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ON THE STABILITY ANALYSIS OF DARBOUX PROBLEM ON BOTH BOUNDED AND UNBOUNDED DOMAINS

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Abstract. In this paper, we first investigate the existence and uniqueness of solution for the Darboux problem with modified argument on both bounded and unbounded domains. Then, we derive different types of the Ulam stability for the proposed problem on these domains. Finally, we present some illustrative examples to support our results.

Keywords: Darboux problem; partial differential equation; Ulam-Hyers stability; Ulam-Hyers-Rassias stability; Wendorff lemma

MSC 2020: 34A12, 35L70, 47H10

1. Introduction

In the literature, existence and uniqueness of solutions for differential equations have been studied extensively by many researchers. More specifically, partial differential equations of order two have been considered in a general frame and with the modification of variables called "modified argument". As an example of this, Ionescu has studied Picard, Cauchy, Darboux and Goursat problems for second order hyperbolic equations with modified argument in his Ph.D. Thesis [7]. Hence, these problems with modified argument are named by these authors such as Darboux-Ionescu. Later Rus considered this type of Darboux-Ionescu in a more general framework and presented an existence and uniqueness result in [21]. Darboux-Ionescu problem has been also studied in [5], [25], [17], [12].

In recent years, besides existence and uniqueness results for differential equations, many authors have focused their attention on the concept of stability in the Ulam-Hyers (UH) and Ulam-Hyers-Rassias (UHR) senses. This type of stability has been discussed for many ordinary and partial differential equations [6], [24], [16], [8], [9], [10], [14], [15], [4], [19], [20]. A somewhat different approach to such stability has

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been proposed in [2]. For further information on this stability concept, we suggest the books [3], [11] to the reader. Also, the stability analysis has been studied in differential equations with modified argument, for some papers see [18], [17], [26].

In this paper, we focus on the hyperbolic partial differential equation with modified argument given by

$$z_{xt}(x,t) = f(x,t,z(g(x,t),h(x,t))), \quad (x,t) \in D, \tag{1.1}$$

with the initial and boundary conditions

$$\begin{cases} z(x,0) = \varphi(x), \\ z(0,t) = \psi(t) \end{cases} \quad \text{such that } \varphi(0) = \psi(0), \tag{1.2}$$

where $D = [0,a] \times [0,b]$, $f \in C(D \times \mathbb{R}, \mathbb{R})$, $g \in C(D,[0,a])$, $h \in C(D,[0,b])$, $\varphi(x)$ and $\psi(t)$ are continuously differentiable mappings defined on $[0,a]$ and $[0,b]$, respectively.

The paper is organized as follows: in Section 2, we introduce some basic concepts and results that are used to obtain the results throughout this paper. Section 3 focuses on the existence and uniqueness of solutions to the proposed equation (1.1)–(1.2) on a bounded domain by using the Bielecki norm, and based on these solutions defined on bounded domains, we extend this result to an unbounded domain. In Section 4, the stability analysis for the equation (1.1) is examined in both bounded and unbounded domains. In bounded domains, we give the UH stability result by using the effectiveness of the Bielecki norm, and then we obtain the stability result in the sense of UHR from Picard operator theory using Wendorff lemma. In addition, we give the stability result also for unbounded domains. Finally, we give some examples to illustrate our theoretical existence, uniqueness and stability results in Section 5.

2. Preliminaries

In this section, we state some basic definitions and lemmas that are used throughout this paper. We first recall the abstract Gronwall lemma and the Picard operator concept from Picard operator theory (Rus [22], [23]).

Definition 2.1. Let $\mathcal{P}\colon X \to X$ be an operator and (X,ϱ) be a metric space. If there exists $z^* \in X$ such that

(i) $F_{\mathcal{P}} = \{z^*\}$ where $F_{\mathcal{P}} = \{z \in X\colon \mathcal{P}(z) = z\}$ is the fixed-point set of $\mathcal{P}$,

(ii) the sequence $(\mathcal{P}^n(z_0))_{n\in\mathbb{N}}$ converges to z^* for all $z_0 \in X$,

then $\mathcal{P}$ is called the Picard operator.

The triple $(X,\varrho,\leqslant)$ is called an ordered metric space if (X,ϱ) is a metric space and $\leqslant$ is a partial order relation on X.

Lemma 2.2. *Let $\mathcal{P}\colon X \to X$ be an increasing Picard operator ($F_{\mathcal{P}} = \{z^*\}$) and $(X, \varrho, \leqslant)$ be an ordered metric space. For $z \in X$, then $z \leqslant \mathcal{P}(z)$ implies $z \leqslant z^*$ while $z \geqslant \mathcal{P}(z)$ implies $z \geqslant z^*$.*

Now we give the following Wendorff lemma, which is the extended form of the Gronwall lemma (see [1], [13]).

Lemma 2.3. *Let $z, h, k \in C([0,a] \times [0,b], \mathbb{R}_+)$, and let $h(x,t)$ be non-decreasing with respect to x and t. Suppose*

$$z(x,t) \leqslant h(x,t) + \int_0^x \int_0^t k(\tau,r) z(\tau,r)\,\mathrm{d}r\,\mathrm{d}\tau, \quad x \in [0,a],\ t \in [0,b].$$

Then

$$z(x,t) \leqslant h(x,t) \exp\biggl(\int_0^x \int_0^t k(\tau,r)\,\mathrm{d}r\,\mathrm{d}\tau\biggr).$$

By a solution to the equation (1.1)–(1.2) we mean a function $z \in C^{1,2}(D,\mathbb{R})$ satisfying the equation (1.1) and the conditions (1.2), where $C^{1,2}(D,\mathbb{R})$ is the space of all functions $z(x,t)\colon D \to \mathbb{R}$ such that $z, z_x, z_t, z_{xt} \in C(D,\mathbb{R})$.

3. Existence results

In this section, we present the main existence and uniqueness results for solution of the proposed equation (1.1)–(1.2) in both bounded and unbounded domains.

Theorem 3.1. *Assume that the following conditions hold:*

(C1) *$f \in C(D \times \mathbb{R}, \mathbb{R})$, $g \in C(D,[0,a])$, $h \in C(D,[0,b])$ with $g(x,t) \leqslant x$ and $h(x,t) \leqslant t$.*

(C2) *There exists a positive constant $\mathcal{L}$ such that*

$$|f(x,t,z) - f(x,t,\tilde{z})| \leqslant \mathcal{L}|z - \tilde{z}|$$

for all $z, \tilde{z} \in \mathbb{R}$ and $(x,t) \in D$.

Then the equation (1.1)–(1.2) has a unique solution.

P r o o f. Under the condition (C1), the proposed equation (1.1)–(1.2) is equivalent to the integral equation

$$z(x,t) = \mu(x,t) + \int_0^x \int_0^t f(\tau, r, z(g(\tau,r), h(\tau,r)))\,\mathrm{d}r\,\mathrm{d}\tau, \tag{3.1}$$

where $\mu(x,t) = \varphi(x) + \psi(t) - \varphi(0)$. Converting this equation to the fixed-point problem, we aim to find the fixed point of the mapping

$$F\colon\ C(D,\mathbb{R}) \to C(D,\mathbb{R})$$

defined by

$$Fz = \text{the right side of the equation (3.1).}$$

We now show that F is a contraction with respect to the Bielecki norm given by

$$\|z\|_B = \max_{(x,t)\in D} |z(x,t)|\mathrm{e}^{-\theta(x+t)} \quad \text{where } \theta > 0. \tag{3.2}$$

For any $z, \vartheta \in C(D,\mathbb{R})$, we have

$$\begin{aligned} |Fz - F\vartheta| &\leqslant \int_0^x \int_0^t |f(\tau,r,z(g(\tau,r),h(\tau,r))) - f(\tau,r,\vartheta(g(\tau,r),h(\tau,r)))|\,\mathrm{d}r\,\mathrm{d}\tau \\ &\leqslant \mathcal{L}\int_0^x \int_0^t \mathrm{e}^{\theta(\tau+r)}(|z(g(\tau,r),h(\tau,r)) - \vartheta(g(\tau,r),h(\tau,r))|\mathrm{e}^{-\theta(\tau+r)})\,\mathrm{d}r\,\mathrm{d}\tau \\ &\leqslant \mathcal{L}\|z-\vartheta\|_B \int_0^x \int_0^t \mathrm{e}^{\theta(\tau+r)}\,\mathrm{d}r\,\mathrm{d}\tau \leqslant \frac{\mathcal{L}}{\theta^2}\|z-\vartheta\|_B \mathrm{e}^{\theta(x+t)}, \end{aligned}$$

which implies that

$$\|Fz - F\vartheta\|_B \leqslant \frac{\mathcal{L}}{\theta^2}\|z-\vartheta\|_B.$$

By choosing $\theta > 0$ sufficiently large such that $\theta^2 > \mathcal{L}$, we obtain that F is a contraction mapping. Thus we ensure the existence of a unique z in $C(D,\mathbb{R})$ such that $z = Fz$ according to the Banach contraction principle. In other words, the equation (1.1)–(1.2) has a unique solution z in $C^{1,2}(D,\mathbb{R})$. $\square$

Now, we will show that the result proved above holds for an unbounded domain. That is, Theorem 3.1 can be also proved if D is replaced by $\mathbb{R}^2_+ = [0,\infty)\times[0,\infty)$ as stated in the following theorem.

Theorem 3.2. *Assume that the following conditions hold:*

(C3) $f \in C(\mathbb{R}^2_+ \times \mathbb{R}, \mathbb{R})$, $g \in C(\mathbb{R}_+,\mathbb{R}_+)$, $h \in C(\mathbb{R}_+,\mathbb{R}_+)$ *with* $g(x,t) \leqslant x$ *and* $h(x,t) \leqslant t$.

(C4) *There exists* $\mathbb{L} \in C(\mathbb{R}^2_+,\mathbb{R}_+)$ *such that*

$$|f(x,t,z) - f(x,t,\tilde{z})| \leqslant \mathbb{L}(x,t)|z-\tilde{z}|$$

for all $z,\tilde{z} \in \mathbb{R}$ *and* $(x,t) \in \mathbb{R}^2_+$.

Then the equation (1.1)–(1.2) *has a unique solution.*

Proof. According to Theorem 3.1, for any $n \in \mathbb{N}$, there exists a unique continuous mapping $z_n\colon D_n \to \mathbb{R}$ such that

$$z_n(x,t) = \mu(x,t) + \int_0^x \int_0^t f(\tau, r, z_n(g(\tau,r), h(\tau,r)))\,\mathrm{d}r\,\mathrm{d}\tau, \tag{3.3}$$

where $D_n = [0,n] \times [0,n]$, since the continuous function $\mathbb{L}$ is bounded on this compact domain. If $(x,t) \in D_n$, the following equality can be easily seen from the uniqueness of z_n,

$$z_n(x,t) = z_{n+i}(x,t) \quad \text{for every } i = 1,2,3,\ldots \tag{3.4}$$

For any $(x,t) \in \mathbb{R}_+^2$, let us define $n(x,t) \in \mathbb{N}$ as

$$n(x,t) = \min\{n \in \mathbb{N};\ (x,t) \in D_n\}.$$

We also define a mapping $z\colon \mathbb{R}_+^2 \to \mathbb{R}$ by

$$z(x,t) = z_{n(x,t)}(x,t). \tag{3.5}$$

To show the continuity of z described above, we choose $n_1 = n(x_1,t_1)$ for an arbitrary $(x_1,t_1) \in \mathbb{R}_+^2$. Then (x_1,t_1) belongs to the interior of D_{n_1+1}. Thus, there exists $\varepsilon > 0$ such that $z(x,t) = z_{n_1+1}(x,t)$ for all $(x,t) \in B_\varepsilon(x_1,t_1)$. Since z_{n_1+1} is continuous at (x_1,t_1), the mapping z is also continuous at this arbitrary point.

Now we show that the mapping z satisfies the equation (3.1). For any $(x,t) \in \mathbb{R}_+^2$, there is an integer $n(x,t)$ such that $(x,t) \in D_{n(x,t)}$. It follows from (3.3) and (3.5) that

$$\begin{aligned} z(x,t) &= z_{n(x,t)}(x,t) \\ &= \mu(x,t) + \int_0^x \int_0^t f(\tau, r, z_{n(x,t)}(g(\tau,r), h(\tau,r)))\,\mathrm{d}r\,\mathrm{d}\tau \\ &= \mu(x,t) + \int_0^x \int_0^t f(\tau, r, z(g(\tau,r), h(\tau,r)))\,\mathrm{d}r\,\mathrm{d}\tau, \end{aligned}$$

where the last equality is obtained, since $n(g(\tau,r), h(\tau,r)) \leqslant n(x,t)$ for any $(\tau,r) \in D_{n(x,t)}$ implies

$$z_{n(x,t)}(g(\tau,r), h(\tau,r)) = z_{n(g(\tau,r),h(\tau,r))}(g(\tau,r), h(\tau,r)) = z(g(\tau,r), h(\tau,r))$$

by using (3.4) and (3.5). To prove the uniqueness, we suppose that ϑ is a continuous mapping which also satisfies (3.1). For an arbitrary $(x,t) \in \mathbb{R}_+^2$, since the restrictions $z|_{D_{n(x,t)}}$ and $\vartheta|_{D_{n(x,t)}}$ both satisfy (3.1) for all $(x,t) \in D_{n(x,t)}$, the uniqueness of $z_{n(x,t)} = z|_{D_{n(x,t)}}$ implies that

$$z(x,t) = z|_{D_{n(x,t)}}(x,t) = \vartheta|_{D_{n(x,t)}}(x,t) = \vartheta(x,t).$$

This completes the proof. $\square$

4. Stability

Definition 4.1. If for $\vartheta \in C^{1,2}(D, \mathbb{R})$ satisfying the inequality

$$|\vartheta_{xt}(x,t) - f(x,t,\vartheta(g(x,t),h(x,t)))| \leqslant k(x,t) \tag{4.1}$$

there exists a solution $z \in C^{1,2}(D, \mathbb{R})$ of the equation (1.1) and a positive number $\mathcal{C}$ with

$$|\vartheta(x,t) - z(x,t)| \leqslant \mathcal{C}k(x,t)$$

for all $(x,t) \in D := [0,a] \times [0,b]$, then we say that the equation (1.1) is UHR stable with respect to $k \in C(D, \mathbb{R}^+)$.

Especially if Definition 4.1 is provided for any positive constant instead of k in the above inequalities, we say that the equation (1.1) is UH stable.

Remark 4.2. A function $\vartheta \in C^{1,2}(D, \mathbb{R})$ satisfies the inequality (4.1) if and only if there is a function $p \in C(D, \mathbb{R})$ such that

$$|p(x,t)| \leqslant k(x,t) \quad \text{and} \quad \vartheta_{xt}(x,t) = f(x,t,\vartheta(g(x,t),h(x,t))) + p(x,t).$$

Remark 4.3. If $\vartheta \in C^{1,2}(D, \mathbb{R})$ satisfies the inequality (4.1), then ϑ satisfies the integral inequality

$$\begin{aligned}\Bigl|\vartheta(x,t) - \vartheta(x,0) - \vartheta(0,t) + \vartheta(0,0) - \int_0^x \int_0^t f(\tau, r, \vartheta(g(\tau,r), h(\tau,r)))\,\mathrm{d}r\,\mathrm{d}\tau\Bigr| \\ \leqslant \int_0^x \int_0^t k(\tau,r)\,\mathrm{d}r\,\mathrm{d}\tau, \quad (x,t) \in D.\end{aligned}$$

Note that if we replace D by $\mathbb{R}_+^2$, analogously we have the aforementioned definitions and remarks.

4.1. Stability results on bounded domain. In the following subsection, we give two stability results for the equation (1.1) on a bounded domain. First, we prove a UH stability result in Theorem 4.4 by using the effectiveness of the Bielecki norm. In Theorem 4.5, we prove a UHR stability result inspired by Otrocol and Ilea's paper [18].

Theorem 4.4. *The equation* (1.1) *is UH stable under the conditions* (C1) *and* (C2).

P r o o f. Let $\varepsilon > 0$ and $\vartheta \in C^{1,2}(D, \mathbb{R})$ satisfy the inequality

$$|\vartheta_{xt}(x,t) - f(x,t,\vartheta(g(x,t),h(x,t)))| \leqslant \varepsilon.$$

If we denote the unique solution of the problem

$$\begin{cases} z_{xt}(x,t) = f(x,t,z(g(x,t),h(x,t))), & (x,t) \in D, \\ z(x,0) = \vartheta(x,0), & x \in [0,a], \\ z(0,t) = \vartheta(0,t), & t \in [0,b], \end{cases} \tag{4.2}$$

by z as in Theorem 3.1, then we have

$$z(x,t) = \vartheta(x,0) + \vartheta(0,t) - \vartheta(0,0) + \int_0^x \int_0^t f(\tau, r, z(g(\tau,r),h(\tau,r)))\,\mathrm{d}r\,\mathrm{d}\tau.$$

From Remark 4.3, we have

$$\begin{aligned} &|\vartheta(x,t) - z(x,t)| \\ &\quad \leqslant \left| \vartheta(x,t) - \vartheta(x,0) - \vartheta(0,t) + \vartheta(0,0) - \int_0^x \int_0^t f(\tau, r, \vartheta(g(\tau,r),h(\tau,r)))\,\mathrm{d}r\,\mathrm{d}\tau \right| \\ &\qquad + \int_0^x \int_0^t |f(\tau,r,\vartheta(g(\tau,r),h(\tau,r))) - f(\tau,r,z(g(\tau,r),h(\tau,r)))|\,\mathrm{d}r\,\mathrm{d}\tau \\ &\quad \leqslant \varepsilon xt + \mathcal{L} \int_0^x \int_0^t |\vartheta(g(\tau,r),h(\tau,r)) - z(g(\tau,r),h(\tau,r))|\,\mathrm{d}r\,\mathrm{d}\tau \\ &\quad \leqslant \varepsilon ab + \mathcal{L}\|\vartheta - z\|_B \int_0^x \int_0^t \mathrm{e}^{\theta(\tau + r)}\,\mathrm{d}r\,\mathrm{d}\tau \\ &\quad \leqslant \varepsilon ab + \frac{\mathcal{L}}{\theta^2}\|\vartheta - z\|_B \mathrm{e}^{\theta(x+t)}, \end{aligned}$$

which implies that

$$\Big(1 - \frac{\mathcal{L}}{\theta^2}\Big)\|\vartheta - z\|_B \leqslant \varepsilon ab.$$

By taking $\theta > 0$ sufficiently large so that $\theta^2 > \mathcal{L}$, we get

$$|\vartheta(x,t) - z(x,t)|\mathrm{e}^{-\theta(x+t)} \leqslant \|\vartheta - z\|_B \leqslant \frac{\varepsilon ab}{1 - \mathcal{L}/\theta^2}.$$

Consequently, we obtain that

$$|\vartheta(x,t) - z(x,t)| \leqslant \mathcal{C}\varepsilon, \quad \mathcal{C} := \frac{ab\mathrm{e}^{\theta(a+b)}}{1 - \mathcal{L}/\theta^2}.$$

for all $(x,t) \in D$. Thus the equation (1.1) is UH stable. □

Theorem 4.5. *Let the conditions* (C1)–(C2) *of Theorem* 3.1 *and the condition:*

(C5) *There exists* $\lambda > 0$ *such that*

$$\int_0^x \int_0^t k(\tau, r)\,\mathrm{d}r\,\mathrm{d}\tau \leqslant \lambda k(x,t), \quad (x,t) \in D,$$

hold. Then the equation (1.1) *is UHR stable with respect to* k.

Proof. Let $\vartheta \in C^{1,2}(D,\mathbb{R})$ satisfy the inequality (4.1) and let z be a solution to the problem (4.2). From Remark 4.3, we have

$$\begin{aligned}
&|\vartheta(x,t) - z(x,t)| \\
&\quad \leqslant \left|\vartheta(x,t) - \vartheta(x,0) - \vartheta(0,t) + \vartheta(0,0) - \int_0^x \int_0^t f(\tau, r, \vartheta(g(\tau,r), h(\tau,r)))\,\mathrm{d}r\,\mathrm{d}\tau\right| \\
&\qquad + \int_0^x \int_0^t |f(\tau, r, \vartheta(g(\tau,r), h(\tau,r))) - f(\tau, r, z(g(\tau,r), h(\tau,r)))|\,\mathrm{d}r\,\mathrm{d}\tau \\
&\quad \leqslant \int_0^x \int_0^t k(\tau,r)\,\mathrm{d}r\,\mathrm{d}\tau + \mathcal{L}\int_0^x \int_0^t |\vartheta(g(\tau,r), h(\tau,r)) - z(g(\tau,r), h(\tau,r))|\,\mathrm{d}r\,\mathrm{d}\tau.
\end{aligned}$$

For $\omega \in C(D,\mathbb{R}^+)$, we define $\mathcal{P}\colon C(D,\mathbb{R}^+) \to C(D,\mathbb{R}^+)$ as

$$\mathcal{P}(\omega)(x,t) = \int_0^x \int_0^t k(\tau,r)\,\mathrm{d}r\,\mathrm{d}\tau + \mathcal{L}\int_0^x \int_0^t \omega(g(\tau,r), h(\tau,r))\,\mathrm{d}r\,\mathrm{d}\tau.$$

Considering the Bielecki norm defined in (3.2), the inequality

$$\|\mathcal{P}\omega - \mathcal{P}\widetilde{\omega}\|_B \leqslant \delta\|\omega - \widetilde{\omega}\|_B \quad \text{where } \delta = \frac{\mathcal{L}}{\theta^2}$$

can be obtained as in Theorem 3.1 for any $\omega, \widetilde{\omega}$ in $C(D,\mathbb{R}^+)$. Taking $\theta > 0$ sufficiently large such that $\delta < 1$, we get that $\mathcal{P}$ is a Picard operator ($F_{\mathcal{P}} = \{\omega^*\}$). Then the following equality holds by the Banach contraction principle

$$\omega^*(x,t) = \int_0^x \int_0^t k(\tau,r)\,\mathrm{d}r\,\mathrm{d}\tau + \mathcal{L}\int_0^x \int_0^t \omega^*(g(\tau,r), h(\tau,r))\,\mathrm{d}r\,\mathrm{d}\tau.$$

Since ω^* is increasing, one gets $\omega^*(g(\tau,r), h(\tau,r)) \leqslant \omega^*(\tau,r)$ due to $g(\tau,r) \leqslant \tau$ and $h(\tau,r) \leqslant r$, which implies

$$\omega^*(x,t) \leqslant \int_0^x \int_0^t k(\tau,r)\,\mathrm{d}r\,\mathrm{d}\tau + \mathcal{L}\int_0^x \int_0^t \omega^*(\tau,r)\,\mathrm{d}r\,\mathrm{d}\tau.$$

Applying Wendorff lemma to the above inequality, we obtain from the condition (C5) that

$$\omega^*(x,t) \leqslant \mathrm{e}^{xt\mathcal{L}} \int_0^x \int_0^t k(\tau,r)\,\mathrm{d}r\,\mathrm{d}\tau \leqslant \lambda \mathrm{e}^{xt\mathcal{L}} k(x,t)$$

for all $(x,t) \in D$. Specifically, we have $\omega \leqslant \mathcal{P}\omega$ choosing $\omega = |\vartheta - z|$. Since $\mathcal{P}$ is an increasing Picard operator, the inequality $\omega \leqslant \omega^*$ is obtained by the abstract Gronwall lemma. As a result, we get

$$|\vartheta(x,t) - z(x,t)| \leqslant \mathcal{C}k(x,t), \quad \mathcal{C} := \lambda \mathrm{e}^{ab\mathcal{L}}.$$

Hence, the equation (1.1) is UHR stable. □

4.2. Stability result on unbounded domain.

Theorem 4.6. *Suppose that* (C3)–(C4) *and the following condition holds:*

(C6) *There exists* $\lambda > 0$ *such that*

$$\int_0^x \int_0^t k(\tau,r)\,\mathrm{d}r\,\mathrm{d}\tau \leqslant \lambda k(x,t), \quad (x,t) \in \mathbb{R}_+^2.$$

If the double integration of $\mathbb{L}$ *in* (C4) *is finite, the equation* (1.1) *is UHR stable with respect to* k.

Proof. Let $\vartheta \in C^{1,2}(\mathbb{R}_+^2, \mathbb{R})$ satisfy the inequality (4.1). One can obtain the unique solution (denoted by z) of the following equation from Theorem 3.2,

$$\begin{cases} z_{xt}(x,t) = f(x,t,z(g(x,t),h(x,t))), & \\ z(x,0) = \vartheta(x,0), & (x,t) \in \mathbb{R}_+^2. \\ z(0,t) = \vartheta(0,t), & \end{cases}$$

For arbitrary $(x,t) \in \mathbb{R}_+^2$, there exists an $n \in \mathbb{N}$ such that $(x,t) \in D_n$. By considering the restrictions on the domain D_n of functions ϑ and z, we have

$$|\vartheta(x,t) - z(x,t)| = |\vartheta|_{D_n}(x,t) - z|_{D_n}(x,t)|$$

(in view of Theorem 4.5)

$$\begin{aligned} &\leqslant \lambda k(x,t) \exp\left(\int_0^x \int_0^t \mathbb{L}(\tau,r)\,\mathrm{d}r\,\mathrm{d}\tau\right) \\ &= \mathcal{C}k(x,t), \end{aligned}$$

where $\mathcal{C} = \lambda \exp\big(\int_0^\infty \int_0^\infty \mathbb{L}(\tau,r)\,\mathrm{d}r\,\mathrm{d}\tau\big)$. Hence, the equation (1.1)–(1.2) is UHR stable. □

5. Examples

We now give some examples to support our theoretical results which are proved in previous sections.

Example 5.1. Consider the following partial differential equation

$$z_{xt}(x,t)=\frac{\sin xt}{1+|z(xt,t^2)|},\quad (x,t)\in[0,6]\times[0,1], \tag{5.1}$$

with the initial and boundary conditions

$$\begin{cases} z(x,0)=\mathrm{e}^x, & x\in[0,6],\\ z(0,t)=\cos t, & t\in[0,1]. \end{cases} \tag{5.2}$$

In Theorem 1.1, we set

$$f(x,t,z(g(x,t),h(x,t)))=\frac{\sin xt}{1+|z(g(x,t),h(x,t))|},\quad (x,t)\in[0,6]\times[0,1],$$

where $g(x,t)=xt$ and $h(x,t)=t^2$.

For every $z,\tilde z\in\mathbb{R}$ and $(x,t)\in[0,6]\times[0,1]$, it is obvious that

$$|f(x,t,z)-f(x,t,\tilde z)|\leqslant|z-\tilde z|.$$

Then, by Theorem 1.1, the equation (5.1)–(5.2) has a unique solution on $[0,6]\times[0,1]$. Moreover, applying Theorem 4.4, we obtain that the equation (5.1) is stable in the sense of UH. We take $k(x,t)=xt$ in the condition (C5) of Theorem 4.5,

$$\int_0^x\int_0^t k(\tau,r)\,\mathrm{d}r\,\mathrm{d}\tau\leqslant\frac{3}{2}xt:=\lambda k(x,t).$$

So the condition (C5) is satisfied with $\lambda=\frac{3}{2}$. Hence, this equation is also UHR stable with respect to k.

Example 5.2. Consider the following partial differential equation

$$z_{xt}(x,t)=\frac{x\cos t+z(x,t\mathrm{e}^{-x})}{\mathrm{e}^{x+t}},\quad (x,t)\in\mathbb{R}_+^2, \tag{5.3}$$

with the initial and boundary conditions

$$\begin{cases} z(x,0)=x^2, & x\in\mathbb{R}_+,\\ z(0,t)=t, & t\in\mathbb{R}_+. \end{cases} \tag{5.4}$$

In Theorem 3.2, we take

$$f(x,t,z(g(x,t),h(x,t))) = \frac{x\cos t + z(g(x,t),h(x,t))}{\mathrm{e}^{x+t}}, \quad (x,t)\in\mathbb{R}_+^2,$$

with $g(x,t)=x$ and $h(x,t)=t\mathrm{e}^{-x}$.

For every $z,\tilde z\in\mathbb{R}$ and $(x,t)\in\mathbb{R}_+^2$, we have

$$|f(x,t,z)-f(x,t,\tilde z)| \leqslant \mathbb{L}(x,t)|z-\tilde z| \quad \text{where } \mathbb{L}(x,t)=\mathrm{e}^{-(x+t)}.$$

Then, it follows from Theorem 3.2 that the equation (5.3)–(5.4) has a unique solution on $\mathbb{R}_+^2$. Also if we denote $k(x,t)=\mathrm{e}^{x+t}$ in the condition (C6) of Theorem 4.6, we have

$$\int_0^x\int_0^t k(\tau,r)\,\mathrm{d}r\,\mathrm{d}\tau \leqslant \mathrm{e}^{x+t} = \lambda k(x,t).$$

Therefore, the condition (C6) is satisfied with $\lambda=1$. Hence, the equation (5.3) is UHR stable with respect to k.

In Examples 5.1 and 5.2, we have illustrated the existence, uniqueness and stability results on both bounded and unbounded domains.

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