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NON-EXISTENCE OF GLOBAL CLASSICAL SOLUTIONS TO 1D COMPRESSIBLE HEAT-CONDUCTING MICROPOLAR FLUID

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Abstract. We study the non-existence of global classical solutions to 1D compressible heat-conducting micropolar fluid without viscosity. We first show that the life span of the classical solutions with decay at far fields must be finite for the 1D Cauchy problem if the initial momentum weight is positive. Then, we present several sufficient conditions for the non-existence of global classical solutions to the 1D initial-boundary value problem on $[0,1]$. To prove these results, some new average quantities are introduced.

Keywords: micoropolar fluid; global classical solution; non-existence

MSC 2020: 35Q35, 35B44

1. Introduction

In 1966, Eringen in [13] introduced the model of micropolar fluid, which can describe many phenomena appeared in a large number of complex fluids such as the suspensions, animal blood, liquid crystals. For more background, we refer to [18] and the references therein. In the last two decades, the model of micropolar fluid received considerable attention and many works about the mathematical analysis results to this model have been published. Here, we only mention some results about the one-dimensional case. The one-dimensional compressible micropolar fluid was first described by Mujaković in [19], and the local existence, the global existence,

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the regularity, the large time behavior and the stability of the solution for the non-isentropic compressible micropolar fluid model were analyzed in [1]–[4], [11], [12], [14], [20]–[29]. The global attractors of the model were first proved in [16]. Recently, the asymptotic behavior of solutions to the initial boundary value problem for the model in a half line was studied in [6], [15]. Moreover, the vanishing coefficients limit of the viscosities to the one-dimensional initial boundary value problem for the model was investigated in [5].

The motion of a viscous non-isentropic compressible micropolar fluids in $\mathbb{R}^3$ can be modeled by the following system with Eulerian description (see [5] for instance):

$$\varrho_t + \operatorname{div}(\varrho\overrightarrow{u}) = 0, \tag{1.1}$$

$$\begin{aligned}\varrho(\overrightarrow{u}_t + \overrightarrow{u}\cdot\nabla\overrightarrow{u}) + \nabla P(\varrho,\theta) = (\mu+\mu_r)\triangle\overrightarrow{u} + (\mu+\lambda-\mu_r)\nabla\operatorname{div}\overrightarrow{u}&\\ +2\mu_r\nabla\times\overrightarrow{\omega},&\end{aligned} \tag{1.2}$$

$$\begin{aligned}\varrho(\overrightarrow{\omega}_t + \overrightarrow{u}\cdot\nabla\overrightarrow{\omega}) + 4\mu_r\overrightarrow{\omega} = (c_d+c_a)\triangle\overrightarrow{\omega} + (c_0+c_d-c_a)\nabla\operatorname{div}\overrightarrow{\omega}&\\ +2\mu_r\nabla\times\overrightarrow{u},&\end{aligned} \tag{1.3}$$

$$\varrho C_v(\theta_t + \overrightarrow{u}\cdot\nabla\theta) + P(\varrho,\theta)\operatorname{div}\overrightarrow{u} = \operatorname{div}(\kappa\nabla\theta) + \Psi, \tag{1.4}$$

where

$$\begin{aligned}\Psi = \lambda(\operatorname{div}\overrightarrow{u})^2 + 2\mu D^* : D^* + 4\mu_r\Big(\frac{1}{2}\nabla\times\overrightarrow{u} - \overrightarrow{\omega}\Big)^2 + c_0(\operatorname{div}\overrightarrow{\omega})^2&\\ + (c_a+c_d)\nabla\overrightarrow{\omega} : \nabla\overrightarrow{\omega} + (c_d-c_a)\nabla\overrightarrow{\omega} : (\nabla\overrightarrow{\omega})^\top&\end{aligned} \tag{1.5}$$

denotes the dissipation function of the mechanical energy per mass unit, and D^* represents the deformation tensor:

$$D^* = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{1.6}$$

Here the fluid density ϱ, the fluid velocity $\overrightarrow{u}$, the microrotation velocity $\overrightarrow{\omega}$ and the temperature θ are unknown variables, $P(\varrho,\theta) = R\varrho\theta$ is the pressure with $R > 0$ being a constant. Furthermore, λ and μ are coefficients of viscosity and μ_r denotes the dynamic of microrotation viscosity, c_0, c_a and c_d are the angular viscosities. $\kappa = \kappa(\theta)$ represents the heat-conduction coefficient. Equations (1.1)–(1.4) indicate the balance laws for mass, momentum, momentum moment and energy, respectively.

We will consider the one-dimensional and non-viscous flow of the fluid described by (1.1)–(1.4). As in [4], we have

$$\varrho(\overrightarrow{x},t) = \varrho(x,t), \tag{1.7}$$

$$\overrightarrow{u}(\overrightarrow{x},t) = (u(x,t),0,0), \tag{1.8}$$

$$\overrightarrow{\omega}(\overrightarrow{x},t) = (\omega(x,t),0,0), \tag{1.9}$$

$$\theta(\overrightarrow{x},t) = \theta(x,t). \tag{1.10}$$

Moreover, we assume that $\mu = \lambda = 0$, $C_v = 1$, $4\mu_r = c_0 + 2c_d = A$ with $A > 0$ being a constant, then system (1.1)–(1.4) becomes the following one-dimensional model in the Eulerian description:

$$\varrho_t + (\varrho u)_x = 0, \tag{1.11}$$

$$(\varrho u)_t + (\varrho u^2)_x + P_x = 0, \tag{1.12}$$

$$(\varrho\omega)_t + (\varrho u\omega)_x + A\omega = A\omega_{xx}, \tag{1.13}$$

$$(\varrho\theta)_t + (\varrho u\theta)_x + Pu_x = (k(\theta)\theta_x)_x + A(\omega_x)^2 + A\omega^2. \tag{1.14}$$

In this paper, we study the non-existence of global classical solutions to system (1.11)–(1.14). As mentioned in [14], when $\omega = 0$, system (1.1)–(1.4) reduces to the classical Navier-Stokes system. If we further assume that $\mu = 0$, then system (1.1)–(1.4) reduces to the compressible Euler equations. It is well known that the smooth solutions of the compressible Euler equations and the compressible Navier-Stokes equations generally blow up in finite time; for the blowup results about the smooth solutions to these two systems we refer to [17], [30]–[33] and the references therein. In particular, Jiu, Wang and Xin in [17] proved the blowup of smooth solutions with decay at far fields to the Cauchy problem for the full compressible Navier-Stokes equations and the isentropic compressible Navier-Stokes equations with constant and degenerate viscosities in $\mathbb{R}^n$ under some restrictions on the initial data. For further generalization of the blow-up results of [17] about the full compressible Navier-Stokes equations and isentropic compressible Navier-Stokes equations with constant viscosities, we can refer to [31]. The main idea of [17], [31] is to establish some relationships between some physical quantities such as mass, momentum, momentum weight, momentum of inertia, kinetic energy, internal energy, total energy and some combined functionals of these quantities. The method of [17], [31] has been applied to the viscous two-phase model (see [10]), the compressible Euler equations with damping (see [8]), the compressible isentropic Navier-Stokes-Poisson equations (see [9]) and the compressible quantum Navier-Stokes equations, see [7].

Motivated by [17], [31], we will prove the non-existence of global classical solutions with decay at far fields to system (1.11)–(1.14) on the whole real line $\mathbb{R}$. As opposed to [7]–[10], [17], [31] for the model of micropolar fluid, the sum of the kinetic energy and the internal energy is not conserved or decreasing, but increasing in time, see (2.22) below. This makes us hard to estimate the upper bound of the momentum of inertia (see (2.33)–(2.34) below), which is crucial in establishing the Ricatti type inequality for the momentum weight, see (2.36) below. To overcome this difficulty, we introduce the micro energy $\frac{1}{2}\int_{\mathbb{R}} \varrho\omega^2 \, \mathrm{d}x$. We find that the sum of the kinetic energy, the internal energy and the micro energy is conserved (see (2.14) below), which enables us to obtain the upper bound of the momentum of inertia. In fact, we can extend the above result to the one-dimensional bounded interval case $[0,1]$ under some boundary conditions. Furthermore, we will present several other sufficient conditions for the non-existence of global classical solutions to the 1D initial-boundary value problem on $[0,1]$ by introducing some new average quantities.

Similarly to [14], we specify exactly a classical solution $(\varrho, u, \omega, \theta)$ to system (1.11)–(1.14) as:

$$\varrho \in C([0,T]; H^2), \quad \varrho_t \in C([0,T]; H^1), \quad \sqrt{\varrho} \in W^{1,\infty}(Q_T), \tag{1.15}$$

$$u \in L^\infty([0,T]; H^2 \cap H_0^1), \quad \omega \in L^\infty([0,T]; H^3 \cap H_0^1), \tag{1.16}$$

$$\sqrt{\varrho} u_t, \sqrt{\varrho}\omega_t \in L^\infty([0,T]; L^2),$$

$$\varrho u_t, \varrho\omega_t \in L^\infty([0,T]; H_0^1), \quad u_t, \omega_t \in L^2([0,T]; H_0^1), \quad \sqrt{\varrho}\theta_t \in L^\infty([0,T]; L^2), \tag{1.17}$$

$$\varrho\theta_t \in L^\infty([0,T]; H^1), \quad \theta \in L^\infty([0,T]; H^3), \quad \theta_t \in L^\infty([0,T]; H^1), \tag{1.18}$$

where $Q_T = [0,T] \times I$ with $I = \mathbb{R}$ or $I = [0,1]$.

This paper is organized as follows. In Section 2, we study the non-existence of global classical solutions to system (1.11)–(1.14) on the whole real line $\mathbb{R}$. In Section 3, we present several sufficient conditions for the non-existence of global classical solutions to the 1D initial-boundary value problem on $[0,1]$.

2. NON-EXISTENCE OF GLOBAL CLASSICAL SOLUTIONS IN $\mathbb{R}$

In this section, we consider the following Cauchy problem:

$$\begin{cases} \varrho_t + (\varrho u)_x = 0, & x \in \mathbb{R}, \\ (\varrho u)_t + (\varrho u^2)_x + P_x = 0, & x \in \mathbb{R}, \\ (\varrho\omega)_t + (\varrho u\omega)_x + A\omega = A\omega_{xx}, & x \in \mathbb{R}, \\ (\varrho\theta)_t + (\varrho u\theta)_x + P u_x = (k(\theta)\theta_x)_x + A(\omega_x)^2 + A\omega^2, & x \in \mathbb{R}, \\ (\varrho, u, \omega, \theta)|_{t=0} = (\varrho_0(x), u_0(x), \omega_0(x), \theta_0(x)), & x \in \mathbb{R}. \end{cases} \tag{2.1}$$

We will use the following physical quantities in this section:

$$I(t) = \frac{1}{2}\int_{\mathbb{R}} x^2 \varrho \,\mathrm{d}x \quad \text{(momentum of inertia)}, \tag{2.2}$$

$$F(t) = \int_{\mathbb{R}} x\varrho u \,\mathrm{d}x \quad \text{(momentum weight)}, \tag{2.3}$$

$$E(t) = \frac{1}{2}\int_{\mathbb{R}} \varrho u^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} \varrho\omega^2 \,\mathrm{d}x + \int_{\mathbb{R}} \varrho\theta \,\mathrm{d}x \quad \text{(total energy)}, \tag{2.4}$$

$$E_k(t) = \frac{1}{2}\int_{\mathbb{R}} \varrho u^2 \,\mathrm{d}x \quad \text{(kinetic energy)}, \tag{2.5}$$

$$E_m(t) = \frac{1}{2}\int_{\mathbb{R}} \varrho\omega^2 \,\mathrm{d}x \quad \text{(micro energy)}, \tag{2.6}$$

$$E_i(t) = \int_{\mathbb{R}} \varrho\theta \,\mathrm{d}x \quad \text{(internal energy)}. \tag{2.7}$$

We assume that

$$0 < I(0), F(0), E(0) < \infty, \quad E_m(0) + E_i(0) > 0. \tag{2.8}$$

We only consider the classical solutions with decay at far fields. Precisely, for any $T > 0$ we require that the solutions $(\varrho, u, \omega, \theta)$ satisfy the following condition:

$$\varrho|u|x^2, \varrho|u|^2|x|, P|x|, \varrho|u|^3, \varrho\theta|u|, k(\theta)|\theta_x|, \varrho|u|\omega^2, |\omega\omega_x| \in L^\infty((0,T); L^1(\mathbb{R})). \tag{2.9}$$

We remark that condition (2.9) guarantees that the integration by parts in our calculations makes sense, see also [7]–[10], [17], [31].

Our result in this section is stated as follows.

Theorem 2.1. *We denote*

$$c_1 = \max\{2, R\}, \quad c_2 = \sqrt{2c_1E(0)I(0) - F(0)^2}. \tag{2.10}$$

Let the initial data $(2.1)_5$ *satisfy* (2.8)–(2.9) *and*

$$\frac{c_2}{F(0)} + \arctan\frac{F(0)}{c_2} < \frac{\pi}{2}. \tag{2.11}$$

If the Cauchy problem (2.1) *has a classical solution such that* (2.9) *holds, then its lifetime* T *is given by*

$$T < \frac{c_2}{c_1E(0)}\tan\Big(\frac{c_2}{F(0)} + \arctan\frac{F(0)}{c_2}\Big) - \frac{F(0)}{c_1E(0)}. \tag{2.12}$$

Remark 2.1. We do not know whether the classical solutions of problem (2.1) satisfying (2.8)–(2.9) locally exist or not. In Theorem 2.1, we only show that if problem (2.1) has a local-in-time classical solution satisfying (2.8)–(2.9), then the life span T of the classical solution satisfies (2.12).

Remark 2.2. Condition (2.11) is used to ensure that the upper bound in (2.12) is positive. In fact, when (2.11) holds, by using the monotonicity of the function $\tan x$ in $(0, \frac{1}{2}\pi)$, we have

$$\frac{c_2}{c_1E(0)}\tan\Big(\frac{c_2}{F(0)} + \arctan\frac{F(0)}{c_2}\Big) > \frac{c_2}{c_1E(0)}\cdot\frac{F(0)}{c_2} = \frac{F(0)}{c_1E(0)}. \tag{2.13}$$

To prove Theorem 2.1, we need the following two lemmas.

Lemma 2.1. *Under the assumptions of Theorem* 2.1, *it holds that*

$$E(t) = E(0). \tag{2.14}$$

Proof. We multiply $(2.1)_1$ and $(2.1)_2$ by $-\frac{1}{2}u^2$ and u, respectively, and sum up the two resultant equations to obtain

$$\frac{1}{2}(\varrho u^2)_t + \frac{1}{2}(\varrho u^3)_x + (R\varrho\theta)_x u = 0. \tag{2.15}$$

Integrating (2.15) over $\mathbb{R}$, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \frac{1}{2}\varrho u^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} (\varrho u^3)_x \,\mathrm{d}x + R\int_{\mathbb{R}} (\varrho\theta)_x u \,\mathrm{d}x = 0. \tag{2.16}$$

By the condition $\varrho|u|^3 \in L^\infty((0,T); L^1(\mathbb{R}))$ in (2.9), we know that

$$\frac{1}{2}\int_{\mathbb{R}} (\varrho u^3)_x \,\mathrm{d}x = 0. \tag{2.17}$$

Using integration by part and the condition $\varrho\theta|u| \in L^\infty((0,T); L^1(\mathbb{R}))$ in (2.9), we get

$$R\int_{\mathbb{R}} (\varrho\theta)_x u \,\mathrm{d}x = -R\int_{\mathbb{R}} \varrho\theta u_x \,\mathrm{d}x. \tag{2.18}$$

We integrate $(2.1)_4$ over $\mathbb{R}$ to have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \varrho\theta \,\mathrm{d}x + R\int_{\mathbb{R}} \varrho\theta u_x \,\mathrm{d}x = A\int_{\mathbb{R}} (\omega_x)^2 \,\mathrm{d}x + A\int_{\mathbb{R}} \omega^2 \,\mathrm{d}x, \tag{2.19}$$

where we have used

$$\int_{\mathbb{R}} (\varrho u\theta)_x \,\mathrm{d}x = 0 \tag{2.20}$$

and

$$\int_{\mathbb{R}} (k(\theta)\theta_x)_x \,\mathrm{d}x = 0 \tag{2.21}$$

due to the condition $\varrho\theta|u|, k(\theta)|\theta_x| \in L^\infty((0,T); L^1(\mathbb{R}))$ in (2.9). Combining (2.16)–(2.19), (2.5) and (2.7), one has

$$\frac{\mathrm{d}}{\mathrm{d}t}[E_k(t)+E_i(t)] = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \frac{1}{2}\varrho u^2 \,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \varrho\theta \,\mathrm{d}x = A\int_{\mathbb{R}} (\omega_x)^2 \,\mathrm{d}x + A\int_{\mathbb{R}} \omega^2 \,\mathrm{d}x. \tag{2.22}$$

Multiplying $(2.1)_1$ and $(2.1)_3$ by $-\frac{1}{2}\omega^2$ and ω, respectively, and summing up the two resultant equations, we obtain

$$\frac{1}{2}(\varrho\omega^2)_t + \frac{1}{2}(\varrho u\omega^2)_x = A\omega_{xx}\omega - A\omega^2. \tag{2.23}$$

We integrate (2.23) over $\mathbb{R}$ to have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \frac{1}{2}\varrho\omega^2 \,\mathrm{d}x = -A\int_{\mathbb{R}} (\omega_x)^2 \,\mathrm{d}x - A\int_{\mathbb{R}} \omega^2 \,\mathrm{d}x, \tag{2.24}$$

where we have used

$$\frac{1}{2}\int_{\mathbb{R}} (\varrho u\omega^2)_x \,\mathrm{d}x = 0 \tag{2.25}$$

and

$$A\int_{\mathbb{R}} \omega_{xx}\omega \,\mathrm{d}x = -A\int_{\mathbb{R}} (\omega_x)^2 \,\mathrm{d}x \tag{2.26}$$

due to the condition $\varrho|u|\omega^2, |\omega\omega_x| \in L^\infty((0,T);L^1(\mathbb{R}))$ in (2.9). It follows from (2.22), (2.24) and (2.4) that

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = 0, \tag{2.27}$$

which implies that (2.14) holds.

Lemma 2.2. *Under the assumptions of Theorem* 2.1, *we have*

$$I'(t) = F(t), \tag{2.28}$$

$$F'(t) = \int_{\mathbb{R}} \varrho u^2 \,\mathrm{d}x + R\int_{\mathbb{R}} \varrho\theta \,\mathrm{d}x, \tag{2.29}$$

$$I(t), F(t) > 0. \tag{2.30}$$

Proof. By (2.2), $(2.1)_1$, the condition $\varrho|u|x^2 \in L^\infty((0,T);L^1(\mathbb{R}))$ in (2.9), integration by part and (2.3), we get

$$I'(t) = \frac{1}{2}\int_{\mathbb{R}} x^2\varrho_t \,\mathrm{d}x = -\frac{1}{2}\int_{\mathbb{R}} x^2(\varrho u)_x \,\mathrm{d}x = \int_{\mathbb{R}} x\varrho u \,\mathrm{d}x = F(t). \tag{2.31}$$

In view of (2.3), $(2.1)_2$, the condition $\varrho|u|^2|x|, P|x| \in L^\infty((0,T);L^1(\mathbb{R}))$ in (2.9) and integration by part, one has

$$F'(t) = \int_{\mathbb{R}} x(\varrho u)_t \,\mathrm{d}x = -\int_{\mathbb{R}} x(\varrho u^2)_x \,\mathrm{d}x - \int_{\mathbb{R}} xP_x \,\mathrm{d}x = \int_{\mathbb{R}} \varrho u^2 \,\mathrm{d}x + R\int_{\mathbb{R}} \varrho\theta \,\mathrm{d}x, \tag{2.32}$$

which together with the condition $F(0) > 0$ in (2.8) lead to $F(t) > 0$. By (2.28), $F(t) > 0$ and the condition $I(0) > 0$ in (2.8), we know that $I(t) > 0$. □

With Lemma 2.1 and Lemma 2.2, we can prove Theorem 2.1.

Proof of Theorem 2.1. By (2.28), (2.29), (2.4), (2.10) and (2.14), we obtain

$$I''(t)=F'(t)=\int_{\mathbb{R}}\varrho u^2\,\mathrm{d}x+R\int_{\mathbb{R}}\varrho\theta\,\mathrm{d}x\leqslant\max\{2,R\}E(t)=c_1E(0). \tag{2.33}$$

We integrate (2.33) over $[0,t]$ twice and use (2.28) to have

$$I(t)\leqslant\frac{1}{2}c_1E(0)t^2+F(0)t+I(0). \tag{2.34}$$

Using the Hölder inequality, one has

$$F(t)^2=\Big(\int_{\mathbb{R}}x\varrho u\,\mathrm{d}x\Big)^2\leqslant\int_{\mathbb{R}}x^2\varrho\,\mathrm{d}x\cdot\int_{\mathbb{R}}\varrho u^2\,\mathrm{d}x=4I(t)E_k(t). \tag{2.35}$$

From (2.29), (2.5), (2.35), (2.34) and (2.10), we can deduce that

$$\begin{aligned}F'(t)\geqslant 2E_k(t)\geqslant\frac{F(t)^2}{2I(t)}&\geqslant\frac{F(t)^2}{c_1E(0)t^2+2F(0)t+2I(0)}\\&=\frac{F(t)^2}{c_1E(0)[(t+F(0)/(c_1E(0)))^2+c_2^2/(c_1^2E(0))^2]}.\end{aligned} \tag{2.36}$$

By (2.10), (2.4), the condition $E_m(0)+E_i(0)>0$ in (2.8) and (2.35), we know that

$$c_2^2=2c_1E(0)I(0)-F(0)^2\geqslant 4E(0)I(0)-F(0)^2>4E_k(0)I(0)-F(0)^2\geqslant 0, \tag{2.37}$$

which implies that

$$c_2>0. \tag{2.38}$$

Dividing (2.36) by $F(t)^2$ and integrating the resultant inequality over $[0,T]$, we obtain

$$\begin{aligned}\frac{1}{F(0)}>\frac{1}{F(0)}-\frac{1}{F(T)}&\geqslant\int_0^T\frac{\mathrm{d}t}{c_1E(0)[(t+F(0)/(c_1E(0)))^2+c_2^2/(c_1^2E(0))^2]}\\&=\frac{1}{c_2}\Big[\arctan\frac{c_1E(0)(T+F(0)/(c_1E(0)))}{c_2}-\arctan\frac{F(0)}{c_2}\Big],\end{aligned} \tag{2.39}$$

where we have used (2.30) and (2.38). We can solve out T by (2.39) as (2.12). We complete the proof of Theorem 2.1. □

3. NON-EXISTENCE OF GLOBAL CLASSICAL SOLUTIONS ON $[0,1]$

In this section, we first consider the following initial boundary value problem:

$$\tag{3.1} \begin{cases} \varrho_t + (\varrho u)_x = 0, & x \in [0,1], \\ (\varrho u)_t + (\varrho u^2)_x + P_x = 0, & x \in [0,1], \\ (\varrho \omega)_t + (\varrho u \omega)_x + A\omega = A\omega_{xx}, & x \in [0,1], \\ (\varrho\theta)_t + (\varrho u \theta)_x + Pu_x = (k(\theta)\theta_x)_x + A(\omega_x)^2 + A\omega^2, & x \in [0,1], \\ (\varrho, u, \omega, \theta)|_{t=0} = (\varrho_0(x), u_0(x), \omega_0(x), \theta_0(x)), & x \in [0,1], \\ (u, \omega, \theta_x)|_{x=0,1} = 0, & \varrho\theta|_{x=1} = 0. \end{cases}$$

The boundary condition $(3.1)_6$ means that the fluid velocity and the microrotation velocity are both zero on the boundaries $x = 0$ and $x = 1$, the temperature is insulate on the boundaries $x = 0$ and $x = 1$, and the fluid density or the temperature is zero on the right boundary.

We will use the following physical quantities in this section:

$$\tag{3.2} m(t) = \int_0^1 \varrho \,\mathrm{d}x \quad \text{(total mass)},$$

$$\tag{3.3} I_1(t) = \frac{1}{2}\int_0^1 x^2 \varrho \,\mathrm{d}x \quad \text{(momentum of inertia)},$$

$$\tag{3.4} F_1(t) = \int_0^1 x\varrho u \,\mathrm{d}x \quad \text{(momentum weight)},$$

$$\tag{3.5} \begin{aligned} E_1(t) &= \frac{1}{2}\int_0^1 \varrho u^2 \,\mathrm{d}x + \frac{1}{2}\int_0^1 \varrho\omega^2 \,\mathrm{d}x + \int_0^1 \varrho\theta \,\mathrm{d}x \\ &= E_{1k}(t) + E_{1m}(t) + E_{1i}(t) \quad \text{(total energy)}, \end{aligned}$$

$$\tag{3.6} I_2(t) = \int_0^1 \varrho\Big(1 - \frac{x}{2}\Big)x \,\mathrm{d}x,$$

$$\tag{3.7} F_2(t) = \int_0^1 \varrho u(1-x) \,\mathrm{d}x,$$

$$\tag{3.8} F_3(t) = \int_0^1 \varrho u \mathrm{e}^{-1/x} \,\mathrm{d}x.$$

For problem (3.1), we can use the similar method as in Section 2 to obtain the following theorem.

Theorem 3.1. *We denote*

$$\tag{3.9} c_1 = \max\{2, R\}, \quad c_3 = \sqrt{2c_1E_1(0)I_1(0) - F_1(0)^2}.$$

Let the initial data $(3.1)_5$ *satisfy*

$$0 < I_1(0), F_1(0), E_1(0) < \infty, \quad E_{1m}(0) + E_{1i}(0) > 0 \tag{3.10}$$

and

$$\frac{c_3}{F_1(0)} + \arctan\frac{F_1(0)}{c_3} < \frac{\pi}{2}. \tag{3.11}$$

If the Cauchy problem (3.1) *has a classical solution, then its lifetime* T_1 *is given by*

$$T_1 < \frac{c_3}{c_1 E_1(0)} \tan\Big(\frac{c_3}{F_1(0)} + \arctan\frac{F_1(0)}{c_3}\Big) - \frac{F_1(0)}{c_1 E_1(0)}. \tag{3.12}$$

We remark that the boundary condition $(3.1)_6$ can guarantee that the integration by parts in our calculations makes sense as in Section 2. Although the proof of Theorem 3.1 is similar to the one of Theorem 2.1, for the sake of completeness, here we provide a sketch of it.

Proof of Theorem 3.1. Similarly to Lemmas 2.1 and 2.2 for problem (3.1) we have

$$E_1(t) = E_1(0), \tag{3.13}$$

$$I_1'(t) = F_1(t), \tag{3.14}$$

$$F_1'(t) = \int_0^1 \varrho u^2 \,\mathrm{d}x + R\int_0^1 \varrho\theta \,\mathrm{d}x, \tag{3.15}$$

$$I_1(t), F_1(t) > 0. \tag{3.16}$$

By (3.14), (3.15), (3.5), (3.13) and (3.9), we obtain

$$I_1''(t) = F_1'(t) = \int_0^1 \varrho u^2 \,\mathrm{d}x + R\int_0^1 \varrho\theta \,\mathrm{d}x \leqslant \max\{2, R\} E_1(t) = c_1 E_1(0). \tag{3.17}$$

We integrate (3.17) over $[0, t]$ twice and use (3.14) to have

$$I_1(t) \leqslant \frac{1}{2} c_1 E_1(0) t^2 + F_1(0) t + I_1(0). \tag{3.18}$$

Using the Hölder inequality, one has

$$F_1(t)^2 = \Big(\int_0^1 x\varrho u \,\mathrm{d}x\Big)^2 \leqslant \int_0^1 x^2 \varrho \,\mathrm{d}x \cdot \int_0^1 \varrho u^2 \,\mathrm{d}x = 4 I_1(t) E_{1k}(t). \tag{3.19}$$

From (3.15), (3.5), (3.19), (3.18) and (3.9), we can deduce that

$$\begin{aligned} F_1'(t) &\geqslant 2E_{1k}(t) \geqslant \frac{F_1(t)^2}{2I_1(t)} \geqslant \frac{F_1(t)^2}{c_1 E_1(0) t^2 + 2F_1(0) t + 2I_1(0)} \\ &= \frac{F_1(t)^2}{c_1 E_1(0)[(t + F_1(0)/(c_1 E_1(0)))^2 + c_3^2/(c_1^2 E_1(0))^2]}. \end{aligned} \tag{3.20}$$

By (3.9), (3.5), the condition $E_{1m}(0)+E_{1i}(0)>0$ in (3.10) and (3.19), we know that

$$c_3^2 = 2c_1E_1(0)I_1(0) - F_1(0)^2 \geqslant 4E_1(0)I_1(0) - F_1(0)^2 > 4E_{1k}(0)I_1(0) - F_1(0)^2 \geqslant 0, \tag{3.21}$$

which implies that

$$c_3 > 0. \tag{3.22}$$

Dividing (3.20) by $F_1(t)^2$ and integrating the resultant inequality over $[0,T_1]$, we obtain

$$\begin{aligned}\frac{1}{F_1(0)} &> \frac{1}{F_1(0)} - \frac{1}{F_1(T_1)} \geqslant \int_0^{T_1} \frac{\mathrm{d}t}{c_1E_1(0)[(t+F_1(0)/(c_1E_1(0)))^2 + c_3^2/(c_1^2E_1(0))^2]}\\ &= \frac{1}{c_3}\Big[\arctan\frac{c_1E_1(0)(T_1+F_1(0)/(c_1E_1(0)))}{c_3} - \arctan\frac{F_1(0)}{c_3}\Big],\end{aligned} \tag{3.23}$$

where we have used (3.16) and (3.22). We can solve out T_1 by (3.23) as (3.12). We complete the proof of Theorem 3.1. ☐

Next, we replace the boundary condition $(3.1)_6$ by

$$u|_{x=0,1} = 0, \quad \varrho\theta|_{x=0} = 0. \tag{BC1}$$

The boundary condition (BC1) means that the fluid velocity is zero on the boundaries $x=0$ and $x=1$, and the fluid density or the temperature is zero on the left boundary. Using the average quantities (3.6) and (3.7), we can obtain the following result about the non-existence of global classical solutions to problem $(3.1)_1$–$(3.1)_5$ and (BC1) on $[0,1]$.

Theorem 3.2. *Let* $I_2(0)>0$ *and* $F_2(0)<0$. *Then the life span* T_2 *of the classical solution to problem* $(3.1)_1$–$(3.1)_5$ *and* (BC1) *satisfies that* $T_2 < -I_2(0)/F_2(0)$.

Proof. By (3.6), $(3.1)_1$, the condition $u|_{x=1}=0$ in (BC1), integration by part and (3.7), one has

$$I_2'(t) = \int_0^1 \varrho_t\Big(1-\frac{x}{2}\Big)x\,\mathrm{d}x = -\int_0^1 (\varrho u)_x\Big(1-\frac{x}{2}\Big)x\,\mathrm{d}x = \int_0^1 \varrho u(1-x)\,\mathrm{d}x = F_2(t). \tag{3.24}$$

In view of (3.7), $(3.1)_2$, the condition $u|_{x=0}=0$, $\varrho\theta|_{x=0}=0$ in (BC1) and integration by part, we have

$$\begin{aligned}F_2'(t) &= \int_0^1 (\varrho u)_t(1-x)\,\mathrm{d}x = -\int_0^1 (\varrho u^2)_x(1-x)\,\mathrm{d}x - \int_0^1 P_x(1-x)\,\mathrm{d}x\\ &= -\int_0^1 \varrho u^2\,\mathrm{d}x - \int_0^1 P\,\mathrm{d}x \leqslant 0,\end{aligned} \tag{3.25}$$

which implies that

$$F_2(t) \leqslant F_2(0). \tag{3.26}$$

Combining (3.24) and (3.26), it holds that

$$I_2'(t) = F_2(t) \leqslant F_2(0). \tag{3.27}$$

We integrate (3.27) over $[0, T_2]$ to obtain

$$0 < I_2(T_2) \leqslant I_2(0) + F_2(0)T_2, \tag{3.28}$$

which together with the conditions $I_2(0) > 0$ and $F_2(0) < 0$ lead to the fact that $T_2 < -I_2(0)/F_2(0)$. □

Finally, we replace the boundary condition (BC1) by

$$u|_{x=0,1} = 0, \quad \varrho\theta|_{x=1} = 0. \tag{BC2}$$

The boundary condition (BC2) means that the fluid velocity is zero on the boundaries $x = 0$ and $x = 1$, and the fluid density or the temperature is zero on the right boundary. Using the average quantities (3.2) and (3.8), we can obtain the following result about the non-existence of global classical solutions to problem $(3.1)_1$–$(3.1)_5$ and (BC2) on $[0, 1]$.

Theorem 3.3. *Let* $m(0) > 0$ *and* $F_3(0) > 0$*. Then the life span* T_3 *of the classical solution to problem* $(3.1)_1$–$(3.1)_5$ *and* (BC2) *satisfies that* $T_3 < m(0)/(\mathrm{e}F_3(0))$.

Proof. By (3.2), $(3.1)_1$, the condition $u|_{x=0,1} = 0$ in (BC2), we know that

$$m'(t) = \int_0^1 \varrho_t \,\mathrm{d}x = -\int_0^1 (\varrho u)_x \,\mathrm{d}x = 0, \tag{3.29}$$

which means that

$$m(t) = m(0). \tag{3.30}$$

In view of (3.8), $(3.1)_2$, the condition $u|_{x=1} = 0$, $\varrho\theta|_{x=1} = 0$ in (BC2) and integration by part, we have

$$\begin{aligned} F_3'(t) &= \int_0^1 (\varrho u)_t \mathrm{e}^{-1/x} \,\mathrm{d}x = -\int_0^1 (\varrho u^2)_x \mathrm{e}^{-1/x} \,\mathrm{d}x - \int_0^1 P_x \mathrm{e}^{-1/x} \,\mathrm{d}x \\ &= \int_0^1 \frac{\varrho u^2}{x^2} \mathrm{e}^{-1/x} \,\mathrm{d}x + R\int_0^1 \frac{\varrho\theta}{x^2} \mathrm{e}^{-1/x} \,\mathrm{d}x \geqslant \int_0^1 \frac{\varrho u^2}{x^2} \mathrm{e}^{-1/x} \,\mathrm{d}x. \end{aligned} \tag{3.31}$$

By the Hölder inequality, it holds that

$$F_3(t)^2 = \Big(\int_0^1 \varrho u \mathrm{e}^{-1/x}\,\mathrm{d}x\Big)^2 \leqslant \int_0^1 \frac{\varrho u^2}{x^2}\mathrm{e}^{-1/x}\,\mathrm{d}x \cdot \int_0^1 x^2\mathrm{e}^{-1/x}\varrho\,\mathrm{d}x. \tag{3.32}$$

By (3.2) and (3.30), we know that

$$\int_0^1 x^2\mathrm{e}^{-1/x}\varrho\,\mathrm{d}x \leqslant \mathrm{e}^{-1}\int_0^1 \varrho\,\mathrm{d}x = \frac{m(t)}{\mathrm{e}} = \frac{m(0)}{\mathrm{e}}. \tag{3.33}$$

Combining (3.31)–(3.33) and the condition $m(0) > 0$, we obtain

$$F_3'(t) \geqslant \frac{\mathrm{e}F_3(t)^2}{m(0)}, \tag{3.34}$$

which together with the condition $F_3(0) > 0$ imply that $F_3(t) > 0$. We divide (3.34) by $F_3(t)^2$ and integrate it over $[0, T_3]$ to have

$$\frac{1}{F_3(0)} > \frac{1}{F_3(0)} - \frac{1}{F_3(T_3)} \geqslant \frac{\mathrm{e}T_3}{m(0)}, \tag{3.35}$$

which means that $T_3 < m(0)/(\mathrm{e}F_3(0))$. ☐

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