

Yuzheng Ma; Yan Ling Shao

Some properties of generalized distance eigenvalues of graphs

*Czechoslovak Mathematical Journal*, Vol. 74 (2024), No. 1, 1–15

Persistent URL: <http://dml.cz/dmlcz/152263>

## Terms of use:

© Institute of Mathematics AS CR, 2024

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

SOME PROPERTIES OF GENERALIZED DISTANCE  
EIGENVALUES OF GRAPHS

YUZHENG MA, YANLING SHAO, Taiyuan

Received April 12, 2021. Published online November 28, 2023.

*Abstract.* Let  $G$  be a simple connected graph with vertex set  $V(G) = \{v_1, v_2, \dots, v_n\}$  and edge set  $E(G)$ , and let  $d_{v_i}$  be the degree of the vertex  $v_i$ . Let  $D(G)$  be the distance matrix and let  $T_r(G)$  be the diagonal matrix of the vertex transmissions of  $G$ . The generalized distance matrix of  $G$  is defined as  $D_\alpha(G) = \alpha T_r(G) + (1 - \alpha)D(G)$ , where  $0 \leq \alpha \leq 1$ . Let  $\lambda_1(D_\alpha(G)) \geq \lambda_2(D_\alpha(G)) \geq \dots \geq \lambda_n(D_\alpha(G))$  be the generalized distance eigenvalues of  $G$ , and let  $k$  be an integer with  $1 \leq k \leq n$ . We denote by  $S_k(D_\alpha(G)) = \lambda_1(D_\alpha(G)) + \lambda_2(D_\alpha(G)) + \dots + \lambda_k(D_\alpha(G))$  the sum of the  $k$  largest generalized distance eigenvalues. The generalized distance spread of a graph  $G$  is defined as  $D_\alpha S(G) = \lambda_1(D_\alpha(G)) - \lambda_n(D_\alpha(G))$ . We obtain some bounds on  $S_k(D_\alpha(G))$  and  $D_\alpha S(G)$  of graph  $G$ , respectively.

*Keywords:* graph; generalized distance matrix; generalized distance eigenvalue; generalized distance spread

*MSC 2020:* 05C50, 05C12, 15A18

## 1. INTRODUCTION

In this paper, all graphs considered are finite, simple and connected. Let  $G$  be such a graph with vertex set  $V(G) = \{v_1, v_2, \dots, v_n\}$  and edge set  $E(G)$ , where  $|V(G)| = n$  and  $|E(G)| = m$ . Let  $d_{v_i}$  denote the degree of vertex  $v_i$ ,  $N(v_i)$  denote the neighbor set of  $v_i$ , and let  $\Delta(G)$  be the maximum degree of all vertices in  $G$ . For  $S \subseteq V(G)$ , we use  $G[S]$  to denote the subgraph of  $G$  induced by  $S$ . The distance between vertices  $v_i$  and  $v_j$  in  $G$  is the length of a shortest path connecting  $v_i$  to  $v_j$ , which is denoted as  $d(v_i, v_j)$ . The diameter of  $G$ , denoted by  $\text{diam}(G)$ , is the maximum distance between any two vertices of  $G$ . The distance matrix (see [5]) of a graph  $G$  is an  $n \times n$  matrix  $D(G) = (d_{ij})$ , where  $d_{ij} = d(v_i, v_j)$  for  $i, j = 1, 2, \dots, n$ .

---

Research supported by Shanxi Scholarship Council of China (No.2022-149).

The transmission of vertex  $v_i$ , denoted by  $\text{Tr}_G(v_i)$  or  $\text{Tr}_i$ , is defined to be the sum of the distances from  $v_i$  to all vertices in  $G$ , that is,  $\text{Tr}_G(v_i) = \text{Tr}_i = \sum_{u \in V(G)} d(u, v_i)$ .

A graph  $G$  is said to be  $k$ -transmission regular if  $\text{Tr}_G(v) = k$  for each  $v \in V(G)$ . The transmission of a vertex  $v$  is also called the *distance degree* or the *first distance degree* of  $v$ . The sequence  $\{\text{Tr}_1, \text{Tr}_2, \dots, \text{Tr}_n\}$  is called the *transmission degree sequence* of  $G$ . The second transmission of vertex  $v_i$ , denoted by  $T_i$ , is defined to be  $T_i = \sum_{j=1}^n d_{ij} \text{Tr}_j$ , and  $\{T_1, T_2, \dots, T_n\}$  is called the *second transmission degree sequence* of  $G$ .

Let  $\text{Tr}(G) = \text{diag}(\text{Tr}_1, \text{Tr}_2, \dots, \text{Tr}_n)$  be the diagonal matrix of vertex transmissions of  $G$ . The adjacency matrix  $A(G)$  of  $G$  is a square matrix of order  $n$  whose  $(i, j)$ -entry is equal to unity if the vertices  $v_i$  and  $v_j$  are adjacent, and is equal to zero otherwise. Let  $\text{Deg}(G) = \text{diag}(d_1, d_2, \dots, d_n)$ , where  $d_i = d_{v_i}$  for  $i = 1, 2, \dots, n$ . The matrices  $L(G) = \text{Deg}(G) - A(G)$  and  $Q(G) = \text{Deg}(G) + A(G)$  are called the *Laplacian matrix* and the *signless Laplacian matrix* of  $G$ , respectively. Aouchiche and Hansen in [2], [3] introduced the distance Laplacian matrix and the distance signless Laplacian matrix of a graph. The matrix  $D_L(G) = \text{Tr}(G) - D(G)$  is called the *distance Laplacian matrix* of  $G$ , while the matrix  $D_Q(G) = \text{Tr}(G) + D(G)$  is called the *distance signless Laplacian matrix* of  $G$ .

In 2019, Cui et al. in [6] proposed the generalized distance matrix  $D_\alpha(G) = \alpha \text{Tr}(G) + (1 - \alpha)D(G)$  by using the convex linear combination of  $\text{Tr}(G)$  and  $D(G)$ , where  $0 \leq \alpha \leq 1$ . As you can see,  $D_0(G) = D(G)$ ,  $2D_{1/2}(G) = D_Q(G)$ ,  $D_1(G) = \text{Tr}(G)$ , and  $D_\alpha(G) - D_\beta(G) = (\alpha - \beta)D_L(G)$ . Since the matrix  $D_\alpha(G)$  is real and symmetric, all its eigenvalues are real, denoted by  $\lambda_1(D_\alpha(G)) \geq \lambda_2(D_\alpha(G)) \geq \dots \geq \lambda_n(D_\alpha(G))$ . The generalized distance spectral radius of  $G$  is defined as  $\varrho(D_\alpha(G)) = \lambda_1(D_\alpha(G))$ . The Wiener index (see [18]) of  $G$  is defined as

$$W(G) = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n d_{ij} = \frac{1}{2} \sum_{v_i, v_j \in V(G)} d_{ij} = \frac{1}{2} \sum_{v_i \in V(G)} \text{Tr}_G(v_i).$$

In [4], Brouwer conjectured that the sum of the  $k$  largest Laplacian eigenvalues of  $G$  is at most

$$S_k(G) \leq e(G) + \binom{k+1}{2}$$

for  $k = 1, 2, \dots, n$ , where  $e(G)$  is the number of edges of  $G$ . Haemers et al. in [8] proved Brouwer's conjecture for  $k = 2$ , and they also showed that if  $G$  is a tree, then the sum of the  $k$  largest Laplacian eigenvalues of  $G$  is at most  $e(G) + 2k - 1$ . Let  $G$  be a connected graph of order  $n$ , and let  $\varrho_1(D(G)) \geq \varrho_2(D(G)) \geq \dots \geq \varrho_n(D(G))$  be the distance eigenvalues of  $G$ . Lin in [11] put forward a concept of the sum of the  $k$  largest

distance eigenvalues of  $G$  as  $S_k(D(G)) = \varrho_1(D(G)) + \varrho_2(D(G)) + \dots + \varrho_k(D(G))$ . For an integer  $1 \leq k \leq n$ , the sum of the  $k$  largest generalized distance eigenvalues of  $D_\alpha(G)$  is defined, that is,

$$S_k(D_\alpha(G)) = \lambda_1(D_\alpha(G)) + \lambda_2(D_\alpha(G)) + \dots + \lambda_k(D_\alpha(G)).$$

Let  $M$  be a symmetric matrix of order  $n$  having eigenvalues  $w_i$ ,  $i = 1, 2, \dots, n$ . If  $w_1$  and  $w_n$  are, respectively, the largest and the smallest eigenvalues of  $M$ , then the spread of  $M$  is defined as  $S(M) = w_1 - w_n$ . From [9], [13], we know that the spread of a matrix is a very interesting topic. The distance spread of  $G$  was considered in [20] and is defined as  $S_D(G) = \varrho_1(D(G)) - \varrho_n(D(G))$ . Various lower and upper bounds for the parameter  $S_D(G)$  can be found in [20]. Motivated by the results obtained for the distance spread, You et al. in [19] put forward the concept of the distance signless Laplacian spread of  $G$  as  $S_{DQ}(G) = q_1(DQ(G)) - q_n(DQ(G))$ , where  $q_1(DQ(G))$  and  $q_n(DQ(G))$  are the largest and smallest distance signless Laplacian eigenvalues of  $G$ , respectively, and obtained various lower and upper bounds for the parameter  $S_{DQ}(G)$  in terms of different graph parameters.

The paper (see [16]) defines the generalized distance spread of a graph  $G$  as the difference between the largest and smallest eigenvalues of  $D_\alpha(G)$ , that is,

$$D_\alpha S(G) = \lambda_1(D_\alpha(G)) - \lambda_n(D_\alpha(G)),$$

and obtains some bounds for the generalized distance spread  $D_\alpha S(G)$  and also obtains the relation between the generalized distance spread  $D_\alpha S(G)$  and the distance spread  $S_D(G)$ . Furthermore, they obtained the lower bounds for  $D_\alpha S(G)$  of bipartite graphs involving different graph parameters and obtained the lower bounds for  $D_\alpha S(G)$  in terms of clique number and independence number of  $G$  and characterized the extremal graphs for some cases. However, the generalized distance Laplacian spread seems less well-known because it was introduced somewhat later, see [16]. Up to now, there are only very limited results on the Laplacian spread. We think the results are only of theoretical interest.

Motivated by these, we study the sum of the  $k$  largest generalized distance eigenvalues  $S_k(D_\alpha(G))$  and the generalized distance spread  $D_\alpha S(G)$  of a graph  $G$  of order  $n$ , where  $1 \leq k \leq n$  and  $0 \leq \alpha \leq 1$ . This paper is organized as follows. In Section 2, several lemmas are given to prove the following theorems. In Section 3, we determine some bounds on  $S_k(D_\alpha(G))$  of  $G$ . In Section 4, we obtain some bounds on generalized distance spread  $D_\alpha S(G)$  of  $G$ .

## 2. LEMMAS

In this section, we state some previously known results that are needed in the next two sections.

**Lemma 2.1** ([15], Cauchy alternating theorem). *Let  $A$  be a real symmetric matrix of order  $n$  and  $B$  be a principal submatrix of order  $m$  of  $A$ . Suppose  $A$  has eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ , and let  $B$  have eigenvalues  $\beta_1 \geq \beta_2 \geq \dots \geq \beta_m$ . Then for all  $k = 1, 2, \dots, m$ ,  $\lambda_{n-m+k} \leq \beta_k \leq \lambda_k$ .*

**Lemma 2.2** ([1], Ramsey theorem). *The Ramsey function  $R(k, l)$  is defined as the minimal integer  $n$  such that any graph on  $n$  vertices contains either a clique of size  $k$  or an independent set of size  $l$ .*

**Lemma 2.3** ([7]). *Let  $G$  be a tree of order  $n \geq 4$ . Then*

$$\varrho(D_\alpha(G)) \geq \frac{\alpha n + 2(n-2) + \sqrt{n^2(2-\alpha)^2 + 4(n-1)(2\alpha-3)}}{2}$$

*with equality holding if and only if  $T \cong S_n$ , where  $S_n$  is a star of order  $n$ .*

**Lemma 2.4** ([7]). *Let  $G$  be a graph with  $n$  vertices and  $m$  edges. Then*

$$\varrho(D_\alpha(G)) \geq \frac{2W(G)}{n}$$

*with equality holding if and only if  $G$  is a transmission regular graph.*

**Lemma 2.5** ([11]). *Let  $G$  be a graph of order  $n$ . If  $\Delta(G) \leq l$  and  $\text{diam}(G) \leq d$ , then*

$$n \leq 1 + l + l(l-1) + l(l-1)^2 + \dots + l(l-1)^{d-1}.$$

**Lemma 2.6** ([11]). *Let  $G$  be a graph with  $n$  vertices and  $m$  edges. Then*

$$W(G) \geq n(n-1) - m.$$

**Lemma 2.7** (Turán theorem). *Let  $G$  be a graph with  $n$  vertices and  $m$  edges without  $k$ -vertex clique. Then*

$$m \leq \frac{(k-2)n^2}{2(k-1)} = \left(1 - \frac{1}{k-1}\right) \frac{n^2}{2}.$$

**Lemma 2.8** ([17]). Let  $G$  be a graph of order  $n$  having transmission degrees  $\text{Tr}_1, \text{Tr}_2, \dots, \text{Tr}_n$  and Wiener index  $W(G)$ . Let  $\lambda_1, \lambda_2, \dots, \lambda_n$  be the eigenvalues of the generalized distance matrix  $D_\alpha(G)$  of graph  $G$ . Then

- (1)  $\sum_{i=1}^n \lambda_i = 2\alpha W(G)$ ;
- (2)  $\sum_{i=1}^n \lambda_i^2 = \alpha^2 \sum_{i=1}^n \text{Tr}_i^2 + (1 - \alpha)^2 \sum_{i=1}^n \sum_{j=1}^n d_{ij}$ .

**Lemma 2.9** ([12]). If  $M$  is an  $n \times n$  nonzero Hermitian matrix, then

$$S(M) \geq \left| \frac{\text{tr}(M^3)}{\text{tr}(M^2)} - \frac{\text{tr}(M)}{n} \right|.$$

**Lemma 2.10** ([14]). Let  $t_i \geq 0$ ,  $T = \sum_{i=1}^n t_i$ ,  $a \leq a_i \leq A$  and  $b \leq b_i \leq B$  for  $i = 1, 2, \dots, n$ . Then

$$\left| \frac{1}{T} \sum_{i=1}^n t_i a_i b_i - \frac{1}{T^2} \sum_{i=1}^n t_i a_i \sum_{i=1}^n t_i b_i \right| \leq \frac{1}{4} (A - a)(B - b).$$

**Lemma 2.11** ([10]). Let  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  be real numbers such that  $a \leq a_i \leq A$  and  $b \leq b_i \leq B$  for  $i = 1, 2, \dots, n$ . Then

$$\left| \frac{1}{n} \sum_{i=1}^n a_i b_i - \frac{1}{n} \sum_{i=1}^n a_i \frac{1}{n} \sum_{i=1}^n b_i \right| \leq \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \left( 1 - \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \right) (A - a)(B - b),$$

where  $\lfloor x \rfloor$  denotes the integer part of  $x$ .

**Lemma 2.12.** Let  $G$  be a graph of order  $n$ , and  $2 \leq k \leq n$ . If  $n$  is sufficiently large with respect to  $k$ , then  $\lambda_k(D_\alpha(G)) \geq k\alpha - 2$ .

*Proof.* We divide the proof into two cases.

*Case 1:*  $\Delta(G) \geq R(k-1, k-1)$ . Let  $v \in V(G)$  with  $d_v = \Delta(G)$ . By Lemma 2.2,  $G[N(v)]$  either contains a clique  $A$  of size  $k-1$  or an independent set  $B$  of size  $k-1$ . If the first case occurs, then  $H = G[A \cup \{v\}] = K_k$ . By Lemma 2.1,  $\lambda_k(D_\alpha(G)) \geq \lambda_k(D_\alpha(H)) = k\alpha - 1$ . If the second case occurs, then  $H = G[B \cup \{v\}] = S_k$ . By a simple calculation, the generalized distance spectrum of  $S_n$  is

$$\left\{ [\alpha(2n-1) - 2]^{(n-2)}, \frac{\alpha n + 2n - 4 \pm \sqrt{s}}{2} \right\},$$

where  $s = n^2(2-\alpha)^2 + 4(n-1)(2\alpha-3)$ . It is easy to verify that  $\frac{1}{2}(\alpha n + 2n - 4 + \sqrt{s})$  is the generalized distance spectral radius of  $S_n$ , and  $\frac{1}{2}(\alpha n + 2n - 4 - \sqrt{s})$  is the least eigenvalue of  $S_n$ .

By Lemma 2.1,

$$\begin{aligned}\lambda_k(D_\alpha(G)) &\geq \lambda_k(D_\alpha(H)) = \frac{k\alpha + 2k - 4 - \sqrt{k^2(2-\alpha)^2 + 4(k-1)(2\alpha-3)}}{2} \\ &\geq k\alpha - 2.\end{aligned}$$

*Case 2:*  $\Delta(G) < R(k-1, k-1)$ . Take  $l = R(k-1, k-1) - 1$  and  $d = k$ . Recall that  $n$  is sufficiently large with respect to  $k$ , so we may let  $n > 1 + l + l(l-1) + l(l-1)^2 + \dots + l(l-1)^{d-1}$ . Then by Lemma 2.5, we have  $\text{diam}(G) \geq d + 1$ . So  $D_\alpha(P_d)$  is a principle submatrix of  $D_\alpha(G)$ . By Lemmas 2.1 and 2.3, we have  $\lambda_k(D_\alpha(G)) \geq \lambda_k(D_\alpha(P_d)) \geq \lambda_k(D_\alpha(K_{1,k-1})) \geq k\alpha - 2$ . This completes the proof.  $\square$

### 3. ON THE SUM OF THE $k$ LARGEST GENERALIZED DISTANCE EIGENVALUES OF A GRAPH

This section mainly introduces some bounds on  $S_k(D_\alpha(G))$ .

**Theorem 3.1.** *Let  $k \geq 2$  be an integer and let  $n$  be sufficiently large with respect to  $k$ . Let  $G$  be a graph of order  $n$ . Then*

$$S_k(D_\alpha(G)) > (k-1) + (k-1)(k\alpha - 1).$$

*Proof.* By a simple calculation, we have that  $\varrho(D_\alpha(K_k)) = k-1$ , and  $\lambda_i(D_\alpha(K_k)) = k\alpha - 1$  for  $i = 2, 3, \dots, k$ . Hence,

$$S_k(D_\alpha(K_k)) = (k-1) + (k-1)(k\alpha - 1).$$

By Lemmas 2.4 and 2.6, we have

$$\varrho(D_\alpha(G)) \geq \frac{2W(G)}{n} = 2(n-1) - \frac{2m}{n}.$$

If  $m \leq \frac{1}{2}n(n-k)$ , then  $\varrho(D_\alpha(G)) \geq n+k-2$ . By Lemma 2.12, if  $n$  is sufficiently large with respect to  $k$ , then  $\lambda_k(D_\alpha(G)) \geq k\alpha - 2$ . So

$$\begin{aligned}S_k(D_\alpha(G)) &= \lambda_1(D_\alpha(G)) + \lambda_2(D_\alpha(G)) + \dots + \lambda_k(D_\alpha(G)) \\ &\geq n+k-2 + (k-1)(k\alpha - 2) = k^2\alpha - k\alpha + n - k \\ &> (k-1) + (k-1)(k\alpha - 1) = S_k(D_\alpha(K_k)).\end{aligned}$$

If  $m > \frac{1}{2}n(n-k)$ , then  $m > (1 - (k-1)^{-1})\frac{1}{2}n^2$  when  $n \geq k^2$  (recall that  $n$  is sufficiently large with respect to  $k$ ). By Lemma 2.7,  $G$  contains  $K_k$ . Thus, we have

$$\lambda_i(D_\alpha(G)) > \lambda_i(D_\alpha(K_k)) = k\alpha - 1 \quad \text{for } i = 1, \dots, k.$$

We obtain  $S_k(D_\alpha(G)) > S_k(D_\alpha(K_k)) = (k-1) + (k-1)(k\alpha - 1)$ . This completes the proof.  $\square$

**Theorem 3.2.** *Let  $G$  be a graph of order  $n$ , and  $1 \leq k \leq n$ . Then*

$$\frac{4\alpha W(G) - \sqrt{t}}{2n} \leq S_k(D_\alpha(G)) \leq \frac{4\alpha W(G) + \sqrt{t}}{2n},$$

where

$$t = 4k(k-n) \left[ \alpha^2 \left( \sum_{i=1}^n \text{Tr}_i \right)^2 - n \left( \alpha^2 \sum_{i=1}^n \text{Tr}_i^2 + (1-\alpha)^2 \sum_{i=1}^n \sum_{j=1}^n d_{ij} \right) \right].$$

*Proof.* Let  $\lambda_1, \lambda_2, \dots, \lambda_n$  be the eigenvalues of the generalized distance matrix  $D_\alpha(G)$  of  $G$ . It is clear that

$$\text{tr}(D_\alpha^2(G)) = \sum_{i=1}^k \lambda_i^2 + \sum_{i=k+1}^n \lambda_i^2 \geq \frac{(\sum_{i=1}^k \lambda_i)^2}{k} + \frac{(\sum_{i=k+1}^n \lambda_i)^2}{n-k}.$$

Let  $S_k(D_\alpha(G)) = \sum_{i=1}^k \lambda_i$  and note that  $\text{tr}(D_\alpha^2(G)) = \text{tr}(D_\alpha^2)$ . Then

$$\text{tr}(D_\alpha^2) \geq \frac{S_k^2(D_\alpha(G))}{k} + \frac{(2\alpha W(G) - S_k(D_\alpha(G)))^2}{n-k},$$

that is,

$$(3.1) \quad nS_k^2(D_\alpha(G)) - 4\alpha W(G)kS_k(D_\alpha(G)) + 4\alpha^2 W^2(G)k - (n-k)k\text{tr}(D_\alpha^2) \leq 0.$$

Letting  $A = n$ ,  $B = -4\alpha W(G)k$  and  $C = 4\alpha^2 W^2(G)k - (n-k)k\text{tr}(D_\alpha^2)$ , by Lemma 2.8, we have

$$\begin{aligned} B^2 - 4AC &= 16\alpha^2 W^2(G)k^2 - 4n[4\alpha^2 W^2(G)k - (n-k)k\text{tr}(D_\alpha^2)] \\ &= 4k(k-n)[4\alpha^2 W^2(G) - n\text{tr}(D_\alpha^2)] = 4k(k-n) \left[ \left( \sum_{i=1}^n \lambda_i \right)^2 - n \sum_{i=1}^n \lambda_i^2 \right] \\ &= 4k(k-n) \left[ \alpha^2 \left( \sum_{i=1}^n \text{Tr}_i \right)^2 - n \left( \alpha^2 \sum_{i=1}^n \text{Tr}_i^2 + (1-\alpha)^2 \sum_{i=1}^n \sum_{j=1}^n d_{ij} \right) \right] \geq 0. \end{aligned}$$

Then we can get

$$\frac{4\alpha W(G) - \sqrt{t}}{2n} \leq S_k(D_\alpha(G)) \leq \frac{4\alpha W(G) + \sqrt{t}}{2n},$$

where

$$t = 4k(k-n) \left[ \alpha^2 \left( \sum_{i=1}^n \text{Tr}_i \right)^2 - n \left( \alpha^2 \sum_{i=1}^n \text{Tr}_i^2 + (1-\alpha)^2 \sum_{i=1}^n \sum_{j=1}^n d_{ij} \right) \right].$$

The theorem holds.  $\square$

#### 4. LOWER BOUNDS ON GENERALIZED DISTANCE SPREAD

In this section, we first provide three lower bounds on  $D_\alpha S(G)$  by using Lemmas 2.9, 2.10 and 2.11. Secondly, we introduce the concept of quotient matrix and obtain a lower bound on  $D_\alpha S(G)$  of a graph with diameter. Finally, we present a lower bound on  $D_\alpha S(G)$  of a cactus with circumference.

**Theorem 4.1.** *Let  $G$  be a graph of order  $n$ , and let the transmission degree sequence of  $G$  be  $\{\text{Tr}_1, \text{Tr}_2, \dots, \text{Tr}_n\}$ . Then*

$$D_\alpha S(G) \geq \left| \frac{a}{b} - \frac{2\alpha W(G)}{n} \right|,$$

where

$$a = \alpha^3 \sum_{i=1}^n \text{Tr}_i^3 + (1-\alpha)^2 \alpha \sum_{i=1}^n \sum_{j=1}^n d_{ij}^2 \text{Tr}_i + (1-\alpha)^3 \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{m=j+1}^n d_{ij} d_{im} d_{jm},$$

$$b = \alpha^2 \sum_{i=1}^n \text{Tr}_i^2 + (1-\alpha)^2 \sum_{i=1}^n \sum_{j=1}^n d_{ij}^2.$$

**Proof.** Denote  $D(G) = D$  and  $\text{Tr}(G) = \text{Tr}$ . Then

$$\begin{aligned} \text{tr}(D_\alpha^3(G)) &= \alpha^3 \text{tr}(\text{Tr}^3) + \alpha^2(1-\alpha) \text{tr}(D\text{Tr}^2) + \alpha^2(1-\alpha) \text{tr}(\text{Tr}D\text{Tr}) \\ &\quad + (1-\alpha)^2 D^2 \text{tr}(D^2\text{Tr}) + \alpha^2(1-\alpha) \text{tr}(\text{Tr}^2D) \\ &\quad + \alpha(1-\alpha) \text{tr}(D\text{Tr}D) + \alpha(1-\alpha) \text{tr}(\text{Tr}D^2) + (1-\alpha)^3 \text{tr}(D^3) \\ &= \alpha^3 \sum_{i=1}^n \text{Tr}_i^3 + (1-\alpha)^2 \alpha \sum_{i=1}^n \sum_{j=1}^n d_{ij}^2 \text{Tr}_i \\ &\quad + (1-\alpha)^3 \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{m=j+1}^n d_{ij} d_{im} d_{jm}. \end{aligned}$$

By Lemma 2.8, we have

$$\begin{aligned}\mathrm{tr}(D_\alpha(G)) &= \sum_{i=1}^n \lambda_i(D_\alpha(G)) = 2\alpha W(G), \\ \mathrm{tr}(D_\alpha^2(G)) &= \sum_{i=1}^n \lambda_i^2(D_\alpha(G)) = \alpha^2 \sum_{i=1}^n \mathrm{Tr}_i^2 + (1-\alpha)^2 \sum_{i=1}^n \sum_{j=1}^n d_{ij}^2.\end{aligned}$$

By Lemma 2.9, the theorem follows.  $\square$

**Theorem 4.2.** *Let  $G$  be a graph of order  $n$ , and let the transmission degree sequence of graph  $G$  be  $\{\mathrm{Tr}_1, \mathrm{Tr}_2, \dots, \mathrm{Tr}_n\}$ . Then*

$$D_\alpha S(G) \geq \sqrt{\left| \frac{2a}{\alpha W(G)} - \frac{b^2}{\alpha^2 W^2(G)} \right|},$$

where

$$\begin{aligned}a &= \alpha^3 \sum_{i=1}^n \mathrm{Tr}_i^3 + (1-\alpha)^2 \alpha \sum_{i=1}^n \sum_{j=1}^n d_{ij}^2 \mathrm{Tr}_i + (1-\alpha)^3 \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{m=j+1}^n d_{ij} d_{im} d_{jm}, \\ b &= \alpha^2 \sum_{i=1}^n \mathrm{Tr}_i^2 + (1-\alpha)^2 \sum_{i=1}^n \sum_{j=1}^n d_{ij}^2.\end{aligned}$$

*Proof.* By the proof of Theorem 4.1, we have

$$\begin{aligned}\sum_{i=1}^n \lambda_i^3(D_\alpha(G)) &= \mathrm{tr}(D_\alpha^3(G)) = \alpha^3 \sum_{i=1}^n \mathrm{Tr}_i^3 + (1-\alpha)^2 \alpha \sum_{i=1}^n \sum_{j=1}^n d_{ij}^2 \mathrm{Tr}_i \\ &\quad + (1-\alpha)^3 \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{m=j+1}^n d_{ij} d_{im} d_{jm},\end{aligned}$$

By Lemma 2.8, we have

$$\begin{aligned}\sum_{i=1}^n \lambda_i^2(D_\alpha(G)) &= \alpha^2 \sum_{i=1}^n \mathrm{Tr}_i^2 + (1-\alpha)^2 \sum_{i=1}^n \sum_{j=1}^n d_{ij}^2, \\ \sum_{i=1}^n \lambda_i(D_\alpha(G)) &= 2\alpha W(G).\end{aligned}$$

Let  $\lambda_1(D_\alpha(G)) \geq \lambda_2(D_\alpha(G)) \geq \dots \geq \lambda_n(D_\alpha(G))$  be the eigenvalues of the generalized distance matrix  $D_\alpha(G)$  of  $G$ . Taking  $a_i = b_i = t_i = \lambda_i(D_\alpha(G))$  for  $1 \leq i \leq n$ ,  $A = B = \lambda_1(D_\alpha(G))$ , and  $a = b = \lambda_n(D_\alpha(G))$ . By Lemmas 2.10 and 2.8, we have

$$\left| \frac{1}{2\alpha W(G)} \sum_{i=1}^n \lambda_i^3(D_\alpha(G)) - \frac{1}{4\alpha^2 W^2(G)} \sum_{i=1}^n \lambda_i^2(D_\alpha(G)) \sum_{i=1}^n \lambda_i^2(D_\alpha(G)) \right| \leq \frac{1}{4} D_\alpha S^2(G).$$

Then

$$\left| \frac{a}{2\alpha W(G)} - \frac{b^2}{4\alpha^2 W^2(G)} \right| \leq \frac{1}{4} D_\alpha S^2(G),$$

where

$$a = \alpha^3 \sum_{i=1}^n \text{Tr}_i^3 + (1-\alpha)^2 \alpha \sum_{i=1}^n \sum_{j=1}^n d_{ij}^2 \text{Tr}_i + (1-\alpha)^3 \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{m=j+1}^n d_{ij} d_{im} d_{jm},$$

$$b = \alpha^2 \sum_{i=1}^n \text{Tr}_i^2 + (1-\alpha)^2 \sum_{i=1}^n \sum_{j=1}^n d_{ij}^2.$$

This completes the proof.  $\square$

**Theorem 4.3.** *Let  $G$  be a graph of order  $n$ , and let the transmission degree sequence of graph  $G$  be  $\{\text{Tr}_1, \text{Tr}_2, \dots, \text{Tr}_n\}$ . Then*

$$D_\alpha S(G) \geq \frac{1}{n\sqrt{\xi(n)}} \sqrt{n \left( \alpha^2 \sum_{i=1}^n \text{Tr}_i^2 + (1-\alpha)^2 \sum_{i=1}^n \sum_{j=1}^n d_{ij}^2 \right) - 4\alpha^2 W^2(G)},$$

where  $\xi(n) = \frac{1}{4}(1 - (1 + (-1)^{n+1})/2n^2)$ .

*Proof.* Let  $\lambda_1(D_\alpha(G)) \geq \lambda_2(D_\alpha(G)) \geq \dots \geq \lambda_n(D_\alpha(G))$  be the eigenvalues of the generalized distance matrix  $D_\alpha(G)$  of  $G$ . By Lemma 2.11, taking  $a_i = b_i = \lambda_i(D_\alpha(G))$  for  $1 \leq i \leq n$ ,  $A = B = \lambda_1(D_\alpha(G))$ , and  $a = b = \lambda_n(D_\alpha(G))$ , we have

$$\left| n \sum_{i=1}^n \lambda_i^2 - \left( \sum_{i=1}^n \lambda_i \right)^2 \right| \leq \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \left( 1 - \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \right) n^2 D_\alpha S^2(G).$$

Since

$$\xi(n) = \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \left( 1 - \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \right) = \frac{1}{4} \left( 1 - \frac{1 + (-1)^{n+1}}{2n^2} \right),$$

and by the proof of Theorem 4.1, we have

$$\sum_{i=1}^n \lambda_i^2(D_\alpha(G)) = \alpha^2 \sum_{i=1}^n \text{Tr}_i^2 + (1-\alpha)^2 \sum_{i=1}^n \sum_{j=1}^n d_{ij}^2,$$

$$\sum_{i=1}^n \lambda_i(D_\alpha(G)) = 2\alpha W(G).$$

Then

$$D_\alpha S(G) \geq \frac{1}{n\sqrt{\xi(n)}} \sqrt{n \left( \alpha^2 \sum_{i=1}^n \text{Tr}_i^2 + (1-\alpha)^2 \sum_{i=1}^n \sum_{j=1}^n d_{ij}^2 \right) - 4\alpha^2 W^2(G)}.$$

The proof is completed.  $\square$

**Definition 4.1** ([16]). Let  $M$  be a real matrix of order  $n$  described in the following block form

$$(4.1) \quad M = \begin{bmatrix} M_{11} & M_{12} & \dots & M_{1t} \\ M_{21} & M_{22} & \dots & M_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ M_{t1} & M_{t2} & \dots & M_{tt} \end{bmatrix},$$

where the diagonal blocks  $M_{ii}$  are  $n_i \times n_i$  matrices for any  $i \in \{1, 2, \dots, t\}$  and  $n = n_1 + \dots + n_t$ . For any  $i, j \in \{1, 2, \dots, t\}$ , let  $b_{ij}$  denote the average row sum of  $M_{ij}$ , that is,  $b_{ij}$  is the sum of all entries in  $M_{ij}$  divided by the number of rows. Then  $B(M) = (b_{ij})$  is called the *quotient matrix* of  $M$ . If in addition for each pair  $i, j$ ,  $M_{ij}$  has constant row sum, then  $B(M)$  is called the *equitable quotient matrix* of  $M$ .

**Lemma 4.1** ([4]). Let  $M$  be a symmetric matrix which has the block form as (4.1) and  $B(M)$  be the quotient matrix of  $M$ . Then the eigenvalues of  $B(M)$  interlace the eigenvalues of  $M$ . That is, if  $a_1 \geq a_2 \geq \dots \geq a_n$  and  $b_1 \geq b_2 \geq \dots \geq b_t$  are the eigenvalues of the matrices  $M$  and  $B(M)$ , respectively, then  $a_i \geq b_i \geq a_{n-t+i}$  for all  $i = 1, 2, \dots, t$ .

**Theorem 4.4.** Let  $G$  be a graph of order  $n$  and diameter  $d$ . Suppose that  $\widetilde{P}_1, \widetilde{P}_2, \dots, \widetilde{P}_k$  are all the diameter paths of  $G$ , and  $S_i = \sum_{v_j \in V(P_i)} \text{Tr}(v_j)$  for  $1 \leq i \leq k$ .

- (1) If  $d = 1$ , then  $D_\alpha S(G) = (1 - \alpha)n$ .
- (2) If  $d \geq 2$ , then

$$(4.2) \quad D_\alpha S(G) \geq \max_{1 \leq i \leq k} \frac{\sqrt{a_i^2 - 12b_i(d+1)(n-d-1)}}{3(d+1)(n-d-1)},$$

where  $a_i = 6(d+1)(S_i - W(G)) - 3\alpha n S_i - (1 - \alpha)nd(d+1)(d+2)$ , and  $b_i = 2(1 - \alpha)d(d+1)(d+2)W(G) + 3S_i(2\alpha W(G) - S_i)$ .

**Proof.** (1) If  $d = 1$ , then  $G$  is  $K_n$ . By direct calculation, the generalized distance spectrum of  $G$  is  $\{n - 1, (n\alpha - 1)^{n-1}\}$ . Thus,  $D_\alpha S(G) = (1 - \alpha)n$ .

(2) If  $d \geq 2$ , for  $1 \leq i \leq k$ , let  $V(\widetilde{P}_i) = \{u_{i1}, u_{i2}, \dots, u_{i,d+1}\}$ , and let  $V(G) \setminus V(\widetilde{P}_i) = \{m_1, m_2, \dots, m_{n-d-1}\}$ . By suitably labelling the vertices of  $G$ , it can be seen that the generalized distance matrix of  $G$  can be written as

$$D_\alpha(G) = \begin{bmatrix} X & (1 - \alpha)Y \\ (1 - \alpha)Y^T & Z \end{bmatrix},$$

where

$$\begin{aligned}
X &= \begin{bmatrix} \alpha \text{Tr}(u_{i1}) & 1 - \alpha & 2(1 - \alpha) & \dots & d(1 - \alpha) \\ 1 - \alpha & \alpha \text{Tr}(u_{i2}) & 1 - \alpha & \dots & (d - 1)(1 - \alpha) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d(1 - \alpha) & (d - 1)(1 - \alpha) & (d - 2)(1 - \alpha) & \dots & \alpha \text{Tr}(u_{i,d+1}) \end{bmatrix}, \\
Y &= \begin{bmatrix} d(u_{i1}, m_1) & \dots & d(u_{i1}, m_{n-d-1}) \\ d(u_{i2}, m_1) & \dots & d(u_{i2}, m_{n-d-1}) \\ \vdots & \ddots & \vdots \\ d(u_{i,d+1}, m_1) & \dots & d(u_{i,d+1}, m_{n-d-1}) \end{bmatrix}, \\
Z &= \begin{bmatrix} \alpha \text{Tr}(m_1) & (1 - \alpha)d(m_1, m_2) & \dots & (1 - \alpha)d(m_1, m_{n-d-1}) \\ (1 - \alpha)d(m_2, m_1) & \alpha \text{Tr}(m_2) & \dots & (1 - \alpha)d(m_2, m_{n-d-1}) \\ \vdots & \vdots & \ddots & \vdots \\ (1 - \alpha)d(m_{n-d-1}, m_1) & (1 - \alpha)d(m_{n-d-1}, m_2) & \dots & \alpha \text{Tr}(m_{n-d-1}) \end{bmatrix}.
\end{aligned}$$

We let  $T = \sum_{v_s, v_j \in V(P_i)} d_{sj}$ . Then when  $d$  is even, we have

$$\begin{aligned}
T &= 2(1 + 2 + \dots + d) + 2[1 + 1 + 2 + 3 + \dots + (d - 1)] + \dots \\
&\quad + 2\left[1 + 1 + 2 + 2 + \dots + \left(\frac{d}{2} - 1\right) + \left(\frac{d}{2} - 1\right) + \frac{d}{2} + \left(\frac{d}{2} + 1\right)\right] \\
&\quad + 2\left[1 + 2 + \dots + \left(\frac{d}{2} - 1\right) + \frac{d}{2}\right] \\
&= d(d + 1) + [(d - 1)d + 2 \times 1] + [(d - 2)(d - 1) + 2 \times 3] + \dots \\
&\quad + \left[\left(\frac{d}{2} + 1\right)\left(\frac{d}{2} + 2\right) + \left(\frac{d}{2} - 1\right)\frac{d}{2}\right] + \frac{d}{2}\left(\frac{d}{2} + 1\right) \\
&= 1^2 + 2^2 + 3^2 + \dots + d^2 + 1 + 2 + 3 + \dots + d \\
&= \frac{d(d + 1)(d + 2)}{3}.
\end{aligned}$$

When  $d$  is odd, we have

$$\begin{aligned}
T &= 2(1 + 2 + \dots + d) + 2[1 + 1 + 2 + 3 + \dots + (d - 1)] + \dots \\
&\quad + 2\left(1 + 1 + 2 + 2 + \dots + \frac{d - 1}{2} + \frac{d - 1}{2} + \frac{d + 1}{2}\right) \\
&= d(d + 1) + [(d - 1)d + 2 \times 1] + [(d - 2)(d - 1) + 2 \times 3] + \dots \\
&\quad + \left(\frac{d + 1}{2}\right)\left(\frac{d + 1}{2} + 1\right) + \left(\frac{d - 1}{2}\right)\left(\frac{d - 1}{2} + 1\right) \\
&= 1^2 + 2^2 + 3^2 + \dots + d^2 + 1 + 2 + 3 + \dots + d \\
&= \frac{d(d + 1)(d + 2)}{3}.
\end{aligned}$$

Now  $V(G)$  is partitioned into two parts which are  $V(\tilde{P}_i)$  and  $V(G) \setminus V(\tilde{P}_i)$ . Corresponding to this partition, the quotient matrix  $B$  of  $D_\alpha(G)$  can be written as

$$B = \begin{bmatrix} \frac{\alpha S_i + \frac{1}{3}(1-\alpha)d(d+1)(d+2)}{d+1} & \frac{(1-\alpha)(S_i - \frac{1}{3}d(d+1)(d+2))}{d+1} \\ \frac{(1-\alpha)(S_i - \frac{1}{3}d(d+1)(d+2))}{n-d-1} & \frac{2W(G) + (\alpha-2)S_i + \frac{1}{3}(1-\alpha)d(d+1)(d+2)}{n-d-1} \end{bmatrix}.$$

The eigenvalues of the matrix  $B$  are

$$\lambda_{1,2}(B) = \frac{-a_i \pm \sqrt{a_i^2 - 12b_i(d+1)(n-d-1)}}{6(d+1)(n-d-1)},$$

where  $a_i = 6(d+1)(S_i - W(G)) - 3\alpha n S_i - (1-\alpha)nd(d+1)(d+2)$  and  $b_i = 2(1-\alpha)d(d+1)(d+2)W(G) + 3S_i(2\alpha W(G) - S_i)$ .

By Lemma 4.5,  $\lambda_1(D_\alpha(G)) \geq \lambda_1(B) \geq \lambda_2(B) \geq \lambda_n(D_\alpha(G))$ . So we get (4.2).  $\square$

**Corollary 4.1.** *Let  $G$  be a graph of order  $n$  and diameter  $d$ . Suppose that  $\tilde{P}_1, \tilde{P}_2, \dots, \tilde{P}_k$  are all the diameter paths of  $G$ , and  $a_i$  and  $b_i$  ( $1 \leq i \leq k$ ) are defined as Theorem 4.6. Then*

$$(1) \lambda_1(D_\alpha(G)) \geq \max_{1 \leq i \leq k} \frac{-a_i + \sqrt{a_i^2 - 12b_i(d+1)(n-d-1)}}{6(d+1)(n-d-1)},$$

$$(2) \lambda_n(D_\alpha(G)) \leq \min_{1 \leq i \leq k} \frac{-a_i - \sqrt{a_i^2 - 12b_i(d+1)(n-d-1)}}{6(d+1)(n-d-1)}.$$

A graph  $G$  is a cactus if any two of its cycles have at most one common vertex. Circumference is the length of the longest cycle of a graph. In the next theorem, we present a lower bound on  $D_\alpha S(G)$  of a cactus with circumference.

**Theorem 4.5.** *Let  $G$  be a cactus on  $n$  vertices with circumference  $l$  ( $l \geq 3$ ) and Wiener index  $W(G)$ . Suppose that  $\tilde{C}_1, \tilde{C}_2, \dots, \tilde{C}_k$  are all the cycles of length  $l$  of  $G$ , and  $S_i = \sum_{v_j \in V(P_i)} \text{Tr}(v_j)$  for  $1 \leq i \leq k$ . Then*

$$(4.3) \quad D_\alpha S(G) \geq \max_{1 \leq i \leq k} \frac{\sqrt{a_i^2 - 16b_i(n-l)l}}{4(n-l)l},$$

where

$$a_i = \begin{cases} -(1-\alpha)nl^3 - 4\alpha n S_i + 8l(S_i - W(G)) & \text{if } l \text{ is even,} \\ (1-\alpha)nl - (1-\alpha)nl^3 - 4\alpha S_i - 8l(W(G) - S_i) & \text{if } l \text{ is odd,} \end{cases}$$

and

$$b_i = \begin{cases} 4S_i(2\alpha W(G) - S_i) + 2(1-\alpha)l^3 W(G) & \text{if } l \text{ is even,} \\ 4S_i(2\alpha W(G) - S_i) + 2(1-\alpha)W(G)(l^3 - l) & \text{if } l \text{ is odd.} \end{cases}$$

Proof. Corresponding to  $\widetilde{C}_i$ ,  $V(G)$  is partitioned into two parts which are  $V(\widetilde{C}_i)$  and  $V(G) \setminus V(\widetilde{C}_i)$ .

*Case 1:  $l$  is even.* Then for any  $v \in V(\widetilde{C}_i)$ , the sum of the distances from vertex  $v$  to all other vertices on cycle  $V(\widetilde{C}_i)$  is  $\frac{1}{4}l^2$ . Corresponding to the above partition, the quotient matrix  $B$  of  $D_\alpha(G)$  is written as

$$B = \begin{bmatrix} \frac{\alpha S_i}{l} + (1-\alpha)\frac{l^2}{4} & (1-\alpha)\left(\frac{S_i}{l} - \frac{l^2}{4}\right) \\ (1-\alpha)\frac{S_i - \frac{1}{4}l^3}{n-l} & \frac{2W(G) - (2-\alpha)S_i + (1-\alpha)\frac{1}{4}l^3}{n-l} \end{bmatrix}.$$

The eigenvalues of the matrix  $B$  are

$$\lambda_{1,2}(B) = \frac{-a_i \pm \sqrt{a_i^2 - 16b_i(n-l)l}}{8(n-l)l},$$

where  $a_i = (1-\alpha)nl^3 - 4\alpha nS_i + 8l(S_i - W(G))$ , and  $b_i = 4S_i(2\alpha W(G) - S_i) + 2(1-\alpha)l^3W(G)$ . Now using Lemma 4.5 we get (4.3).

*Case 2:  $l$  is odd.* Then for any  $v \in V(\widetilde{C}_i)$ , the sum of the distances from vertex  $v$  to all other vertices on cycle  $V(\widetilde{C}_i)$  is  $\frac{1}{4}(l^2-1)$ . Corresponding to the above partition, the quotient matrix of  $D_\alpha(G)$  is written as

$$B = \begin{bmatrix} \frac{\alpha S_i}{l} + (1-\alpha)\frac{l^2-1}{4} & (1-\alpha)\left(\frac{S_i}{l} - \frac{l^2-1}{4}\right) \\ (1-\alpha)\frac{S_i - \frac{1}{4}(l^2-1)l}{n-l} & \frac{2W(G) - (2-\alpha)S_i + (1-\alpha)\frac{1}{4}(l^3-l)}{n-l} \end{bmatrix}.$$

The eigenvalues of the matrix  $B$  are

$$\lambda_{1,2}(B) = \frac{-a_i \pm \sqrt{a_i^2 - 16b_i(n-l)l}}{8(n-l)l},$$

where  $a_i = (1-\alpha)nl - (1-\alpha)nl^3 - 4\alpha S_i - 8l(W(G) - S_i)$  and  $b_i = 4S_i(2\alpha W(G) - S_i) + 6(1-\alpha)W(G)(l^3 - l)$ . Now using Lemma 4.5 we get (4.3).  $\square$

**Corollary 4.2.** *Let  $G$  be a cactus of order  $n$  and circumference  $l$  ( $l \geq 3$ ). Suppose that  $\widetilde{C}_1, \widetilde{C}_2, \dots, \widetilde{C}_k$  are all the cycles of length  $l$  of  $G$ , and  $a_i$  and  $b_i$  ( $1 \leq i \leq k$ ) are defined as Theorem 4.8. Then*

- (1)  $\lambda_1(D_\alpha(G)) \geq \max_{1 \leq i \leq k} \frac{-a_i + \sqrt{a_i^2 - 16b_i(n-l)l}}{8(n-l)l};$
- (2)  $\lambda_n(D_\alpha(G)) \leq \min_{1 \leq i \leq k} \frac{-a_i - \sqrt{a_i^2 - 16b_i(n-l)l}}{8(n-l)l}.$

**Acknowledgments.** The authors thank to the reviewers for their constructive suggestions for better presentation of this article.

### References

- [1] *M. Ajtai, J. Komlós, E. Szemerédi*: A note on Ramsey numbers. *J. Comb. Theory, Ser. A* *29* (1980), 354–360. zbl MR doi
- [2] *M. Aouchiche, P. Hansen*: Two Laplacians for the distance matrix of a graph. *Linear Algebra Appl.* *439* (2013), 21–33. zbl MR doi
- [3] *M. Aouchiche, P. Hansen*: Distance spectra of graphs: A survey. *Linear Algebra Appl.* *458* (2014), 301–386. zbl MR doi
- [4] *A. E. Brouwer, W. H. Haemers*: *Spectra of Graphs*. Universitext. Springer, Berlin, 2012. zbl MR doi
- [5] *F. Buckley, F. Harary*: *Distance in Graphs*. Addison-Wesley, Redwood, 1990. zbl MR
- [6] *S.-Y. Cui, J.-X. He, G.-X. Tian*: The generalized distance matrix. *Linear Algebra Appl.* *563* (2019), 1–23. zbl MR doi
- [7] *S.-Y. Cui, G.-X. Tian, L. Zheng*: On the generalized distance spectral radius of graphs. Available at <https://arxiv.org/abs/1901.07695> (2019), 13 pages. doi
- [8] *W. H. Haemers, A. Mohammadian, B. Tayfeh-Rezaie*: On the sum of Laplacian eigenvalues of graphs. *Linear Algebra Appl.* *432* (2010), 2214–2221. zbl MR doi
- [9] *C. R. Johnson, R. Kumar, H. Wolkowicz*: Lower bounds for the spread of a matrix. *Linear Algebra Appl.* *71* (1985), 161–173. zbl MR doi
- [10] *X. Li, E. N. Mohapatra, R. S. Rodriguez*: Grüss-type inequalities. *J. Math. Anal. Appl.* *267* (2002), 434–443. zbl MR doi
- [11] *H. Lin*: On the sum of  $k$  largest distance eigenvalues of graphs. *Discrete Appl. Math.* *259* (2019), 153–159. zbl MR doi
- [12] *J. K. Merikoski, R. Kumar*: Characterizations and lower bounds for the spread of a normal matrix. *Linear Algebra Appl.* *364* (2003), 13–31. zbl MR doi
- [13] *L. Mirsky*: The spread of a matrix. *Mathematica, Lond.* *3* (1956), 127–130. zbl MR doi
- [14] *B. G. Pachpatte*: *Analytic Inequalities: Recent Advances*. Atlantis Studies in Mathematics 3. Atlantis Press, Paris, 2012. zbl MR doi
- [15] *B. N. Parlett*: *The Symmetric Eigenvalue Problem*. Classics in Applied Mathematics 20. SIAM, Philadelphia, 1998. zbl MR doi
- [16] *S. Pirzada, H. A. Ganie, A. Alhevaz, M. Baghipur*: On spectral spread of generalized distance matrix of a graph. *Linear Multilinear Algebra* *70* (2022), 2819–2835. zbl MR doi
- [17] *S. Pirzada, H. A. Ganie, B. A. Rather, R. UlShaban*: On generalized distance energy of graphs. *Linear Algebra Appl.* *603* (2020), 1–19. zbl MR doi
- [18] *H. Wiener*: Structural determination of paraffin boiling points. *J. Am. Chem. Soc.* *69* (1947), 17–20. doi
- [19] *L. You, L. Ren, G. Yu*: Distance and distance signless Laplacian spread of connected graphs. *Discrete Appl. Math.* *223* (2017), 140–147. zbl MR doi
- [20] *G. Yu, H. Zhang, H. Lin, Y. Wu, J. Shu*: Distance spectral spread of a graph. *Discrete Appl. Math.* *160* (2012), 2474–2478. zbl MR doi

*Authors' addresses:* Yuzheng Ma, School of Data Science and Technology, North University of China, Xueyuan Road 3, Taiyuan, Shanxi 030051, P.R. China, e-mail: 1002631851@qq.com; Yanling Shao (corresponding author), School of Mathematics, North University of China, Xueyuan Road 3, Taiyuan, Shanxi 030051, P.R. China, e-mail: ylshao@nuc.edu.cn.