac{x}{\hat{\varepsilon}_i}\Big) \\
&\qquad \times \Big(\sum_{l=2}^{\lambda+1} \frac{r_\varepsilon}{\check{\varepsilon}_{l-1}} c_1(t) c_2\Big(\frac{t}{\check{\varepsilon}_1}\Big) \ldots \partial_{s_{l-1}} c_l\Big(\frac{t}{\check{\varepsilon}_{l-1}}\Big) \ldots c_{\lambda+1}\Big(\frac{t}{\check{\varepsilon}_\lambda}\Big)\Big) \\
&\qquad + a\Big(\frac{x}{\hat{\varepsilon}^n}, \frac{t}{\check{\varepsilon}^m}, \nabla u^\varepsilon(x,t)\Big) \\
&\qquad \times \sum_{j=2}^{i+1} \frac{r_\varepsilon}{\hat{\varepsilon}_{j-1}} v_1(x) v_2\Big(\frac{x}{\hat{\varepsilon}_1}\Big) \ldots \nabla_{y_{j-1}} v_j\Big(\frac{x}{\hat{\varepsilon}_{j-1}}\Big) \ldots v_{i+1}\Big(\frac{x}{\hat{\varepsilon}_i}\Big) \\
&\qquad \times c_1(t) c_2\Big(\frac{t}{\check{\varepsilon}_1}\Big) \ldots c_{\lambda+1}\Big(\frac{t}{\check{\varepsilon}_\lambda}\Big)\Big) \, \mathrm{d}x \, \mathrm{d}t = 0
\end{aligned}$$

require some more attention. Rewriting we obtain

$$
\begin{aligned}
(3.7)\qquad \lim_{\varepsilon\to 0}\int_{\Omega_T}\Bigg(&-\frac{1}{\hat{\varepsilon}_i}u^{\varepsilon}(x,t)\sum_{l=2}^{\lambda+1}\frac{r_\varepsilon\hat{\varepsilon}_i}{\check{\varepsilon}_{l-1}}v_1(x)v_2\Big(\frac{x}{\hat{\varepsilon}_1}\Big)\dots v_{i+1}\Big(\frac{x}{\hat{\varepsilon}_i}\Big)\\
&\times c_1(t)c_2\Big(\frac{t}{\check{\varepsilon}_1}\Big)\dots\partial_{s_{l-1}}c_l\Big(\frac{t}{\check{\varepsilon}_{l-1}}\Big)\dots c_{\lambda+1}\Big(\frac{t}{\check{\varepsilon}_\lambda}\Big)\\
&+a\Big(\frac{x}{\hat{\varepsilon}^n},\frac{t}{\check{\varepsilon}^m},\nabla u^{\varepsilon}(x,t)\Big)\\
&\times\sum_{j=2}^{i+1}\frac{r_\varepsilon}{\hat{\varepsilon}_{j-1}}v_1(x)v_2\Big(\frac{x}{\hat{\varepsilon}_1}\Big)\dots\nabla_{y_{j-1}}v_j\Big(\frac{x}{\hat{\varepsilon}_{j-1}}\Big)\dots v_{i+1}\Big(\frac{x}{\hat{\varepsilon}_i}\Big)\\
&\times c_1(t)c_2\Big(\frac{t}{\check{\varepsilon}_1}\Big)\dots c_{\lambda+1}\Big(\frac{t}{\check{\varepsilon}_\lambda}\Big)\Bigg)\,\mathrm{d}x\,\mathrm{d}t=0,
\end{aligned}
$$

where we have factored out $1/\hat{\varepsilon}_i$ from the first sum to make it obvious that it is possible to pass to the limit by means of very weak $(i+1,\lambda+1)$-scale convergence. Suppose that $\{r_\varepsilon\hat{\varepsilon}_i/\check{\varepsilon}_\lambda\}$ and $\{r_\varepsilon/\hat{\varepsilon}_i\}$ are bounded. This implies that

$$
\frac{r_\varepsilon\hat{\varepsilon}_i}{\check{\varepsilon}_{\lambda-j}}\to 0,\quad j=1,\dots,\lambda-1\quad\text{and}\quad\frac{r_\varepsilon}{\hat{\varepsilon}_{i-j}}\to 0,\quad j=1,\dots,i-1
$$

as $\varepsilon\to 0$ due to the fact that the scales are separated. Hence, under these assumptions, (3.7) turns into

$$
\begin{aligned}
(3.8)\qquad &\lim_{\varepsilon\to 0}\int_{\Omega_T}\Bigg(-\frac{1}{\hat{\varepsilon}_i}u^{\varepsilon}(x,t)\frac{r_\varepsilon\hat{\varepsilon}_i}{\check{\varepsilon}_\lambda}v_1(x)v_2\Big(\frac{x}{\hat{\varepsilon}_1}\Big)\dots v_{i+1}\Big(\frac{x}{\hat{\varepsilon}_i}\Big)c_1(t)c_2\Big(\frac{t}{\check{\varepsilon}_1}\Big)\dots\partial_{s_\lambda}c_{\lambda+1}\Big(\frac{t}{\check{\varepsilon}_\lambda}\Big)\\
&\quad+a\Big(\frac{x}{\hat{\varepsilon}^n},\frac{t}{\check{\varepsilon}^m},\nabla u^{\varepsilon}(x,t)\Big)\frac{r_\varepsilon}{\hat{\varepsilon}_i}v_1(x)v_2\Big(\frac{x}{\hat{\varepsilon}_1}\Big)\dots v_i\Big(\frac{x}{\hat{\varepsilon}_{i-1}}\Big)\nabla_{y_i}v_{i+1}\Big(\frac{x}{\hat{\varepsilon}_i}\Big)\\
&\quad\times c_1(t)c_2\Big(\frac{t}{\check{\varepsilon}_1}\Big)\dots c_{\lambda+1}\Big(\frac{t}{\check{\varepsilon}_\lambda}\Big)\Bigg)\,\mathrm{d}x\,\mathrm{d}t=0,
\end{aligned}
$$

which will be our springboard when deriving the independencies of the local time variables in the corrector functions as well as the local problems. This will be done for the two different cases of nonresonance and resonance.

Case 1: Nonresonance ($\varrho_i=0$). First we derive the independencies for $d_i>0$. Let λ successively be $m,\dots,m-d_i+1$. If $r_\varepsilon=\check{\varepsilon}_\lambda/\hat{\varepsilon}_i$, we have that

$$
\frac{r_\varepsilon\hat{\varepsilon}_i}{\check{\varepsilon}_\lambda}=1
$$

and, from the chosen values of λ and the meaning of d_i,

$$
(3.9)\qquad \frac{r_\varepsilon}{\hat{\varepsilon}_i}=\frac{\check{\varepsilon}_\lambda}{(\hat{\varepsilon}_i)^2}\to 0
$$

as $\varepsilon \to 0$. Hence, we may use (3.8) for this choice of r_ε and we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Omega_T}\Big(&-\frac{1}{\hat{\varepsilon}_i}u^\varepsilon(x,t)v_1(x)v_2\Big(\frac{x}{\hat{\varepsilon}_1}\Big)\dots v_{i+1}\Big(\frac{x}{\hat{\varepsilon}_i}\Big)c_1(t)c_2\Big(\frac{t}{\check{\varepsilon}_1}\Big)\dots\partial_{s_\lambda}c_{\lambda+1}\Big(\frac{t}{\check{\varepsilon}_\lambda}\Big)\\&+a\Big(\frac{x}{\hat{\varepsilon}^n},\frac{t}{\check{\varepsilon}^m},\nabla u^\varepsilon(x,t)\Big)\frac{\check{\varepsilon}_\lambda}{(\hat{\varepsilon}_i)^2}v_1(x)v_2(\frac{x}{\hat{\varepsilon}_1})\dots v_i\Big(\frac{x}{\hat{\varepsilon}_{i-1}}\Big)\nabla_{y_i}v_{i+1}\Big(\frac{x}{\hat{\varepsilon}_i}\Big)\\&\times c_1(t)c_2\Big(\frac{t}{\check{\varepsilon}_1}\Big)\dots c_{\lambda+1}\Big(\frac{t}{\check{\varepsilon}_\lambda}\Big)\Big)\,\mathrm{d}x\,\mathrm{d}t=0.\end{aligned}$$

We let ε tend to zero and obtain, due to (3.9) and Theorem 2.8, that

$$\begin{aligned}\int_{\Omega_T}\int_{\mathcal{Y}_{i,\lambda}}&-u_i(x,t,y^i,s^\lambda)v_1(x)v_2(y_1)\dots v_{i+1}(y_i)\\&\times c_1(t)c_2(s_1)\dots\partial_{s_\lambda}c_{\lambda+1}(s_\lambda)\,\mathrm{d}y^i\,\mathrm{d}s^\lambda\,\mathrm{d}x\,\mathrm{d}t=0\end{aligned}$$

and by the Variational Lemma we have

$$\int_{S_\lambda}-u_i(x,t,y^i,s^\lambda)\partial_{s_\lambda}c_{\lambda+1}(s_\lambda)\,\mathrm{d}s_\lambda=0$$

almost everywhere for all $c_{\lambda+1}\in C_\sharp^\infty(S_\lambda)$. This means that u_i is independent of $s_{m-d_i+1},\dots,s_m$. We proceed by deriving the local problems and for this purpose we choose $r_\varepsilon=\hat{\varepsilon}_i$ and $\lambda=m-d_i$, where $d_i\geqslant 0$. Since $\varrho_i=0$, we conclude that

$$\frac{r_\varepsilon\hat{\varepsilon}_i}{\check{\varepsilon}_\lambda}=\frac{(\hat{\varepsilon}_i)^2}{\check{\varepsilon}_{m-d_i}}\to 0$$

as $\varepsilon\to 0$ and

$$\frac{r_\varepsilon}{\hat{\varepsilon}_i}=1,$$

which means that (3.8) is valid and we get

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Omega_T}\Big(&-\frac{1}{\hat{\varepsilon}_i}u^\varepsilon(x,t)\frac{(\hat{\varepsilon}_i)^2}{\check{\varepsilon}_{m-d_i}}v_1(x)v_2\Big(\frac{x}{\hat{\varepsilon}_1}\Big)\dots v_{i+1}\Big(\frac{x}{\hat{\varepsilon}_i}\Big)\\&\times c_1(t)c_2\Big(\frac{t}{\check{\varepsilon}_1}\Big)\dots\partial_{s_{m-d_i}}c_{m-d_i+1}\Big(\frac{t}{\check{\varepsilon}_{m-d_i}}\Big)\\&+a\Big(\frac{x}{\hat{\varepsilon}^n},\frac{t}{\check{\varepsilon}^m},\nabla u^\varepsilon(x,t)\Big)v_1(x)v_2\Big(\frac{x}{\hat{\varepsilon}_1}\Big)\dots v_i\Big(\frac{x}{\hat{\varepsilon}_{i-1}}\Big)\nabla_{y_i}v_{i+1}\Big(\frac{x}{\hat{\varepsilon}_i}\Big)\\&\times c_1(t)c_2\Big(\frac{t}{\check{\varepsilon}_1}\Big)\dots c_{m-d_i+1}\Big(\frac{t}{\check{\varepsilon}_{m-d_i}}\Big)\Big)\,\mathrm{d}x\,\mathrm{d}t=0.\end{aligned}$$

As $\varepsilon\to 0$, we obtain

$$\begin{aligned}\int_{\Omega_T}\int_{\mathcal{Y}_{n,m}}&a_0(x,t,y^n,s^m)\cdot v_1(x)v_2(y_1)\dots v_i(y_{i-1})\nabla_{y_i}v_{i+1}(y_i)\\&\times c_1(t)c_2(s_1)\dots c_{m-d_i+1}(s_{m-d_i})\,\mathrm{d}y^n\,\mathrm{d}s^m\,\mathrm{d}x\,\mathrm{d}t=0\end{aligned}$$

and, finally,

$$\int_{S_{m-d_i+1}} \cdots \int_{S_m} \int_{Y_i} \cdots \int_{Y_n} a_0(x,t,y^n,s^m) \times \nabla_{y_i} v_{i+1}(y_i)\,\mathrm{d}y_n \ldots \mathrm{d}y_i\,\mathrm{d}s_m \ldots \mathrm{d}s_{m-d_i+1} = 0 \tag{3.10}$$

almost everywhere for all $v_{i+1} \in C^\infty_\sharp(Y_i)/\mathbb{R}$ and by density all $v_{i+1} \in H^1_\sharp(Y_i)/\mathbb{R}$, which is the weak form of the local problem in this nonresonant case.

Case 2: Resonance ($\varrho_i = D_i$). As in the first case we begin with the independencies for $d_i > 0$. Again, let λ successively be $m, \ldots, m-d_i+1$. Now choose $r_\varepsilon = \check{\varepsilon}_\lambda/\hat{\varepsilon}_i$ directly implying that

$$\frac{r_\varepsilon \hat{\varepsilon}_i}{\check{\varepsilon}_\lambda} = 1 \quad \text{and} \quad \frac{r_\varepsilon}{\hat{\varepsilon}_i} = \frac{\check{\varepsilon}_\lambda}{(\hat{\varepsilon}_i)^2} \to 0$$

when $\varepsilon \to 0$, by the restriction of λ and the definition of d_i. Thus, (3.8) turns into

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_{\Omega_T} \Big(&-\frac{1}{\hat{\varepsilon}_i} u^\varepsilon(x,t) v_1(x) v_2\Big(\frac{x}{\hat{\varepsilon}_1}\Big) \ldots v_{i+1}\Big(\frac{x}{\hat{\varepsilon}_i}\Big) c_1(t) c_2\Big(\frac{t}{\check{\varepsilon}_1}\Big) \ldots \partial_{s_\lambda} c_{\lambda+1}\Big(\frac{t}{\check{\varepsilon}_\lambda}\Big) \\ &+ a\Big(\frac{x}{\hat{\varepsilon}^n}, \frac{t}{\check{\varepsilon}^m}, \nabla u^\varepsilon(x,t)\Big) \frac{\check{\varepsilon}_\lambda}{(\hat{\varepsilon}_i)^2} v_1(x) v_2\Big(\frac{x}{\hat{\varepsilon}_1}\Big) \ldots v_i\Big(\frac{x}{\hat{\varepsilon}_{i-1}}\Big) \nabla_{y_i} v_{i+1}\Big(\frac{x}{\hat{\varepsilon}_i}\Big) \\ &\times c_1(t) c_2\Big(\frac{t}{\check{\varepsilon}_1}\Big) \ldots c_{\lambda+1}\Big(\frac{t}{\check{\varepsilon}_\lambda}\Big)\Big)\,\mathrm{d}x\,\mathrm{d}t = 0\end{aligned}$$

and a passage to the limit gives

$$\begin{aligned}\int_{\Omega_T} \int_{\mathcal{Y}_{i,\lambda}} &- u_i(x,t,y^i,s^\lambda) v_1(x) v_2(y_1) \ldots v_{i+1}(y_i) \\ &\times c_1(t) c_2(s_1) \ldots \partial_{s_\lambda} c_{\lambda+1}(s_\lambda)\,\mathrm{d}y^i\,\mathrm{d}s^\lambda\,\mathrm{d}x\,\mathrm{d}t = 0.\end{aligned}$$

Hence,

$$\int_{S_\lambda} -u_i(x,t,y^i,s^\lambda) \partial_{s_\lambda} c_{\lambda+1}(s_\lambda)\,\mathrm{d}s_\lambda = 0$$

almost everywhere for all $c_{\lambda+1} \in C^\infty_\sharp(S_\lambda)$, and thus, u_i is independent of s_λ. To extract the local problem we choose $r_\varepsilon = \hat{\varepsilon}_i$ and $\lambda = m-d_i$, where $d_i \geqslant 0$, which gives

$$\frac{r_\varepsilon \hat{\varepsilon}_i}{\check{\varepsilon}_\lambda} = \frac{(\hat{\varepsilon}_i)^2}{\check{\varepsilon}_{m-d_i}} \to \varrho_i$$

as $\varepsilon \to 0$ and

$$\frac{r_\varepsilon}{\hat{\varepsilon}_i} = 1$$

and from (3.8) we then have

$$\lim_{\varepsilon\to 0}\int_{\Omega_T}\Big(-\frac{1}{\hat{\varepsilon}_i}u^\varepsilon(x,t)\frac{(\hat{\varepsilon}_i)^2}{\check{\varepsilon}_{m-d_i}}v_1(x)v_2\Big(\frac{x}{\hat{\varepsilon}_1}\Big)\dots v_{i+1}\Big(\frac{x}{\hat{\varepsilon}_i}\Big)$$
$$\times c_1(t)c_2\Big(\frac{t}{\check{\varepsilon}_1}\Big)\dots\partial_{s_{m-d_i}}c_{m-d_i+1}\Big(\frac{t}{\check{\varepsilon}_{m-d_i}}\Big)$$
$$+a\Big(\frac{x}{\hat{\varepsilon}^n},\frac{t}{\check{\varepsilon}^m},\nabla u^\varepsilon(x,t)\Big)v_1(x)v_2\Big(\frac{x}{\hat{\varepsilon}_1}\Big)\dots v_i\Big(\frac{x}{\hat{\varepsilon}_{i-1}}\Big)\nabla_{y_i}v_{i+1}\Big(\frac{x}{\hat{\varepsilon}_i}\Big)$$
$$\times c_1(t)c_2\Big(\frac{t}{\check{\varepsilon}_1}\Big)\dots c_{m-d_i+1}\Big(\frac{t}{\check{\varepsilon}_{m-d_i}}\Big)\Big)\,\mathrm{d}x\,\mathrm{d}t=0.$$

Letting ε tend to zero and applying Theorem 2.8 we obtain

$$\int_{\Omega_T}\int_{\mathcal{Y}_{n,m}}(-\varrho_i u_i(x,t,y^i,s^{m-d_i})v_1(x)v_2(y_1)\dots v_{i+1}(y_i)$$
$$\times c_1(t)c_2(s_1)\dots\partial_{s_{m-d_i}}c_{m-d_i+1}(s_{m-d_i})$$
$$+a_0(x,t,y^n,s^m)v_1(x)v_2(y_1)\dots v_i(y_{i-1})\nabla_{y_i}v_{i+1}(y_i)$$
$$\times c_1(t)c_2(s_1)\dots c_{m-d_i+1}(s_{m-d_i}))\,\mathrm{d}y^n\,\mathrm{d}s^m\,\mathrm{d}x\,\mathrm{d}t=0$$

and hence, we end up with

(3.11)
$$\int_{S_{m-d_i}}\dots\int_{S_m}\int_{Y_i}\dots\int_{Y_n}-\varrho_i u_i(x,t,y^i,s^{m-d_i})v_{i+1}(y_i)\partial_{s_{m-d_i}}c_{m-d_i+1}(s_{m-d_i})$$
$$+a_0(x,t,y^n,s^m)\nabla_{y_i}v_{i+1}(y_i)c_{m-d_i+1}(s_{m-d_i})\,\mathrm{d}y_n\dots\mathrm{d}y_i\,\mathrm{d}s_m\dots\mathrm{d}s_{m-d_i}=0$$

almost everywhere for all $c_{m-d_i+1}\in C^\infty_\sharp(S_{m-d_i})$ and $v_{i+1}\in C^\infty_\sharp(Y_i)/\mathbb{R}$ and by density all $v_{i+1}\in H^1_\sharp(Y_i)/\mathbb{R}$, the weak form of the local problem in this second case.

Main step 4: Characterization of a_0. What remains is to characterize a_0 and to this end we use perturbed test functions, see [8] and [9], according to

$$p^k(x,t,y^n,s^m)=p^{k,0}(x,t)+\sum_{j=1}^n p^{k,j}(x,t,y^j,s^{m-d_j})+\delta g(x,t,y^n,s^m),$$

where $p^{k,0}\in D(\Omega_T)^N$, $p^{k,j}\in D(\Omega_T;C^\infty_\sharp(\mathcal{Y}_{j,m-d_j}))^N$ for $j=1,\dots,n$, $g\in D(\Omega_T;C^\infty_\sharp(\mathcal{Y}_{n,m}))^N$ and δ is a positive real number. We choose these sequences such that

$$p^{k,0}(x,t)\to\nabla u(x,t)\quad\text{in } L^2(\Omega_T)^N,$$
$$p^{k,j}(x,t,y^j,s^{m-d_j})\to\nabla_{y_j}u_j(x,t,y^j,s^{m-d_j})\quad\text{in } L^2(\Omega_T\times\mathcal{Y}_{j,m-d_j})^N$$

and such that they converge almost everywhere to the same limits as $k \to \infty$, see p. 88 in [16]. We introduce the notation

$$p_\varepsilon^k(x,t) = p^k\Big(x,t,\frac{x}{\hat{\varepsilon}^n},\frac{t}{\check{\varepsilon}^m}\Big).$$

Using structure condition (iv), we get

$$\Big(a\Big(\frac{x}{\hat{\varepsilon}^n},\frac{t}{\check{\varepsilon}^m},\nabla u^\varepsilon\Big) - a\Big(\frac{x}{\hat{\varepsilon}^n},\frac{t}{\check{\varepsilon}^m},p_\varepsilon^k\Big)\Big)\cdot(\nabla u^\varepsilon(x,t) - p_\varepsilon^k(x,t)) \geqslant 0$$

and integration and expansion lead to

$$\begin{aligned}(3.12)\quad \int_{\Omega_T}\Big(a\Big(\frac{x}{\hat{\varepsilon}^n},\frac{t}{\check{\varepsilon}^m},\nabla u^\varepsilon\Big)\cdot\nabla u^\varepsilon(x,t) - a\Big(\frac{x}{\hat{\varepsilon}^n},\frac{t}{\check{\varepsilon}^m},\nabla u^\varepsilon\Big)\cdot p_\varepsilon^k(x,t)\\ - a\Big(\frac{x}{\hat{\varepsilon}^n},\frac{t}{\check{\varepsilon}^m},p_\varepsilon^k\Big)\cdot\nabla u^\varepsilon(x,t) + a\Big(\frac{x}{\hat{\varepsilon}^n},\frac{t}{\check{\varepsilon}^m},p_\varepsilon^k\Big)\cdot p_\varepsilon^k(x,t)\Big)\,\mathrm{d}x\,\mathrm{d}t \geqslant 0.\end{aligned}$$

From, e.g., Theorem 30.A (c) in [28] it follows that the weak form (3.4) is equivalent to

$$\begin{aligned}\int_0^T \langle\partial_t u^\varepsilon(t), w(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,\mathrm{d}t + \int_{\Omega_T} a\Big(\frac{x}{\hat{\varepsilon}^n},\frac{t}{\check{\varepsilon}^m},\nabla u^\varepsilon\Big)\cdot\nabla w(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ = \int_{\Omega_T} f(x,t)w(x,t)\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

for all $w \in L^2(0,T;H_0^1(\Omega))$. This means that we may replace vc with u^ε in (3.4) via the above form and get another way of expressing the first term in (3.12) and thus we obtain

$$\begin{aligned}\int_{\Omega_T}\Big(f(x,t)u^\varepsilon(x,t) - a\Big(\frac{x}{\hat{\varepsilon}^n},\frac{t}{\check{\varepsilon}^m},\nabla u^\varepsilon\Big)\cdot p_\varepsilon^k(x,t) - a\Big(\frac{x}{\hat{\varepsilon}^n},\frac{t}{\check{\varepsilon}^m},p_\varepsilon^k\Big)\cdot\nabla u^\varepsilon(x,t)\\ + a\Big(\frac{x}{\hat{\varepsilon}^n},\frac{t}{\check{\varepsilon}^m},p_\varepsilon^k\Big)\cdot p_\varepsilon^k(x,t)\Big)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\langle\partial_t u^\varepsilon(t),u^\varepsilon(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,\mathrm{d}t \geqslant 0.\end{aligned}$$

We note that p^k, $a(y^n,s^m,p^k)$ and their product are admissible test functions and since

$$-\liminf_{\varepsilon\to 0}\int_0^T\langle\partial_t u^\varepsilon(t),u^\varepsilon(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,\mathrm{d}t \leqslant -\int_0^T\langle\partial_t u(t),u(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,\mathrm{d}t$$

(see pp. 12–13 in [19]), we get, up to a subsequence, that

$$\int_{\Omega_T}\int_{\mathcal{Y}_{n,m}}\Big(f(x,t)u(x,t)-a_0(x,t,y^n,s^m)\cdot p^k(x,t,y^n,s^m) - a(y^n,s^m,p^k)\cdot\Big(\nabla u(x,t)+\sum_{j=1}^{n}\nabla_{y_j}u_j(x,t,y^j,s^{m-d_j})\Big) + a(y^n,s^m,p^k)\cdot p^k(x,t,y^n,s^m)\,\mathrm{d}y^n\,\mathrm{d}s^m\Big)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\langle\partial_t u(t),u(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,\mathrm{d}t\geqslant 0 \tag{3.13}$$

when ε tends to zero. From the choice of p^k we have that

$$p^k(x,t,y^n,s^m)\to\nabla u(x,t)+\sum_{j=1}^{n}\nabla_{y_j}u_j(x,t,y^j,s^{m-d_j})+\delta g(x,t,y^n,s^m)$$

in $L^2(\Omega_T\times\mathcal{Y}_{n,m})^N$ and almost everywhere in $\Omega_T\times\mathcal{Y}_{n,m}$ as $k\to\infty$. Furthermore,

$$a(y^n,s^m,p^k)\to a\Big(y^n,s^m,\nabla u+\sum_{j=1}^{n}\nabla_{y_j}u_j+\delta g\Big)$$

almost everywhere in $\Omega_T\times\mathcal{Y}_{n,m}$ and hence,

$$a(y^n,s^m,p^k)\cdot p^k(x,t,y^n,s^m)\to a\Big(y^n,s^m,\nabla u+\sum_{j=1}^{n}\nabla_{y_j}u_j+\delta g\Big)\times\Big(\nabla u(x,t)+\sum_{j=1}^{n}\nabla_{y_j}u_j(x,t,y^j,s^{m-d_j})+\delta g(x,t,y^n,s^m)\Big)$$

almost everywhere in $\Omega_T\times\mathcal{Y}_{n,m}$. We proceed by letting k tend to infinity in (3.13) and we go through the details for the fourth term. We will use Lebesgue's generalized majorized convergence theorem (Theorem (19a) in the appendix of [28]). Choosing $\xi=p^k$ in (3.5) we have that

$$|a(y^n,s^m,p^k)|\leqslant C_1(1+|p^k(x,t,y^n,s^m)|). \tag{3.14}$$

Successively applying the Cauchy-Schwarz inequality and (3.14), we get

$$\begin{aligned}|a(y^n,s^m,p^k)\cdot p^k(x,t,y^n,s^m)| &\leqslant |a(y^n,s^m,p^k)||p^k(x,t,y^n,s^m)|\\ &\leqslant C_1(1+|p^k(x,t,y^n,s^m)|)|p^k(x,t,y^n,s^m)|\\ &= C_1(|p^k(x,t,y^n,s^m)|+|p^k(x,t,y^n,s^m)|^2).\end{aligned}$$

Letting $k \to \infty$, we have

$$\int_{\Omega_T} \int_{\mathcal{Y}_{n,m}} |p^k(x,t,y^n,s^m)| + |p^k(x,t,y^n,s^m)|^2 \, \mathrm{d}y^n \, \mathrm{d}s^m \, \mathrm{d}x \, \mathrm{d}t$$
$$\to \int_{\Omega_T} \int_{\mathcal{Y}_{n,m}} \left| \nabla u(x,t) + \sum_{j=1}^{n} \nabla_{y_j} u_j(x,t,y^j,s^{m-d_j}) + \delta g(x,t,y^n,s^m) \right|$$
$$+ \left| \nabla u(x,t) + \sum_{j=1}^{n} \nabla_{y_j} u_j(x,t,y^j,s^{m-d_j}) + \delta g(x,t,y^n,s^m) \right|^2 \mathrm{d}y^n \, \mathrm{d}s^m \, \mathrm{d}x \, \mathrm{d}t$$

and hence we conclude that

$$\int_{\Omega_T} \int_{\mathcal{Y}_{n,m}} a(y^n,s^m,p^k) \cdot p^k(x,t,y^n,s^m) \, \mathrm{d}y^n \, \mathrm{d}s^m \, \mathrm{d}x \, \mathrm{d}t$$
$$\to \int_{\Omega_T} \int_{\mathcal{Y}_{n,m}} a\Big(y^n, s^m, \nabla u + \sum_{j=1}^{n} \nabla_{y_j} u_j + \delta g \Big)$$
$$\times \Big(\nabla u(x,t) + \sum_{j=1}^{n} \nabla_{y_j} u_j(x,t,y^j,s^{m-d_j}) + \delta g(x,t,y^n,s^m) \Big) \mathrm{d}y^n \, \mathrm{d}s^m \, \mathrm{d}x \, \mathrm{d}t.$$

Thus, as k tends to infinity in (3.13) we find that

$$\int_{\Omega_T} \int_{\mathcal{Y}_{n,m}} \Big(f(x,t)u(x,t) - a_0(x,t,y^n,s^m)$$
$$\times \Big(\nabla u(x,t) + \sum_{j=1}^{n} \nabla_{y_j} u_j(x,t,y^j,s^{m-d_j}) + \delta g(x,t,y^n,s^m) \Big)$$
$$- a\Big(y^n, s^m, \nabla u + \sum_{j=1}^{n} \nabla_{y_j} u_j + \delta g \Big)$$
$$\times \Big(\nabla u(x,t) + \sum_{j=1}^{n} \nabla_{y_j} u_j(x,t,y^j,s^{m-d_j}) \Big)$$
$$+ a\Big(y^n, s^m, \nabla u + \sum_{j=1}^{n} \nabla_{y_j} u_j + \delta g \Big)$$
$$\times \Big(\nabla u(x,t) + \sum_{j=1}^{n} \nabla_{y_j} u_j(x,t,y^j,s^{m-d_j}) + \delta g(x,t,y^n,s^m) \Big) \Big) \mathrm{d}y^n \, \mathrm{d}s^m \, \mathrm{d}x \, \mathrm{d}t$$
$$- \int_0^T \langle \partial_t u(t), u(t) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \, \mathrm{d}t \geqslant 0,$$

where some terms vanish directly and we have

$$\begin{aligned}(3.15)\qquad &\int_{\Omega_T}\int_{\mathcal{Y}_{n,m}}\Bigg(f(x,t)u(x,t)-a_0(x,t,y^n,s^m)\\ &\times\Bigg(\nabla u(x,t)+\sum_{j=1}^{n}\nabla_{y_j}u_j(x,t,y^j,s^{m-d_j})+\delta g(x,t,y^n,s^m)\Bigg)\\ &+a\Bigg(y^n,s^m,\nabla u+\sum_{j=1}^{n}\nabla_{y_j}u_j+\delta g\Bigg)\cdot\delta g(x,t,y^n,s^m)\Bigg)\,\mathrm{d}y^n\,\mathrm{d}s^m\,\mathrm{d}x\,\mathrm{d}t\\ &-\int_0^T\langle\partial_t u(t),u(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,\mathrm{d}t\geqslant 0.\end{aligned}$$

Replacing vc by u in (3.6), we get

(3.16)

$$\begin{aligned}&\int_0^T\langle\partial_t u(t),u(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,\mathrm{d}t+\int_{\Omega_T}\Bigg(\int_{\mathcal{Y}_{n,m}}a_0(x,t,y^n,s^m)\,\mathrm{d}y^n\,\mathrm{d}s^m\Bigg)\cdot\nabla u(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad=\int_{\Omega_T}f(x,t)u(x,t)\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

and with (3.16) in (3.15) we obtain

(3.17)

$$\begin{aligned}&\int_{\Omega_T}\int_{\mathcal{Y}_{n,m}}\Bigg(\sum_{j=1}^{n}-a_0(x,t,y^n,s^m)\cdot\nabla_{y_j}u_j(x,t,y^j,s^{m-d_j})\\ &\qquad-a_0(x,t,y^n,s^m)\cdot\delta g(x,t,y^n,s^m)\\ &\qquad+a\Bigg(y^n,s^m,\nabla u+\sum_{j=1}^{n}\nabla_{y_j}u_j+\delta g\Bigg)\cdot\delta g(x,t,y^n,s^m)\Bigg)\,\mathrm{d}y^n\,\mathrm{d}s^m\,\mathrm{d}x\,\mathrm{d}t\geqslant 0.\end{aligned}$$

Using the local problems (3.10) and (3.11) we will eliminate the first n terms in (3.17). We study them one at a time by letting j successively be $1,\dots,n$. If $\varrho_j=0$, we use the local problem (3.10) with $i=j$ and the term in question vanishes directly. If $\varrho_j\neq 0$, then $u_j\in L^2(\Omega_T\times\mathcal{Y}_{j-1,m-d_j-1},W_\sharp^{1,2}(S_{m-d_j};H_\sharp^1(Y_j)/\mathbb{R},L_\sharp^2(Y_j)/\mathbb{R}))$, which implies that $u_j(x,t,y^{j-1},s^{m-d_j-1})\in W_\sharp^{1,2}(S_{m-d_j};H_\sharp^1(Y_j)/\mathbb{R},L_\sharp^2(Y_j)/\mathbb{R})$. Then, by density (see Proposition 4.6 in [20]) we may replace vc by u_j in (3.11). Thus, from (3.11) with $i=j$, we obtain that

$$\begin{aligned}&\int_{S_{m-d_j}}\cdots\int_{S_m}\int_{Y_j}\cdots\int_{Y_n}-a_0(x,t,y^n,s^m)\\ &\qquad\times\nabla_{y_j}u_j(x,t,y^j,s^{m-d_j})\,\mathrm{d}y_n\ldots\,\mathrm{d}y_j\,\mathrm{d}s_m\ldots\,\mathrm{d}s_{m-d_j}\\ &\quad=\int_{S_{m-d_j}}\int_{Y_j}-\varrho_j u_j(x,t,y^j,s^{m-d_j})\partial_{s_{m-d_j}}u_j(x,t,y^j,s^{m-d_j})\,\mathrm{d}y_j\,\mathrm{d}s_{m-d_j}\\ &\quad=-\varrho_j\langle\partial_{s_{m-d_j}}u_j,u_j\rangle_{L_\sharp^2(S_{m-d_j};(H_\sharp^1(Y_j)/\mathbb{R})'),L_\sharp^2(S_{m-d_j};H_\sharp^1(Y_j)/\mathbb{R})}.\end{aligned}$$

By Corollary 4.1 in [20] the duality pairing above equals zero, which yields that the term in question in (3.17) vanishes. What remains of (3.17) is

$$\int_{\Omega_T}\int_{\mathcal{Y}_{n,m}}\Big(-a_0(x,t,y^n,s^m)+a\Big(y^n,s^m,\nabla u+\sum_{j=1}^n\nabla_{y_j}u_j+\delta g\Big)\Big) \\ \times\delta g(x,t,y^n,s^m)\,\mathrm{d}y^n\,\mathrm{d}s^m\,\mathrm{d}x\,\mathrm{d}t\geqslant 0.$$

Dividing by δ and letting δ tend to zero, we deduce that

$$a_0(x,t,y^n,s^m)=a\Big(y^n,s^m,\nabla u+\sum_{j=1}^n\nabla_{y_j}u_j\Big).$$

Finally, by the uniqueness of u, the entire sequence converges and the proof is complete. □

Remark 3.4. The existence of a unique solution to the limit problem, i.e., the homogenized problem, follows from G-convergence, see [23]. See also [13]. A detailed study regarding the uniqueness and regularity of the solution to a monotone parabolic local problem can be found in [25]. Taking one spatial and one temporal microscopic scale, both powers of ε, the local problem (3.3) is a special case of the local problem studied in [25] and the procedure in [25] can be applied in like manner for the case with multiple scales and scales which are not necessarily powers of ε.

We conclude the paper by applying the main result to a nonlinear parabolic problem with a specific choice of fixed scales.

Example 3.5. Consider the (3,4)-scaled special case of (3.1) given by

$$\begin{aligned}\partial_t u^\varepsilon(x,t)-\nabla\cdot a\Big(\frac{x}{2\sqrt{\varepsilon}},\frac{x}{\varepsilon^2},\frac{t}{\mathrm{e}^\varepsilon-1},\frac{t}{\ln(1+\varepsilon^2)},\frac{t}{\varepsilon^3\ln(1+1/\varepsilon)},\nabla u^\varepsilon(x,t)\Big)&\\ =f(x,t)\quad&\text{in }\Omega_T,\\ u^\varepsilon(x,t)=0\quad&\text{on }\partial\Omega\times(0,T),\\ u^\varepsilon(x,0)=u^0(x)\quad&\text{in }\Omega.\end{aligned}$$

According to Example 2.4 the list of spatial scales and the list of temporal scales are jointly well-separated.

Identifying d_i and ϱ_i (defined in connection to Theorem 3.1, see also Remark 3.2) for $i=1$ we have $d_1=2$ and $\varrho_1=4$ and for $i=2$ we obtain $d_2=0$ and $\varrho_2=0$. Now, by Theorem 3.1 we have

$$\begin{aligned}u^\varepsilon(x,t)&\to u(x,t)\quad\text{in }L^2(\Omega_T),\\ u^\varepsilon(x,t)&\rightharpoonup u(x,t)\quad\text{in }L^2(0,T;H_0^1(\Omega))\end{aligned}$$

and

$$\nabla u^\varepsilon(x,t) \overset{3,4}{\rightharpoonup} \nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s_1) + \nabla_{y_2} u_2(x,t,y^2,s^3),$$

where $u \in W^{1,2}(0,T; H_0^1(\Omega), L^2(\Omega))$, $u_1 \in L^2(\Omega_T; W_\sharp^{1,2}(S_1; H_\sharp^1(Y_1)/\mathbb{R}, L_\sharp^2(Y_1)/\mathbb{R}))$ and $u_2 \in L^2(\Omega_T \times \mathcal{Y}_{1,3}; H_\sharp^1(Y_2)/\mathbb{R})$. Here u is the unique solution to the homogenized problem

$$\begin{aligned} \partial_t u(x,t) - \nabla \cdot b(x,t,\nabla u(x,t)) &= f(x,t) && \text{in } \Omega_T, \\ u(x,t) &= 0 && \text{on } \partial\Omega \times (0,T), \\ u(x,0) &= u^0(x) && \text{in } \Omega \end{aligned}$$

with

$$\begin{aligned} b(x,t,\nabla u(x,t)) = \int_{\mathcal{Y}_{2,3}} a\Big(&y^2, s^3, \nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s_1) \\ &+ \nabla_{y_2} u_2(x,t,y^2,s^3)\Big)\,\mathrm{d}y^2\,\mathrm{d}s^3 \end{aligned}$$

and we have the two local problems

$$\begin{aligned} 4\partial_{s_1} u_1(x,t,y_1,s_1) - \nabla_{y_1} \cdot \int_{S_2}\int_{S_3}\int_{Y_2} a\Big(&y^2, s^3, \nabla u(x,t) \\ &+ \nabla_{y_1} u_1(x,t,y_1,s_1) + \nabla_{y_2} u_2(x,t,y^2,s^3)\Big)\,\mathrm{d}y_2\,\mathrm{d}s_3\,\mathrm{d}s_2 = 0 \end{aligned}$$

and

$$-\nabla_{y_2} \cdot a(y^2, s^3, \nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s_1) + \nabla_{y_2} u_2(x,t,y^2,s^3)) = 0.$$

A c k n o w l e d g m e n t. We would like to thank the anonymous referee for valuable comments that have improved the quality of the paper.

References

[1] *G. Allaire*: Homogenization and two-scale convergence. SIAM J. Math. Anal. *23* (1992), 1482–1518. zbl MR doi

[2] *G. Allaire, M. Briane*: Multiscale convergence and reiterated homogenisation. Proc. R. Soc. Edinb., Sect. A *126* (1996), 297–342. zbl MR doi

[3] *M. Amar, D. Andreucci, D. Bellaveglia*: The time-periodic unfolding operator and applications to parabolic homogenization. Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat., IX. Ser., Rend. Lincei, Mat. Appl. *28* (2017), 663–700. zbl MR doi

[4] *M. Amar, D. Andreucci, R. Gianni, C. Timofte*: Homogenization results for a class of parabolic equations with a non-local interface condition via time-periodic unfolding. NoDEA, Nonlinear Differ. Equ. Appl. *26* (2019), Article ID 52, 28 pages. zbl MR doi

[5] *A. Bensoussan, J.-L. Lions, G. Papanicoloau*: Asymptotic Analysis for Periodic Structures. Studies in Mathematics and Its Applications 5. North-Holland Publishing, Amsterdam, 1978. zbl MR doi
[6] *D. Cioranescu, P. Donato*: An Introduction to Homogenization. Oxford Lecture Series in Mathematics and Its Applications 17. Oxford University Press, New York, 1999. zbl MR
[7] *T. Danielsson, P. Johnsen*: Homogenization of linear parabolic equations with three spatial and three temporal scales for certain matchings between the microscopic scales. Math. Bohem. *146* (2021), 483–511. zbl MR doi
[8] *L. C. Evans*: The perturbed test function method for viscosity solutions of nonlinear PDE. Proc. R. Soc. Edinb., ASect. A *111* (1989), 359–375. zbl MR doi
[9] *L. C. Evans*: Periodic homogenisation of certain fully nonlinear partial differential equations. Proc. R. Soc. Edinb., Sect. A *120* (1992), 245–265. zbl MR doi
[10] *L. Flodén, A. Holmbom, M. Olsson, J. Persson*: Very weak multiscale convergence. Appl. Math. Lett. *23* (2010), 1170–1173. zbl MR doi
[11] *L. Flodén, A. Holmbom, M. Olsson Lindberg, J. Persson*: Two-scale convergence: Some remarks and extensions. Pure Appl. Math. Q. *9* (2013), 461–486. zbl MR doi
[12] *L. Flodén, A. Holmbom, M. Olsson Lindberg, J. Persson*: Homogenization of parabolic equations with an arbitrary number of scales in both space and time. J. Appl. Math. *2014* (2014), Article ID 101685, 16 pages. zbl MR doi
[13] *L. Flodén, M. Olsson*: Reiterated homogenization of some linear and nonlinear monotone parabolic operators. Can. Appl. Math. Q. *14* (2006), 149–183. zbl MR
[14] *L. Flodén, M. Olsson*: Homogenization of some parabolic operators with several time scales. Appl. Math., Praha *52* (2007), 431–446. zbl MR doi
[15] *A. Holmbom*: Homogenization of parabolic equations: An alternative approach and some corrector-type results. Appl. Math., Praha *42* (1997), 321–343. zbl MR doi
[16] *A. Kufner, O. John, S. Fučík*: Function Spaces. Monographs and Textbooks on Mechanics of Solids and Fluids. Mechanics: Analysis 3. Noordhoff, Leyden, 1977. zbl MR
[17] *D. Lukkassen, G. Nguetseng, P. Wall*: Two-scale convergence. Int. J. Pure Appl. Math. *2* (2002), 35–86. zbl MR
[18] *G. Nguetseng*: A general convergence result for a functional related to the theory of homogenization. SIAM J. Math. Anal. *20* (1989), 608–623. zbl MR doi
[19] *G. Nguetseng, J. L. Woukeng*: Deterministic homogenization of parabolic monotone operators with time dependent coefficients. Electron. J. Differ. Equ. *2004* (2004), Article ID 82, 23 pages. zbl MR
[20] *G. Nguetseng, J. L. Woukeng*: Σ-convergence of nonlinear parabolic operators. Nonlinear Anal., Theory Methods Appl., Ser. A *66* (2007), 968–1004. zbl MR doi
[21] *J. Persson*: Homogenization of monotone parabolic problems with several temporal scales. Appl. Math., Praha *57* (2012), 191–214. zbl MR doi
[22] *J. Persson*: Selected Topics in Homogenization: Doctoral Thesis. Mid Sweden University, Østersund, 2012.
[23] *N. Svanstedt*: *G*-convergence of parabolic operators. Nonlinear Anal., Theory Methods Appl. *36* (1999), 807–843. zbl MR doi
[24] *N. Svanstedt, N. Wellander, J. Wyller*: A numerical algorithm for nonlinear parabolic equations with highly oscillating coefficients. Numer. Methods Partial Differ. Equations *12* (1996), 423–440. zbl MR doi
[25] *N. Svanstedt, J. L. Woukeng*: Periodic homogenization of strongly nonlinear reaction-diffusion equations with large reaction terms. Appl. Anal. *92* (2013), 1357–1378. zbl MR doi
[26] *J. L. Woukeng*: Periodic homogenization of nonlinear non-monotone parabolic operators with three time scales. Ann. Mat. Pura Appl. (4) *189* (2010), 357–379. zbl MR doi

[27] *J. L. Woukeng*: Σ-convergence and reiterated homogenization of nonlinear parabolic operators. Commun. Pure Appl. Anal. *9* (2010), 1753–1789. zbl MR doi

[28] *E. Zeidler*: Nonlinear Functional Analysis and Its Applications. II/B. Nonlinear Monotone Operators. Springer, New York, 1990. zbl MR doi

[29] *V. V. Zhikov*: On an extension of the method of two-scale convergence and its applications. Sb. Math. *191* (2000), 973–1014. zbl MR doi

Authors' address: *Tatiana Danielsson*, *Liselott Flodén* (corresponding author), *Pernilla Johnsen*, *Marianne Olsson Lindberg*, Department of Engineering, Mathematics and Science Education, Mid Sweden University, Kunskapens väg 8, S-83125 Östersund, Sweden, e-mail: danielsson.math.stat@gmail.com, liselott.floden@miun.se, pernilla.johnsen@miun.se, marianne.olssonlindberg@miun.se.