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# STRONG CONVERGENCE FOR WEIGHTED SUMS OF WOD RANDOM VARIABLES AND ITS APPLICATION IN THE EV REGRESSION MODEL

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*Abstract.* The strong convergence for weighted sums of widely orthant dependent (WOD) random variables is investigated. As an application, we further investigate the strong consistency of the least squares estimator in EV regression model for WOD random variables. A simulation study is carried out to confirm the theoretical results.

*Keywords*: errors-in-variables regression model; least squares estimator; widely orthant dependent; strong consistency

*MSC 2020*: 62F12, 60F15

## 1. Introduction

As we know the limiting behavior for weighted sums of random variables is very important for probability theory and statistics as many useful linear statistics, like least squares estimators, nonparametric regression function estimators and jackknife estimators, are of the form of weighted sums. Therefore, the study of the limiting behavior for weighted sums is of great importance, especially studies on strong convergence for $\sum\limits_{i=1}^{n} a_{ni}X_i$, where $\{X_n, n \geqslant 1\}$ is a sequence of random variables defined on a fixed probability space $(\Omega, \mathscr{F}, P)$ and $\{a_{ni}, i \geqslant 1, n \geqslant 1\}$ is a triangular array of constants.

There is a lot of literature on the limit properties of weighted sums when $\{X_n, n \geqslant 1\}$ are assumed to be independent random variables. For instance, Teicher [18] established the following result: Let $\{X, X_n, n \geqslant 1\}$ be a sequence of

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independent identically distributed (i.i.d.) random variables with $EX = 0$. If $\max_{1\leqslant i\leqslant n}|a_{ni}| = O(1/(n^{1/p}\log n))$ and $E|X|^p < \infty$ for some $1 \leqslant p \leqslant 2$, then $\sum_{i=1}^{n} a_{ni}X_i$ converges to zero a.s. Choi and Sung [5] proved the result of Teicher [18] for $p = 1$ under the weaker condition $\max_{1\leqslant i\leqslant n}|a_{ni}| = O(1/(n^{1/p}(\log n)^{1-1/p}))$. Sung [17] weakened the Teicher' condition $\max_{1\leqslant i\leqslant n}|a_{ni}| = O(1/(n^{1/p}\log n))$ for the case $p = 2$. However, scholars later found that the independence assumption gradually cannot satisfy the requirements of statistics, therefore many dependence structures were put forward to meet the requirements of it, and many classical results for weighted sums under independence assumptions were extended to various dependence assumptions even though many of them could not be so optimal as those of the independent case. Xu and Miao [27] extended the results of Choi and Sung [5] and Sung [17] for i.i.d. random variables to the case of identically distributed negatively associated (NA) random variables. Deng et al. [8] generalized the result of Xu and Miao [27] for identically distributed NA random variables to the case of negatively orthant dependent (NOD) random variables.

In this paper, we will further investigate the strong convergence for weighted sums of WOD random variables. First let us recall some concepts of dependence structures. The concept of WOD random variables was introduced by Wang et al. [19] as follows.

**Definition 1.1.** For the random variables $\{X_n, n \geqslant 1\}$, if there exists a finite real sequence $\{g_U(n), n \geqslant 1\}$ satisfying for each $n \geqslant 1$ and for all $x_i \in (-\infty, \infty)$, $1 \leqslant i \leqslant n$,

$$P(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) \leqslant g_U(n)\prod_{i=1}^{n} P(X_i > x_i), \tag{1.1}$$

then we say that the $\{X_n, n \geqslant 1\}$ are widely upper orthant dependent (WUOD); if there exists a finite real sequence $\{g_L(n), n \geqslant 1\}$ satisfying for each $n \geqslant 1$ and for all $x_i \in (-\infty, \infty)$, $1 \leqslant i \leqslant n$,

$$P(X_1 \leqslant x_1, X_2 \leqslant x_2, \ldots, X_n \leqslant x_n) \leqslant g_L(n)\prod_{i=1}^{n} P(X_i \leqslant x_i), \tag{1.2}$$

then we say that the $\{X_n, n \geqslant 1\}$ are widely lower orthant dependent (WLOD); if they are both WUOD and WLOD, then we say that the $\{X_n, n \geqslant 1\}$ are WOD, and $g_U(n)$, $g_L(n)$, $n \geqslant 1$ are called dominating coefficients. Denote $g(n) = \max\{g_U(n), g_L(n)\}$ for $n \geqslant 1$.

An array $\{X_{ni}, i \geqslant 1, n \geqslant 1\}$ of random variables is called row-wise WOD if for every $n \geqslant 1$, $\{X_{ni}, i \geqslant 1\}$ is a sequence of WOD random variables.

By the definition of WOD random variables, we can suppose that $g_U(n) \geqslant 1$ and $g_L(n) \geqslant 1$. If both (1.1) and (1.2) hold with $g_U(n) = g_L(n) = M$ for all $n \geqslant 1$, where $M$ is a positive constant, then the random variables are said to be extended negatively dependent (END). If both (1.1) and (1.2) hold with $g_U(n) = g_L(n) = 1$ for all $n \geqslant 1$, then the random variables are said to be NOD. Obviously, NA random variables are NOD. Moreover, Hu [11] pointed out that negatively superadditive dependent (NSD) random variables are NOD. Hence, the class of WOD random variables includes independent random variables, END random variables, NOD random variables, NSD random variables and NA random variables as special cases. Hence, studying the limit behaviour of WOD random variables is highly desirable and of considerable significance in theory and application.

Some applications in different fields for WOD sequence have been found. See, many authors have been devoted to the study of risk theory and renewal theory of WOD random variables. For more details, we may refer to Wang and Cheng [22], Wang et al. [23], Wang et al. [19], Chen et al. [4]. Many authors have been devoted to the study of the limiting theory of WOD random variables. For more details, we can refer to Shen [15], Wang et al. [21], Xi et al. [26], Wang and Wang [24].

For convenience, we give the definition of stochastic domination here.

**Definition 1.2.** A sequence $\{X_n, n \geqslant 1\}$ of random variables is said to be stochastically dominated by a random variable $X$ if there exists a positive constant $C$ such that

$$P(|X_n| > x) \leqslant CP(|X| > x)$$

for all $x \geqslant 0$ and $n \geqslant 1$.

Our main result for weighted sums of WOD random variables is presented as follows.

**Theorem 1.1.** *Let $\{X_n, n \geqslant 1\}$ be a sequence of WOD random variables which is stochastically dominated by a random variable $X$. Suppose that $E|X|^{2p} < \infty$ for some $p > 0$ and $EX_n = 0$ if $p > 1$. Assume further that there exists some $\vartheta \geqslant 0$ such that $g(n) = O(n^{\vartheta})$. Let $\{a_{ni}, i \geqslant 1, n \geqslant 1\}$ be an array of constants satisfying*

$$\max_{1 \leqslant i \leqslant n} |a_{ni}| = O(n^{-1/p}), \quad \text{and} \quad \begin{cases} \sum_{i=1}^{n} |a_{ni}|^p = O(n^{-\delta}) \text{ for some } \delta > 0 & \text{if } 0 < p \leqslant 2, \\ \sum_{i=1}^{n} |a_{ni}|^2 = o((\log n)^{-1}) & \text{if } p > 2. \end{cases}$$

*Then*

$$\sum_{i=1}^{n} a_{ni} X_i \to 0 \text{ a.s.}$$

R e m a r k 1.1. Compared with Yi et al. [28], they obtained the strong law of large numbers for weighted sums satisfying $\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n)$, and showed the strong law of large numbers under the moment conditions that $E|X|^{\beta} < \infty$, where $1/p = 1/\alpha + 1/\beta$ for $1 \leqslant p < 2$ and $0 < \alpha$, $\beta < \infty$. But the weighted sums and the moment conditions of our result are quite different from that of Yi et al. [28]. According to the proof of Theorem 2.4 obtained by Yi et al. [28] and the properties of stochastic domination, the condition of identical distribution in Theorem 2.4 can be weakened to stochastic domination in this paper.

The rest of this paper is organized as follows. Some important lemmas are presented in Section 2. The proof is provided in Section 3 and an application of our main result is stated in Section 4. In Section 5 we give a simulation study. Throughout this paper, $C$ represents a positive constant whose value may vary in different places. Let $I(A)$ be the indicator function of the set $A$. $a_n = O(b_n)$ stands for $|a_n| \leqslant C|b_n|$ for all $n \geqslant 1$. Denote $\log x = \ln \max(x, \mathrm{e})$.

## 2. Preliminary lemmas

To prove the main results of the paper, we need the following important lemmas. The first one is a basic property for WOD random variables, which was presented by Wang et al. [21].

**Lemma 2.1.** *Let $\{X_n, n \geqslant 1\}$ be a sequence of WOD random variables.*

(i) *If $\{f_n(\cdot), n \geqslant 1\}$ are all nondecreasing (or all nonincreasing), then $\{f_n(X_n), n \geqslant 1\}$ are still WOD.*

(ii) *For each $n \geqslant 1$ and any $t \in \mathbb{R}$,*

$$E \exp\left\{t \sum_{i=1}^{n} X_i\right\} \leqslant g(n) \prod_{i=1}^{n} E \exp\{tX_i\}.$$

**Lemma 2.2.** *Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ be both WOD random variables with the dominating coefficients $g_1(n)$ and $g_2(n)$, respectively. If random variables $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ are independent, then for any $\beta \in \mathbb{R}$, $X_1 + \beta Y_1$, $X_2 + \beta Y_2, \ldots, X_n + \beta Y_n$ are also WOD random variables with the same dominating coefficients $g_1(n)g_2(n)$.*

Proof. For any $t_1, t_2, \ldots, t_n \in \mathbb{R}$, by the independence of $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$, Lemma 2.1 and the definition of WOD random variables, it follows

$$\begin{aligned}
&P(X_1 + \beta Y_1 > t_1, \ldots, X_n + \beta Y_n > t_n)\\
&\quad = \int \ldots \int I(x_1 + \beta y_1 > t_1, \ldots, x_n + \beta y_n > t_n)\\
&\qquad \times \mathrm{d}F_{X_1,\ldots,X_n,Y_1,\ldots,Y_n}(x_1, \ldots, x_n, y_1, \ldots, y_n)\\
&\quad = \int \ldots \int P(X_1 + \beta y_1 > t_1, \ldots, X_n + \beta y_n > t_n)\,\mathrm{d}F_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n)\\
&\quad \leqslant g_1(n) \int \ldots \int P(X_1 + \beta y_1 > t_1) \ldots P(X_n + \beta y_n > t_n)\,\mathrm{d}F_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n)\\
&\quad = g_1(n) E\Big(\prod_{i=1}^{n} \int I(x_i + \beta Y_i > t_i)\,\mathrm{d}F_{X_i}(x_i)\Big)\\
&\quad \leqslant g_1(n) g_2(n) \prod_{i=1}^{n} E \int I(x_i + \beta Y_i > t_i)\,\mathrm{d}F_{X_i}(x_i)\\
&\quad = g_1(n) g_2(n) \prod_{i=1}^{n} P(X_i + \beta Y_i > t_i).
\end{aligned}$$

Similarly, we have

$$P(X_1 + \beta Y_1 \leqslant t_1, \ldots, X_n + \beta Y_n \leqslant t_n) \leqslant g_1(n) g_2(n) \prod_{i=1}^{n} P(X_i + \beta Y_i \leqslant t_i).$$

Hence, by the definition of WOD random variables, we can see that $X_1 + \beta Y_1$, $X_2 + \beta Y_2, \ldots, X_n + \beta Y_n$ are WOD random variables with dominating coefficients $g_1(n) g_2(n)$. The proof is completed. □

The following lemma is a Fuk-Nagaev type inequality for WOD random variable which is due to Shen et al. [16].

**Lemma 2.3.** *Let $\{X_n, n \geqslant 1\}$ be a sequence of WOD random variables with dominating coefficients $g(n)$ for $n \geqslant 1$. If $EX_n = 0$ and $EX_n^2 < \infty$ for $n \geqslant 1$, then for any $x > 0$ and $y > 0$,*

$$P\Big(\Big|\sum_{i=1}^{n} X_i\Big| > x\Big) \leqslant 2P\Big(\max_{1 \leqslant i \leqslant n} |X_i| > y\Big) + 2g(n)\exp\Big\{-\frac{x^2}{2(xy + M_n)}\Big\},$$

*where $M_n = \sum_{i=1}^{n} EX_i^2$.*

By the definition of stochastic domination and integration by parts, we can get the following property for stochastic domination. For the details of the proof, one can refer to Wu [25].

**Lemma 2.4.** *Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables which is stochastically dominated by a random variable $X$. Then, for any $\alpha > 0$ and $b > 0$, the following two statements hold:*

$$E|X_n|^\alpha I(|X_n| \leqslant b) \leqslant C_1(E|X|^\alpha I(|X| \leqslant b) + b^\alpha P(|X| > b)),$$
$$E|X_n|^\alpha I(|X_n| > b) \leqslant C_2 E|X|^\alpha I(|X| > b),$$

*where $C_1$ and $C_2$ are positive constants. Consequently, $E|X_n|^\alpha \leqslant CE|X|^\alpha$, where $C$ is a positive constant.*

## 3. Proof of the main result

With the lemmas in Section 2, we will present the proof of the main result.

P r o o f of Theorem 1.1. Without loss of generality, we can assume that $a_{ni} > 0$, and note that $a_{ni} = a_{ni}^+ - a_{ni}^-$. For any $\varrho > 0$, choose a small $\varpi > 0$ and large $N \geqslant 1$ ($\varpi$ and $N$ be specified later).

Denote for $n \geqslant 1$ and $1 \leqslant i \leqslant n$ that

$$\begin{aligned}
X_{ni}^{(1)} &\doteq -a_{ni}^{-1}n^{-\varpi}I(a_{ni}X_i < -n^{-\varpi}) \\
&\quad + X_i I(|a_{ni}X_i| \leqslant n^{-\varpi}) + a_{ni}^{-1}n^{-\varpi}I(a_{ni}X_i > n^{-\varpi}), \\
X_{ni}^{(2)} &\doteq (X_i - a_{ni}^{-1}n^{-\varpi})I(n^{-\varpi} < a_{ni}X_i < \varrho/N), \\
X_{ni}^{(3)} &\doteq (X_i + a_{ni}^{-1}n^{-\varpi})I(-n^{-\varpi} > a_{ni}X_i > -\varrho/N), \\
X_{ni}^{(4)} &\doteq (X_i - a_{ni}^{-1}n^{-\varpi})I(a_{ni}X_i \geqslant \varrho/N) + (X_i + a_{ni}^{-1}n^{-\varpi})I(a_{ni}X_i \leqslant -\varrho/N), \\
S_n(k) &\doteq \sum_{i=1}^n a_{ni}X_{ni}^{(k)}, \quad k = 1, 2, 3, 4.
\end{aligned}$$

For fixed $n \geqslant 1$, it is easy to obtain $\sum\limits_{i=1}^n a_{ni}X_i = S_n(1) + S_n(2) + S_n(3) + S_n(4)$, and it follows from Lemma 2.1 and the definition of $X_{ni}^{(1)}$ that $\{a_{ni}X_{ni}^{(1)} - Ea_{ni}X_{ni}^{(1)}, 1 \leqslant i \leqslant n\}$ are still WOD random variables.

In what follows, we consider two cases.

*Case I.* $p > 2$. In order to obtain $\sum\limits_{i=1}^n a_{ni}X_i \to 0$ a.s., it suffices to show $S_n(k) \to 0$ a.s. for $k = 1, 2, 3, 4$.

Firstly, $S_n(1) \to 0$ a.s. is valid if we can show that $ES_n(1) \to 0$ and $S_n(1) - ES_n(1) \to 0$ a.s. as $n \to \infty$.

Taking $\varpi$ sufficiently small such that $0 < \varpi < (1-2/p)/(p-1)$, so $(p-1)\varpi - 1 + 2/p < 0$, then we have by $EX_n = 0$, Markov's inequality and Lemma 2.4 that

$$
\begin{aligned}
(3.1)\quad |ES_n(1)| &\leqslant n^{-\varpi}\sum_{i=1}^{n} P(|a_{ni}X_i| > n^{-\varpi}) + \Big|\sum_{i=1}^{n} a_{ni}EX_iI(|a_{ni}X_i| \leqslant n^{-\varpi})\Big| \\
&= n^{-\varpi}\sum_{i=1}^{n} P(|a_{ni}X_i| > n^{-\varpi}) + \Big|\sum_{i=1}^{n} a_{ni}EX_iI(|a_{ni}X_i| > n^{-\varpi})\Big| \\
&\leqslant n^{(p-1)\varpi}\sum_{i=1}^{n} E|a_{ni}X_i|^p + n^{(p-1)\varpi}\sum_{i=1}^{n} E|a_{ni}X_i|^pI(|a_{ni}X_i| > n^{-\varpi}) \\
&\leqslant Cn^{(p-1)\varpi}\Big(\max_{1\leqslant i\leqslant n}|a_{ni}|\Big)^{p-2}\sum_{i=1}^{n} a_{ni}^2 \\
&\leqslant Cn^{(p-1)\varpi-1+2/p}(\log n)^{-1} \to 0 \quad \text{as } n \to \infty,
\end{aligned}
$$

which yields that $ES_n(1) \to 0$ as $n \to \infty$.

In order to show $S_n(1) - ES_n(1) \to 0$ a.s., we only need to prove that for any $\varrho > 0$,

$$
(3.2)\qquad \sum_{n=1}^{\infty} P(|S_n(1) - ES_n(1)| > \varrho) < \infty.
$$

From the definition of $X_{ni}^{(1)}$,

$$
(3.3)\qquad \max_{1\leqslant i\leqslant n} |a_{ni}X_{ni}^{(1)} - Ea_{ni}X_{ni}^{(1)}| \leqslant 2n^{-\varpi},
$$

and by Lemma 2.4, we have

$$
(3.4)\qquad M_n \doteq \sum_{i=1}^{n} E(a_{ni}X_{ni}^{(1)} - Ea_{ni}X_{ni}^{(1)})^2 \leqslant \sum_{i=1}^{n} E|a_{ni}X_{ni}^{(1)}|^2 \leqslant \sum_{i=1}^{n} E|a_{ni}X_i|^2.
$$

Noting that $g(n) = O(n^{\vartheta})$ for some $\vartheta \geqslant 0$, then using Lemma 2.3 with $x = \varrho$ and $y = 2n^{-\varpi}$, we derive by (3.3), (3.4) and Lemma 2.3 that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} P(|S_n(1) - ES_n(1)| > \varrho) \\
&\qquad = \sum_{n=1}^{\infty} P\Big(\Big|\sum_{n=1}^{n} a_{ni}X_{ni}^{(1)} - Ea_{ni}X_{ni}^{(1)}\Big| > \varrho\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant 2\sum_{n=1}^{\infty} P\Big(\max_{1\leqslant i\leqslant n} |a_{ni}X_{ni}^{(1)} - Ea_{ni}X_{ni}^{(1)}| > 2n^{-\varpi}\Big)\\
&\quad + 2\sum_{n=1}^{\infty} g(n)\exp\Big\{-\frac{\varrho^2}{2(2\varrho n^{-\varpi} + o((\log n)^{-1}))}\Big\}\\
&\leqslant C\sum_{n=1}^{\infty} g(n)\exp\{-(2+\vartheta)\log n\}\\
&\leqslant C\sum_{n=1}^{\infty} \exp\{-2\log n\} < \infty,
\end{aligned}
$$

which yields (3.2). The proof of $S_n(1) \to 0$ a.s. is finished. □

Next, we will consider $S_n(2)$. It follows from the definition of $X_{ni}^{(2)}$ that $0 \leqslant a_{ni}X_{ni}^{(2)} < \varrho/N$. Then $|S_n(2)| = \sum_{i=1}^{n} a_{ni}X_{ni}^{(2)} > \varrho$ gives that there are at least $N$ $i$'s such that $a_{ni}X_{ni}^{(2)} \neq 0$. Noting that $g(n) = O(n^{\vartheta})$ for some $\vartheta \geqslant 0$, we take $\varpi$ sufficiently small and $N \geqslant 1$ sufficiently large such that $1 - 2/p - p\varpi > 0$ and $(1-2/p-p\varpi)N \geqslant \vartheta+1$. Hence, we have by Markov's inequality and Lemma 2.4 that

$$
\begin{aligned}
&P(|S_n(2)| > \varrho)\\
&\leqslant P(\text{there are at least } N\ i'\text{s such that } a_{ni}X_{ni}^{(2)} \neq 0)\\
&\leqslant \sum_{1\leqslant i_1<i_2<\ldots<i_N\leqslant n} P(a_{n1}X_{ni_1}^{(2)} \neq 0, a_{ni_2}X_{ni_2}^{(2)} \neq 0, \ldots, a_{ni_N}X_{ni_N}^{(2)} \neq 0)\\
&\leqslant \sum_{1\leqslant i_1<i_2<\ldots<i_N\leqslant n} P(a_{n1}X_{i_1} > n^{-\varpi}, a_{ni_2}X_{i_2} > n^{-\varpi}, \ldots, a_{ni_N}X_{i_N} > n^{-\varpi})\\
&\leqslant g(n)\sum_{1\leqslant i_1<i_2<\ldots<i_N\leqslant n} P(a_{n1}X_{i_1} > n^{-\varpi})P(a_{ni_2}X_{i_2} > n^{-\varpi})\ldots P(a_{ni_N}X_{i_N} > n^{-\varpi})\\
&\leqslant g(n)\Big(\sum_{i=1}^{n} P(a_{ni}X_i > n^{-\varpi})\Big)^N \leqslant g(n)\Big(\sum_{i=1}^{n} P(|a_{ni}X_i| > n^{-\varpi})\Big)^N\\
&\leqslant g(n)\Big(n^{p\varpi}\sum_{i=1}^{n} E|a_{ni}X_i|^p\Big)^N \leqslant g(n)\Big(n^{p\varpi}\Big(\max_{1\leqslant i\leqslant n}|a_{ni}|\Big)^{p-2}\sum_{i=1}^{n} a_{ni}^2\Big)^N\\
&\leqslant g(n)n^{-(1-2/p-p\varpi)N}(\log n)^{-N} \leqslant Cn^{\vartheta-(1-2/p-p\varpi)N}(\log n)^{-N},
\end{aligned}
$$

which yields

$$
\sum_{n=1}^{\infty} P(|S_n(2)| > \varrho) < \infty.
$$

So, we have $S_n(2) \to 0$ a.s.

Based on $-\varrho/N < a_{ni}X_{ni}^{(3)} \leqslant 0$, then $|S_n(3)| = -\sum_{i=1}^{n} a_{ni}X_{ni}^{(3)} > \varrho$ proves that there are at least $N$ $i$'s such that $a_{ni}X_{ni}^{(3)} \neq 0$. Similarly to the proof of $S_n(2) \to 0$ a.s., we obtain $S_n(3) \to 0$ a.s. immediately.

Finally, we will prove $S_n(4) \to 0$ a.s. By Lemma 2.4 and $E|X|^{2p} < \infty$, we can obtain

$$\begin{aligned}\sum_{n=1}^{\infty} P(|S_n(4)| > \varrho) &\leqslant \sum_{n=1}^{\infty}\sum_{i=1}^{n} P(|a_{ni}X_i| \geqslant \varrho/N) \leqslant C\sum_{n=1}^{\infty} nP(|X| > Cn^{1/p}) \\ &\leqslant C\sum_{k=1}^{\infty} k^2 P(Ck^{1/p} < |X| \leqslant C(k+1)^{1/p}) \leqslant CE|X|^{2p} < \infty,\end{aligned}$$

which implies that $S_n(4) \to 0$ a.s.

*Case II.* $0 < p \leqslant 2$. We first need to prove $ES_n(1) \to 0$. If $p > 1$, taking $\varpi$ sufficiently small such that $0 < \varpi < (1 - 2/p)/(p-1)$, which means that $(p-1)\varpi - 1 + 2/p < 0$, by $EX_n = 0$, Markov's inequality and Lemma 2.4, we have

$$\begin{aligned}|ES_n(1)| &\leqslant n^{-\varpi}\sum_{i=1}^{n} P(|a_{ni}X_i| > n^{-\varpi}) + \left|\sum_{i=1}^{n} a_{ni}EX_iI(|a_{ni}X_i| \leqslant n^{-\varpi})\right| \\ &\leqslant n^{(p-1)\varpi}\sum_{i=1}^{n} E|a_{ni}X_i|^p + n^{(p-1)\varpi}\sum_{i=1}^{n} E|a_{ni}X_i|^p I(|a_{ni}X_i| > n^{-\varpi}) \\ &\leqslant Cn^{(p-1)\varpi}\Big(\max_{1\leqslant i\leqslant n}|a_{ni}|\Big)^{p-2}\sum_{i=1}^{n} a_{ni}^2 \\ &\leqslant Cn^{(p-1)\varpi-1+2/p}(\log n)^{-1} \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

So, we can obtain $ES_n(1) \to 0$ immediately.

If $0 < p \leqslant 1$, we conclude from Lemma 2.4 that for any $\varpi > 0$,

$$\begin{aligned}|ES_n(1)| &\leqslant n^{-\varpi}\sum_{i=1}^{n} P(|a_{ni}X_i| > n^{-\varpi}) + \sum_{i=1}^{n} E|a_{ni}X_i|I(|a_{ni}X_i| \leqslant n^{-\varpi}) \\ &\leqslant n^{(p-1)\varpi}\sum_{i=1}^{n} E|a_{ni}X_i|^p + n^{(p-1)\varpi}\sum_{i=1}^{n} E|a_{ni}X_i|^p I(|a_{ni}X_i| \leqslant n^{-\varpi}) \\ &\leqslant Cn^{(p-1)\varpi-\delta} \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Consequently, for all $0 < p \leqslant 2$, we get $ES_n(1) \to 0$.

Now, we prove $S_n(1) - ES_n(1) \to 0$ a.s. Recall that $|a_{ni}X_{ni}^{(1)}| \leqslant n^{-\varpi}$ and

$$M_n \doteq \sum_{i=1}^{n} E(a_{ni}X_{ni}^{(1)} - Ea_{ni}X_{ni}^{(1)})^2 \leqslant \sum_{i=1}^{n} E|a_{ni}X_{ni}^{(1)}|^2.$$

Noting that $g(n) = O(n^{\vartheta})$ for some $\vartheta \geqslant 0$, then using Lemma 2.3 with $x = \varrho$ and $y = 2n^{-\varpi}$, we have by Lemma 2.3 that

$$\begin{aligned}
\sum_{n=1}^{\infty} P(|S_n(1) - ES_n(1)| > \varrho) &\leqslant 2\sum_{n=1}^{\infty} P\Big(\max_{1\leqslant i\leqslant n} |a_{ni}X_{ni}^{(1)} - Ea_{ni}X_{ni}^{(1)}| > 2n^{-\varpi}\Big) \\
&\quad + 2\sum_{n=1}^{\infty} g(n)\exp\Big\{-\frac{\varrho^2}{2(2\varrho n^{-\varpi} + o((\log n)^{-1}))}\Big\} \\
&\leqslant C\sum_{n=1}^{\infty} g(n)\exp\{-(2+\vartheta)\log n\} \\
&\leqslant C\sum_{n=1}^{\infty} \exp\{-2\log n\} < \infty,
\end{aligned}$$

which ensures that $S_n(1) - ES_n(1) \to 0$ a.s., and thus $S_n(1) \to 0$ a.s.

For $S_n(2)$, it follows from the definition of $X_{ni}^{(2)}$ that $0 \leqslant a_{ni}X_{ni}^{(2)} < \varrho/N$. Then $|S_n(2)| = \sum\limits_{i=1}^{n} a_{ni}X_{ni}^{(2)} > \varrho$ gives that there are at least $N$ $i$'s such that $a_{ni}X_{ni}^{(2)} \neq 0$. Noting that $g(n) = O(n^{\vartheta})$ for some $\vartheta \geqslant 0$, we take $\varpi$ sufficiently small and $N \geqslant 1$ sufficiently large such that $\delta - p\varpi > 0$ and $(\delta - p\varpi)N \geqslant \vartheta + 1$. Hence, we have by Markov's inequality and Lemma 2.4 that

$$\begin{aligned}
&P(|S_n(2)| > \varrho) \\
&\leqslant P(\text{there are at least } N \ i'\text{s such that } a_{ni}X_{ni}^{(2)} \neq 0) \\
&\leqslant \sum_{1\leqslant i_1<i_2<\ldots<i_N\leqslant n} P(a_{n1}X_{ni_1}^{(2)} \neq 0, a_{ni_2}X_{ni_2}^{(2)} \neq 0, \ldots, a_{ni_N}X_{ni_N}^{(2)} \neq 0) \\
&\leqslant \sum_{1\leqslant i_1<i_2<\ldots<i_N\leqslant n} P(a_{n1}X_{i_1} > n^{-\varpi}, a_{ni_2}X_{i_2} > n^{-\varpi}, \ldots, a_{ni_N}X_{i_N} > n^{-\varpi}) \\
&\leqslant g(n) \sum_{1\leqslant i_1<i_2<\ldots<i_N\leqslant n} P(a_{n1}X_{i_1} > n^{-\varpi})P(a_{ni_2}X_{i_2} > n^{-\varpi}) \ldots P(a_{ni_N}X_{i_N} > n^{-\varpi}) \\
&\leqslant g(n)\Big(\sum_{i=1}^{n} P(a_{ni}X_i > n^{-\varpi})\Big)^N \leqslant g(n)\Big(\sum_{i=1}^{n} P(|a_{ni}X_i| > n^{-\varpi})\Big)^N \\
&\leqslant g(n)\Big(n^{p\varpi}\sum_{i=1}^{n} E|a_{ni}X_i|^p\Big)^N \leqslant g(n)n^{(p\varpi-\delta)N} \leqslant Cn^{\vartheta+(p\varpi-\delta)N},
\end{aligned}$$

which yields

$$\sum_{n=1}^{\infty} P(|S_n(2)| > \varrho) < \infty.$$

So, we have $S_n(2) \to 0$ a.s.

Based on $-\varrho/N < a_{ni}X_{ni}^{(3)} \leqslant 0$, then $|S_n(3)| = -\sum_{i=1}^{n} a_{ni}X_{ni}^{(3)} > \varrho$ proves that there are at least $N$ $i$'s such that $a_{ni}X_{ni}^{(3)} \neq 0$. Similarly to the proof of $S_n(2) \to 0$ a.s., we obtain $S_n(3) \to 0$ a.s. immediately.

For $S_n(4)$, by Lemma 2.4 and $E|X|^{2p} < \infty$, we can obtain

$$\begin{aligned}\sum_{n=1}^{\infty} P(|S_n(4)| > \varrho) &\leqslant \sum_{n=1}^{\infty}\sum_{i=1}^{n} P(|a_{ni}X_i| \geqslant \varrho/N) \leqslant C\sum_{n=1}^{\infty} nP(|X| > Cn^{1/p}) \\ &\leqslant C\sum_{k=1}^{\infty} k^2 P(Ck^{1/p} < |X| \leqslant C(k+1)^{1/p}) \leqslant CE|X|^{2p} < \infty,\end{aligned}$$

which implies that $S_n(4) \to 0$ a.s. From the statements above, the proof is completed. □

## 4. Application of the main result

In this section, we will present an application of our main result established in Section 1.

Consider the following simple linear EV model

$$\eta_i = \theta + \beta x_i + \varepsilon_i, \quad \xi_i = x_i + \delta_i, \quad 1 \leqslant i \leqslant n, \tag{4.1}$$

where $\eta_i$ are the response variables, $\theta$, $\beta$, $x_i$ are unknown constants (parameters), and $\varepsilon_i$ are random errors with mean zero. Due to the measuring mechanism or the nature of the environment, the variables $x_i$ are measured with error and are not directly observable. Instead, $x_i$ are observed through $\xi_i = x_i + \delta_i$, where $\delta_i$ are the measurement errors with mean zero and $\{\varepsilon_i, i \geqslant 1\}$ is independent of $\{\delta_i, i \geqslant 1\}$. Meanwhile, $\{\varepsilon_i, i \geqslant 1\}$ and $\{\delta_i, i \geqslant 1\}$ are both mean zero WOD random variables with identical distribution.

The EV regression model (4.1) was first proposed by Allen [1] and subsequently has been studied by many authors. For instance, Deaton [7] corrected it for the effects of sampling error and found it is somewhat more practical than the ordinary regression model, which includes many usual parametric, partially linear and nonparametric regression models. For more details about the EV regression model, we can refer to Fuller [10], Carroll et al. [2], and so on. As we all know, many authors focused on the estimation of the involved parameter in the EV model. Moreover, the limiting behaviors for the LS estimators of $\beta$ and $\theta$ in the EV model have been extensively studied. For example, for the case that the errors are sequences of independent random variables, Cui [6] for asymptotic normality, Miao et al. [13] for the central limit theorem, Chen et al. [3] for strong and weak consistency, and so on. Under the case that the errors are sequences of dependent random variables, Fazekas and Kukush [9] for asymptotic normality, Miao et al. [14] for the strong consistency, Wang et al. [20] for asymptotic properties, and so on. Inspired by the above literatures, the main purpose of this section is to further study the strong consistency for the LS estimator in the EV regression model under WOD errors by using the result obtained in Section 1.

From (4.1) we have

$$\eta_i = \theta + \beta\xi_i + \nu_i, \quad \nu_i = \varepsilon_i - \beta\delta_i, \quad 1 \leqslant i \leqslant n. \tag{4.2}$$

Considering formally (4.2) as a usual regression model of $\eta_i$ on $\xi_i$, we get the LS estimators of $\beta$ and $\theta$ as

$$\widehat{\beta}_n = \frac{\sum\limits_{i=1}^{n}(\xi_i - \overline{\xi}_n)(\eta_i - \overline{\eta}_n)}{\sum\limits_{i=1}^{n}(\xi_i - \overline{\xi}_n)^2}, \quad \widehat{\theta}_n = \overline{\eta}_n - \widehat{\beta}_n\overline{\xi}_n, \tag{4.3}$$

where $\overline{\xi}_n = n^{-1}\sum\limits_{i=1}^{n}\xi_i$, and other similar notations, such as $\overline{\eta}_n$, $\overline{\delta}_n$, $\overline{x}_n$ and $\overline{\varepsilon}_n$ are defined in the same way. Denote $S_n = \sum\limits_{i=1}^{n}(x_i - \overline{x}_n)^2$ for each $n \geqslant 1$. Based on the notations above, we can get that

$$\widehat{\beta}_n - \beta = \frac{\sum_{i=1}^{n}(\delta_i - \overline{\delta}_n)\varepsilon_i + \sum_{i=1}^{n}(x_i - \overline{x}_n)(\varepsilon_i - \beta\delta_i) - \beta\sum_{i=1}^{n}(\delta_i - \overline{\delta}_n)^2}{\sum_{i=1}^{n}(\xi_i - \overline{\xi}_n)^2}, \tag{4.4}$$

and

$$\widehat{\theta}_n - \theta = (\beta - \widehat{\beta}_n)\overline{x}_n + (\beta - \widehat{\beta}_n)\overline{\delta}_n + \overline{\varepsilon}_n - \beta\overline{\delta}_n. \tag{4.5}$$

These relations play important roles in the proofs of the main results in this section.

As an application of Theorem 1.1, we obtain the following strong consistency of the LS estimator in a multiple linear EV regression model.

**Theorem 4.1.** *Let $E|\varepsilon_1|^{4p} < \infty$ and $E|\delta_1|^{4p} < \infty$ for some $p \geqslant 1$. Suppose that there exists some $\kappa > 0$ such that $\max\limits_{1\leqslant i\leqslant n}|x_i - \overline{x}_n|/(\sqrt{S_n}n^{\kappa-1/p}) \leqslant C$, $n^{\kappa}/\sqrt{S_n} \leqslant C$ and $n^{1-\kappa}/\sqrt{S_n} \to 0$. Assume further that $\kappa > 1/p - 1/2$ if $1 \leqslant p \leqslant 2$. Then*

$$\frac{\sqrt{S_n}}{n^{\kappa}}(\widehat{\beta}_n - \beta) \to 0 \text{ a.s.}$$

**Theorem 4.2.** *Suppose that the conditions of Theorem* 4.1 *are satisfied. Assume further that there exist some* $0 < \gamma < \min(1-1/p, 1/2)$ *and* $p > 1$ *such that* $n^{\kappa+\gamma}/\sqrt{S_n}|\overline{x}_n| \leqslant C$. *Then*

$$n^{\gamma}(\widehat{\theta}_n - \theta) \to 0 \text{ a.s.}$$

Remark 4.1. The conditions of our Theorems 4.1 and 4.2 are relatively general, which has been used by Wang et al. [20]. It is easy to see that $n^{1-\kappa}/\sqrt{S_n} \to 0$ implies $n^{\kappa}/\sqrt{S_n} \leqslant C$ when $0 < \kappa \leqslant 1/2$ and $n^{\kappa}/\sqrt{S_n} \leqslant C$ implies $n^{1-\kappa}/\sqrt{S_n} \to 0$ when $\kappa > 1/2$. Compared with Theorem 2.3 in Miao et al. [14], we can see the condition in $\sqrt{S_n}/n^{1-\kappa+\gamma} \to \infty$ $(\gamma > 0)$ for $p = 2$ of Miao et al. [14] is stronger than $n^{1-\kappa}/\sqrt{S_n} \to 0$ of this paper.

Proof of Theorem 4.1. Noting that $E\delta_1^2 < \infty$ by $E|\delta_1|^{4p} < \infty$ for $p \geqslant 1$, it follows

$$\begin{aligned}\frac{1}{\sqrt{S_n}n^{\kappa}}\sum_{i=1}^{n}(\delta_i - \overline{\delta}_n)^2 &\leqslant \frac{1}{\sqrt{S_n}n^{\kappa}}\sum_{i=1}^{n}(\delta_i^2 - E\delta_i^2) + \frac{1}{\sqrt{S_n}n^{\kappa}}\sum_{i=1}^{n}E\delta_i^2\\ &= \frac{1}{\sqrt{S_n}n^{\kappa}}\sum_{i=1}^{n}(\delta_i^2 I(\delta_i \geqslant 0) - E\delta_i^2 I(\delta_i \geqslant 0))\\ &\quad + \frac{1}{\sqrt{S_n}n^{\kappa}}\sum_{i=1}^{n}(\delta_i^2 I(\delta_i < 0) - E\delta_i^2 I(\delta_i < 0)) + \frac{n^{1-\kappa}}{\sqrt{S_n}}E\delta_1^2\\ &\doteq I_{n1} + I_{n2} + \frac{n^{1-\kappa}}{\sqrt{S_n}}E\delta_1^2.\end{aligned}$$

By Lemma 2.2, we easily see that $\{\delta_i^2 I(\delta_i \geqslant 0) - E\delta_i^2 I(\delta_i \geqslant 0), i \geqslant 1\}$ and $\{\delta_i^2 I(\delta_i < 0) - E\delta_i^2 I(\delta_i < 0), i \geqslant 1\}$ are still WOD random variables. For $I_{n1}$, denote $a_{ni} = 1/(\sqrt{S_n}n^{\kappa})$ and $X_i = \delta_i^2 I(\delta_i \geqslant 0) - E\delta_i^2 I(\delta_i \geqslant 0)$.

It follows from $E|\delta_1|^{4p} < \infty$ that $E|X_1|^{2p} \leqslant E|\delta_1|^{4p} < \infty$. Since $n^{1-\kappa}/\sqrt{S_n} \to 0$, one has

$$\max_{1\leqslant i\leqslant n}|a_{ni}| = \frac{1}{\sqrt{S_n}n^{\kappa}} \leqslant \frac{n^{1-\kappa}}{\sqrt{S_n}}\cdot\frac{1}{n^{1/p}} = O(n^{-1/p}),$$

and

$$\begin{cases}\displaystyle\sum_{i=1}^{n}|a_{ni}|^p = \frac{n}{S_n^{p/2}n^{p\kappa}} = n^{1-p}\cdot\left(\frac{n^{1-\kappa}}{\sqrt{S_n}}\right)^p = O(n^{-(p-1)}) & \text{if } 1 < p \leqslant 2,\\ \displaystyle\sum_{i=1}^{n}a_{ni}^2 = \frac{n^{2-2\kappa}}{S_n}\cdot\frac{1}{n} = o((\log n)^{-1}) & \text{if } p > 2,\end{cases}$$

if $p = 1$, by $n^{\kappa}/\sqrt{S_n} \leqslant C$, we have

$$\sum_{i=1}^{n}|a_{ni}| = \frac{n^{1-\kappa}}{\sqrt{S_n}} = n^{1-2\kappa}\cdot\frac{n^{\kappa}}{\sqrt{S_n}} = O(n^{-(2\kappa-1)}),$$

where $\kappa > 1/2$, since $\kappa > 1/p - 1/2$ if $1 \leqslant p \leqslant 2$. Thus, applying Theorem 1.1, we can prove $I_{n1} \to 0$ a.s., we can also prove $I_{n2} \to 0$ a.s. in the similar way to that of $I_{n1}$. Moreover, recall that $E|\delta_1|^{4p} < \infty$ implies $E\delta_1^2 < \infty$, then

$$\frac{1}{\sqrt{S_n}n^{\kappa}} \sum_{i=1}^{n} (\delta_i - \overline{\delta}_n)^2 \to 0 \text{ a.s.} \tag{4.6}$$

Like with the arguments in the proof of (4.6), we can obtain

$$\frac{1}{\sqrt{S_n}n^{\kappa}} \sum_{i=1}^{n} (\varepsilon_i - \overline{\varepsilon}_n)^2 \to 0 \text{ a.s.} \tag{4.7}$$

Note that

$$\left|\sum_{i=1}^{n} (\delta_i - \overline{\delta}_n)\varepsilon_i\right| = \left|\sum_{i=1}^{n} (\delta_i - \overline{\delta}_n)(\varepsilon_i - \overline{\varepsilon}_n)\right| \leqslant \frac{1}{2}\left(\sum_{i=1}^{n} (\delta_i - \overline{\delta}_n)^2 + \sum_{i=1}^{n} (\varepsilon_i - \overline{\varepsilon}_n)^2\right).$$

So, from (4.6) and (4.7), we can obtain

$$\frac{1}{\sqrt{S_n}n^{\kappa}} \sum_{i=1}^{n} (\delta_i - \overline{\delta}_n)\varepsilon_i \to 0 \text{ a.s.} \tag{4.8}$$

Denote $a_{ni} = (x_i - \overline{x}_n)/(\sqrt{S_n}n^{\kappa})$, $X_i = \varepsilon_i - \beta\delta_i$. We have by Lemma 2.2 that $\{\varepsilon_i - \beta\delta_i, i \geqslant 1\}$ is still a sequence of WOD random variables. Meanwhile, recall that $E|\varepsilon_1|^{4p} < \infty$ and $E|\delta_1|^{4p} < \infty$ implies

$$E|X_1|^{2p} = E|\varepsilon_1 - \beta\delta_1|^{2p} \leqslant CE|\varepsilon_1|^{2p} + CE|\delta_1|^{2p} < \infty.$$

Since $\max\limits_{1 \leqslant i \leqslant n} |x_i - \overline{x}_n|/(\sqrt{S_n}n^{\kappa - 1/p}) \leqslant C$, we have

$$\max_{1 \leqslant i \leqslant n} |a_{ni}| = \max_{1 \leqslant i \leqslant n} \frac{|x_i - \overline{x}_n|}{\sqrt{S_n}n^{\kappa - 1/p}} \cdot \frac{1}{n^{1/p}} = O(n^{-1/p}).$$

If $1 \leqslant p \leqslant 2$, note that $\kappa > 1/p - 1/2$, it follows

$$\begin{aligned}\sum_{i=1}^{n} |a_{ni}|^p &= \frac{1}{S_n^{p/2} n^{p\kappa}} \sum_{i=1}^{n} |x_i - \overline{x}_n|^p \leqslant \frac{1}{S_n^{p/2} n^{p\kappa}} \cdot n\left(\frac{1}{n}\sum_{i=1}^{n} |x_i - \overline{x}_n|^2\right)^{p/2} \\ &= n^{1 - p/2 - p\kappa} = O(n^{-(p\kappa + p/2 - 1)}),\end{aligned}$$

if $p > 2$,

$$\sum_{i=1}^{n} a_{ni}^2 = \frac{1}{n^{2\kappa}} = o((\log n)^{-1}).$$

Hence, applying Theorem 1.1, we can prove that

$$\frac{1}{\sqrt{S_n}n^{\kappa}}\sum_{i=1}^{n}(x_i-\overline{x}_n)(\varepsilon_i-\beta\delta_i)\to 0 \text{ a.s.} \tag{4.9}$$

For any $\lambda>0$ we conclude that

$$\begin{aligned}\sum_{i=1}^{n}|x_i-\overline{x}_n|\cdot|\delta_i-\overline{\delta}_n| &\leqslant \sqrt{\sum_{i=1}^{n}(x_i-\overline{x}_n)^2}\cdot\sqrt{\sum_{i=1}^{n}(\delta_i-\overline{\delta}_n)^2}\\ &= \sqrt{\lambda\sum_{i=1}^{n}(x_i-\overline{x}_n)^2}\cdot\sqrt{\frac{1}{\lambda}\sum_{i=1}^{n}(\delta_i-\overline{\delta}_n)^2}\\ &\leqslant \frac{\lambda\sum_{i=1}^{n}(x_i-\overline{x}_n)^2+1/\lambda\sum_{i=1}^{n}(\delta_i-\overline{\delta}_n)^2}{2}\\ &= \frac{\lambda}{2}S_n+\frac{1}{2\lambda}\sum_{i=1}^{n}(\delta_i-\overline{\delta}_n)^2,\end{aligned}$$

which means that

$$\begin{aligned}\left|\sum_{i=1}^{n}(\xi_i-\overline{\xi}_n)^2-S_n\right| &= \left|2\sum_{i=1}^{n}(x_i-\overline{x}_n)(\delta_i-\overline{\delta}_n)+\sum_{i=1}^{n}(\delta_i-\overline{\delta}_n)^2\right|\\ &\leqslant 2\sum_{i=1}^{n}|x_i-\overline{x}_n|\cdot|\delta_i-\overline{\delta}_n|+\sum_{i=1}^{n}(\delta_i-\overline{\delta}_n)^2\\ &\leqslant \lambda S_n+\frac{1+\lambda}{\lambda}\sum_{i=1}^{n}(\delta_i-\overline{\delta}_n)^2.\end{aligned} \tag{4.10}$$

Obviously, by $n^{\kappa}/\sqrt{S_n}\leqslant C$, (4.6) and (4.10), it follows

$$\begin{aligned}\left|\frac{1}{S_n}\sum_{i=1}^{n}(\xi_i-\overline{\xi}_n)^2-1\right| &\leqslant \lambda+\frac{1+\lambda}{\lambda}\frac{1}{S_n}\sum_{i=1}^{n}(\delta_i-\overline{\delta}_n)^2\\ &\leqslant \lambda+\frac{1+\lambda}{\lambda}\cdot\frac{n^{\kappa}}{\sqrt{S_n}}\cdot\frac{1}{\sqrt{S_n}n^{\kappa}}\sum_{i=1}^{n}(\delta_i-\overline{\delta}_n)^2\to\lambda \text{ a.s.},\end{aligned}$$

since $\lambda>0$ is arbitrary, so that $\left|S_n^{-1}\sum_{i=1}^{n}(\xi_i-\overline{\xi}_n)^2-1\right|\to 0$ a.s., which yields

$$\frac{1}{S_n}\sum_{i=1}^{n}(\xi_i-\overline{\xi}_n)^2\to 1 \text{ a.s.} \tag{4.11}$$

According to relation (4.4), the desired result follows by (4.6), (4.8), (4.9), and (4.11). The proof is completed. □

P r o o f of Theorem 4.2. It is observed that

$$n^{\gamma}(\overline{\varepsilon}_n - \beta\overline{\delta}_n) = \frac{1}{n^{1-\gamma}}\sum_{i=1}^{n}(\varepsilon_i - \beta\delta_i).$$

Denote $a_{ni} = 1/n^{1-\gamma}$, $X_i = \varepsilon_i - \beta\delta_i$. Recall that $0 < \gamma < \min(1-1/p, 1/2)$ and $p > 1$, we get

$$E|X_1|^{2p} = E|\varepsilon_1 - \beta\delta_1|^{2p} \leqslant CE|\varepsilon_1|^{2p} + CE|\delta_1|^{2p} < \infty,$$
$$\max_{1\leqslant i\leqslant n}|a_{ni}| = n^{-(1-\gamma)} = O(n^{-1/p}),$$

noting that $p - 1 - p\gamma > 0$, we have

$$\begin{cases} \displaystyle\sum_{i=1}^{n}|a_{ni}|^p = \frac{n}{n^{p(1-\gamma)}} = O(n^{-(p-1-p\gamma)}) & \text{if } 1 < p \leqslant 2,\\ \displaystyle\sum_{i=1}^{n}a_{ni}^2 = \frac{1}{n^{1-2\gamma}} = o((\log n)^{-1}) & \text{if } p > 2. \end{cases}$$

Hence, applying Theorem 1.1, we can prove that

$$n^{\gamma}(\overline{\varepsilon}_n - \beta\overline{\delta}_n) \to 0 \text{ a.s.} \tag{4.12}$$

It can be checked that

$$n^{\gamma}(\beta - \widehat{\beta}_n)\overline{x}_n = \frac{\sqrt{S_n}}{n^{\kappa}}(\beta - \widehat{\beta}_n)\cdot\frac{n^{\kappa+\gamma}}{\sqrt{S_n}}\overline{x}_n,$$

hence, by $n^{\kappa+\gamma}/\sqrt{S_n}|\overline{x}_n| \leqslant C$ and Theorem 4.1, we have

$$n^{\gamma}(\beta - \widehat{\beta}_n)\overline{x}_n \to 0 \text{ a.s.} \tag{4.13}$$

Like with the arguments in the proof of (4.12), we can obtain

$$\frac{1}{n^{1-\gamma}}\sum_{i=1}^{n}\delta_i \to 0 \text{ a.s.} \tag{4.14}$$

Note that

$$n^{\gamma}(\beta - \widehat{\beta}_n)\overline{\delta}_n = \frac{\sqrt{S_n}}{n^{\kappa}}(\beta - \widehat{\beta}_n)\cdot\frac{n^{\kappa}}{\sqrt{S_n}}\cdot\frac{1}{n^{1-\gamma}}\sum_{i=1}^{n}\delta_i,$$

thus, it follows from $n^{\kappa}/\sqrt{S_n} \leqslant C$, Theorem 4.1 and (4.14) that

$$n^{\gamma}(\beta - \widehat{\beta}_n)\overline{\delta}_n \to 0 \text{ a.s.} \tag{4.15}$$

According to relation (4.5), the desired result follows by (4.12), (4.13), and (4.15). The proof is completed. □

## 5. Simulation study

In this section, we will carry out a simulation to study the numerical performance of the strong consistency result that we established in Theorems 4.1 and 4.2. The data are generated from model (4.1). For any fixed $n \geqslant 3$, let $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)^\top \sim N(\mathbf{0}, \Sigma)$, and $(\delta_1, \delta_2, \dots, \delta_n)^\top \sim N(\mathbf{0}, \Sigma)$, where $\mathbf{0}$ represents zero column vector, $\Sigma$ is

$$\Sigma = \begin{pmatrix} 1 & -\chi & 0 & \dots & 0 & 0 & 0 \\ -\chi & 1 & -\chi & \dots & 0 & 0 & 0 \\ 0 & -\chi & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -\chi & 0 \\ 0 & 0 & 0 & \dots & -\chi & 1 & -\chi \\ 0 & 0 & 0 & \dots & 0 & -\chi & 1 \end{pmatrix}_{n\times n}, \quad \chi = 0.1.$$

By Joag-Dev and Proschan [12], we can see that $\{\varepsilon_i, i \geqslant 1\}$ and $\{\delta_i, i \geqslant 1\}$ are both NA vectors for each $n \geqslant 3$ with finite moment of any order, which are special case of WOD vectors. Here taking $\kappa = 0.45$, $\gamma = 0.05$ and $x_i = i/n^{0.2}$ for all $1 \leqslant i \leqslant n$, it is easy to show that the conditions required in Theorems 4.1 and 4.2 are satisfied. For example, $n^{\kappa+\gamma}/\sqrt{S_n}|\overline{x}_n| = \sqrt{3(n+1)/(n-1)}$. Considering the different values of $\beta$ and $\theta$, we will choose two cases as follows.

*Case I.* $\beta = 2$ and $\theta = 3$. By taking the sample sizes $n$ as $n = 100$, 300, 700, 1200, 1800 respectively, we compute $\widehat{\beta}_n - \beta$ and $\widehat{\theta}_n - \theta$ 300 times and plot the boxplots of them in Figure 1. We find that the differences $\widehat{\beta}_n - \beta$ and $\widehat{\theta}_n - \theta$ approach to the zero line and the ranges of $\widehat{\beta}_n - \beta$ and $\widehat{\theta}_n - \theta$ decrease as $n$ increases.

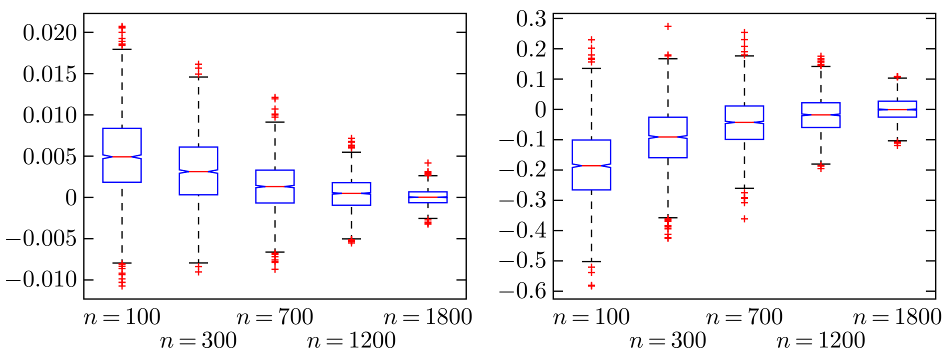


Figure 1. The boxplots for $\widehat{\beta}_n - \beta$ (left) and $\widehat{\theta}_n - \theta$ (right) with $\beta = 2$ and $\theta = 3$.

*Case II.* $\beta = 3$ *and* $\theta = 4$. We also compute $\widehat{\beta}_n - \beta$ and $\widehat{\theta}_n - \theta$ 200 times and plot the boxplots of them in Figure 2. We also see that the performances of $\widehat{\beta}_n - \beta$ and

$\widehat{\theta}_n - \theta$ fluctuate to zero uniformly as $n$ increases. These provide verifications to our theoretical results.

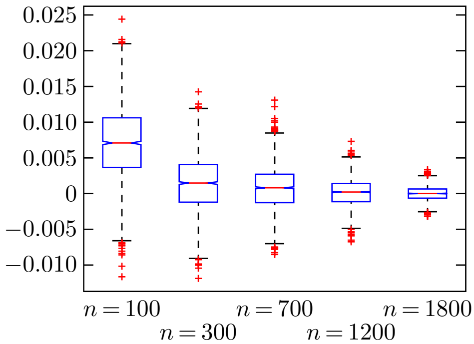


Figure 2. The boxplots for $\widehat{\beta}_n - \beta$ (left) and $\widehat{\theta}_n - \theta$ (right) with $\beta = 3$ and $\theta = 4$.

A c k n o w l e d g e m e n t s. The authors want to express their gratitude to the referee for the very helpful comments and suggestions.

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