Czechoslovak Mathematical Journal

Xiaohui Zhang; Wei Wang; Juzhen Chen

Parametric representations of BiHom-Hopf algebras

*Czechoslovak Mathematical Journal*, Vol. 74 (2024), No. 1, 45–86

Persistent URL: `http://dml.cz/dmlcz/152268`

**Czechoslovak Mathematical Journal, 74 (149) (2024), 45–86**

# PARAMETRIC REPRESENTATIONS OF BIHOM-HOPF ALGEBRAS

Xiaohui Zhang, Qufu, Wei Wang, Juzhen Chen, Nanjing

Received May 19, 2022.

*Abstract.* The main purpose of the present paper is to study representations of BiHom-Hopf algebras. We first introduce the notion of BiHom-Hopf algebras, and then discuss BiHom-type modules, Yetter-Dinfeld modules and Drinfeld doubles with parameters. We get some new $n$-monoidal categories via the category of BiHom-(co)modules and the category of BiHom-Yetter-Drinfeld modules. Finally, we obtain a center construction type theorem on BiHom-Hopf algebras.

*Keywords*: BiHom-Hopf algebra; BiHom-Yang-Baxter equation; $n$-monoidal category; Drinfeld double

*MSC 2020*: 16T99, 16T25, 16W10

## 1. Introduction

Hom-type algebras first appeared in describing the $q$-deformations of Witt and Virasoro algebras, see [11], [12]. In 2008, Makhlouf and Silvestrov introduced the definition of Hom-associative algebras (see [22]), where the associativity of a Hom-algebra is twisted by an endomorphism (here we call it the Hom-structure map). They are rapidly developing into a hot topic in algebra theory. The generalized notions, including Hom-bialgebras, Hom-Hopf algebras are developed in [17], [20]–[24], [27]–[30].

An interesting question of Hom-type algebras is how to explain them based on the theory of monoidal categories. In 2011, in order to provide a categorical approach to Hom-type algebras, the notions of Hom-categories and monoidal Hom-Hopf algebras were introduced by Caenepeel and Goyvaerts, see [3]. Note that a monoidal Hom-bialgebra is a bimonoid in the Hom-category, and a Hom-bialgebra (with the

---

The work was partially supported by the Natural Science Foundation of Shandong Province (No. ZR2023MA008), and the National Natural Science Foundation of China (No. 12271292).

DOI: 10.21136/CMJ.2023.0213-22

bijective Hom-structure map) is a bimonoid in a (strict) duoidal category, see [31]. That is the main difference between Hom-bialgebras and monoidal Hom-bialgebras. Further results on monoidal Hom-Hopf algebras can be found in [9], [32], [33] and references cited therein. One of the important features of (monoidal) Hom-bialgebras is that they have parametric (co)representations, which means that there exist infinite monoidal structures in the category of Hom-(co)modules, see [29]. Although they are monoidal isomorphic to each other, it still provides examples of higher monoidal categories.

BiHom-(co)algebras and BiHom-bialgebras were investigated by Graziani et al. in [8]. It is a more generalized definition. Precisely, a BiHom-bialgebra is a special bialgebra on which the associative law and unit law are twisted by two automorphisms ($\alpha$ and $\beta$), coassociative law and counit law are twisted by the other two automorphisms ($\varphi$ and $\psi$). It becomes a usual Hom-bialgebra when $\alpha = \beta = \varphi = \psi$, and becomes a monoidal Hom-bialgebra when $\alpha^{-1} = \beta^{-1} = \varphi = \psi$. Further research on BiHom-type algebras could be found in [6], [10], [15], [16], [31] and so on.

Although the description of BiHom-(co)algebras and BiHom-bialgebras in [8] is detailed, the discussion of BiHom-Hopf algebras is not clear enough. Note that the antipode $S$ of the Hopf algebra $H$ is the convolution inverse of $\mathrm{id}_H$ in $\mathrm{End}(H)$, and is both an anti-algebra map and an anti-coalgebra map. The above properties still hold in the setting of Hom-Hopf algebras and monoidal Hom-Hopf algebras. But in the case of BiHom-Hopf algebras, it is different. We find that the antipode (the convolution inverse of $\mathrm{id}_H$) of a BiHom-Hopf algebra is neither anti-algebra map nor anti-coalgebra map, but a map which satisfies some similar properties, see Proposition 2.8. Moreover, the existence of $S$ is closely connected with the Hom-structure maps, see Proposition 2.9. Based on these properties, some classical theory of Hopf algebras, such as Yetter-Drinfeld modules and the Center Construction Theorem, could be possible to develop in the setting of BiHom-Hopf algebras.

Note that the definition of BiHom-Hopf algebras in [8] is slightly different from ours. Recall from [8], Definition 6.9, the antipode $S$ is defined by $S\beta\varphi(h_1)\alpha\psi(h_2) = \varepsilon(h)1_H = \beta\varphi(h_1)S\alpha\psi(h_2)$ for any $h \in H$. Although it may induce some similar properties and theories, we choose the traditional definition of antipode (the convolution inverse of $\mathrm{id}_H$) to complete our theoretical analysis.

In order to give a categorical interview of the quantum double for a Hopf algebra, Drinfeld introduced the notion of the center $\mathscr{Z}(\mathcal{C})$ of a monoidal category $\mathcal{C}$, see [13], [18]. Note that if $\mathcal{C}$ is the representation category of a Hopf algebra $H$, then $\mathscr{Z}(\mathcal{C})$ is isomorphic with the category of $D(H)$-modules, where $D(H)$ means the Drinfeld double of $H$. Furthermore, $\mathscr{Z}(\mathcal{C})$ is also isomorphic with the Yetter-Drinfeld categories of $H$, see [14], [25]. Generalized results have been obtained by many scholars. The theory of center constructions on quasi-Hopf algebras was developed in [19], [34]. In [26], Nenciu discussed the theory of center constructions on weak Hopf algebras, see also [4]. In 2014, Chen and Zhang researched the center constructions on monoidal Hom-Hopf algebras, see [5]. In 2019, Zhang, Guo and Wang discussed the Drinfeld codoubles of Hom-Hopf algebras, see [30]. The natural consideration is to ask how this result appears in BiHom-Hopf algebras. That is the motivation of our paper.

The paper is organized as follows. In Section 2 we first recall some notions of category centers and BiHom-bialgebras, and then introduce the definition of BiHom-Hopf algebras. In Section 3, we describe the monoidal structure of the category of BiHom-modules, and show it forms an $n$-monoidal category. We also prove the quasi-triangular structures can provide new solutions of the BiHom-Yang-Baxter equation. In Section 4, we mainly introduce the parametric BiHom-Yetter-Drinfeld category, and show that it is a full braided category of the center of the category of BiHom-modules. In Section 5, we discuss the parametric Drinfeld doubles of BiHom-Hopf algebras, and then prove that its representation category is braided isomorphic to the BiHom-Yetter-Drinfeld category.

## 2. PRELIMINARIES

Throughout the paper, $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}, \ldots$ always mean integers in $\mathbb{Z}$. Let $\Bbbk$ be a fixed field and $\mathrm{char}(\Bbbk) = 0$, and $\mathrm{Vec}_{\Bbbk}$ be the category of finite dimensional $\Bbbk$-spaces. All algebras are supposed to be over $\Bbbk$. For the comultiplication $\Delta$ of a $\Bbbk$-space $C$, we use Sweedler-Heyneman's notation $\Delta(c) = c_1 \otimes c_2$ for any $c \in C$. When we say "BiHom-algebra" or "BiHom-coalgebra", we mean the unital BiHom-algebra and counital BiHom-coalgebra. We always assume that the Hom-structure maps are invertible.

**2.1. Higher monoidal category.** Recall from [1], Definition 6.1 that a 2-*monoidal category* (or equally, a *duoidal category*, or a *nonstrict* 2*-fold monoidal category*) is a category $\mathcal{M}$ equipped with two monoidal structures $(\mathcal{M}, \otimes_1, I_1, \mathbf{a}_1, \mathbf{l}_1, \mathbf{r}_1)$ and $(\mathcal{M}, \otimes_2, I_2, \mathbf{a}_2, \mathbf{l}_2, \mathbf{r}_2)$, along with a natural transformation (called the *interchange law*)

$$\zeta_{A,B,C,D} \colon (A \otimes_2 B) \otimes_1 (C \otimes_2 D) \to (A \otimes_1 C) \otimes_2 (B \otimes_1 D) \quad \forall A, B, C, D \in \mathcal{M}$$

and three morphisms

$$I_1 \xrightarrow{\omega} I_1 \otimes_2 I_1 \ , \quad I_2 \otimes_1 I_2 \xrightarrow{\varpi} I_2 \ , \quad I_1 \xrightarrow{\tau} I_2 \ ,$$

such that the axioms below are satisfied.

*Associativity.* For any $A$, $B$, $C$, $D$, $E$, $F$ in $\mathcal{M}$, the following diagrams commute.

$$\begin{tikzcd}
((A \otimes_2 B) \otimes_1 (C \otimes_2 D)) \otimes_1 (E \otimes_2 F) \arrow[rr, "\mathbf{a}_1"] \arrow[d, "\zeta \otimes_1 \mathrm{id}"'] & {} \arrow[d, phantom, "\text{(A1)}"] & (A \otimes_2 B) \otimes_1 ((C \otimes_2 D) \otimes_1 (E \otimes_2 F)) \arrow[d, "\mathrm{id} \otimes_1 \zeta"] \\
((A \otimes_1 C) \otimes_2 (B \otimes_1 D)) \otimes_1 (E \otimes_2 F) \arrow[d, "\zeta"'] & {} & (A \otimes_2 B) \otimes_1 ((C \otimes_1 E) \otimes_2 (D \otimes_1 F)) \arrow[d, "\zeta"] \\
((A \otimes_1 C) \otimes_1 E) \otimes_2 ((B \otimes_1 D) \otimes_1 F) \arrow[rr, "\mathbf{a}_1 \otimes_2 \mathbf{a}_1"'] & & (A \otimes_1 (C \otimes_1 E)) \otimes_2 (B \otimes_1 (D \otimes_1 F)),
\end{tikzcd}$$

$$\begin{tikzcd}
((A \otimes_2 B) \otimes_2 C) \otimes_1 ((D \otimes_2 E) \otimes_2 F) \arrow[rr, "\mathbf{a}_2 \otimes_1 \mathbf{a}_2"] \arrow[d, "\zeta"'] & {} \arrow[d, phantom, "\text{(A2)}"] & (A \otimes_2 (B \otimes_2 C)) \otimes_1 (D \otimes_2 (E \otimes_2 F)) \arrow[d, "\zeta"] \\
((A \otimes_2 B) \otimes_1 (D \otimes_2 E)) \otimes_2 (C \otimes_1 F) \arrow[d, "\zeta \otimes_2 \mathrm{id}"'] & {} & (A \otimes_1 D) \otimes_2 ((B \otimes_2 C) \otimes_1 (E \otimes_2 F)) \arrow[d, "\mathrm{id} \otimes_2 \zeta"] \\
((A \otimes_1 D) \otimes_2 (B \otimes_1 E)) \otimes_2 (C \otimes_1 F) \arrow[rr, "\mathbf{a}_2"'] & & (A \otimes_1 D) \otimes_2 ((B \otimes_1 E) \otimes_2 (C \otimes_1 F)).
\end{tikzcd}$$

*Unitality.* For any $A$, $B$ in $\mathcal{M}$, the following diagrams commute.

$$\begin{tikzcd}
I_1 \otimes_1 (A \otimes_2 B) \arrow[rr, "\omega \otimes_1 \mathrm{id}"] \arrow[d, "\mathbf{l}_1"'] & {} \arrow[d, phantom, "\text{(U1)}"] & (I_1 \otimes_2 I_1) \otimes_1 (A \otimes_2 B) \arrow[d, "\zeta"] \\
A \otimes_2 B & {} & (I \otimes_1 A) \otimes_2 (I_1 \otimes_1 B), \arrow[ll, "\mathbf{l}_1 \otimes_2 \mathbf{l}_1"]
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes_2 B) \otimes_1 I_1 \arrow[rr, "\mathrm{id} \otimes_1 \omega"] \arrow[d, "\mathbf{r}_1"'] & {} \arrow[d, phantom, "\text{(U2)}"] & (A \otimes_2 B) \otimes_1 (I_1 \otimes_2 I_1) \arrow[d, "\zeta"] \\
A \otimes_2 B & {} & (A \otimes_1 I_1) \otimes_2 (B \otimes_1 I_1), \arrow[ll, "\mathbf{r}_1 \otimes_2 \mathbf{r}_1"]
\end{tikzcd}$$

$$\begin{tikzcd}
(I_2 \otimes_1 I_2) \otimes_2 (A \otimes_1 B) \arrow[rr, "\varpi \otimes_2 \mathrm{id}"] & {} \arrow[d, phantom, "\text{(U3)}"] & I_2 \otimes_2 (A \otimes_1 B) \arrow[d, "\mathbf{l}_2"] \\
(I_2 \otimes_2 A) \otimes_1 (I_2 \otimes_2 B) \arrow[u, "\zeta"] \arrow[rr, "\mathbf{l}_2 \diamond \mathbf{l}_2"'] & {} & A \otimes_1 B,
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes_1 B) \otimes_2 (I_2 \otimes_1 I_2) \arrow[rr, "\mathrm{id} \otimes_2 \varpi"] & {} \arrow[d, phantom, "\text{(U4)}"] & (A \otimes_1 B) \otimes_2 I_2 \arrow[d, "\mathbf{r}_2"] \\
(A \otimes_2 I_2) \otimes_1 (B \otimes_2 I_2) \arrow[u, "\zeta"] \arrow[rr, "\mathbf{r}_2 \diamond \mathbf{r}_2"'] & {} & A \otimes_1 B
\end{tikzcd}$$

*Compatibility of the units.* The monoidal units $I_1$ and $I_2$ satisfy:

- ▷ $(I_2, \varpi, \tau)$ is a monoid in $(\mathcal{M}, \otimes_1, I_1, \mathbf{a}_1, \mathbf{l}_1, \mathbf{r}_1)$,
- ▷ $(I_1, \omega, \tau)$ is a comonoid in $(\mathcal{M}, \otimes_2, I_2, \mathbf{a}_2, \mathbf{l}_2, \mathbf{r}_2)$.

Now assume that there exists a category $\mathcal{M}$ with $n$ monoidal structures. We denote such a structure by the tuple

$$(\mathcal{M}, \otimes_i, I_i, \mathbf{a}_i, \mathbf{l}_i, \mathbf{r}_i)_{\{i=1,2,\ldots,n\}},$$

where $\otimes_1, \otimes_2, \ldots, \otimes_n$ are the monoidal products and $I_1, I_2, \ldots, I_n$ are the respective unit objects. We write $\mathcal{M}_i$ for $(\mathcal{M}, \otimes_i, I_i, \mathbf{a}_i, \mathbf{l}_i, \mathbf{r}_i)$, $\mathcal{M}_{i,j}$ for $(\mathcal{M}, \otimes_i, I_i, \otimes_j, I_j)$, $\mathcal{M}_{i,j,k}$ for $(\mathcal{M}, \otimes_i, I_i, \otimes_j, I_j, \otimes_k, I_k)$, and so on, for short.

Recall from Definition 7.24 of [1], that we call $\mathcal{M} = (\mathcal{M}, \otimes_i, I_i, \mathbf{a}_i, \mathbf{l}_i, \mathbf{r}_i)_{\{i=1,2,\ldots,n\}}$ an *$n$-monoidal category* (or equally, a *nonstrict $n$-fold monoidal category*) if the following formulae are satisfied:

- ▷ for any $1 \leqslant i < j \leqslant n$, $\mathcal{M}_{i,j}$ forms a 2-monoidal category;
- ▷ for any $1 \leqslant i < j < k \leqslant n$, $\mathcal{M}_{i,j,k}$ forms a 3-*monoidal category*, namely, the Diagrams I. commute.

**Remark 2.1.**

(1) If $\mathbf{l}_i = \mathbf{r}_i = \mathrm{id}$ for any $i = 1, 2, \ldots, n$, then the above $n$-monoidal category is exactly the $n$-fold monoidal category defined in [7].

(2) If $\mathcal{M}_i$ is a strict monoidal category for any $i = 1, 2, \ldots, n$, and $I_1 = I_2 = \ldots = I_n$, then the above $n$-monoidal category is exactly the $n$-fold monoidal category defined in [2].

**2.2. The center of a monoidal category.** Suppose that $(\mathcal{C}, \otimes, I, \mathbf{a}, \mathbf{l}, \mathbf{r})$ is a monoidal category. Recall from [14], that a new category $\mathcal{Z}(\mathcal{C})$ is defined as follows (and called the *center of* $\mathcal{C}$):

- ▷ the object of $\mathcal{Z}(\mathcal{C})$ is a pair $(V, T_{-,V})$, where $V \in \mathcal{C}$ and $T_{-,V}$ is a family of natural isomorphisms

$$T_{X,V} \colon X \otimes V \to V \otimes X,$$

where $X \in \mathcal{C}$, such that

$$T_{I,V} = I \otimes V \xrightarrow{\mathbf{l}_V} V \xrightarrow{\mathbf{r}_V^{-1}} V \otimes I, \tag{C1}$$

$$\begin{tikzcd}
& ((A \otimes_k A') \otimes_j (B \otimes_k B')) \otimes_i ((C \otimes_k C') \otimes_j (D \otimes_k D')) \arrow[dl, "\zeta_{j,k} \otimes_i \zeta_{j,k}"'] \arrow[dr, "\zeta_{i,j}"] & \\
((A \otimes_j B) \otimes_k (A' \otimes_j B')) \otimes_i ((C \otimes_j D) \otimes_k (C' \otimes_j D')) \arrow[d, "\zeta_{i,k}"'] & \text{(A3)} & ((A \otimes_k A') \otimes_i (C \otimes_k C')) \otimes_j ((B \otimes_k B') \otimes_i (D \otimes_k D')) \arrow[d, "\zeta_{i,k} \otimes_j \zeta_{i,k}"] \\
((A \otimes_j B) \otimes_i (C \otimes_j D)) \otimes_k ((A' \otimes_j B') \otimes_i (C' \otimes_j D')) \arrow[dr, "\zeta_{i,j} \otimes_k \zeta_{i,j}"'] & & ((A \otimes_i C) \otimes_k (A' \otimes_i C')) \otimes_j ((B \otimes_i D) \otimes_k (B' \otimes_i D')) \arrow[dl, "\zeta_{j,k}"] \\
& ((A \otimes_i C) \otimes_j (B \otimes_i D)) \otimes_k ((A' \otimes_i C') \otimes_j (B' \otimes_i D')) &
\end{tikzcd}$$

$$\begin{tikzcd}
I_i \arrow[r, "\omega_{i,j}"] \arrow[d, "\omega_{i,k}"'] & I_i \otimes_j I_i \arrow[r, "\omega_{i,k} \otimes_j \omega_{i,k}"'] & (I_i \otimes_k I_i) \otimes_j (I_i \otimes_k I_i) \arrow[d, "\zeta_{j,k}"] \\
I_i \otimes_k I_i \arrow[rr, "\omega_{i,j} \otimes_k \omega_{i,j}"'] & \text{(U5)} & (I_i \otimes_j I_i) \otimes_k (I_i \otimes_j I_i),
\end{tikzcd}$$

$$\begin{tikzcd}
I_j \otimes_i I_j \arrow[rr, "\omega_{j,k} \otimes_i \omega_{j,k}"] \arrow[d, "\varpi_{i,j}"'] & \text{(U6)} & (I_j \otimes_k I_j) \otimes_i (I_j \otimes_k I_j) \arrow[d, "\zeta_{i,k}"] \\
I_i \arrow[r, "\omega_{j,k}"'] & I_j \otimes_k I_j & (I_j \otimes_i I_j) \otimes_k (I_j \otimes_i I_j), \arrow[l, "\varpi_{i,j} \otimes_j \varpi_{i,j}"]
\end{tikzcd}$$

$$\begin{tikzcd}
(I_k \otimes_j I_k) \otimes_i (I_k \otimes_j I_k) \arrow[r, "\varpi_{j,k} \otimes_i \varpi_{j,k}"'] \arrow[d, "\zeta_{i,j}"'] & I_k \otimes_i I_k \arrow[r, "\varpi_{i,k}"] & I_k \\
(I_k \otimes_i I_k) \otimes_j (I_k \otimes_i I_k) \arrow[rr, "\varpi_{i,k} \otimes_j \varpi_{i,k}"'] & \text{(U7)} & I_k \otimes_j I_k, \arrow[u, "\varpi_{jk}"]
\end{tikzcd}$$

$$\begin{tikzcd}
I_i \arrow[r, "\omega_{i,k}"] \arrow[d, "\tau_{i,j}"'] & I_i \otimes_k I_i \arrow[d, "\tau_{i,j} \otimes_k \tau_{i,j}"] \\
I_j \arrow[r, "\omega_{j,k}"'] & I_j \otimes_k I_j,
\end{tikzcd}
\quad \text{(U8)}$$

$$\begin{tikzcd}
I_j \otimes_i I_j \arrow[r, "\varpi_{i,j}"] \arrow[d, "\tau_{j,k} \otimes_i \tau_{j,k}"'] & I_j \arrow[d, "\tau_{j,k}"] \\
I_k \otimes_i I_k \arrow[r, "\varpi_{i,k}"'] & I_k,
\end{tikzcd}
\quad \text{(U9)}$$

$$\begin{tikzcd}
I_i \arrow[r, "\omega_{i,j}"] \arrow[d, "\tau_{i,k}"'] & I_i \otimes_j I_i \arrow[d, "\tau_{i,k} \otimes_j \tau_{i,k}"] \\
I_k & I_j \otimes_k I_j, \arrow[l, "\varpi_{j,k}"]
\end{tikzcd}
\quad \text{(U10)}$$

$$\begin{tikzcd}
& I_i \arrow[dl, "\tau_{i,j}"'] \arrow[dr, "\tau_{i,k}"] & \\
I_j \arrow[rr, "\tau_{j,k}"'] & \text{(U11)} & I_k.
\end{tikzcd}$$

Diagrams I.

and the diagram

(C2)
$$\begin{tikzcd}
X\otimes(Y\otimes V) \arrow[r, "\mathrm{id}\otimes T_{Y,V}"] \arrow[d, "\mathbf{a}^{-1}_{X,Y,V}"'] & X\otimes(V\otimes Y) \arrow[r, "\mathbf{a}^{-1}_{X,V,Y}"] & (X\otimes V)\otimes Y \arrow[d, "T_{X,V}\otimes\mathrm{id}"] \\
(X\otimes Y)\otimes V \arrow[r, "T_{X\otimes Y,V}"] & V\otimes(X\otimes Y) \arrow[r, "\mathbf{a}^{-1}_{X,Y,V}"] & (V\otimes X)\otimes Y
\end{tikzcd}$$

commutes for all objects $X, Y \in \mathcal{C}$;

- ▷ the morphism from $(V, T_{\_,V})$ to $(W, T_{\_,W})$ is a morphism $f\colon V \to W$ in $\mathcal{C}$, such that for any $X \in \mathcal{C}$ we have the commutative diagram

(C3)
$$\begin{tikzcd}
X\otimes V \arrow[r, "T_{X,V}"] \arrow[d, "\mathrm{id}\otimes f"'] & V\otimes X \arrow[d, "f\otimes\mathrm{id}"] \\
X\otimes W \arrow[r, "T_{X,W}"] & W\otimes X.
\end{tikzcd}$$

Note that $\mathcal{Z}(\mathcal{C})$ is a braided monoidal category with the following structures:

- ▷ the tensor product of $(V, T_{\_,V})$ and $(W, T_{\_,W})$ is $(V\otimes W, T_{\_,V\otimes W})$, where

$$T_{X,V\otimes W} := \mathbf{a}^{-1}_{V,W,X} \circ (\mathrm{id}_V \otimes T_{X,W}) \circ \mathbf{a}_{V,X,W} \circ (T_{X,V}\otimes \mathrm{id}_W) \circ \mathbf{a}^{-1}_{X,V,W};$$

- ▷ the unit object is $(I, T_{\_,I})$, where

$$T_{X,I} := \ X\otimes I \xrightarrow{\mathbf{r}_X} X \xrightarrow{\mathbf{l}^{-1}_X} I\otimes X$$

for any $X \in \mathcal{C}$;

- ▷ the braiding is given by

$$T_{V,W}\colon\ (V, T_{\_,V})\otimes(W, T_{\_,W}) \to (W, T_{\_,W})\otimes(V, T_{\_,V}) \quad \text{for any } V, W \in \mathcal{C}.$$

It is a direct computation to check that $(\mathcal{Z}(\mathcal{C}), \otimes, (I, T_{\_,I}), \mathbf{a}, \mathbf{l}, \mathbf{r}, T)$ is a braided category.

**2.3. BiHom-bialgebras.** In this section, we review several definitions and notations related to BiHom-bialgebras.

Recall from Section 5.2 of [31], that if $\mathcal{C} = (\mathcal{C}, \otimes, I, \mathbf{a}, \mathbf{l}, \mathbf{r})$ is a monoidal category, $\alpha = \{\alpha_X\colon X \to X\}_{X\in \mathrm{Obj}\mathcal{C}}$, $\beta = \{\beta_X\colon X \to X\}_{X\in\mathrm{Obj}\mathcal{C}}$ are two families of isomorphisms (not necessarily natural) satisfying the *monoidal condition*, i.e.,

$\alpha_{X\otimes Y} = \alpha_X \otimes \alpha_Y$ and $\beta_{X\otimes Y} = \beta_X \otimes \beta_Y$ for all $X, Y \in \mathcal{C}$, then we can define a new monoidal category $\mathcal{C}^{\{\alpha,\beta\}} = (\mathcal{C}, \otimes, I, \mathbf{a}^{\alpha,\beta}, \mathbf{l}^{\alpha,\beta}, \mathbf{r}^{\alpha,\beta})$ as follows:

▷ the objects in $\mathcal{C}^{\{\alpha,\beta\}}$ are the same with $\mathcal{C}$;

▷ the morphism $f\colon X \to Y$ in $\mathcal{C}^{\{\alpha,\beta\}}$ is a morphism $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ such that

$$\alpha_Y \circ f = f \circ \alpha_X, \quad \beta_Y \circ f = f \circ \beta_X;$$

▷ the tensor product and the unit object in $\mathcal{C}^{\{\alpha,\beta\}}$ are the same with $\mathcal{C}$;

▷ the associativity constraint $\mathbf{a}^{\alpha,\beta}$ and the unity constraints $\mathbf{l}^{\alpha,\beta}$, $\mathbf{r}^{\alpha,\beta}$ are given by

$$\begin{gathered}\mathbf{a}^{\alpha,\beta} = ((\alpha \otimes \mathrm{id}) \otimes \beta^{-1}) \circ \mathbf{a}\colon ({\_} \otimes {\_}) \otimes {\_} \Rightarrow {\_} \otimes ({\_} \otimes {\_});\\ \mathbf{l}^{\alpha,\beta} = \beta \circ \mathbf{l}\colon I \otimes {\_} \Rightarrow {\_}, \quad \mathbf{r}^{\alpha,\beta} = \alpha \circ \mathbf{r}\colon {\_} \otimes I \Rightarrow {\_}.\end{gathered}$$

Now let $\alpha$, $\beta$, $\varphi$, $\psi$ (the *Hom-structure maps*) be families of isomorphisms (not necessarily natural) in $\mathrm{Vec}_{\Bbbk}$ satisfying the monoidal condition, and commute with each other. Recall from Theorem 5.12 in [31], that there exists a 2-monoidal category $\mathrm{Vec}_{\Bbbk}^{\{\alpha,\beta,\varphi^{-1},\psi^{-1}\}} = (\mathrm{Vec}_{\Bbbk}^{\{\alpha,\beta\}}, \otimes, \mathrm{Vec}_{\Bbbk}^{\{\varphi^{-1},\psi^{-1}\}}, \otimes, \zeta, \omega, \varpi, \tau)$, where $\omega = \varpi = \tau = \mathrm{id}_{\Bbbk}$, and

$$\begin{aligned}\zeta_{A,B,A',B'}\colon\ & (A \otimes B) \otimes (A' \otimes B') \to (A \otimes A') \otimes (B \otimes B'),\\ & (a \otimes b) \otimes (a' \otimes b') \mapsto (a \otimes a') \otimes (b \otimes b') \quad \forall A, B, A', B' \in \mathrm{Vec}_{\Bbbk}.\end{aligned}$$

A *BiHom-algebra over* $\Bbbk$ is a monoid in $\mathrm{Vec}_{\Bbbk}^{\{\alpha,\beta\}}$. Namely, a BiHom-algebra $A$ is a 5-tuple $(A, \mu_A, 1_H, \alpha_A, \beta_A)$, where $A$ is a $k$-linear space, $1_A \in A$ is an element (the *unit*), $\alpha_A, \beta_A\colon A \to A$ are both bijective linear maps, $\mu_A\colon A \otimes A \to A$ is a linear map with the notation $\mu(a \otimes b) = ab$, satisfying the following conditions for all $a, b, c \in A$:

$$\begin{gathered}\alpha_A(1_A) = \beta_A(1_A) = 1_A, \quad a1_A = \alpha_A(a), \quad 1_A a = \beta_A(a), \quad \alpha_A(a)(bc) = (ab)\beta_A(c),\\ \alpha_A \circ \beta_A = \beta_A \circ \alpha_A, \quad \alpha_A(ab) = \alpha_A(a)\alpha_A(b), \quad \beta_A(ab) = \beta_A(a)\beta_A(b).\end{gathered}$$

**Remark 2.2.**

(1) Note that the second line of the above identities can be derived from the first line, see [31], Proposition 2.9.

(2) If a BiHom-algebra $A$ is commutative, then we immediately get that $\alpha_A = \beta_A$.

**Example 2.3.**

(1) If $A = (A, \mu_A, 1_A)$ is an associative algebra and $\alpha, \beta : A \to A$ are both algebra isomorphisms, then $(A, \mu_A \circ (\alpha \otimes \beta), 1_A, \alpha, \beta)$ is a BiHom-algebra.

(2) If $A = (A, \mu_A, 1_H, \alpha_A, \beta_A)$ is a BiHom-algebra, then $A^{\mathrm{op}} = (A, \mu_A^{\mathrm{op}}, 1_A, \beta_A, \alpha_A)$ is also a BiHom-algebra.

(3) If $\alpha = \beta$, then the BiHom-algebra becomes a Hom-algebra.

A *BiHom-coalgebra over* $\Bbbk$ is a comonoid in $\mathrm{Vec}_{\Bbbk}^{\{\varphi^{-1},\psi^{-1}\}}$. Precisely, a BiHom-coalgebra $C$ is a 5-tuple $(C,\Delta_C,\varepsilon_C,\varphi_C,\psi_C)$, in which $C$ is a linear space, $\varphi_C,\psi_C\colon C\to C$ are linear isomorphisms, $\varepsilon_C\colon C\to k$ and $\Delta_C\colon C\to C\otimes C$ are linear maps, such that

$$c_1\varepsilon_C(c_2)=\varphi_C(c),\quad \varepsilon_C(c_1)c_2=\psi_C(c),$$
$$\varepsilon_C(\varphi_C(c))=\varepsilon_C(\psi_C(c))=\varepsilon_C(c),\quad \varphi_C(c_1)\otimes\Delta_C(c_2)=\Delta_C(c_1)\otimes\psi_C(c_2),$$
$$\varphi_C\circ\psi_C=\psi_C\circ\varphi_C,\ \Delta_C(\varphi_C(c))=\varphi_C(c_1)\otimes\varphi_C(c_2),\ \Delta_C(\psi_C(c))=\psi_C(c_1)\otimes\psi_C(c_2).$$

**Remark 2.4.**

(1) Note that the third line of the above identities can be derived from the first two lines, see [31], Proposition 2.11.

(2) If a BiHom-coalgebra $C$ is cocommutative, then we immediately get that $\varphi_C=\psi_C$.

**Example 2.5.**

(1) If $(C,\Delta_C,\varepsilon_C)$ is a coassociative coalgebra, $\varphi,\psi\colon C\to C$ are both coalgebra isomorphisms, then $(C,(\varphi\otimes\psi)\circ\Delta_C,\varepsilon_C,\varphi,\psi)$ is a BiHom-coalgebra.

(2) If $C=(C,\Delta_C,\varepsilon_C,\varphi,\psi)$ is a BiHom-coalgebra, then $C^{\mathrm{cop}}=(C,\Delta_C^{\mathrm{cop}},\varepsilon_C,\psi,\varphi)$ is also a BiHom-coalgebra.

(3) If $\varphi=\psi$, then the BiHom-coalgebra becomes a Hom-coalgebra.

A *BiHom-bialgebra over* $\Bbbk$ is a bimonoid in the duoidal category $\mathrm{Vec}_{\Bbbk}^{\{\alpha,\beta,\varphi^{-1},\psi^{-1}\}}$. Namely, a BiHom-bialgebra $H$ is a 9-tuple $(H,\mu_H,1_H,\Delta_H,\varepsilon_H,\alpha_H,\beta_H,\varphi_H,\psi_H)$ with the property that $(H,\mu_H,1_H,\ \alpha_H,\ \beta_H)$ is a BiHom-algebra, $(H,\Delta_H,\varepsilon_H,\varphi_H,\psi_H)$ is a BiHom-coalgebra, and $\Delta_H,\varepsilon_H$ are all morphisms of BiHom-algebras preserving unit, i.e., for all $h,g\in H$,

$$\Delta_H(hg)=h_1g_1\otimes h_2g_2,\quad \varepsilon_H(hg)=\varepsilon_H(h)\varepsilon_H(g),\quad \Delta_H(1_H)=1_H\otimes 1_H,\varepsilon_H(1_H)=1_{\Bbbk}.$$

Moreover, it is easy to check that $\alpha_H$, $\beta_H$ are BiHom-coalgebra maps, $\varphi_H$, $\psi_H$ are BiHom-algebra maps, and they commute with each other, see [31], Proposition 2.14.

**Example 2.6.**

(1) If $(H,\mu_H,1_H,\Delta_H,\varepsilon_H)$ is a bialgebra and $\alpha,\beta,\varphi,\psi\colon H\to H$ are all bialgebra isomorphisms, then $H^{\mathrm{Bi}}=(H,\mu_H\circ(\alpha\otimes\beta),1_H,(\varphi\otimes\psi)\circ\Delta_H,\varepsilon_H,\alpha,\beta,\varphi,\psi)$ is a BiHom-bialgebra.

(2) If $H=(H,\mu_H,1_H,\Delta_H,\varepsilon_H,\alpha,\beta,\varphi,\psi)$ is a BiHom-bialgebra, then $H^{\mathrm{op}}=(H,\mu_H^{\mathrm{op}},1_H,\Delta_H,\varepsilon_H,\beta,\ \alpha,\varphi,\psi)$, $H^{\mathrm{cop}}=(H,\mu_H,1_H,\Delta_H^{\mathrm{cop}},\varepsilon_H,\alpha,\beta,\psi,\varphi)$ and $H^{\mathrm{op,cop}}=(H,\mu_H^{\mathrm{op}},1_H,\Delta_H^{\mathrm{cop}},\varepsilon_H,\beta,\alpha,\psi,\varphi)$ are all BiHom-bialgebras.

(3) If $H = (H, \mu_H, 1_H, \Delta_H, \varepsilon_H, \alpha, \beta, \varphi, \psi)$ is a finite dimensional BiHom-bialgebra, $H^* = \hom(H, \Bbbk)$. Define the multiplication $\star$, the comultiplication $\Delta_{H^*}$ (with the notation $\Delta_{H^*}(p) = p_1 \otimes p_2$) and $\varepsilon_{H^*}$ by

$$(p \star q)(h) = p(\alpha^{-1}\varphi^{-1}(h_1))q(\beta^{-1}\psi^{-1}(h_2)), \quad \varepsilon_{H^*}(p) = p(1_H),$$
$$(p_1 \otimes p_2)(h \otimes g) = p(\alpha^{-1}\psi^{-1}(h)\beta^{-1}\varphi^{-1}(g)),$$

where $p, q \in H^*$, $h, g \in H$. Define $\alpha_{H^*}$, $\beta_{H^*}$, $\varphi_{H^*}$, $\psi_{H^*}$ by

$$\alpha_{H^*}(p) = p \circ \alpha^{-1}, \quad \beta_{H^*}(p) = p \circ \beta^{-1}, \quad \varphi_{H^*}(p) = p \circ \psi^{-1}, \quad \psi_{H^*}(p) = p \circ \varphi^{-1}.$$

Then $H^* = (H^*, \star, \varepsilon_H, \Delta_{H^*}, \varepsilon_{H^*}, \alpha_{H^*}, \beta_{H^*}, \varphi_{H^*}, \psi_{H^*})$ is a BiHom-bialgebra.

Proof. We leave most of the details to the reader and only show $\Delta_{H^*}(p * q) = \Delta_{H^*}(p) * \Delta_{H^*}(q)$ for any $p, q \in H^*$. For all $a, b \in H$, we compute

$$\begin{aligned}
&(p_1 \star q_1 \otimes p_2 \star q_2)(a \otimes b)\\
&\quad = p_1(\alpha^{-1}\varphi^{-1}(a_1))q_1(\beta^{-1}\psi^{-1}(a_2))p_2(\alpha^{-1}\varphi^{-1}(b_1))q_2(\beta^{-1}\psi^{-1}(b_2))\\
&\quad = p(\alpha^{-2}\varphi^{-1}\psi^{-1}(a_1)\alpha^{-1}\beta^{-1}\varphi^{-2}(b_1))q(\alpha^{-1}\beta^{-1}\psi^{-2}(a_2)\beta^{-2}\varphi^{-1}\psi^{-1}(b_2))\\
&\quad = (p \star q)(\alpha^{-1}\psi^{-1}(a)\beta^{-1}\varphi^{-1}(b)) = (p \star q)_1(a) \otimes (p \star q)_2(b).
\end{aligned}$$

Therefore, $\Delta_{H^*}$ preserves the multiplication, as needed. □

(4) Let $H$ be a finite dimensional BiHom-bialgebra and $H^* = \hom(H, \Bbbk)$. We present another BiHom-bialgebra structure on $H^*$. Define the multiplication $*$ (the usual convolution product) and the comultiplication ${\Delta'}_{H^*}$ (with the notation $\Delta'_{H^*}(p) = p_{(1)} \otimes p_{(2)}$) by

$$(p * q)(h) = p(h_1)q(h_2), \quad (p_{(1)} \otimes p_{(2)})(h \otimes g) = p(hg),$$

where $p, q \in H^*$, $h, g \in H$. Define $\alpha'_{H^*}$, $\beta'_{H^*}$, $\varphi'_{H^*}$, $\psi'_{H^*}$ by

$${\alpha'}_{H^*}(p) = p \circ \alpha, \quad {\beta'}_{H^*}(p) = p \circ \beta, \quad {\varphi'}_{H^*}(p) = p \circ \varphi, {\psi'}_{H^*}(p) = p \circ \psi,$$

where $p \in H^*$. Then we immediately get that

$$H^* = (H^*, *, \varepsilon_H, {\Delta'}_{H^*}, \varepsilon_{H^*}, {\alpha'}_{H^*}, {\beta'}_{H^*}, {\varphi'}_{H^*}, {\psi'}_{H^*})$$

is also a BiHom-bialgebra.

(5) If $\alpha = \beta = \varphi = \psi$, then the BiHom-bialgebra becomes a Hom-bialgebra. If $\alpha^{-1} = \beta^{-1} = \varphi = \psi$, then the BiHom-bialgebra becomes a monoidal Hom-bialgebra.

**2.4. BiHom-Hopf algebras.** In this section we will present the definition of BiHom-Hopf algebras which is little different from [8], Definition 6.9.

**Definition 2.7.** Let $H = (H, \mu, 1_H, \Delta, \varepsilon, \alpha, \beta, \varphi, \psi)$ be a BiHom-bialgebra. If there exists $S\colon H \to H$ (the *antipode*) such that $S$ commutes with $\alpha$, $\beta$, $\varphi$, $\psi$, and satisfies, for any $h \in H$,

$$h_1 S(h_2) = S(h_1) h_2 = \varepsilon(h) 1_H,$$

then $(H, S)$ is called a *BiHom-Hopf algebra*.

**Proposition 2.8.** *If $H$ is a BiHom-Hopf algebra, then for any $a, b \in H$, the antipode $S$ satisfies*

$$S(ab) = S\alpha^{-1}\beta(b) S\alpha\beta^{-1}(a), \quad S(1_H) = 1_H, \tag{2.1}$$

$$\Delta(S(a)) = S\varphi\psi^{-1}(a_2) \otimes S\varphi^{-1}\psi(a_1), \quad \varepsilon \circ S = \varepsilon. \tag{2.2}$$

Proof. We only prove (2.2). Consider the morphisms $\Theta, \Lambda \in \hom(H, H \otimes H)$ given by

$$\Theta(a) = S\varphi\psi^{-1}(a_2) \otimes S\varphi^{-1}\psi(a_1), \quad \Lambda(a) = \underline{S(a)}_1 \otimes \underline{S(a)}_2 \quad \forall\, a \in H.$$

Now we show that $\Lambda$ is the left convolution inverse, and $\Theta$ is the right convolution inverse of the comultiplication $\Delta$. We compute

$$(\Lambda * \Delta)(a) = (\underline{S(a_1)}_1 \otimes \underline{S(a_1)}_2)(a_{21} \otimes a_{22}) = \underline{S(a_1)a_2}_1 \otimes \underline{S(a_1)a_2}_2 = \varepsilon(a) 1_H \otimes 1_H$$

and

$$\begin{aligned}(\Delta * \Theta)(a) &= a_{11} S\varphi\psi^{-1}(a_{22}) \otimes a_{12} S\varphi^{-1}\psi(a_{21}) \\ &= \varphi(a_1) S\varphi\psi^{-1}(a_{22}) \otimes \varphi^{-1}(a_{211}) S\varphi^{-1}(a_{212}) \\ &= \varphi(a_1) S\varphi(a_2) \otimes 1_H = \varepsilon(a) 1_H \otimes 1_H,\end{aligned}$$

which implies the first formula of (2.2). To prove the second one, we have

$$\varepsilon(S(a)) = \varepsilon(S\psi^{-1}(\varepsilon(a_1)a_2)) = \varepsilon(a_1)\varepsilon(S(a_2)) = \varepsilon(a_1 S(a_2)) = \varepsilon(a),$$

hence, (2.2) holds. ☐

**Proposition 2.9.** *If $H$ is a BiHom-Hopf algebra, then*

(1) *the antipode $S$ satisfies*

$$S\alpha^2\varphi^2 = S\beta^2\psi^2; \tag{2.3}$$

(2) *if $S$ is a bijective map, then*

$$\alpha^2\varphi^2 = \beta^2\psi^2. \tag{2.4}$$

Proof. We only need to check (2.3). Indeed, for any $a \in H$, we have

$$\begin{aligned}S(a) &= \varepsilon(a_1)S(\alpha^{-1}\psi^{-1}(a_2)1_H) \overset{(2.1)}{=} \varepsilon(a_1)1_H S\beta^{-1}\psi^{-1}(a_2)\\ &= (S\beta^{-2}\psi^{-2}(a_{11})\beta^{-2}\psi^{-2}(a_{12}))S\beta^{-1}\psi^{-1}(a_2)\\ &= S\alpha\beta^{-2}\varphi\psi^{-2}(a_1)(\beta^{-2}\psi^{-2}(a_{21})S\beta^{-2}\psi^{-2}(a_{22})) = S\alpha^2\beta^{-2}\varphi^2\psi^{-2}(a),\end{aligned}$$

which implies (2.3). □

**Proposition 2.10.** *Assume that $H$ is a BiHom-Hopf algebra with the bijective antipode $S$. Then for any $a, b \in H$, $S^{-1}$ satisfies*

$$S^{-1}(ab) = S^{-1}\alpha^{-1}\beta(b)S^{-1}\alpha\beta^{-1}(a), S^{-1}(1_H) = 1_H, \tag{2.5}$$

$$\Delta(S^{-1}(a)) = S^{-1}\varphi\psi^{-1}(a_2) \otimes S^{-1}\varphi^{-1}\psi(a_1), \quad \varepsilon \circ S^{-1} = \varepsilon, \tag{2.6}$$

$$S^{-1}\alpha^{-2}\beta^2(a_2)a_1 = a_2S^{-1}\alpha^2\beta^{-2}(a_1) = \varepsilon(a)1_H. \tag{2.7}$$

Proof. We only check (2.5). Actually, we have

$$S(S^{-1}(ab)) = ab = S(S^{-1}(a))S(S^{-1}(b)) \overset{(2.1)}{=} S(S^{-1}\alpha^{-1}\beta(b)S^{-1}\alpha\beta(b)^{-1}),$$

which implies the conclusion. □

**Remark 2.11.**

(1) Obviously (2.7) is equal to

$$S^{-1}\varphi^2\psi^{-2}(a_2)a_1 = a_2S^{-1}\varphi^{-2}\psi^2(a_1) = \varepsilon(a)1_H. \tag{2.8}$$

(2) The antipode of a BiHom-bialgebra $H$ may not exist. Namely, some special BiHom-bialgebras cannot become BiHom-Hopf algebras, see Example 2.12 (3) and (4) below.

**Example 2.12.**

(1) If $(H, S, \mu_H, 1_H, \Delta_H, \varepsilon_H)$ is a bialgebra, $\alpha, \beta, \varphi, \psi\colon H \to H$ are all Hopf algebra isomorphisms and satisfy $S\alpha^2\varphi^2 = S\beta^2\psi^2$, then

$$H^{\mathrm{BiH}} = (H, S, \mu_H \circ (\alpha \otimes \beta), 1_H, (\varphi \otimes \psi) \circ \Delta_H, \varepsilon_H, \alpha, \beta, \varphi, \psi)$$

is a BiHom-Hopf algebra.

(2) Let $H_4$ be Sweedler's 4-dimensional Hopf algebra $H_4 = \Bbbk\{1_H, g, x, y\colon g^2 = 1_H, x^2 = 0, y = gx = -xg\}$ with the following structures:

$$\begin{aligned}&\Delta(g) = g \otimes g, \Delta(x) = x \otimes 1_H + g \otimes x, \quad \Delta(y) = y \otimes g + 1_H \otimes y,\\ &\varepsilon(g) = 1, \quad \varepsilon(x) = \varepsilon(y) = 0, \quad S(g) = g, \quad S(x) = -y, S(y) = x.\end{aligned}$$

Recall from [25], Example 10.1.17 that $H_4$ is a quasitriangular Hopf algebra with the $R$-matrix

$$\mathrm{R}_\kappa = \frac{1}{2}(1_H \otimes 1_H + 1_H \otimes g + g \otimes 1_H - g \otimes g) + \frac{\kappa}{2}(x \otimes x - x \otimes y + y \otimes x + y \otimes y),$$

where $\kappa \in \Bbbk$.

Note that any bialgebra isomorphism $\alpha\colon H_4 \to H_4$ takes the form

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & b & a \end{pmatrix},$$

where $a, b \in \Bbbk$ and $ab \neq 0$. Moreover, $\alpha$ is a Hopf algebra map if and only if $b = 0$.

It is easy to check that the group of the bialgebra automorphisms $\mathrm{Aut}(H_4)$ is an Abelian group. From Example 2.6 (1) it follows that for any $\alpha, \beta, \varphi, \psi \in \mathrm{Aut}(H_4)$, $H_4^{\mathrm{Bi}} = (H_4, \mu \circ (\alpha \otimes \beta), 1_{H_4}, (\varphi \otimes pr) \circ \Delta, \varepsilon, \alpha, \beta, \varphi, \psi)$ is a BiHom-bialgebra.

Moreover, assume that

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \tag{2.9}$$

$$\varphi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{pmatrix}, \quad \psi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & d \end{pmatrix},$$

where $a, b, c, d \in \Bbbk$. Obviously they are all Hopf algebra isomorphisms. If $a^2c^2 = b^2d^2$, then from (1), $H_4^{\mathrm{BiH}} = (H_4^{\mathrm{Bi}}, S)$ is a BiHom-Hopf algebra.

(3) If $H = (H, S, \mu, 1_H, \Delta, \varepsilon, \alpha, \beta, \varphi, \psi)$ is a BiHom-Hopf algebra, then recall from Example 2.6 (2) that $H^{\mathrm{op}}$, $H^{\mathrm{cop}}$ and $H^{\mathrm{op,cop}}$ are all BiHom-bialgebras, and $(H^{\mathrm{op,cop}}, S, \mu, 1_H, \Delta, \varepsilon, \beta, \alpha, \psi, \varphi)$ is a BiHom-Hopf algebra. Moreover, neither $H^{\mathrm{op}}$ nor $H^{\mathrm{cop}}$ have antipode.

(4) If $H = (H, S, \mu, 1_H, \Delta, \varepsilon, \alpha, \beta, \varphi, \psi)$ is a BiHom-Hopf algebra, then recall from Example 2.6 (3) that $H^* = (H^*, \star, \varepsilon, \Delta_{H^*}, \varepsilon_{H^*}, (\alpha^{-1})^*, (\beta^{-1})^*, (\psi^{-1})^*, (\varphi^{-1})^*)$ is a BiHom-bialgebra. Clearly there is no $S_{H^*} \in \hom(H^*, H^*)$ such that $(H^*, S_{H^*})$ becomes a BiHom-Hopf algebra.

(5) Under the same consideration above, if $S$ is a bijective map, then it is not hard to check that $H^{*\mathrm{cop}} = (H^*, \star, \varepsilon, \Delta^{\mathrm{cop}}_{H^*}, \varepsilon_{H^*}, (S^{-1})^*, (\alpha^{-1})^*, (\beta^{-1})^*, (\varphi^{-1})^*, (\psi^{-1})^*)$ and $H^{*\mathrm{op}} = (H^*, \star^{\mathrm{op}}, \varepsilon, \Delta_{H^*}, \varepsilon_{H^*}, (S^{-1})^*, (\beta^{-1})^*, (\alpha^{-1})^*, (\psi^{-1})^*, (\varphi^{-1})^*)$ are all BiHom-Hopf algebras.

(6) If $H = (H, S, \mu, 1_H, \Delta, \varepsilon, \alpha, \beta, \varphi, \psi)$ is a BiHom-Hopf algebra, then recall from Example 2.6 (4) that $H^* = (H^*, *, \varepsilon, \Delta'_{H^*}, \varepsilon_{H^*}, \alpha^*, \beta^*, \varphi^*, \psi^*)$ is a BiHom-bialgebra. Moreover, $(H^*, S^*)$ is a BiHom-Hopf algebra.

(7) If $\alpha = \beta$ and $\varphi = \psi$, then the BiHom-Hopf algebra becomes the so-called monoidal BiHom-Hopf algebra. If $\alpha = \beta = \varphi = \psi$, then the BiHom-Hopf algebra becomes the usual Hom-Hopf algebra. Similarly, if $\alpha^{-1} = \beta^{-1} = \varphi = \psi$, then the BiHom-Hopf algebra becomes the usual monoidal Hom-Hopf algebra.

## 3. The BiHom-type representations with parameters

In this section, we always assume that $H = (H, S, \mu, 1_H, \Delta, \varepsilon, \alpha, \beta, \varphi, \psi)$ is a BiHom-Hopf algebra.

**3.1. The category of BiHom-modules.** Recall that a $\Bbbk$-space $M$ is caled a *left BiHom-module of* $H$ (shortly an $H$*-BiHom-module*) if there exist $\Bbbk$-linear isomorphisms $\alpha_M, \beta_M, \varphi_M, \psi_M \colon M \to M$ (the *Hom-structure maps*) and an $H$ action $\theta_M \colon H \otimes M \to M$ (with the notation $\theta_M(h \otimes m) = h \cdot m$), such that for any $h, g \in H$, $m \in M$,

$$\alpha_M,\ \beta_M,\ \varphi_M,\ \psi_M \quad \text{commute with each other,}$$
$$\alpha(h) \cdot \alpha_M(m) = \alpha_M(h \cdot m), \quad \beta(h) \cdot \beta_M(m) = \beta_M(h \cdot m),$$
$$\varphi(h) \cdot \varphi_M(m) = \varphi_M(h \cdot m), \quad \psi(h) \cdot \psi_M(m) = \psi_M(h \cdot m),$$
$$\alpha(h) \cdot (g \cdot m) = (hg) \cdot \beta_M(m), \quad 1_H \cdot m = \beta_M(m).$$

If $(M, \alpha_M, \beta_M, \varphi_M, \psi_M)$ and $(N, \alpha_N, \beta_N, \varphi_N, \psi_N)$ are left $H$-BiHom-modules with $H$-actions $\theta_M$ and $\theta_N$, respectively, a *morphism of* $H$*-BiHom-modules* $f \in \hom_{\Bbbk}(M, N)$ is an $H$-linear map satisfying the conditions

$$\alpha_N \circ f = f \circ \alpha_M, \quad \beta_N \circ f = f \circ \beta_M, \quad \varphi_N \circ f = f \circ \varphi_M, \quad \psi_N \circ f = f \circ \psi_M.$$

The category of $H$-BiHom-modules and morphisms is denoted by ${}_H\mathcal{BM}$.

**Remark 3.1.**

(1) Obviously $H \in \mathrm{Obj}({}_H\mathcal{BM})$.

(2) If $\alpha = \beta = \varphi = \psi$ and $\alpha_M = \beta_M = \varphi_M = \psi_M$, then the BiHom-modules become the Hom-modules of Hom-bialgebra $H$.

(3) The definition of the right BiHom-module of $H$ can be given in a similar way.

For any integers $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{g}, \mathfrak{h}, \mathfrak{i}, \mathfrak{j} \in \mathbb{Z}$, $M, N, P \in {}_H\mathcal{BM}$, consider the following structures:

▷ the tensor product of $(M, \alpha_M, \beta_M, \varphi_M, \psi_M)$ and $(N, \alpha_N, \beta_N, \varphi_N, \psi_N)$ is

$$(M \otimes N, \alpha_{M\otimes N}, \alpha_M \otimes \alpha_N, \beta_M \otimes \beta_N, \varphi_M \otimes \varphi_N, \psi_M \otimes \psi_N),$$

where the $H$-action on $M \otimes N$ is given by

$$h \cdot (m \otimes n) = \alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\varphi^{\mathfrak{c}}\psi^{\mathfrak{d}}(h_1) \cdot m \otimes \alpha^{\mathfrak{g}}\beta^{\mathfrak{h}}\varphi^{\mathfrak{i}}\psi^{\mathfrak{j}}(h_2) \cdot n,$$

where $m \in M$, $n \in N$, $h \in H$;

▷ the unit object is $(\Bbbk, \mathrm{id}_\Bbbk, \mathrm{id}_\Bbbk, \mathrm{id}_\Bbbk, \mathrm{id}_\Bbbk)$ with the trivial module action;

▷ for any $m \in M$, $n \in N$, $p \in P$, the associativity and the unit constraints are given by

$$\mathbf{a}_{M,N,P}((m \otimes n) \otimes p) = \alpha_M^{-\mathfrak{a}}\beta_M^{-\mathfrak{b}}\varphi_M^{-\mathfrak{c}-1}\psi_M^{-\mathfrak{d}}(m) \otimes (n \otimes \alpha_P^{\mathfrak{g}}\beta_P^{\mathfrak{h}}\varphi_P^{\mathfrak{i}}\psi_P^{\mathfrak{j}+1}(p));$$
$$\mathbf{l}_M(1_\Bbbk \otimes m) = \alpha_M^{-\mathfrak{g}}\beta_M^{-\mathfrak{h}}\varphi_M^{-\mathfrak{i}}\psi_M^{-\mathfrak{j}-1}(m), \quad \mathbf{r}_M(m \otimes 1_\Bbbk) = \alpha_M^{-\mathfrak{a}}\beta_M^{-\mathfrak{b}}\varphi_M^{-\mathfrak{c}-1}\psi_M^{-\mathfrak{d}}(m).$$

**Theorem 3.2.** *${}_H\mathcal{BM}$ forms a monoidal category under the above structures.*

Proof. We only show the linearity of $\mathbf{a}$ and the others are left to reader. For any $h \in H$, we have

$$\begin{aligned}
&\mathbf{a}_{M,N,P}(h \cdot ((m \otimes n) \otimes p)) \\
&\quad = \mathbf{a}_{M,N,P}((\alpha^{2\mathfrak{a}}\beta^{2\mathfrak{b}}\varphi^{2\mathfrak{c}}\psi^{2\mathfrak{d}}(h_{11}) \cdot m \otimes \alpha^{\mathfrak{a}+\mathfrak{g}}\beta^{\mathfrak{b}+\mathfrak{h}}\varphi^{\mathfrak{c}+\mathfrak{i}}\psi^{\mathfrak{d}+\mathfrak{j}}(h_{21}) \cdot n) \\
&\qquad \otimes \alpha^{\mathfrak{g}}\beta^{\mathfrak{h}}\varphi^{\mathfrak{i}}\psi^{\mathfrak{j}}(h_2) \cdot p) \\
&\quad = \alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\varphi^{\mathfrak{c}}\psi^{\mathfrak{d}}(h_1) \cdot \alpha_M^{-\mathfrak{a}}\beta_M^{-\mathfrak{b}}\varphi_M^{-\mathfrak{c}-1}\psi_M^{-\mathfrak{d}}(m) \otimes (\alpha^{\mathfrak{a}+\mathfrak{g}}\beta^{\mathfrak{b}+\mathfrak{h}}\varphi^{\mathfrak{c}+\mathfrak{i}}\psi^{\mathfrak{d}+\mathfrak{j}}(h_{12}) \cdot n \\
&\qquad \otimes \alpha^{2\mathfrak{g}}\beta^{2\mathfrak{h}}\varphi^{2\mathfrak{i}}\psi^{2\mathfrak{j}}(h_{22}) \cdot \alpha_P^{\mathfrak{g}}\beta_P^{\mathfrak{h}}\varphi_P^{\mathfrak{i}}\psi_P^{\mathfrak{j}+1}(p)) \\
&\quad = h \cdot \mathbf{a}_{M,N,P}(((m \otimes n) \otimes p)),
\end{aligned}$$

which implies the conclusion. □

**Remark 3.3.** We write ${}_H\mathcal{BM}$ by ${}_H\mathcal{BM}^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}}$ if the monoidal structure of ${}_H\mathcal{BM}$ is given as above.

Now, if we assume that the antipode $S$ is a bijective map, then we have the following result.

**Theorem 3.4.** *${}_H\mathcal{BM}^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}}$ is a rigid category.*

Proof. Suppose that $(M, \alpha_M, \beta_M, \varphi_M, \psi_M)$ is an $H$-BiHom- module, $M^* = \mathrm{hom}(M, \Bbbk)$. Define the following structures:

▷ $\theta_{M^*}\colon\ H \otimes M^* \to M^*$ is given by

$$(h \cdot f)(m) = f(S(h)) \cdot \beta_M^{-2}(m) \quad \forall\, h \in H,\ m \in M,\ f \in M^*;$$

▷ the Hom-structure maps of $M^*$ are given by

$$\alpha_{M^*} = (\alpha_M^{-1})^*, \quad \beta_{M^*} = (\beta_M^{-1})^*, \quad \varphi_{M^*} = (\varphi_M^{-1})^*, \quad \psi_{M^*} = (\psi_M^{-1})^*;$$

▷ the evaluation $\mathrm{ev}_M\colon\ M^* \otimes M \to \Bbbk$ and coevaluation $\mathrm{coev}_M\colon\ \Bbbk \to M \otimes M^*$ are given by

$$\begin{aligned}
\mathrm{ev}_M(f \otimes m) &= f(\alpha_M^{\mathfrak{a}-\mathfrak{g}-1}\beta_M^{\mathfrak{b}-\mathfrak{h}+2}\varphi_M^{\mathfrak{c}-\mathfrak{i}}\psi_M^{\mathfrak{d}-\mathfrak{j}}(m)),\\
\mathrm{coev}_M(1_\Bbbk) &= \sum \alpha_M^{\mathfrak{a}-\mathfrak{g}-1}\beta_M^{\mathfrak{b}-\mathfrak{h}}\varphi_M^{\mathfrak{c}-\mathfrak{i}}\psi_M^{\mathfrak{d}-\mathfrak{j}}(e_i) \otimes e^i,
\end{aligned}$$

where $e_i$ and $e^i$ are dual bases of $M$ and $M^*$, respectively.

Next we show that $(M^*, \mathrm{ev}_M, \mathrm{coev}_M)$ is the left dual of $M$. Firstly, it is easy to check that $(M^*, \alpha_{M^*}, \beta_{M^*}, \varphi_{M^*}, \psi_{M^*})$ is an $H$-BiHom-module. Secondly, since we have

$$\begin{aligned}
\mathrm{ev}_M(h \cdot (f \otimes m)) &= (\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\varphi^{\mathfrak{c}}\psi^{\mathfrak{d}}(h_1) \cdot f)(\alpha_M^{\mathfrak{a}-\mathfrak{g}-1}\beta_M^{\mathfrak{b}-\mathfrak{h}+2}\varphi_M^{\mathfrak{c}-\mathfrak{i}}\psi_M^{\mathfrak{d}-\mathfrak{j}}\\
&\quad \times (\alpha^{\mathfrak{g}}\beta^{\mathfrak{h}}\varphi^{\mathfrak{i}}\psi^{\mathfrak{j}}(h_2) \cdot m))\\
&= f((S\alpha^{\mathfrak{a}-1}\beta^{\mathfrak{b}}\varphi^{\mathfrak{c}}\psi^{\mathfrak{d}}(h_1)\alpha^{\mathfrak{a}-1}\beta^{\mathfrak{b}}\varphi^{\mathfrak{c}}\psi^{\mathfrak{d}}(h_2))\\
&\quad \times \alpha_M^{\mathfrak{a}-\mathfrak{g}-1}\beta_M^{\mathfrak{b}-\mathfrak{h}+1}\varphi_M^{\mathfrak{c}-\mathfrak{i}}\psi_M^{\mathfrak{d}-\mathfrak{j}}(m))\\
&= h \cdot \mathrm{ev}_M(f \otimes m)
\end{aligned}$$

for any $h \in H$, $m \in M$, $f \in M^*$, $\mathrm{ev}_M$ is $H$-linear. Similarly, $\mathrm{coev}_M$ is also an $H$-linear map.

At last, we obtain

$$\begin{aligned}
&((\mathrm{id}_M \otimes \mathrm{ev}_M) \circ \mathbf{a}_{M,M^*,M} \circ (\mathrm{coev}_M \otimes \mathrm{id}_M))(1_\Bbbk \otimes m)\\
&\quad = \sum (\mathrm{id}_M \otimes \mathrm{ev}_M)(\alpha_M^{-\mathfrak{g}-1}\beta_M^{-\mathfrak{h}}\varphi_M^{-\mathfrak{i}-1}\psi_M^{-\mathfrak{j}}(e_i) \otimes (e^i \otimes (\alpha_M^{\mathfrak{g}}\beta_M^{\mathfrak{h}}\varphi_M^{\mathfrak{i}}\psi_M^{\mathfrak{j}+1}(m))))\\
&\quad = \alpha_M^{\mathfrak{a}-\mathfrak{g}-2}\beta_M^{\mathfrak{b}-\mathfrak{h}+2}\varphi_M^{\mathfrak{c}-\mathfrak{i}-1}\psi_M^{\mathfrak{d}-\mathfrak{j}+1}(m)\\
&\quad \overset{(2.4)}{=} \alpha_M^{\mathfrak{a}-\mathfrak{g}}\beta_M^{\mathfrak{b}-\mathfrak{h}}\varphi_M^{\mathfrak{c}-\mathfrak{i}+1}\psi_M^{\mathfrak{d}-\mathfrak{j}-1}(m) = (\mathbf{r}_M^{-1} \circ \mathbf{l}_M)(1_\Bbbk \otimes m)
\end{aligned}$$

and similarly we can get

$$(\mathrm{ev}_M \otimes \mathrm{id}_{M^*}) \circ \mathbf{a}_{M^*,M,M^*}^{-1} \circ (\mathrm{id}_{M^*} \otimes \mathrm{coev}_M) = \mathbf{l}_{M^*}^{-1} \circ \mathbf{r}_{M^*},$$

which implies that ${}_H\mathcal{BM}_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}}^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}$ is a left rigid category.

Similarly, one can show that ${}_H\mathcal{BM}^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}}$ is a right rigid category. Indeed, the right dual of $M$ is ${}^*M = \hom(M, \Bbbk)$, where

- ▷ $\theta_{{}^*M}\colon H \otimes {}^*M \to {}^*M$ is given by

$$(h \cdot f)(m) = f(S^{-1}(h)) \cdot \beta_M^{-2}(m) \quad \forall\, h \in H,\ m \in M,\ f \in {}^*M;$$

- ▷ the Hom-structure maps of ${}^*M$ are given by

$$\alpha_{{}^*M} = (\alpha_M^{-1})^*, \quad \beta_{{}^*M} = (\beta_M^{-1})^*, \quad \varphi_{{}^*M} = (\varphi_M^{-1})^*, \quad \psi_{{}^*M} = (\psi_M^{-1})^*;$$

- ▷ the evaluation $\widetilde{\mathrm{ev}}_M\colon M \otimes {}^*M \to \Bbbk$ and coevaluation $\widetilde{\mathrm{coev}}_M\colon \Bbbk \to {}^*M \otimes M$ are given by

$$\begin{aligned}\widetilde{\mathrm{ev}}_M(m \otimes f) &= f(\alpha_M^{-\mathfrak{a}+\mathfrak{g}+1}\beta_M^{-\mathfrak{b}+\mathfrak{h}}\varphi_M^{-\mathfrak{c}+\mathfrak{i}}\psi_M^{-\mathfrak{d}+\mathfrak{j}}(m)),\\ \widetilde{\mathrm{coev}}_M(1_\Bbbk) &= \sum e^i \otimes \alpha_M^{-\mathfrak{a}+\mathfrak{g}+1}\beta_M^{-\mathfrak{b}+\mathfrak{h}-2}\varphi_M^{-\mathfrak{c}+\mathfrak{i}}\psi_M^{-\mathfrak{d}+\mathfrak{j}}(e_i),\end{aligned}$$

where $e_i$ and $e^i$ are dual bases of $M$ and ${}^*M$, respectively.

□

**Proposition 3.5.** *For any* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{g}, \mathfrak{h}, \mathfrak{i}, \mathfrak{j}, \mathfrak{a}', \mathfrak{b}', \mathfrak{c}', \mathfrak{d}', \mathfrak{g}', \mathfrak{h}', \mathfrak{i}', \mathfrak{j}' \in \mathbb{Z}$, ${}_H\mathcal{BM}^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}}$ *is monoidal isomorphic to* ${}_H\mathcal{BM}^{\mathfrak{a}',\mathfrak{b}',\mathfrak{c}',\mathfrak{d}'}_{\mathfrak{g}',\mathfrak{h}',\mathfrak{i}',\mathfrak{j}'}$.

Proof. Define the functor $\mathscr{S} = (\mathscr{S}, \mathscr{S}_2, \mathscr{S}_0)\colon {}_H\mathcal{BM}^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}} \to {}_H\mathcal{BM}^{\mathfrak{a}',\mathfrak{b}',\mathfrak{c}',\mathfrak{d}'}_{\mathfrak{g}',\mathfrak{h}',\mathfrak{i}',\mathfrak{j}'}$ by

$$\mathscr{F}(M) = M \quad \text{as } H\text{-BiHom-module}, \quad \mathscr{F}(f) = f,$$

where $(M, \alpha_M, \beta_M, \varphi_M, \psi_M) \in {}_H\mathcal{BM}$, $f \in \mathrm{Mor}({}_H\mathcal{BM})$. Further, $\mathscr{S}_0 = \mathrm{id}$, $\mathscr{S}_{2M,N}$ is given by

$$\mathscr{S}_{2M,N}(m \otimes n) = \alpha_M^{\mathfrak{a}-\mathfrak{a}'}\beta_M^{\mathfrak{b}-\mathfrak{b}'}\varphi_M^{\mathfrak{c}-\mathfrak{c}'}\psi_M^{\mathfrak{d}-\mathfrak{d}'}(m) \otimes \alpha_N^{\mathfrak{g}-\mathfrak{g}'}\beta_N^{\mathfrak{h}-\mathfrak{h}'}\varphi_N^{\mathfrak{i}-\mathfrak{i}'}\psi_N^{\mathfrak{j}-\mathfrak{j}'}(n)$$

for any $M, N \in {}_H\mathcal{BM}$, $m \in M$, $n \in N$. It is direct to check that $\mathscr{S} = (\mathscr{S}, \mathscr{S}_2, \mathscr{S}_0)$ is a monoidal isomorphic functor. □

**Theorem 3.6.** *Let* $n \in \mathbb{Z}^+$. *If* $H$ *is a cocommutative BiHom-bialgebra, then* ${}_H\mathcal{BM}$ *forms an* $n$*-monoidal category.*

Proof. Let $\mathfrak{a}_i, \mathfrak{b}_i, \mathfrak{c}_i, \mathfrak{d}_i, \mathfrak{g}_i, \mathfrak{h}_i, \mathfrak{i}_i, \mathfrak{j}_i \in \mathbb{Z}$, where $i = 1, 2, \ldots, n$. We use the notation ${}_H\mathcal{BM}^{\mathfrak{a}_i,\mathfrak{b}_i,\mathfrak{c}_i,\mathfrak{d}_i}_{\mathfrak{g}_i,\mathfrak{h}_i,\mathfrak{i}_i,\mathfrak{j}_i} = ({}_H\mathcal{BM}^{\mathfrak{a}_i,\mathfrak{b}_i,\mathfrak{c}_i,\mathfrak{d}_i}_{\mathfrak{g}_i,\mathfrak{h}_i,\mathfrak{i}_i,\mathfrak{j}_i}, \otimes_i, \Bbbk, \mathbf{a}_i, \mathbf{l}_i, \mathbf{r}_i)$. For any $i, j \in \{1, 2, \ldots, n\}$, we define the following maps in ${}_H\mathcal{BM}$:

- ▷ the interchange law is

$$\begin{aligned}\zeta_{i,j}A, B, C, D\colon\ &(A \otimes_i B) \otimes_j (C \otimes_i D) \to (A \otimes_j C) \otimes_i (B \otimes_j D),\\ &(a \otimes_i b) \otimes_j (c \otimes_i d) \mapsto (a \otimes_j c) \otimes_i (b \otimes_j d),\end{aligned}$$

  where $A, B, B, C, D \in {}_H\mathcal{BM}$;

- ▷ since $(\Bbbk, \mathrm{id}_\Bbbk, \mathrm{id}_\Bbbk, \mathrm{id}_\Bbbk, \mathrm{id}_\Bbbk)$ is the unit object of all the monoidal structures, $\omega$, $\varpi$ and $\tau$ are given by

$$\omega_{i,j} = \varpi_{i,j} = \tau_{i,j} = \mathrm{id}_\Bbbk.$$

Clearly $\zeta_{i,j}$ is a natural isomorphism in ${}_H\mathcal{BM}$ since $H$ is cocommutative.

Moreover, note that if the diagrams (A1)–(A3) and diagrams (U1)–(U11) are satisfied, then $({}_H\mathcal{BM}^{\mathfrak{a}_i,\mathfrak{b}_i,\mathfrak{c}_i,\mathfrak{d}_i}_{\mathfrak{g}_i,\mathfrak{h}_i,\mathfrak{i}_i,\mathfrak{j}_i}, \otimes_i, \Bbbk, \mathbf{a}_i, \mathbf{l}_i, \mathbf{r}_i)$ forms an $n$-monoidal category. ☐

**Remark 3.7.** Let $H$ be cocommutative. Since $\zeta$, $\omega$, $\varpi$, $\tau$ given above are all isomorphisms, then for the given integers $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{i}$, $\mathfrak{j}$ and $\mathfrak{a}'$, $\mathfrak{b}'$, $\mathfrak{c}'$, $\mathfrak{d}'$, $\mathfrak{g}'$, $\mathfrak{h}'$, $\mathfrak{i}'$, $\mathfrak{j}'$, ${}_H\mathcal{BM} = ({}_H\mathcal{BM}^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}}, {}_H\mathcal{BM}^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}})$ forms a strong 2-monoidal category (see [1], Definition 6.3) for the definition of strong 2-monoidal category).

### 3.2. Quasitriangular BiHom-Hopf algebras and BiHom-Type QYBE.

Let $\mathrm{R}$ and $\mathrm{R}'$ be two elements in $H \otimes H$. For any $M, N \in {}_H\mathcal{BM}^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}}$, we can define families of maps $\mathbf{C}\colon \otimes \Rightarrow \otimes^{\mathrm{op}}$ and $\mathbf{C}'\colon \otimes^{\mathrm{op}} \Rightarrow \otimes$ as follows:

- ▷ $\mathbf{C}_{M,N}\colon M \otimes N \to N \otimes M$ is given by

$$\begin{aligned}m \otimes n \mapsto \sum &\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\varphi^{\mathfrak{c}}\psi^{\mathfrak{d}}(\mathrm{R}^{(2)}) \cdot \alpha_N^{\mathfrak{a}-\mathfrak{g}}\beta_N^{\mathfrak{b}-\mathfrak{h}-1}\varphi_N^{\mathfrak{c}-\mathfrak{i}+1}\psi_N^{\mathfrak{d}-\mathfrak{j}-1}(n)\\ &\otimes \alpha^{\mathfrak{g}}\beta^{\mathfrak{h}}\varphi^{\mathfrak{i}}\psi^{\mathfrak{j}}(\mathrm{R}^{(1)}) \cdot \alpha_M^{-\mathfrak{a}+\mathfrak{g}}\beta_M^{-\mathfrak{b}+\mathfrak{h}-1}\varphi_M^{-\mathfrak{c}+\mathfrak{i}-1}\psi_M^{-\mathfrak{d}+\mathfrak{j}+1}(m).\end{aligned} \tag{3.1}$$

- ▷ $\mathbf{C}'_{M,N}\colon N \otimes M \to M \otimes N$ is given by

$$\begin{aligned}n \otimes m \mapsto \sum &\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\varphi^{\mathfrak{c}+1}\psi^{\mathfrak{d}-1}(\mathrm{R}'^{(1)}) \cdot \alpha_M^{\mathfrak{a}-\mathfrak{g}}\beta_M^{\mathfrak{b}-\mathfrak{h}-1}\varphi_M^{\mathfrak{c}-\mathfrak{i}+1}\psi_M^{\mathfrak{d}-\mathfrak{j}-1}(m)\\ &\otimes \alpha^{\mathfrak{g}}\beta^{\mathfrak{h}}\varphi^{\mathfrak{i}-1}\psi^{\mathfrak{j}+1}(\mathrm{R}'^{(2)}) \cdot \alpha_N^{-\mathfrak{a}+\mathfrak{g}}\beta_N^{-\mathfrak{b}+\mathfrak{h}-1}\varphi_N^{-\mathfrak{c}+\mathfrak{i}-1}\psi_N^{-\mathfrak{d}+\mathfrak{j}+1}(n).\end{aligned} \tag{3.2}$$

**Definition 3.8.** Let $(H,\alpha,\beta,\varphi,\psi)$ be a BiHom-bialgebra. If there exists an invertible element $\mathrm{R}\in H\otimes H$, such that the conditions
(Q1) $(\alpha\otimes\alpha)\mathrm{R}=(\beta\otimes\beta)\mathrm{R}=(\varphi\otimes\varphi)\mathrm{R}=(\psi\otimes\psi)\mathrm{R}=\mathrm{R}$;
(Q2) $\sum\mathrm{R}^{(1)}\varphi^{-1}\psi(h_1)\otimes\mathrm{R}^{(2)}\varphi\psi^{-1}(h_2)=\sum\alpha^{-1}\beta(h_2)\mathrm{R}^{(1)}\otimes\alpha^{-1}\beta(h_1)\mathrm{R}^{(2)}$;
(Q3) $\sum\mathrm{R}_1^{(1)}\otimes\mathrm{R}_2^{(1)}\otimes\mathrm{R}^{(2)}=\sum\alpha\varphi(\dot{\mathrm{R}}^{(1)})\otimes\beta\psi(\mathrm{R}^{(1)})\otimes\dot{\mathrm{R}}^{(2)}\mathrm{R}^{(2)}$;
(Q4) $\sum R^{(1)}\otimes R_1^{(2)}\otimes R_2^{(2)}=\sum\dot{\mathrm{R}}^{(1)}\mathrm{R}^{(1)}\otimes\beta\varphi(\mathrm{R}^{(2)})\otimes\alpha\psi(\dot{\mathrm{R}}^{(2)})$
hold for any $h\in H$, where $\dot{\mathrm{R}}=\mathrm{R}=\sum\mathrm{R}^{(1)}\otimes\mathrm{R}^{(2)}=\sum\dot{\mathrm{R}}^{(1)}\otimes\dot{\mathrm{R}}^{(2)}$, then R is called an *R-matrix* or a *quasitriangular structure* of $H$, $(H,\mathrm{R})$ is called a *quasitriangular BiHom-bialgebra.*

Assume that R and R′ are two elements in $H\otimes H$, $\mathbf{C}$ and $\mathbf{C}'$ are defined as in (3.1) and (3.2). Then we have the following results.

**Lemma 3.9.** $\mathbf{C}$ *is a natural transformation in* ${}_H\mathcal{BM}^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}}$ *if and only if* R *satisfies* (Q1) *and* (Q2).

Proof. Firstly, it is easy to check that $\mathbf{C}$ is compatible with the Hom-structure maps of BiHom modules if and only if R is an invariant under $\alpha$, $\beta$, $\varphi$, $\psi$.

Secondly, if (Q2) holds, then for any $M,N\in{}_H\mathcal{BM}^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}}$, we have

$$\begin{aligned}
&\mathbf{C}_{M,N}(h\cdot(m\otimes n))\\
&\quad=\ \sum\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\varphi^{\mathfrak{c}}\psi^{\mathfrak{d}}(\mathrm{R}^{(2)})\cdot\alpha_N^{\mathfrak{a}-\mathfrak{g}}\beta_N^{\mathfrak{b}-\mathfrak{h}-1}\varphi_N^{\mathfrak{c}-\mathfrak{i}+1}\psi_N^{\mathfrak{d}-\mathfrak{j}-1}(\alpha^{\mathfrak{g}}\beta^{\mathfrak{h}}\varphi^{\mathfrak{i}}\psi^{\mathfrak{j}}(h_2)\cdot n)\\
&\qquad\otimes\alpha^{\mathfrak{g}}\beta^{\mathfrak{h}}\varphi^{\mathfrak{i}}\psi^{\mathfrak{j}}(\mathrm{R}^{(1)})\cdot\alpha_M^{-\mathfrak{a}+\mathfrak{g}}\beta_M^{-\mathfrak{b}+\mathfrak{h}-1}\varphi_M^{-\mathfrak{c}+\mathfrak{i}-1}\psi_M^{-\mathfrak{d}+\mathfrak{j}+1}(\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\varphi^{\mathfrak{c}}\psi^{\mathfrak{d}}(h_1)\cdot m)\\
&\quad\overset{(\mathrm{Q1})}{=}\ \sum(\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\varphi^{\mathfrak{c}}\psi^{\mathfrak{d}}(\mathrm{R}^{(2)})\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}-1}\varphi^{\mathfrak{c}+1}\psi^{\mathfrak{d}-1}(h_2))\cdot\alpha_N^{\mathfrak{a}-\mathfrak{g}}\beta_N^{\mathfrak{b}-\mathfrak{h}}\varphi_N^{\mathfrak{c}-\mathfrak{i}+1}\psi_N^{\mathfrak{d}-\mathfrak{j}-1}(n)\\
&\qquad\otimes(\alpha^{\mathfrak{g}}\beta^{\mathfrak{h}}\varphi^{\mathfrak{i}}\psi^{\mathfrak{j}}(\mathrm{R}^{(1)})\alpha^{\mathfrak{g}}\beta^{\mathfrak{h}-1}\varphi^{\mathfrak{i}-1}\psi^{\mathfrak{j}+1}(h_1))\cdot\alpha_M^{-\mathfrak{a}+\mathfrak{g}}\beta_M^{-\mathfrak{b}+\mathfrak{h}}\varphi_M^{-\mathfrak{c}+\mathfrak{i}-1}\psi_M^{-\mathfrak{d}+\mathfrak{j}+1}(m)\\
&\quad\overset{(\mathrm{Q2})}{=}\ \sum\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\varphi^{\mathfrak{c}}\psi^{\mathfrak{d}}(\alpha^{-1}(h_1)\mathrm{R}^{(2)})\cdot\alpha_N^{\mathfrak{a}-\mathfrak{g}}\beta_N^{\mathfrak{b}-\mathfrak{h}}\varphi_N^{\mathfrak{c}-\mathfrak{i}+1}\psi_N^{\mathfrak{d}-\mathfrak{j}-1}(n)\\
&\qquad\otimes\alpha^{\mathfrak{g}}\beta^{\mathfrak{h}}\varphi^{\mathfrak{i}}\psi^{\mathfrak{j}}(\alpha^{-1}(h_2)\mathrm{R}^{(1)})\cdot\alpha_M^{-\mathfrak{a}+\mathfrak{g}}\beta_M^{-\mathfrak{b}+\mathfrak{h}}\varphi_M^{-\mathfrak{c}+\mathfrak{i}-1}\psi_M^{-\mathfrak{d}+\mathfrak{j}+1}(m)\\
&\quad=\ h\cdot\mathbf{C}_{M,N}((m\otimes n)),
\end{aligned}$$

where $m\in M$, $n\in N$. Thus, $\mathbf{C}$ is $H$-linear.

Finally, if $\mathbf{C}$ is $H$-linear, then we have $\mathbf{C}_{H,H}(h\cdot(1_H\otimes 1_H))=h\cdot\mathbf{C}_{H,H}((1_H\otimes 1_H))$, which implies (Q2). □

**Lemma 3.10.** $\mathbf{C}'$ *is the inverse of* $\mathbf{C}$ *if and only if* R′ *is the inverse of* R.

Proof. $\Leftarrow$: If $\mathrm{R}'$ is the inverse of $\mathrm{R}$, then for any $M, N \in {}_H\mathcal{BM}^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}}$, we have

$$\begin{aligned}
&\mathbf{C'}_{M,N}\mathbf{C}_{M,N}(m \otimes n)\\
&\quad= \sum \alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\varphi^{\mathfrak{c}+1}\psi^{\mathfrak{d}-1}(\mathrm{R}'^{(1)}) \cdot (\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}-1}\varphi^{\mathfrak{c}+1}\psi^{\mathfrak{d}-1}(\mathrm{R}^{(1)}) \cdot \beta_M^{-2}(m))\\
&\quad\qquad \otimes \alpha^{\mathfrak{g}}\beta^{\mathfrak{h}}\varphi^{\mathfrak{i}-1}\psi^{\mathfrak{j}+1}(\mathrm{R}'^{(2)}) \cdot (\alpha^{\mathfrak{g}}\beta^{\mathfrak{h}-1}\varphi^{\mathfrak{i}-1}\psi^{\mathfrak{j}+1}(\mathrm{R}^{(2)}) \cdot \beta_N^{-2}(n))\\
&\quad\overset{(\mathrm{Q1})}{=} 1_H \cdot \beta_M^{-1}(m) \otimes 1_H \cdot \beta_N^{-1}(n) = m \otimes n,
\end{aligned}$$

where $m \in M$, $n \in N$. Similarly, we can get $\mathbf{C}_{M,N}\mathbf{C}'_{M,N}(n \otimes m) = n \otimes m$. This means that $\mathbf{C}'$ is the inverse of $\mathbf{C}$.

$\Rightarrow$: Conversely, if $\mathbf{C}'$ is the inverse of $\mathbf{C}$, then $\mathbf{C}'_{H,H}\mathbf{C}_{H,H} = \mathrm{id} = \mathbf{C}_{H,H}\mathbf{C}'_{H,H}$, which implies $\mathrm{R}'\mathrm{R} = \mathrm{R}\mathrm{R}' = 1_H \otimes 1_H$. $\square$

**Lemma 3.11.** *$\mathbf{C}$ satisfies $\mathbf{a}^{-1}_{P,M,N} \circ \mathbf{C}_{M\otimes N,P} \circ \mathbf{a}^{-1}_{M,N,P} = (\mathbf{C}_{M,P} \otimes \mathrm{id}_N) \circ \mathbf{a}^{-1}_{M,P,N} \circ (\mathrm{id}_M \otimes \mathbf{C}_{N,P})$ if and only if* (Q3) *holds.*

Proof. $\Leftarrow$: If (Q3) holds, then for any $M, N, P \in {}_H\mathcal{BM}^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}}$, we have

$$\begin{aligned}
&(\mathbf{a}^{-1}_{P,M,N} \circ \mathbf{C}_{M\otimes N,P} \circ \mathbf{a}^{-1}_{M,N,P})(m \otimes (n \otimes p))\\
&\quad= \sum \mathbf{a}^{-1}_{P,M,N}(\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\varphi^{\mathfrak{c}}\psi^{\mathfrak{d}}(\mathrm{R}^{(2)}) \cdot \alpha_P^{\mathfrak{a}-2\mathfrak{g}}\beta_P^{\mathfrak{b}-2\mathfrak{h}-1}\varphi_P^{\mathfrak{c}-2\mathfrak{i}+1}\psi_P^{\mathfrak{d}-2\mathfrak{j}-2}(p)\\
&\quad\qquad \otimes (\alpha^{\mathfrak{a}+\mathfrak{g}}\beta^{\mathfrak{b}+\mathfrak{h}}\varphi^{\mathfrak{c}+\mathfrak{i}}\psi^{\mathfrak{d}+\mathfrak{j}}(\mathrm{R}_1^{(1)})\alpha_M^{\mathfrak{g}}\beta_M^{\mathfrak{h}-1}\varphi_M^{\mathfrak{i}}\psi_M^{\mathfrak{j}+1}(m) \otimes \alpha^{2\mathfrak{g}}\beta^{2\mathfrak{h}}\varphi^{2\mathfrak{i}}\psi^{2\mathfrak{j}}(\mathrm{R}_2^{(1)})\\
&\quad\qquad \times \alpha_N^{\mathfrak{a}-\mathfrak{g}}\beta_N^{\mathfrak{b}-\mathfrak{h}-1}\varphi_N^{\mathfrak{c}-\mathfrak{i}+1}\psi_N^{\mathfrak{d}-\mathfrak{j}-1}(n)))\\
&\quad\overset{(\mathrm{Q1,Q3})}{=} \sum((\alpha^{\mathfrak{a}-1}\beta^{\mathfrak{b}}\varphi^{\mathfrak{c}}\psi^{\mathfrak{d}}(\dot{\mathrm{R}}^{(2)})\alpha^{2\mathfrak{a}-\mathfrak{g}}\beta^{2\mathfrak{b}-\mathfrak{h}-1}\varphi^{2\mathfrak{c}-\mathfrak{i}+1}\psi^{2\mathfrak{d}-\mathfrak{j}-1}(\mathrm{R}^{(2)}))\\
&\quad\qquad \times \alpha_P^{2\mathfrak{a}-2\mathfrak{g}}\beta_P^{2\mathfrak{b}-2\mathfrak{h}-1}\varphi_P^{2\mathfrak{c}-2\mathfrak{i}+2}\psi_P^{2\mathfrak{d}-2\mathfrak{j}-2}(p)\\
&\quad\qquad \otimes \alpha^{\mathfrak{g}}\beta^{\mathfrak{h}}\varphi^{\mathfrak{i}}\psi^{\mathfrak{j}}(\dot{\mathrm{R}}^{(1)})\alpha_M^{\mathfrak{g}}\beta_M^{\mathfrak{h}-1}\varphi_M^{\mathfrak{i}}\psi_M^{\mathfrak{j}+1}(m))\\
&\quad\qquad \otimes (\psi^{-1}(\mathrm{R}^{(1)}) \cdot \alpha_N^{-\mathfrak{a}}\beta_N^{-\mathfrak{b}-1}\varphi_N^{\mathfrak{c}-1}\psi_N^{-\mathfrak{d}}(n))\\
&\quad= \sum(\mathbf{a}^{-1}_{M,P,N} \circ (\mathrm{id}_M \otimes \mathbf{C}_{N,P})(m \otimes (\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\varphi^{\mathfrak{c}}\psi^{\mathfrak{d}}(\mathrm{R}^{(2)})\\
&\quad\qquad \times \alpha_P^{\mathfrak{a}-\mathfrak{g}}\beta_P^{\mathfrak{b}-\mathfrak{h}-1}\varphi_P^{\mathfrak{c}-\mathfrak{i}+1}\psi_P^{\mathfrak{d}-\mathfrak{j}-1}(p)\\
&\quad\qquad \otimes \alpha^{\mathfrak{g}}\beta^{\mathfrak{h}}\varphi^{\mathfrak{i}}\psi^{\mathfrak{j}}(\mathrm{R}^{(1)}) \cdot \alpha_N^{-\mathfrak{a}+\mathfrak{g}}\beta_N^{-\mathfrak{b}+\mathfrak{h}-1}\varphi_N^{-\mathfrak{c}+\mathfrak{i}-1}\psi_N^{-\mathfrak{d}+\mathfrak{j}+1}(n)))\\
&\quad= ((\mathbf{C}_{M,P} \otimes \mathrm{id}_N) \circ \mathbf{a}^{-1}_{M,P,N} \circ (\mathrm{id}_M \otimes \mathbf{C}_{N,P}))(m \otimes (n \otimes p)),
\end{aligned}$$

where $m \in M$, $n \in N$, $p \in P$, as needed.

$\Rightarrow$: Conversely, we have

$$\begin{aligned}
&(\mathbf{a}^{-1}_{H,H,H} \circ \mathbf{C}_{H\otimes H,H} \circ \mathbf{a}^{-1}_{H,H,H})(1_H \otimes (1_H \otimes 1_H))\\
&\qquad((\mathbf{C}_{H,H} \otimes \mathrm{id}_H) \circ \mathbf{a}^{-1}_{H,H,H} \circ (\mathrm{id}_H \otimes \mathbf{C}_{H,H}))(1_H \otimes (1_H \otimes 1_H)),
\end{aligned}$$

which implies (Q3). $\square$

In a similar manner to Lemma 3.11, we have:

**Lemma 3.12.** $\mathbf{C}$ *satisfies* $\mathbf{a}_{N,P,M}\circ\mathbf{C}_{M,N\otimes P}\circ\mathbf{a}_{M,N,P}=(\mathrm{id}_N\otimes\mathbf{C}_{M,P})\circ\mathbf{a}_{N,M,P}\circ(\mathbf{C}_{M,N}\otimes\mathrm{id}_P)$ *if and only if* (Q4) *holds.*

Combing Lemma 3.9 to Lemma 3.12, we immediately get the following result.

**Theorem 3.13.** *Let* $H=(H,\alpha,\beta,\varphi,\psi)$ *be a BiHom-bialgebra,* $\mathrm{R}$, $\mathrm{R}'$ *be elements in* $H\otimes H$, $\mathbf{C}$, $\mathbf{C}'$ *be families of maps defined in the way of* (3.1) *and* (3.2) *for any* $\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d},\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}\in\mathbb{Z}$. *Then* $\mathbf{C}$ *is a braiding in* ${}_H\mathcal{BM}_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}}^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}$ *with the inverse* $\mathbf{C}'$ *if and only if* $\mathrm{R}$ *is a quasitriangular structure of* $H$ *with the inverse element* $\mathrm{R}'$.

Proof. Straightforward. □

**Corollary 3.14.** *If* $\mathrm{R}$ *is a quasitriangular structure of* $H$, *then for any* $\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d},\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j},\mathfrak{a}',\mathfrak{b}',\mathfrak{c}',\mathfrak{d}',\mathfrak{g}',\mathfrak{h}',\mathfrak{i}',\mathfrak{j}'\in\mathbb{Z}$, ${}_H\mathcal{BM}_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}}^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}$ *is braided isomorphic to* ${}_H\mathcal{BM}_{\mathfrak{g}',\mathfrak{h}',\mathfrak{i}',\mathfrak{j}'}^{\mathfrak{a}',\mathfrak{b}',\mathfrak{c}',\mathfrak{d}'}$.

Proof. Obviously the functor $\mathscr{S}$ defined in Proposition 3.5 is a braided monoidal functor, as needed. □

**Example 3.15.**

(1) If $(H,\mu_H,1_H,\Delta_H,\varepsilon_H,\mathrm{R})$ is a quasitriangular bialgebra, $\alpha,\beta,\varphi,\psi\colon H\to H$ are all bialgebra isomorphisms and $\mathrm{R}$ is an invariant under $\alpha$, $\beta$, $\varphi$, $\psi$, then $H^{\mathrm{Bi}}=(H,\mu_H\circ(\alpha\otimes\beta),1_H,(\varphi\otimes\psi)\circ\Delta_H,\varepsilon_H,\mathrm{R},\alpha,\beta,\varphi,\psi)$ is a quasitriangular BiHom-bialgebra.

(2) If $H=(H,\mu_H,1_H,\Delta_H,\varepsilon_H,\mathrm{R},\alpha,\beta,\varphi,\psi)$ is a quasitriangular BiHom-bialgebra, then $H^{\mathrm{cop}}=(H,\mu_H,1_H,\Delta_H^{\mathrm{cop}},\varepsilon_H,\mathrm{R}_{21},\alpha,\beta,\psi,\varphi)$ is also a quasitriangular BiHom-bialgebra.

(3) Under the consideration of Example 2.12 (2), it is easy to check that $H_4^{\mathrm{Bi}}=(H_4,\mu\circ(\alpha\otimes\beta),1_{H_4},(\varphi\otimes\psi)\circ\Delta,\varepsilon,\mathrm{R}^{\mathrm{Bi}},\alpha,\beta,\varphi,\psi)$ is a quasitriangular BiHom-bialgebra, where $\mathrm{R}^{\mathrm{Bi}}$ takes one of following forms:

▷ when $a,b,c,d=\pm1$,

$$\mathrm{R}^{\mathrm{Bi}}=\frac{1}{2}(1_H\otimes1_H+1_H\otimes g+g\otimes1_H-g\otimes g)+\frac{\kappa}{2}(x\otimes x-x\otimes y+y\otimes x+y\otimes y)\quad\forall\,\kappa\in\Bbbk,$$

where $a$, $b$, $c$, $d$ are defined in (2.9);

▷ in other cases, for $a,b,c,d\neq\pm1$, $\mathrm{R}^{\mathrm{Bi}}=\frac{1}{2}(1_H\otimes1_H+1_H\otimes g+g\otimes1_H-g\otimes g)$.

(4) If $\alpha=\beta=\varphi=\psi$, then the quasitriangular BiHom-bialgebra

$$H=(H,\mathrm{R},\mu,1_H,\Delta,\varepsilon,\alpha,\beta,\varphi,\psi)$$

becomes the usual quasitriangular Hom-bialgebra. Similarly, if $\alpha^{-1}=\beta^{-1}=\varphi=\psi$, then $H$ becomes the usual quasitriangular monoidal Hom-bialgebra.

**Theorem 3.16.** *If* R *is a quasitriangular structure of a BiHom-bialgebra $H$, then* R *satisfies the BiHom-Yang-Baxter equation*

$$\mathrm{R}_{12}((\psi\otimes\mathrm{id}\otimes\mathrm{id})\mathrm{R}_{13}(\mathrm{id}\otimes\varphi\otimes\mathrm{id})\mathrm{R}_{23})=((\mathrm{id}\otimes\varphi\otimes\mathrm{id})\mathrm{R}_{23}(\psi\otimes\mathrm{id}\otimes\mathrm{id})\mathrm{R}_{13})\mathrm{R}_{12}.$$

Proof. Recall that $\mathrm{R}_{12}=\sum\mathrm{R}^{(1)}\otimes\mathrm{R}^{(2)}\otimes 1_H$, $\mathrm{R}_{13}=\sum\mathrm{R}^{(1)}\otimes 1_H\otimes\mathrm{R}^{(2)}$, and $\mathrm{R}_{23}=\sum 1_H\otimes\mathrm{R}^{(1)}\otimes\mathrm{R}^{(2)}$. We have

$$\begin{aligned}
&\sum\ddot{\mathrm{R}}^{(1)}\alpha\psi(\dot{\mathrm{R}}^{(1)})\otimes\ddot{\mathrm{R}}^{(2)}\beta\varphi(\mathrm{R}^{(1)})\otimes\beta(\dot{\mathrm{R}}^{(2)})\beta(\mathrm{R}^{(2)})\\
&\overset{(\mathrm{Q1})}{=}\sum\ddot{\mathrm{R}}^{(1)}\beta^{-1}\varphi^{-1}\psi(\dot{\mathrm{R}}^{(1)})\otimes\ddot{\mathrm{R}}^{(2)}\beta^{-1}\varphi\psi^{-1}(\mathrm{R}^{(1)})\otimes\alpha^{-1}\varphi^{-1}(\dot{\mathrm{R}}^{(2)})\beta^{-1}\psi^{-1}(\mathrm{R}^{(2)})\\
&\overset{(\mathrm{Q3})}{=}\sum\ddot{\mathrm{R}}^{(1)}\beta^{-1}\varphi^{-1}\psi(\mathrm{R}^{(1)}_1)\otimes\ddot{\mathrm{R}}^{(2)}\beta^{-1}\varphi\psi^{-1}(\mathrm{R}^{(1)}_2)\otimes\mathrm{R}^{(2)}\\
&\overset{(\mathrm{Q2})}{=}\sum\alpha^{-1}(\mathrm{R}^{(1)}_2)\ddot{\mathrm{R}}^{(1)}\otimes\alpha^{-1}(\mathrm{R}^{(1)}_1)\ddot{\mathrm{R}}^{(2)}\otimes\mathrm{R}^{(2)}\\
&\overset{(\mathrm{Q3})}{=}\sum\alpha^{-1}(\mathrm{R}^{(1)})\ddot{\mathrm{R}}^{(1)}\otimes\alpha^{-1}(\dot{\mathrm{R}}^{(1)})\ddot{\mathrm{R}}^{(2)}\otimes\alpha^{-1}\varphi^{-1}(\dot{\mathrm{R}}^{(2)})\beta^{-1}\psi^{-1}(\mathrm{R}^{(2)})\\
&\overset{(\mathrm{Q1})}{=}\sum\beta\psi(\mathrm{R}^{(1)})\ddot{\mathrm{R}}^{(1)}\otimes\alpha\varphi(\dot{\mathrm{R}}^{(1)})\ddot{\mathrm{R}}^{(2)}\otimes\alpha(\dot{\mathrm{R}}^{(2)})\alpha(\mathrm{R}^{(2)}),
\end{aligned}$$

which implies the conclusion. □

**3.3. The dual case: coquasitriangular BiHom-Hopf algebras.** The results in this section are dual to the corresponding results above, so we will not give the complete proof.

Let $(H,\alpha,\beta,\varphi,\psi)$ be a BiHom-bialgebra. Recall from [31], Definition 5.3 that a *right $H$-BiHom-comodule* is a 5-tuple $(M,\alpha_M,\beta_M,\varphi_M,\psi_M)$, where $M$ is a linear space, $\alpha_M,\beta_M,\varphi_M,\psi_M\colon M\to M$ are linear isomorphisms and we have a linear map (called a *coaction*) $\varrho\colon M\to M\otimes H$, with the notation $\varrho(m)=m_0\otimes m_1$ for all $m\in M$, such that the following conditions are satisfied:

$$\begin{gathered}
\varphi_M,\ \psi_M,\ \alpha_M,\ \beta_M\ \text{ commute with each other,}\\
(\alpha_M\otimes\alpha)\circ\varrho=\varrho\circ\alpha_M,\quad(\beta_M\otimes\beta)\circ\varrho=\varrho\circ\beta_M,\\
(\varphi_M\otimes\varphi)\circ\varrho=\varrho\circ\varphi_M,\quad(\psi_M\otimes\psi)\circ\varrho=\varrho\circ\psi_M,\\
\varphi_M(m_0)\otimes m_{11}\otimes m_{12}=m_{00}\otimes m_{01}\otimes\psi(m_1),m_0\varepsilon(m_1)=\varphi_M(m).
\end{gathered}$$

If $(M,\alpha_M,\beta_M,\varphi_M,\psi_M)$ and $(N,\alpha_N,\beta_N,\varphi_N,\psi_N)$ are right $H$-BiHom-comodules with coactions $\varrho_M$ and $\varrho_N$, respectively, a *morphism of right $H$-BiHom-comodules* $f\colon M\to N$ is a linear map satisfying the conditions

$$\begin{gathered}
\alpha_N\circ f=f\circ\alpha_M,\quad\beta_N\circ f=f\circ\beta_M,\quad\varphi_N\circ f=f\circ\varphi_M,\\
\psi_N\circ f=f\circ\psi_M,\quad\varrho_N\circ f=(f\otimes\mathrm{id}_H)\circ\varrho_M.
\end{gathered}$$

The category of $H$-BiHom-comodules and $H$-colinear morphisms is denoted by $\mathcal{BM}^H$.

**Theorem 3.17.** *If $(H, \alpha, \beta, \varphi, \psi)$ is a BiHom-Hopf algebra with bijective antipode $S$, then $\mathcal{BM}^H$ is a rigid category.*

Proof. For any $\mathfrak{k}, \mathfrak{l}, \mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q}, \mathfrak{r}, \mathfrak{s} \in \mathbb{Z}$, define the following monoidal structures on $\mathcal{BM}^H$ by:

▷ the tensor product of $H$-BiHom-comodules

$$(U, \alpha_U, \beta_U, \varphi_U, \psi_U) \quad \text{and} \quad (V, \alpha_V, \beta_V, \varphi_V, \psi_V)$$

is $(U \otimes V, \alpha_U \otimes \alpha_V, \beta_U \otimes \beta_V, \varphi_U \otimes \varphi_V, \psi_U \otimes \psi_V)$ with the $H$-coaction $\varrho^{U \otimes V}$:

$$u \otimes v \mapsto u_{(0)} \otimes v_{(0)} \otimes \alpha^{\mathfrak{k}} \beta^{\mathfrak{l}} \varphi^{\mathfrak{m}} \psi^{\mathfrak{n}}(u_{(1)}) \alpha^{\mathfrak{p}} \beta^{\mathfrak{q}} \varphi^{\mathfrak{r}} \psi^{\mathfrak{s}}(v_{(1)});$$

▷ the unit object is $(\mathbb{k}, \mathrm{id}_{\mathbb{k}}, \mathrm{id}_{\mathbb{k}}, \mathrm{id}_{\mathbb{k}}, \mathrm{id}_{\mathbb{k}})$ with the trivial coaction;
▷ the associativity constraints $\mathbf{a}$ and the unit constraint $\mathbf{l}$ and $\mathbf{r}$ are given by

$$\mathbf{a}_{U,V,W}((u \otimes v) \otimes w) = \alpha_U^{\mathfrak{k}+1} \beta_U^{\mathfrak{l}} \varphi_U^{\mathfrak{m}} \psi_U^{\mathfrak{n}}(u) \otimes (v \otimes \alpha_W^{-\mathfrak{p}} \beta_W^{-\mathfrak{q}-1} \varphi_W^{-\mathfrak{r}} \psi_W^{-\mathfrak{s}}(w));$$
$$\mathbf{r}_U(u \otimes 1_k) = \alpha^{\mathfrak{k}+1} \beta_U^{\mathfrak{l}} \varphi_U^{\mathfrak{m}} \psi_{(}^{\mathfrak{n}} u), \quad \mathbf{l}_U(1_k \otimes u) = \alpha^{\mathfrak{p}} \beta_U^{\mathfrak{q}+1} \varphi_U^{\mathfrak{r}} \psi_{(}^{\mathfrak{s}} u).$$

After a direct computation we can get that $(\mathcal{BM}^H, \otimes, (\mathbb{k}, \mathrm{id}, \mathrm{id}, \mathrm{id}, \mathrm{id}), \mathbf{a}, \mathbf{l}, \mathbf{r})$ forms a monoidal category. We denote this category by $(\mathcal{BM}^H)^{\mathfrak{k},\mathfrak{l},\mathfrak{m},\mathfrak{n}}_{\mathfrak{p},\mathfrak{q},\mathfrak{r},\mathfrak{s}}$.

Moreover, if $(U, \alpha_U, \beta_U, \varphi_U, \psi_U) \in \mathcal{BM}^H$, $U^* = \hom(U, \mathbb{k})$, define the following structures:

▷ $\varrho^{U^*}\colon U^* \to U^* \otimes H$ is given by

$$f_0(u) \otimes f_1 = f(\varphi_U^{-2}(u_0)) \cdot S(u_1) \quad \forall\, u \in U, f \in U^*;$$

▷ the Hom-structure maps of $U^*$ are given by

$$\alpha_{U^*} = (\alpha_U^{-1})^*, \quad \beta_{U^*} = (\beta_U^{-1})^*, \quad \varphi_{U^*} = (\varphi_U^{-1})^*, \quad \psi_{U^*} = (\psi_U^{-1})^*;$$

▷ the evaluation $\mathrm{ev}_U\colon U^* \otimes U \to \mathbb{k}$ and coevaluation $\mathrm{coev}_U\colon \mathbb{k} \to U \otimes U^*$ are given by

$$\mathrm{ev}_U(f \otimes u) = f(\alpha_U^{-\mathfrak{k}+\mathfrak{p}} \beta_U^{-\mathfrak{l}+\mathfrak{q}} \varphi_U^{-\mathfrak{m}+\mathfrak{r}} \psi_U^{-\mathfrak{n}+\mathfrak{s}-1}(u)),$$
$$\mathrm{coev}_U(1_{\mathbb{k}}) = \sum \alpha_U^{-\mathfrak{k}+\mathfrak{p}} \beta_U^{-\mathfrak{l}+\mathfrak{q}} \varphi_U^{-\mathfrak{m}+\mathfrak{r}+2} \psi_U^{-\mathfrak{n}+\mathfrak{s}-1}(e_i) \otimes e^i,$$

where $e_i$ and $e^i$ are dual bases of $U$ and $U^*$, respectively. Clearly $(U^*, \mathrm{ev}_M, \mathrm{coev}_M)$ is the left dual of $U$ and hence $(\mathcal{BM}^H)^{\mathfrak{k},\mathfrak{l},\mathfrak{m},\mathfrak{n}}_{\mathfrak{p},\mathfrak{q},\mathfrak{r},\mathfrak{s}}$ is a left rigid category. Similarly, it is also a rigid category.

□

**Definition 3.18.** $(H,\alpha,\beta,\varphi,\psi)$ is a BiHom-bialgebra, if there is a bilinear map $\sigma\colon H\otimes H\to\Bbbk$, such that $\sigma$ is invertible, under the convolution invertible, and the following formulae are satisfied:
(CQ1) $\sigma(\alpha(a),\alpha(b))=\sigma(\beta(a),\beta(b))=\sigma(\varphi(a),\varphi(b))=\sigma(\psi(a),\psi(b))=\sigma(a,b)$;
(CQ2) $\sigma(a_1,b_1)\varphi\psi^{-1}(a_2)\varphi\psi^{-1}(b_2)=\alpha^{-1}\beta(b_1)\alpha\beta^{-1}(a_1)\sigma(a_2,b_2)$;
(CQ3) $\sigma(\alpha\beta(a),bc)=\sigma(\alpha(a_1),\varphi(c))\sigma(\beta(a_2),\psi(b))$;
(CQ4) $\sigma(ab,\varphi\psi(c))=\sigma(\alpha(a),\psi(c_1))\sigma(\beta(b),\varphi(c_2))$,
then $\sigma$ is called a *coquasitriangular form* of $H$, $(H,\sigma)$ is called a *coquasitriangular BiHom-bialgebra.*

**Remark 3.19.** Obviously (CQ3) is equivalent to the one of the next identities:

$$\begin{aligned}\sigma(\alpha\varphi(a),bc)&=\sigma(\alpha(a_1),\beta(c))\sigma(\alpha(a_2),\psi(b)),\\ \sigma(\varphi\psi(a),bc)&=\sigma(\psi(a_1),\beta(c))\sigma(\varphi(a_2),\alpha(b)),\\ \sigma(\beta\psi(a),bc)&=\sigma(\psi(a_1),\varphi(c))\sigma(\beta(a_2),\alpha(b)).\end{aligned}$$

Equation (CQ4) is equivalent to one of the following identities:

$$\begin{aligned}\sigma(ab,\alpha\psi(c))&=\sigma(\varphi(a),\psi(c_1))\sigma(\beta(b),\alpha(c_2)),\\ \sigma(ab,\alpha\beta(c))&=\sigma(\varphi(a),\beta(c_1))\sigma(\psi(b),\alpha(c_2)),\\ \sigma(ab,\beta\varphi(c))&=\sigma(\alpha(a),\beta(c_1))\sigma(\psi(b),\varphi(c_2)).\end{aligned}$$

For any bilinear form $\sigma\in\hom(H\otimes H,\Bbbk)$, $U,V\in(\mathcal{BM}^H)^{\mathfrak{k},\mathfrak{l},\mathfrak{m},\mathfrak{n}}_{\mathfrak{p},\mathfrak{q},\mathfrak{r},\mathfrak{s}}$, define the families of maps $\mathbf{C}_{U,V}\colon U\otimes V\to V\otimes U$ by

$$\begin{aligned}u\otimes v\mapsto{}&\alpha_V^{-\mathfrak{k}+\mathfrak{p}-1}\beta_V^{-\mathfrak{l}+\mathfrak{q}+1}\varphi_V^{-\mathfrak{m}+\mathfrak{r}-1}\psi_V^{-\mathfrak{n}+\mathfrak{s}}(v_0)\\ &\otimes\alpha_U^{\mathfrak{k}-\mathfrak{p}+1}\beta_U^{\mathfrak{l}-\mathfrak{q}-1}\varphi_U^{\mathfrak{m}-\mathfrak{r}-1}\psi_U^{\mathfrak{n}-\mathfrak{s}}(u_0)\sigma(\alpha^{\mathfrak{k}}\beta^{\mathfrak{l}}\varphi^{\mathfrak{m}}\psi^{\mathfrak{n}}(u_{(1)})\alpha^{\mathfrak{p}}\beta^{\mathfrak{q}}\varphi^{\mathfrak{r}}\psi^{\mathfrak{s}}(v_{(1)})).\end{aligned}$$

**Theorem 3.20.** *Under the condition above, $\sigma$ is a coquasitriangular form of $H$ if and only if $(\mathcal{BM}^H)^{\mathfrak{k},\mathfrak{l},\mathfrak{m},\mathfrak{n}}_{\mathfrak{p},\mathfrak{q},\mathfrak{r},\mathfrak{s}}$ is a braided category with the braiding* $\mathbf{C}$.

Proof. Dual to Theorem 3.13. ☐

**Proposition 3.21.** *If $H$ is a BiHom-bialgebra, then for any $\mathfrak{k},\mathfrak{l},\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q},\mathfrak{r},\mathfrak{s},\mathfrak{k}',\mathfrak{l}',\mathfrak{m}',\mathfrak{n}',\mathfrak{p}',\mathfrak{q}',\mathfrak{r}',\mathfrak{s}'\in\mathbb{Z}$, $(\mathcal{BM}^H)^{\mathfrak{k},\mathfrak{l},\mathfrak{m},\mathfrak{n}}_{\mathfrak{p},\mathfrak{q},\mathfrak{r},\mathfrak{s}}$ is monoidal isomorphic to $(\mathcal{BM}^H)^{\mathfrak{k}',\mathfrak{l}',\mathfrak{m}',\mathfrak{n}'}_{\mathfrak{p}',\mathfrak{q}',\mathfrak{r}',\mathfrak{s}'}$. Furthermore, if $H$ is a coquasitriangular BiHom-bialgebra, then $(\mathcal{BM}^H)^{\mathfrak{k},\mathfrak{l},\mathfrak{m},\mathfrak{n}}_{\mathfrak{p},\mathfrak{q},\mathfrak{r},\mathfrak{s}}$ is braided isomorphic to $(\mathcal{BM}^H)^{\mathfrak{k}',\mathfrak{l}',\mathfrak{m}',\mathfrak{n}'}_{\mathfrak{p}',\mathfrak{q}',\mathfrak{r}',\mathfrak{s}'}$.*

Proof. Dual to Theorem 3.5 and Proposition 3.14. ☐

**Theorem 3.22.** *Let $n\in\mathbb{Z}^+$. If $H$ is a commutative BiHom-bialgebra, then $\mathcal{BM}^H$ forms an $n$-monoidal category.*

Proof. Dual to Theorem 3.6. ☐

## 4. THE YETTER-DRINFELD CATEGORY OF BIHOM-HOPF ALGEBRAS

**4.1. The parametric BiHom-Yetter-Drinfeld modules.** From now on we always assume that $(H,S)$ is a BiHom-Hopf algebra and $S$ is a bijective map.

**Definition 4.1.** For any $\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q}\in\mathbb{Z}$, a $\Bbbk$-space $U$ is called an $(\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q})$th *BiHom-Yetter-Drinfeld module* of $H$, if there exist morphisms $\alpha_U,\beta_U,\varphi_U,\psi_U\in\mathrm{Aut}(U)$ such that $(U,\alpha_U,\beta_U,\varphi_U,\psi_U)$ is both a left $H$-BiHom-module and a right $H$-BiHom-comodule, and the compatibility condition

$$\underline{h_2\cdot u}_0\otimes\underline{h_2\cdot u}_1\alpha^{\mathfrak{m}}\beta^{\mathfrak{n}}\varphi^{\mathfrak{p}}\psi^{\mathfrak{q}}(h_1)=\psi(h_1)\cdot u_0\otimes\alpha^{\mathfrak{m}-1}\beta^{\mathfrak{n}+1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-1}(h_2)\alpha(u_1)\tag{4.1}$$

is satisfied for all $u\in U$ and $h\in H$. We denote by ${}_H\mathcal{YD}^H(\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q})$ the category of $(\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q})$th BiHom-Yetter-Drinfeld modules, morphisms being $H$-linear and $H$-colinear.

**Proposition 4.2.** *Equation* (4.1) *is equivalent to the equation*

$$\begin{aligned}\varrho(h\cdot u)=\psi^{-1}(h_{(21)})\cdot u_0&\otimes(\alpha^{\mathfrak{m}-3}\beta^{\mathfrak{n}+1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-3}(h_{22})\alpha^{-1}(u_1))\\&\times S^{-1}\alpha^{\mathfrak{m}}\beta^{\mathfrak{n}-1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-2}(h_1),\end{aligned}\tag{4.2}$$

*where $u\in U$, $h\in H$.*

Proof. For any $u\in U$, $h\in H$, in order to prove (4.2), we have

$$\begin{aligned}&\underline{h_{22}\cdot u}_0\otimes(\underline{h_{22}\cdot u}_1\alpha^{\mathfrak{m}}\beta^{\mathfrak{n}}\varphi^{\mathfrak{p}}\psi^{\mathfrak{q}}(h_{21}))S^{-1}\alpha^{\mathfrak{m}+2}\beta^{\mathfrak{n}-1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}}(h_1)\\&\quad=\underline{\psi(h_2)\cdot u}_0\otimes\alpha(\underline{\psi(h_2)\cdot u}_1)(\alpha^{\mathfrak{m}}\beta^{\mathfrak{n}}\varphi^{\mathfrak{p}}\psi^{\mathfrak{q}}(h_{12})S^{-1}\alpha^{\mathfrak{m}+2}\beta^{\mathfrak{n}-2}\varphi^{\mathfrak{p}}\psi^{\mathfrak{q}}(h_{11}))\\&\quad\overset{(2.7)}{=}\underline{\psi^2(h)\cdot u}_0\otimes\alpha^2(\underline{\psi^2(h)\cdot u}_1)\end{aligned}$$

and then

$$\begin{aligned}\varrho(h\cdot u)&=\underline{\psi^{-2}(h_{22})\cdot u}_0\otimes\alpha^{-2}(\underline{\psi^{-2}(h_{22})\cdot u}_1\alpha^{\mathfrak{m}}\beta^{\mathfrak{n}}\varphi^{\mathfrak{p}}\psi^{\mathfrak{q}-2}(h_{21}))\\&\quad\times S^{-1}\alpha^{\mathfrak{m}}\beta^{\mathfrak{n}-1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-2}(h_1)\\&\overset{(4.1)}{=}\psi^{-1}(h_{21})\cdot u_0\otimes\alpha^{-2}(\alpha^{\mathfrak{m}-1}\beta^{\mathfrak{n}+1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-3}(h_{22})\alpha(u_1))\\&\quad\times S^{-1}\alpha^{\mathfrak{m}}\beta^{\mathfrak{n}-1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-2}(h_1),\end{aligned}$$

hence, the conclusion holds. Conversely, if (4.2) holds, we have

$$\begin{aligned}&(h_2\cdot u)_0\otimes(h_2\cdot u)_1\alpha^{\mathfrak{m}}\beta^{\mathfrak{n}}\varphi^{\mathfrak{p}}\psi^{\mathfrak{q}}(h_1)\\&\overset{(4.2)}{=}h_{(21)}\cdot u_0\otimes(\alpha^{\mathfrak{m}-2}\beta^{\mathfrak{n}+1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-2}(h_{22})u_1)\\&\qquad\times(S^{-1}\alpha^{\mathfrak{m}}\beta^{\mathfrak{n}-1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-2}(h_{12})\alpha^{\mathfrak{m}}\beta^{\mathfrak{n}-1}\varphi^{\mathfrak{p}-1}\psi^{\mathfrak{q}}(h_{11}))\\&\overset{(2.8)}{=}\psi(h_1)\cdot u_0\otimes\alpha^{\mathfrak{m}-1}\beta^{\mathfrak{n}+1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-1}(h_2)\alpha(u_1),\end{aligned}$$

thus, we get (4.1). $\square$

**Example 4.3.**

(1) For any integers $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{p}$, $\mathfrak{q}$, define the right $H$-coaction on $H \otimes H$ by

$$\varrho(a \otimes b) = a_1 \otimes \varphi(b) \otimes a_2 \quad \text{for any } a, b \in H$$

and define the left $H$-action $\xrightarrow{\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q}}$ on $H \otimes H$ by

$$\begin{aligned} h \xrightarrow{\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q}} (a \otimes b) = {} & (\alpha^{\mathfrak{m}-3}\beta^{\mathfrak{n}+1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-4}(h_{22})\alpha^{-1}(a)) \\ & \times S^{-1}\alpha^{\mathfrak{m}}\beta^{\mathfrak{n}-1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-3}(h_1) \otimes h_{21}b, \end{aligned}$$

then $(H \otimes H, \xrightarrow{\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q}}, \varrho, \alpha \otimes \alpha, \beta \otimes \beta, \varphi \otimes \varphi, \psi \otimes \psi)$ is an $(\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q})$th BiHom-Yetter-Drinfeld module.

Proof. We only check the Hom-associative law of the $H$-action and (4.2). For any $h, g, a, b \in H$, we have

$$\begin{aligned} & \alpha(h) \xrightarrow{\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q}} (g \xrightarrow{\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q}} (a \otimes b)) \\ & \quad = (\alpha^{\mathfrak{m}-2}\beta^{\mathfrak{n}+1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-4}(h_{22})((\alpha^{\mathfrak{m}-4}\beta^{\mathfrak{n}+1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-4}(g_{22})\alpha^{-2}(a)) \\ & \qquad \times S^{-1}\alpha^{\mathfrak{m}-1}\beta^{\mathfrak{n}-1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-3}(g_1))) \\ & \qquad \times S^{-1}\alpha^{\mathfrak{m}+1}\beta^{\mathfrak{n}-1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-3}(h_1) \otimes \alpha(h_{21})(g_{21}b) \\ & \quad \overset{(2.5)}{=} (\alpha^{\mathfrak{m}-3}\beta^{\mathfrak{n}+1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-4}(h_{22}g_{22})\alpha^{-1}\beta(a)) \\ & \qquad \times S^{-1}\alpha^{\mathfrak{m}}\beta^{\mathfrak{n}-1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-3}(h_1g_1) \otimes (h_{21}g_{21})\beta(b) \\ & \quad = (hg) \xrightarrow{\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q}} \beta(a \otimes b), \end{aligned}$$

thus, the Hom-associative law holds. Furthermore, we have

$$\begin{aligned} & \varrho(h \xrightarrow{\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q}} (a \otimes b)) \\ & \quad \overset{(2.6)}{=} (\alpha^{\mathfrak{m}-3}\beta^{\mathfrak{n}+1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-4}(h_{221})\alpha^{-1}(a_1))S^{-1}\alpha^{\mathfrak{m}}\beta^{\mathfrak{n}-1}\varphi^{\mathfrak{p}+2}\psi^{\mathfrak{q}-4}(h_{12}) \\ & \qquad \otimes \varphi(h_{21})\varphi(b) \otimes (\alpha^{\mathfrak{m}-3}\beta^{\mathfrak{n}+1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-4}(h_{222})\alpha^{-1}(a_2)) \\ & \qquad \times S^{-1}\alpha^{\mathfrak{m}}\beta^{\mathfrak{n}-1}\varphi^{\mathfrak{p}}\psi^{\mathfrak{q}-2}(h_{11}) \\ & \quad = (\alpha^{\mathfrak{m}-3}\beta^{\mathfrak{n}+1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-5}(h_{2122})\alpha^{-1}(a_1))S^{-1}\alpha^{\mathfrak{m}}\beta^{\mathfrak{n}-1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-4}(h_{211}) \\ & \qquad \otimes \psi^{-1}(h_{2121})\varphi(b) \otimes (\alpha^{\mathfrak{m}-3}\beta^{\mathfrak{n}+1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-3}(h_{22})\alpha^{-1}(a_2)) \\ & \qquad \times S^{-1}\alpha^{\mathfrak{m}}\beta^{\mathfrak{n}-1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-2}(h_1) \\ & \quad = \psi^{-1}(h_{21}) \xrightarrow{\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q}} \underline{a \otimes b}_0 \otimes (\alpha^{\mathfrak{m}-3}\beta^{\mathfrak{n}+1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-3}(h_{22})\alpha^{-1}(\underline{a \otimes b}_1)) \\ & \qquad \times S^{-1}\alpha^{\mathfrak{m}}\beta^{\mathfrak{n}-1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-2}(h_1), \end{aligned}$$

thus, (4.2) is satisfied. □

(2) Similarly, define the $H$-coaction on $H\otimes H$ by

$$\varrho'(a\otimes b)=\psi^{-1}(a_{21})\otimes b_1\otimes(\alpha^{-3}\beta^{2}\psi^{-1}(a_{22})b_2)S^{-1}(a_1)\quad\text{for any } a,b\in H$$

and define the left $H$-action on $H\otimes H$ by

$$h\rightharpoonup(a\otimes b)=\alpha^{\mathfrak{m}-1}\beta^{\mathfrak{n}-1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-2}(h)a\otimes\psi(b),$$

then $(H\otimes H,\rightharpoonup,\varrho',\alpha\otimes\alpha,\beta\otimes\beta,\varphi\otimes\varphi,\psi\otimes\psi)$ is an $(\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q})$th BiHom-Yetter-Drinfeld module.

(3) If $\alpha=\beta=\varphi=\psi$, i.e., $H$ is a Hom-Hopf algebra, then the above $(\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q})$th BiHom-Yetter-Drinfeld modules become the usual Yetter-Drinfeld modules of $H$. For example, take $\alpha=\beta=\varphi=\psi$ and $\mathfrak{m}=2$, $\mathfrak{n}=\mathfrak{p}=\mathfrak{q}$, then it becomes the left-right Hom-Yetter-Drinfeld modules defined in [21], Definition 5.2.

(4) Similarly, if $\alpha^{-1}=\beta^{-1}=\varphi=\psi$, i.e., $H$ is a monoidal Hom-Hopf algebra, then the above $(\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q})$th BiHom-Yetter-Drinfeld modules become the usual Yetter-Drinfeld modules over $H$. For example, take $\alpha^{-1}=\beta^{-1}=\varphi=\psi$ and $\mathfrak{m}=\mathfrak{n}=\mathfrak{p}=\mathfrak{q}=0$, then they become the Hom-Yetter-Drinfeld modules defined in [9], Definition 3.2.

For any $\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d},\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}\in\mathbb{Z}$, we are able to define the monoidal structures in ${}_H\mathcal{YD}^H(\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q})$ as follows:

- ▷ the monoidal product of $(U,\alpha_U,\beta_U,\varphi_U,\psi_U)$ and $(V,\alpha_V,\beta_V,\varphi_V,\psi_V)$ is $(U\otimes V,$ $\alpha_U\otimes\alpha_V,\beta_U\otimes\beta_V,\varphi_U\otimes\varphi_V,\psi_U\otimes\psi_V)$, where the BiHom-module and BiHom-comodule structures are given by

$$\begin{aligned}h\cdot(u\otimes v)&=\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\varphi^{\mathfrak{c}}\psi^{\mathfrak{d}}(h_1)\cdot u\otimes\alpha^{\mathfrak{g}}\beta^{\mathfrak{h}}\varphi^{\mathfrak{i}}\psi^{\mathfrak{j}}(h_2)\cdot v,\\ \varrho^{U\otimes V}(u\otimes v)&=u_0\otimes v_0\otimes\alpha^{-\mathfrak{g}-1}\beta^{-\mathfrak{h}}\varphi^{-\mathfrak{i}}\psi^{-\mathfrak{j}-1}(v_1)\alpha^{-\mathfrak{a}}\beta^{-\mathfrak{b}-1}\varphi^{-\mathfrak{c}-1}\psi^{-\mathfrak{d}}(u_1);\end{aligned}$$

- ▷ the unit object is $(\Bbbk,\mathrm{id},\mathrm{id},\mathrm{id},\mathrm{id})$ with the trivial action and trivial coaction.

**Theorem 4.4.** *${}_H\mathcal{YD}^H(\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q})$ is a rigid braided category.*

Proof. Firstly, for any $u\in U$, $v\in V$, we have

$$\begin{aligned}\varrho(h\cdot(u\otimes v))&=\underline{\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\varphi^{\mathfrak{c}}\psi^{\mathfrak{d}}(h_1)\cdot u}_{0}\otimes\underline{\alpha^{\mathfrak{g}}\beta^{\mathfrak{h}}\varphi^{\mathfrak{i}}\psi^{\mathfrak{j}}(h_2)\cdot v}_{0}\otimes\alpha^{-\mathfrak{g}-1}\beta^{-\mathfrak{h}}\varphi^{-\mathfrak{i}}\psi^{-\mathfrak{j}-1}\\&\quad\times(\underline{\alpha^{\mathfrak{g}}\beta^{\mathfrak{h}}\varphi^{\mathfrak{i}}\psi^{\mathfrak{j}}(h_2)\cdot v}_{1})\alpha^{-\mathfrak{a}}\beta^{-\mathfrak{b}-1}\varphi^{-\mathfrak{c}-1}\psi^{-\mathfrak{d}}(\underline{\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\varphi^{\mathfrak{c}}\psi^{\mathfrak{d}}(h_1)\cdot u}_{1})\\&=\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\varphi^{\mathfrak{c}}\psi^{\mathfrak{d}-1}(h_{121})\cdot u_0\otimes\alpha^{\mathfrak{g}}\beta^{\mathfrak{h}}\varphi^{\mathfrak{i}}\psi^{\mathfrak{j}-1}(h_{221})\cdot v_0\\&\quad\otimes((\alpha^{\mathfrak{m}-4}\beta^{\mathfrak{n}+1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-4}(h_{222})\alpha^{-\mathfrak{g}-2}\beta^{-\mathfrak{h}}\varphi^{-\mathfrak{i}}\psi^{-\mathfrak{j}-1}(v_1))\\&\quad\times S^{-1}\alpha^{\mathfrak{m}-1}\beta^{\mathfrak{n}-1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-3}(h_{21}))\\&\quad\times((\alpha^{\mathfrak{m}-3}\beta^{\mathfrak{n}}\varphi^{\mathfrak{p}}\psi^{\mathfrak{q}-3}(h_{122})\alpha^{-\mathfrak{a}-1}\beta^{-\mathfrak{b}-1}\varphi^{-\mathfrak{c}-1}\psi^{-\mathfrak{d}}(u_1))\\&\quad\times S^{-1}\alpha^{\mathfrak{m}}\beta^{\mathfrak{n}-2}\varphi^{\mathfrak{p}}\psi^{\mathfrak{q}-2}(h_{11}))\end{aligned}$$

$$\begin{aligned}
&= \alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\varphi^{\mathfrak{c}}\psi^{\mathfrak{d}-1}(h_{121})\cdot u_0 \otimes \alpha^{\mathfrak{g}}\beta^{\mathfrak{h}}\varphi^{\mathfrak{i}}\psi^{\mathfrak{j}}(h_{21})\cdot v_0\\
&\quad \otimes (\alpha^{\mathfrak{m}-3}\beta^{\mathfrak{n}+1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-3}(h_{22})\alpha^{-\mathfrak{g}-1}\beta^{-\mathfrak{h}}\varphi^{-\mathfrak{i}}\psi^{-\mathfrak{j}-1}(v_1))\\
&\quad \times (((S^{-1}\alpha^{\mathfrak{m}-3}\beta^{\mathfrak{n}-1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-5}(h_{1222})\alpha^{\mathfrak{m}-3}\beta^{\mathfrak{n}-1}\varphi^{\mathfrak{p}-1}\psi^{\mathfrak{q}-3}(h_{1221}))\\
&\quad \times \alpha^{-\mathfrak{a}-1}\beta^{-\mathfrak{b}-1}\varphi^{-\mathfrak{c}-1}\psi^{-\mathfrak{d}}(u_1))S^{-1}\alpha^{\mathfrak{m}}\beta^{\mathfrak{n}-2}\varphi^{\mathfrak{p}}\psi^{\mathfrak{q}-2}(h_{11}))\\
&\overset{(2.8)}{=} \alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\varphi^{\mathfrak{c}+1}\psi^{\mathfrak{d}-1}(h_{12})\cdot u_0 \otimes \alpha^{\mathfrak{g}}\beta^{\mathfrak{h}}\varphi^{\mathfrak{i}}\psi^{\mathfrak{j}}(h_{21})\cdot v_0\\
&\quad \otimes (\alpha^{\mathfrak{m}-3}\beta^{\mathfrak{n}+1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-3}(h_{22})\alpha^{-\mathfrak{g}-1}\beta^{-\mathfrak{h}}\varphi^{-\mathfrak{i}}\psi^{-\mathfrak{j}-1}(v_1))\\
&\quad \times (\alpha^{-\mathfrak{a}-1}\beta^{-\mathfrak{b}}\varphi^{-\mathfrak{c}-1}\psi^{-\mathfrak{d}}(u_1)S^{-1}\alpha^{\mathfrak{m}}\beta^{\mathfrak{n}-2}\varphi^{\mathfrak{p}}\psi^{\mathfrak{q}-2}(h_{11}))\\
&= \alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\varphi^{\mathfrak{c}}\psi^{\mathfrak{d}-1}(h_{211})\cdot u_0 \otimes \alpha^{\mathfrak{g}}\beta^{\mathfrak{h}}\varphi^{\mathfrak{i}}\psi^{\mathfrak{j}-1}(h_{212})\cdot v_0\\
&\quad \otimes (\alpha^{\mathfrak{m}-3}\beta^{\mathfrak{n}+1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-3}(h_{22})\\
&\quad \times (\alpha^{-\mathfrak{g}-2}\beta^{-\mathfrak{h}}\varphi^{-\mathfrak{i}}\psi^{-\mathfrak{j}-1}(v_1)\alpha^{-\mathfrak{a}-1}\beta^{-\mathfrak{b}-1}\varphi^{-\mathfrak{c}-1}\psi^{-\mathfrak{d}}(u_1)))\\
&\quad \times S^{-1}\alpha^{\mathfrak{m}}\beta^{\mathfrak{n}-1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-2}(h_1)\\
&= \psi^{-1}(h_{21})\cdot \underline{u\otimes v}_0 \otimes (\alpha^{\mathfrak{m}-3}\beta^{\mathfrak{n}+1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-3}(h_{22})\alpha^{-1}(\underline{u\otimes v}_1))\\
&\quad \times S^{-1}\alpha^{\mathfrak{m}}\beta^{\mathfrak{n}-1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-2}(h_1),
\end{aligned}$$

which implies (4.2). Hence, $U\otimes V \in {}_H\mathcal{YD}^H(\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q})$.

Secondly, define the the the associativity **a** and the unit constraints **l**, **r** by

$$\begin{aligned}
\mathbf{a}_{U,V,W}((u\otimes v)\otimes w) &= \alpha_U^{-\mathfrak{a}}\beta_U^{-\mathfrak{b}}\varphi_U^{-\mathfrak{c}-1}\psi_U^{-\mathfrak{d}}(u)\otimes (v\otimes \alpha_W^{\mathfrak{g}}\beta_W^{\mathfrak{h}}\varphi_W^{\mathfrak{i}}\psi_W^{\mathfrak{j}+1}(w)),\\
\mathbf{l}_U(1_{\Bbbk}\otimes u) &= \alpha_U^{-\mathfrak{g}}\beta_U^{-\mathfrak{h}}\varphi_U^{-\mathfrak{i}}\psi_U^{-\mathfrak{j}-1}(u),\quad \mathbf{r}_U(u\otimes 1_{\Bbbk}) = \alpha_U^{-\mathfrak{a}}\beta_U^{-\mathfrak{b}}\varphi_U^{-\mathfrak{c}-1}\psi_U^{-\mathfrak{d}}(u),
\end{aligned}$$

where $U,V,W \in {}_H\mathcal{YD}^H(\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q})$, then it is not hard to check that

$$({}_H\mathcal{YD}^H(\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q}),\otimes,\Bbbk,\mathbf{a},\mathbf{l},\mathbf{r})$$

is a monoidal category.

Thirdly, define $\mathbf{t}\colon \otimes \Rightarrow \otimes^{\mathrm{op}}$ by

$$\begin{aligned}
\mathbf{t}_{U,V}(u\otimes v) = \alpha_V^{\mathfrak{a}-\mathfrak{g}}\beta_V^{\mathfrak{b}-\mathfrak{h}}\varphi_V^{\mathfrak{c}-\mathfrak{i}}\psi_V^{\mathfrak{d}-\mathfrak{j}-1}(v_0)\otimes \alpha^{-\mathfrak{m}+1}\beta^{-\mathfrak{n}-1}\varphi^{-\mathfrak{p}-1}\psi^{-\mathfrak{q}+1}(v_1)\\
\times \alpha_U^{-\mathfrak{a}+\mathfrak{g}}\beta_U^{-\mathfrak{b}+\mathfrak{h}-1}\varphi_U^{-\mathfrak{c}+\mathfrak{i}-1}\psi_U^{-\mathfrak{d}+\mathfrak{j}+1}(u).
\end{aligned}$$

Obviously **t** is compatible with the Hom-structure maps. Since we have

$$\begin{aligned}
\mathbf{t}_{U,V}(h\cdot(u\otimes v)) &= \alpha_V^{\mathfrak{a}-\mathfrak{g}}\beta_V^{\mathfrak{b}-\mathfrak{h}}\varphi_V^{\mathfrak{c}-\mathfrak{i}}\psi_V^{\mathfrak{d}-\mathfrak{j}-1}(\underline{\alpha^{\mathfrak{g}}\beta^{\mathfrak{h}}\varphi^{\mathfrak{i}}\psi^{\mathfrak{j}}(h_2)\cdot v}_0)\\
&\quad \otimes \alpha^{-\mathfrak{m}+1}\beta^{-\mathfrak{n}-1}\varphi^{-\mathfrak{p}-1}\psi^{-\mathfrak{q}+1}(\underline{\alpha^{\mathfrak{g}}\beta^{\mathfrak{h}}\varphi^{\mathfrak{i}}\psi^{\mathfrak{j}}(h_2)\cdot v}_1)\\
&\quad \times \alpha_U^{-\mathfrak{a}+\mathfrak{g}}\beta_U^{-\mathfrak{b}+\mathfrak{h}-1}\varphi_U^{-\mathfrak{c}+\mathfrak{i}-1}\psi_U^{-\mathfrak{d}+\mathfrak{j}+1}(\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\varphi^{\mathfrak{c}}\psi^{\mathfrak{d}}(h_1)\cdot u)
\end{aligned}$$

$$\begin{aligned}
&= \alpha_V^{\mathfrak{a}-\mathfrak{g}}\beta_V^{\mathfrak{b}-\mathfrak{h}}\varphi_V^{\mathfrak{c}-\mathfrak{i}}\psi_V^{\mathfrak{d}-\mathfrak{j}-1}(\underline{\alpha^{\mathfrak{g}}\beta^{\mathfrak{h}}\varphi^{\mathfrak{i}}\psi^{\mathfrak{j}}(h_2)\cdot v}_0)\\
&\quad \otimes \alpha^{-\mathfrak{m}}\beta^{-\mathfrak{n}-1}\varphi^{-\mathfrak{p}-1}\psi^{-\mathfrak{q}+1}\\
&\quad \times (\underline{\alpha^{\mathfrak{g}}\beta^{\mathfrak{h}}\varphi^{\mathfrak{i}}\psi^{\mathfrak{j}}(h_2)v}_1\alpha^{\mathfrak{m}+\mathfrak{g}}\beta^{\mathfrak{n}+\mathfrak{h}}\varphi^{\mathfrak{p}+\mathfrak{i}}\psi^{\mathfrak{q}+\mathfrak{j}}(h_1))\\
&\quad \times \alpha_U^{-\mathfrak{a}+\mathfrak{g}}\beta_U^{-\mathfrak{b}+\mathfrak{h}}\varphi_U^{-\mathfrak{c}+\mathfrak{i}-1}\psi_U^{-\mathfrak{d}+\mathfrak{j}+1}(u)\\
&\overset{(4.1)}{=} \alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\varphi^{\mathfrak{c}}\psi^{\mathfrak{d}}(h_1)\cdot\alpha_V^{\mathfrak{a}-\mathfrak{g}}\beta_V^{\mathfrak{b}-\mathfrak{h}}\varphi_V^{\mathfrak{c}-\mathfrak{i}}\psi_V^{\mathfrak{d}-\mathfrak{j}-1}(v_0)\\
&\quad \otimes (\alpha^{\mathfrak{g}-1}\beta^{\mathfrak{h}}\varphi^{\mathfrak{i}}\psi^{\mathfrak{j}}(h_2)\alpha^{-\mathfrak{m}+1}\beta^{-\mathfrak{n}-1}\varphi^{-\mathfrak{p}-1}\psi^{-\mathfrak{q}+1}(v_1))\\
&\quad \times \alpha_U^{-\mathfrak{a}+\mathfrak{g}}\beta_U^{-\mathfrak{b}+\mathfrak{h}}\varphi_U^{-\mathfrak{c}+\mathfrak{i}-1}\psi_U^{-\mathfrak{d}+\mathfrak{j}+1}(u)\\
&= h\cdot\mathbf{t}_{U,V}((u\otimes v)),
\end{aligned}$$

where **t** is $H$-linear. Similarly we have

$$(\mathbf{t}_{U,V}\otimes\mathrm{id}_H)\circ\varrho^{U\otimes V}=\varrho^{V\otimes U}\circ\mathbf{t}_{U,V},$$

which implies that **t** is $H$-colinear. Moreover, we also have

$$\begin{aligned}
&((\mathrm{id}_V\otimes\mathbf{t}_{U,W})\circ\mathbf{a}_{V,U,W}\circ(\mathbf{t}_{U,V}\otimes\mathrm{id}_W))((u\otimes v)\otimes w)\\
&\qquad = (\mathrm{id}_V\otimes\mathbf{t}_{U,W})(\alpha_V^{-\mathfrak{g}}\beta_V^{-\mathfrak{h}}\varphi_V^{-\mathfrak{i}-1}\psi_V^{-\mathfrak{j}-1}(v_0)\\
&\qquad\quad \otimes (\alpha^{-\mathfrak{m}+1}\beta^{-\mathfrak{n}-1}\varphi^{-\mathfrak{p}-1}\psi^{-\mathfrak{q}+1}(n_1)\alpha_U^{-\mathfrak{a}+\mathfrak{g}}\beta_U^{-\mathfrak{b}+\mathfrak{h}-1}\\
&\qquad\quad \times \varphi_U^{-\mathfrak{c}+\mathfrak{i}-1}\psi_U^{-\mathfrak{d}+\mathfrak{j}+1}(u)\otimes\alpha_W^{\mathfrak{g}}\beta_W^{\mathfrak{h}}\varphi_W^{\mathfrak{i}}\psi_W^{\mathfrak{j}+1}(w)))\\
&\qquad = \alpha_V^{-\mathfrak{g}}\beta_V^{-\mathfrak{h}}\varphi_V^{-\mathfrak{i}-1}\psi_V^{-\mathfrak{j}-1}(v_0)\\
&\qquad\quad \otimes (\alpha_W^{\mathfrak{a}}\beta_W^{\mathfrak{b}}\varphi_W^{\mathfrak{c}}\psi_W^{\mathfrak{d}}(w_0)\otimes(\alpha^{-\mathfrak{m}+\mathfrak{g}}\beta^{-\mathfrak{n}+\mathfrak{h}-1}\varphi^{-\mathfrak{p}+\mathfrak{i}-1}\psi^{-\mathfrak{q}+\mathfrak{j}+2}(w_1)\\
&\qquad\quad \times \alpha^{-\mathfrak{a}+\mathfrak{g}-\mathfrak{m}+1}\beta^{-\mathfrak{b}+\mathfrak{h}-\mathfrak{n}-2}\varphi^{-\mathfrak{c}+\mathfrak{i}-\mathfrak{p}-2}\psi^{-\mathfrak{d}+\mathfrak{j}-\mathfrak{q}+2}(v_1))\\
&\qquad\quad \times \alpha_U^{-2\mathfrak{a}+2\mathfrak{g}}\beta_U^{-2\mathfrak{b}+2\mathfrak{h}-1}\varphi_U^{-2\mathfrak{c}+2\mathfrak{i}-2}\psi_U^{-2\mathfrak{d}+2\mathfrak{j}+2}(u))\\
&\qquad = \mathbf{a}_{V,W,U}(\alpha_V^{\mathfrak{a}-\mathfrak{g}}\beta_V^{\mathfrak{b}-\mathfrak{h}}\varphi_V^{\mathfrak{c}-\mathfrak{i}}\psi_V^{\mathfrak{d}-\mathfrak{j}-1}(v_0)\otimes\alpha_W^{\mathfrak{a}}\beta_W^{\mathfrak{b}}\varphi_W^{\mathfrak{c}}\psi_W^{\mathfrak{d}}(w_0))\\
&\qquad\quad \otimes \alpha^{-\mathfrak{m}+1}\beta^{-\mathfrak{n}-1}\varphi^{-\mathfrak{p}-1}\psi^{-\mathfrak{q}+1}(\alpha^{-1}(w_1)\alpha^{-\mathfrak{a}}\beta^{-\mathfrak{b}-1}\varphi^{-\mathfrak{c}-1}\psi^{-\mathfrak{d}}(v_1))\\
&\qquad\quad \times \alpha_U^{-\mathfrak{a}+\mathfrak{g}}\beta_U^{-\mathfrak{b}+\mathfrak{h}-1}\varphi_U^{-\mathfrak{c}+\mathfrak{i}-1}\psi_U^{-\mathfrak{d}+\mathfrak{j}+1}(\alpha_U^{-\mathfrak{a}}\beta_U^{-\mathfrak{b}}\varphi_U^{-\mathfrak{c}-1}\psi_U^{-\mathfrak{d}}(u))\\
&\qquad = (\mathbf{a}_{V,W,U}\circ\mathbf{t}_{U,V\otimes W}\circ\mathbf{a}_{U,V,W})((u\otimes v)\otimes w),
\end{aligned}$$

and similarly we can get

$$\mathbf{a}_{W,U,V}^{-1}\circ\mathbf{t}_{U\otimes V,W}\circ\mathbf{a}_{U,V,W}^{-1}=(\mathbf{t}_{U,W}\otimes\mathrm{id}_V)\circ\mathbf{a}_{U,W,V}^{-1}\circ(\mathrm{id}_U\otimes\mathbf{t}_{V,W}).$$

Note that the inverse of **t** is $\mathbf{t}'\colon \otimes^{\mathrm{op}}\Rightarrow\otimes$, where

$$\begin{aligned}
\mathbf{t}'_{U,V}(v\otimes u) = S\alpha^{-\mathfrak{m}+2}\beta^{-\mathfrak{n}-2}\varphi^{-\mathfrak{p}}\psi^{-\mathfrak{q}}(v_1)\cdot\alpha_U^{\mathfrak{a}-\mathfrak{g}}\beta_U^{\mathfrak{b}-\mathfrak{h}-1}\varphi_U^{\mathfrak{c}-\mathfrak{i}+1}\psi_U^{\mathfrak{d}-\mathfrak{j}-1}(u)\\
\otimes\alpha_V^{-\mathfrak{a}+\mathfrak{g}}\beta_V^{-\mathfrak{b}+\mathfrak{h}}\varphi_V^{-\mathfrak{c}+\mathfrak{i}-2}\psi_V^{-\mathfrak{d}+\mathfrak{j}+1}(v_0).
\end{aligned}$$

This means that $({}_H\mathcal{YD}^H(\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q}),\otimes,\Bbbk,a,\mathbf{l},\mathbf{r},\mathbf{t})$ is a braided category.

Finally, set $U^* = \hom_{\Bbbk}(U, \Bbbk)$ with the BiHom-module and BiHom-comodule structure

$$\theta_{U^*}\colon\ U^* \otimes H \to U^*, \quad (f \cdot h)(u) := f(S(h) \cdot \beta_U^{-2}(u)),$$
$$\varrho^{U^*}\colon\ U^* \to U^* \otimes H, \quad f_{(0)}(u) \otimes f_{(1)} := f(\varphi_U^{-2}(u_0)) \otimes S^{-1}(\beta^{-1}\varphi^{-1}(u_1))$$

and the Hom-structure maps

$$\alpha_{U^*} = (\alpha_U^{-1})^*, \quad \beta_{U^*} = (\beta_U^{-1})^*, \quad \varphi_{U^*} = (\varphi_U^{-1})^*, \quad \psi_{U^*} = (\psi_U^{-1})^*,$$

then $U^*$ is an object in ${}_H\mathcal{YD}^H(\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q})$. Furthermore, we define the evaluation $\mathrm{ev}_U\colon\ U^* \otimes U \to \Bbbk$ and coevaluation $\mathrm{coev}_U\colon\ \Bbbk \to U \otimes U^*$ as

$$\mathrm{ev}_U(f \otimes u) = f(\alpha_U^{\mathfrak{a}-\mathfrak{g}-1}\beta_U^{\mathfrak{b}-\mathfrak{h}+2}\varphi_U^{\mathfrak{c}-\mathfrak{i}}\psi_U^{\mathfrak{d}-\mathfrak{j}}(u)),$$
$$\mathrm{coev}_U(1_{\Bbbk}) = \sum \alpha_U^{\mathfrak{a}-\mathfrak{g}-1}\beta_U^{\mathfrak{b}-\mathfrak{h}}\varphi_U^{\mathfrak{c}-\mathfrak{i}}\psi_U^{\mathfrak{d}-\mathfrak{j}}(e_i) \otimes e^i,$$

where $e_i$ and $e^i$ are dual bases of $U$ and $U^*$, respectively. It is easy to check that $({}_H\mathcal{YD}^H(\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q}), \otimes, \Bbbk, \mathbf{a}, \mathbf{l}, \mathbf{r}, \mathbf{t})$ is a left rigid category. Similarly, it is also a right rigid category. $\square$

**Remark 4.5.** We denote $(\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q})$th BiHom-Yetter-Drinfeld category (under the monoidal structures given above) by $({}_H\mathcal{YD}^H(\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q}))_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}}^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}$.

**Proposition 4.6.** *For any* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{g}, \mathfrak{h}, \mathfrak{i}, \mathfrak{j}, \mathfrak{a}', \mathfrak{b}', \mathfrak{c}', \mathfrak{d}', \mathfrak{g}', \mathfrak{h}', \mathfrak{i}', \mathfrak{j}' \in \mathbb{Z}$ *we have that* $({}_H\mathcal{YD}^H(\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q}))_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}}^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}$ *is braided isomorphic to* $({}_H\mathcal{YD}^H(\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q}))_{\mathfrak{g}',\mathfrak{h}',\mathfrak{i}',\mathfrak{j}'}^{\mathfrak{a}',\mathfrak{b}',\mathfrak{c}',\mathfrak{d}'}$.

Proof. Define the functor $\mathscr{S} = (\mathscr{S}, \mathscr{S}_2, \mathscr{S}_0)\colon\ ({}_H\mathcal{YD}^H(\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q}))_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}}^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}} \to ({}_H\mathcal{YD}^H(\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q}))_{\mathfrak{g}',\mathfrak{h}',\mathfrak{i}',\mathfrak{j}'}^{\mathfrak{a}',\mathfrak{b}',\mathfrak{c}',\mathfrak{d}'}$ by

$$\mathscr{F}(U) = U \quad \text{as a BiHom-Yetter-Drinfeld-module}, \quad \mathscr{F}(f) = f,$$

where $(U, \alpha_U, \beta_U, \varphi_U, \psi_U) \in {}_H\mathcal{YD}^H(\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q})$, $f \in \mathrm{Mor}({}_H\mathcal{YD}^H(\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q}))$, and $\mathscr{S}_{2U,U}$ is given by

$$\mathscr{S}_{2U,V}(u \otimes v) = \alpha_U^{\mathfrak{a}-\mathfrak{a}'}\beta_U^{\mathfrak{b}-\mathfrak{b}'}\varphi_U^{\mathfrak{c}-\mathfrak{c}'}\psi_U^{\mathfrak{d}-\mathfrak{d}'}(u) \otimes \alpha_V^{\mathfrak{g}-\mathfrak{g}'}\beta_V^{\mathfrak{h}-\mathfrak{h}'}\varphi_V^{\mathfrak{i}-\mathfrak{i}'}\psi_V^{\mathfrak{j}-\mathfrak{j}'}(v)$$

for any $U, V \in {}_H\mathcal{YD}^H(\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q})$, $u \in U$, $v \in V$. Obviously $\mathscr{S} = (\mathscr{S}, \mathscr{S}_2, \mathscr{S}_0)$ is a braided isomorphic functor. $\square$

**Theorem 4.7.** *Let* $n \in \mathbb{Z}^+$. *If* $H$ *is both commutative and cocommutative, then* ${}_H\mathcal{YD}^H(\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q})$ *forms an* $n$*-monoidal category.*

Proof. It is similar to that of Theorem 3.6. Indeed, for $\mathfrak{a}_i, \mathfrak{b}_i, \mathfrak{c}_i, \mathfrak{d}_i, \mathfrak{g}_i, \mathfrak{h}_i, \mathfrak{i}_i, \mathfrak{j}_i \in \mathbb{Z}$, define the following morphisms in ${}_H\mathcal{YD}^H(\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q}) = (({}_H\mathcal{YD}^H(\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q}))^{\mathfrak{a}_i, \mathfrak{b}_i, \mathfrak{c}_i, \mathfrak{d}_i}_{\mathfrak{g}_i, \mathfrak{h}_i, \mathfrak{i}_i, \mathfrak{j}_i}, \otimes_i, \Bbbk, \mathbf{a}_i, \mathbf{l}_i, \mathbf{r}_i)$:

- ▷ the interchange law is

$$\zeta_{i,j}A, B, C, D\colon\ (A \otimes_i B) \otimes_j (C \otimes_i D) \to (A \otimes_j C) \otimes_i (B \otimes_j D),$$
$$(a \otimes_i b) \otimes_j (c \otimes_i d) \mapsto (a \otimes_j c) \otimes_i (b \otimes_j d),$$

where $A, B, B, C, D \in {}_H\mathcal{YD}^H(\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q})$;

- ▷ $\omega$, $\varpi$ and $\tau$ are given by

$$\omega_{i,j} = \varpi_{i,j} = \tau_{i,j} = \mathrm{id}_{\Bbbk}.$$

It is easy to check that $(({}_H\mathcal{YD}^H(\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q}))^{\mathfrak{a}_i, \mathfrak{b}_i, \mathfrak{c}_i, \mathfrak{d}_i}_{\mathfrak{g}_i, \mathfrak{h}_i, \mathfrak{i}_i, \mathfrak{j}_i}, \otimes_i, \Bbbk, \mathbf{a}_i, \mathbf{l}_i, \mathbf{r}_i)$ forms an $n$-monoidal category. □

**4.2. The center of the category of BiHom-modules.** In this section, we discuss the relations between BiHom-type Yetter-Dinfeld categories and the center of the category of BiHom-modules.

**Theorem 4.8.** *For any* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{g}, \mathfrak{h}, \mathfrak{i}, \mathfrak{j}, \mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q} \in \mathbb{Z}$, $({}_H\mathcal{YD}^H(\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q}))^{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}}_{\mathfrak{g}, \mathfrak{h}, \mathfrak{i}, \mathfrak{j}}$ *is a full braided subcategory of* $\mathcal{Z}({}_H\mathcal{BM}^{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}}_{\mathfrak{g}, \mathfrak{h}, \mathfrak{i}, \mathfrak{j}})$.

Proof. For any $V \in ({}_H\mathcal{YD}^H(\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q}))^{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}}_{\mathfrak{g}, \mathfrak{h}, \mathfrak{i}, \mathfrak{j}}$, $X \in {}_H\mathcal{BM}^{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}}_{\mathfrak{g}, \mathfrak{h}, \mathfrak{i}, \mathfrak{j}}$, define $T_{X,V}\colon X \otimes V \to V \otimes X$ by

$$\begin{aligned} T_{X,V}(x \otimes v) = {}& \alpha_V^{\mathfrak{a}-\mathfrak{g}} \beta_V^{\mathfrak{b}-\mathfrak{h}} \varphi_V^{\mathfrak{c}-\mathfrak{i}} \psi_V^{\mathfrak{d}-\mathfrak{j}-1}(v_0) \otimes \alpha^{-\mathfrak{m}+1} \beta^{-\mathfrak{n}-1} \varphi^{-\mathfrak{p}-1} \psi^{-\mathfrak{q}+1}(v_1) \\ & \times \alpha_X^{-\mathfrak{a}+\mathfrak{g}} \beta_X^{-\mathfrak{b}+\mathfrak{h}-1} \varphi_X^{-\mathfrak{c}+\mathfrak{i}-1} \psi_X^{-\mathfrak{d}+\mathfrak{j}+1}(x). \end{aligned}$$

Note that $T_{X,V}$ is a morphism in ${}_H\mathcal{BM}$, and $T_{\_,V}$ is natural. Obviously the diagrams (C1) and (C2) hold. Moreover, if $f\colon V \to W$ is a morphism in ${}^H\mathcal{YD}_H(\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q})$, then the diagram (C3) holds since $f$ is $H$-linear. Thus, $({}_H\mathcal{YD}^H(\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q}))^{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}}_{\mathfrak{g}, \mathfrak{h}, \mathfrak{i}, \mathfrak{j}}$ is a braided subcategory of $\mathcal{Z}({}_H\mathcal{BM}^{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}}_{\mathfrak{g}, \mathfrak{h}, \mathfrak{i}, \mathfrak{j}})$.

Next we show that $({}_H\mathcal{YD}^H(\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q}))^{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}}_{\mathfrak{g}, \mathfrak{h}, \mathfrak{i}, \mathfrak{j}}$ is a full subcategory. Actually, if $V, W \in {}_H\mathcal{YD}^H(\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q})$ and $f\colon V \to W$ is a morphism in $\mathcal{Z}({}_H\mathcal{BM}^{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}}_{\mathfrak{g}, \mathfrak{h}, \mathfrak{i}, \mathfrak{j}})$, then for any $v \in V$, $1_H \in H$, we have $(T_{H,W} \circ (\mathrm{id}_H \otimes f))(1_H \otimes v) = ((f \otimes \mathrm{id}_H) \circ T_{H,V}) \times (1_H \otimes v)$, i.e.,

$$\begin{aligned} & \alpha_W^{\mathfrak{a}-\mathfrak{g}} \beta_W^{\mathfrak{b}-\mathfrak{h}} \varphi_W^{\mathfrak{c}-\mathfrak{i}} \psi_W^{\mathfrak{d}-\mathfrak{j}}(f(v_0)) \otimes \alpha^{-\mathfrak{m}+2} \beta^{-\mathfrak{n}-1} \varphi^{-\mathfrak{p}-1} \psi^{-\mathfrak{q}+1}(f(v_0)) \\ & \qquad = f(\alpha_V^{\mathfrak{a}-\mathfrak{g}} \beta_V^{\mathfrak{b}-\mathfrak{h}} \varphi_V^{\mathfrak{c}-\mathfrak{i}} \psi_V^{\mathfrak{d}-\mathfrak{j}}(v_0)) \otimes \alpha^{-\mathfrak{m}+2} \beta^{-\mathfrak{n}-1} \varphi^{-\mathfrak{p}-1} \psi^{-\mathfrak{q}+1}(v_0), \end{aligned}$$

and hence, $f$ is $H$-colinear, which implies $f \in \mathrm{Mor}({}^H\mathcal{YD}_H(\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q}))$. □

Let $\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d},\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}\in\mathbb{Z}$. For any $U\in\mathscr{Z}({}_H\mathcal{BM}^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}})$, if we define the $H$-coaction $\varrho^U\colon U\to U\otimes H$ (with the notation $\varrho^U(u)=u_{(0)}\otimes u_{(1)}$) as

$$(4.3)\quad u_{(0)}\otimes u_{(1)}=\sum\alpha_U^{-\mathfrak{a}+\mathfrak{g}}\beta_U^{-\mathfrak{b}+\mathfrak{h}}\varphi_U^{-\mathfrak{c}+\mathfrak{i}}\psi_U^{-\mathfrak{d}+\mathfrak{j}+1}(u_T)\otimes 1_H{}^T\in U\otimes H\quad\forall\, u\in U,$$

where $\sum u_T\otimes 1_H{}^T=T_{H,U}(1_H\otimes u)$, then it is easy to check that $\varrho$ is compatible with the Hom-structure maps. Furthermore, we have the following results.

**Lemma 4.9.** *For any $u\in U$, $u_{(0)}\varepsilon(u_{(1)})=\varphi_U(u)$.*

Proof. Obviously $\varepsilon$ is $H$-linear. Thus, we have the commutative diagram

$$\begin{tikzcd}
H\otimes U \arrow[rr, "T_{H,U}"] \arrow[d, "\mathrm{id}\otimes\varepsilon"'] & & U\otimes H \arrow[d, "\mathrm{id}\otimes\varepsilon"] \\
\Bbbk\otimes U \arrow[r, "\mathbf{l}_U"'] \arrow[rr, bend left, "T_{\Bbbk,U}"] & U \arrow[r, "\mathbf{r}_U^{-1}"'] & U\otimes\Bbbk,
\end{tikzcd}$$

which implies $\sum u_T\otimes\varepsilon(1_H{}^T)=\mathbf{r}_U^{-1}\mathbf{l}_U(1_\Bbbk\otimes u)$, and hence the conclusion holds. $\square$

**Lemma 4.10.** *For any $u\in U$, we have*

$$(4.4)\quad u_{(0)}\otimes\alpha(u_{(1)})=u_{(0)}\otimes\beta(u_{(1)})=u_{(0)}\otimes\varphi(u_{(1)})=u_{(0)}\otimes\psi(u_{(1)})=u_{(0)}\otimes u_{(1)}.$$

Proof. For any $\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q}\in\mathbb{Z}$, we have

$$\begin{aligned}
&((\mathrm{id}_U\otimes\alpha^{\mathfrak{m}}\beta^{\mathfrak{n}}\varphi^{\mathfrak{p}}\psi^{\mathfrak{q}})\circ\varrho)(u)\\
&\qquad=(\alpha_U^{-\mathfrak{a}+\mathfrak{g}}\beta_U^{-\mathfrak{b}+\mathfrak{h}}\varphi_U^{-\mathfrak{c}+\mathfrak{i}}\psi_U^{-\mathfrak{d}+\mathfrak{j}+1}\otimes\alpha^{\mathfrak{m}}\beta^{\mathfrak{n}}\varphi^{\mathfrak{p}}\psi^{\mathfrak{q}})T_{H,U}(1_H\otimes u)\\
&\qquad=T_{H,U}(\alpha^{\mathfrak{m}}\beta^{\mathfrak{n}}\varphi^{\mathfrak{p}}\psi^{\mathfrak{q}}(1_H)\otimes\alpha_U^{-\mathfrak{a}+\mathfrak{g}}\beta_U^{-\mathfrak{b}+\mathfrak{h}}\varphi_U^{-\mathfrak{c}+\mathfrak{i}}\psi_U^{-\mathfrak{d}+\mathfrak{j}+1}(u))\\
&\qquad=(\alpha_U^{-\mathfrak{a}+\mathfrak{g}}\beta_U^{-\mathfrak{b}+\mathfrak{h}}\varphi_U^{-\mathfrak{c}+\mathfrak{i}}\psi_U^{-\mathfrak{d}+\mathfrak{j}+1}\otimes\mathrm{id}_H)T_{H,U}(1_H\otimes u)=\varrho(u).
\end{aligned}$$

That completes the proof. $\square$

**Corollary 4.11.** *For any $(U,T_{-,U})\in\mathscr{Z}({}_H\mathcal{BM}^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}})$, $(U,\varrho)$ is an $H$-BiHom-comodule, where $\varrho$ is defined in* (4.3).

Proof. We only need to prove the Hom-coassociative law. Define $\widehat{\Delta}_H\colon H\to H\otimes H$ by

$$\widehat{\Delta}_H(h)=\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\varphi^{\mathfrak{c}}\psi^{\mathfrak{d}}(h_1)\otimes\alpha^{\mathfrak{g}}\beta^{\mathfrak{h}}\varphi^{\mathfrak{i}}\psi^{\mathfrak{j}}(h_2),$$

where $h \in H$. Obviously $\widehat{\Delta}_H \in \mathrm{Mor}({}_H\mathcal{BM}^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}})$. Hence, the diagram

$$\begin{tikzcd}
\Bbbk \otimes U \arrow[r, "\mathrm{id}\otimes \mathbf{l}_U^{-1}"] \arrow[d, "\eta_H\otimes\mathrm{id}"'] \arrow[dr, "\mathbf{r}_{\Bbbk}^{-1}\otimes\mathrm{id}"] & \Bbbk \otimes (\Bbbk \otimes U) \arrow[r, "\eta_H\otimes\eta_H\otimes\mathrm{id}"] & H \otimes (H \otimes U) \arrow[r, "\mathrm{id}\otimes T_{H,U}"] \arrow[d, "\mathbf{a}^{-1}"] & H \otimes (U \otimes H) \arrow[d, "\mathbf{a}^{-1}"] \\
H \otimes U \arrow[d, "T_{H,U}"'] \arrow[rr, bend right, "\widehat{\Delta}_H\otimes\mathrm{id}"] & (\Bbbk \otimes \Bbbk) \otimes U \arrow[r, "\eta_H\otimes\eta_H\otimes\mathrm{id}"] & (H \otimes H) \otimes U \arrow[d, "T_{H\otimes H,U}"] & (H \otimes U) \otimes H \arrow[d, "T_{H,U}\otimes\mathrm{id}"] \\
U \otimes H \arrow[rr, "\mathrm{id}\otimes\widehat{\Delta}_H"'] & & U \otimes (H \otimes H) \arrow[r, "\mathbf{a}^{-1}"'] & (U \otimes H) \otimes H
\end{tikzcd}$$

is commutative. From Lemma 4.10, for any $u \in U$, we have

$$\sum \alpha_U^{\mathfrak{a}}\beta_U^{\mathfrak{b}}\varphi_U^{\mathfrak{c}+1}\psi_U^{\mathfrak{d}}(u_T) \otimes \underline{1_H{}^T}_1 \otimes \underline{1_H{}^T}_2 = \sum \underline{\alpha_U^{\mathfrak{g}}\beta_U^{\mathfrak{h}}\varphi_U^{\mathfrak{i}}\psi_U^{\mathfrak{j}+1}(u)}_{TT'} \otimes 1_H{}^{T'} \otimes 1_H{}^T,$$

which implies the conclusion. □

**Remark 4.12.** Note that for any $\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q} \in \mathbb{Z}$, $U \in {}_H\mathcal{YD}^H(\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q})$, (4.4) is not always satisfied. That implies that the objects in $\mathcal{Z}({}_H\mathcal{BM}^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}})$ are not objects in $({}_H\mathcal{YD}^H(\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q}))^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}}$ in general, and consequently, $({}_H\mathcal{YD}^H(\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q}))^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}}$ is a real subcategory of $\mathcal{Z}({}_H\mathcal{BM}^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}})$.

## 5. The Drinfeld doubles of BiHom-Hopf algebras

Let $H$ be a BiHom-Hopf algebra with bijective antipode $S$. In this section, we always assume that $\sum e_i \otimes e^i$ and $\sum o_i \otimes o^i$ are dual bases of $H$ and $H^*$, respectively.

**5.1. The BiHom-type Drinfeld doubles with parameters.** Recall from Example 2.6 (3) and Example 2.12 (5) that

$$H^* = (H^*, \star, \varepsilon, \Delta_{H^*}, \varepsilon_{H^*}, (\alpha^{-1})^*, (\beta^{-1})^*, (\psi^{-1})^*, (\varphi^{-1})^*)$$

is a BiHom-bialgebra,

$$H^{*\mathrm{cop}} = (H^*, \star, \varepsilon, \Delta_{H^*}^{\mathrm{cop}}, \varepsilon_{H^*}, (S^{-1})^*, (\beta^{-1})^*, (\alpha^{-1})^*, (\varphi^{-1})^*, (\psi^{-1})^*)$$

is a BiHom-Hopf algebra.

**Theorem 5.1.** *For any $\mathfrak{r},\mathfrak{s},\mathfrak{t},\mathfrak{u} \in \mathbb{Z}$, the $(\mathfrak{r},\mathfrak{s},\mathfrak{t},\mathfrak{u})$th Drinfeld double*

$$D(H)^{\{\mathfrak{r},\mathfrak{s},\mathfrak{t},\mathfrak{u}\}} = H \otimes H^{*\mathrm{cop}}$$

*of $H$, in a form containing $H$ and $H^{*\mathrm{cop}}$, is a BiHom-Hopf algebra with the following structures:*

▷ *the multiplication is given by*

$$\begin{aligned}(a\otimes p)(b\otimes q) = {}& a\varphi^{-1}\psi^{-1}(b_{21})\\ &\otimes p((\alpha^{\mathfrak{r}-3}\beta^{\mathfrak{s}+2}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}-1}(b_{22})\alpha^{-1}\beta^{-1}(-))S^{-1}\alpha^{\mathfrak{r}}\beta^{\mathfrak{s}}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}}(b_1))\star q,\end{aligned}$$

*where $p, q \in H^{*\mathrm{cop}}$, $a, b \in H$;*

▷ *the comultiplication is given by*

$$\overline{\Delta}(a\otimes p) = (a_1\otimes p_2)\otimes(a_2\otimes p_1);$$

▷ *the unit element is $1_H\otimes\varepsilon$ and the counit is given by*

$$\overline{\varepsilon}(a\otimes p) = \varepsilon(a)p(1_H);$$

▷ *the Hom-structure maps are given by*

$$\overline{\alpha} = \alpha\otimes(\alpha^{-1})^*,\quad \overline{\beta} = \beta\otimes(\beta^{-1})^*,\quad \overline{\varphi} = \varphi\otimes(\varphi^{-1})^*,\quad \overline{\psi} = \psi\otimes(\psi^{-1})^*;$$

▷ *the antipode $S$ is given by*

$$\begin{aligned}\overline{S}(a\otimes p) = {}& S\varphi^{-1}\psi^{-1}(a_{21})\\ &\otimes p((S^{-1}\alpha^{\mathfrak{r}-1}\beta^{\mathfrak{s}}\varphi^{\mathfrak{t}+1}\psi^{\mathfrak{u}-2}(a_{22})S^{-1}\alpha^{-1}\beta^{-1}(-))\alpha^{\mathfrak{r}}\beta^{\mathfrak{s}}\varphi^{\mathfrak{t}-1}\psi^{\mathfrak{u}+1}(a_1)).\end{aligned}$$

*Furthermore, $D(H)^{\{\mathfrak{r},\mathfrak{s},\mathfrak{t},\mathfrak{u}\}}$ has a quasitriangular structure*

$$\overline{\mathrm{R}} = \sum(\alpha^{-\mathfrak{r}}\beta^{-\mathfrak{s}-1}\varphi^{-\mathfrak{t}-1}\psi^{-\mathfrak{u}}(e_i)\otimes\varepsilon)\otimes(1_H\otimes e^i) \in D(H)^{\{\mathfrak{r},\mathfrak{s},\mathfrak{t},\mathfrak{u}\}}\otimes D(H)^{\{\mathfrak{r},\mathfrak{s},\mathfrak{t},\mathfrak{u}\}}.$$

Proof. It is straightforward to check that Hom-structure maps of $D(H)^{\{\mathfrak{r},\mathfrak{s},\mathfrak{t},\mathfrak{u}\}}$ commute with each others and are compatible with the multiplication, the unit and $\overline{\Delta}$, $\overline{\varepsilon}$.

Obviously $1_H\otimes\varepsilon$ is the unit object. For any $x\in H$, we have

$$\begin{aligned}&(((a\otimes p)(b\otimes q))(\beta(c)\otimes\overline{\beta}(f)))(x)\\ &\quad= (a\varphi^{-1}\psi^{-1}(b_{21}))\beta\varphi^{-1}\psi^{-1}(c_{21})\\ &\qquad\otimes((p((\alpha^{\mathfrak{r}-3}\beta^{\mathfrak{s}+2}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}-1}(b_{22})\alpha^{-1}\beta^{-1}(-))S^{-1}\alpha^{\mathfrak{r}}\beta^{\mathfrak{s}}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}}(b_1))\star q)\\ &\qquad\times((\alpha^{\mathfrak{r}-3}\beta^{\mathfrak{s}+3}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}-1}(c_{22})\alpha^{-1}\beta^{-1}(-))S^{-1}\alpha^{\mathfrak{r}}\beta^{\mathfrak{s}+1}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}}(c_1))\star\overline{\alpha}(f))(x)\\ &\quad\overset{(2.6)}{=} (a\varphi^{-1}\psi^{-1}(b_{21}))\beta\varphi^{-1}\psi^{-1}(c_{21})\otimes p((\alpha^{\mathfrak{r}-3}\beta^{\mathfrak{s}+2}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}-1}(b_{22})\\ &\qquad\times((\alpha^{\mathfrak{r}-5}\beta^{\mathfrak{s}+2}\varphi^{\mathfrak{t}-1}\psi^{\mathfrak{u}-1}(c_{221})\alpha^{-4}\beta^{-2}\varphi^{-2}(x_{11}))S^{-1}\alpha^{\mathfrak{r}-2}\beta^{\mathfrak{s}}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}-1}(c_{12})))\\ &\qquad\times S^{-1}\alpha^{\mathfrak{r}}\beta^{\mathfrak{s}}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}}(b_1))q((\alpha^{\mathfrak{r}-3}\beta^{\mathfrak{s}+2}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}-2}(c_{222})\alpha^{-2}\beta^{-2}\varphi^{-1}\psi^{-1}(x_{12}))\\ &\qquad\times S^{-1}\alpha^{\mathfrak{r}}\beta^{\mathfrak{s}}\varphi^{\mathfrak{t}-1}\psi^{\mathfrak{u}}(c_{11}))f(\beta^{-1}\psi^{-1}(x_2)),\end{aligned}$$

and

$$\begin{aligned}
&((\alpha(a)\otimes\overline{\alpha}(p))((b\otimes q)(c\otimes f)))(x)\\
&\quad=\alpha(a)(\varphi^{-1}\psi^{-1}(b_{21})\varphi^{-2}\psi^{-2}(c_{2121}))\otimes\overline{\alpha}(p)(((\alpha^{\mathfrak{r}-3}\beta^{\mathfrak{s}+2}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}-1}(b_{22})\alpha^{\mathfrak{r}-3}\beta^{\mathfrak{s}+2}\\
&\qquad\times\varphi^{\mathfrak{t}-1}\psi^{\mathfrak{u}-2}(c_{2122}))\alpha^{-2}\beta^{-1}\varphi^{-1}(x_1))(S^{-1}\alpha^{\mathfrak{r}-1}\beta^{\mathfrak{s}+1}\varphi^{\mathfrak{t}-1}\psi^{\mathfrak{u}-1}(c_{211})\\
&\qquad\times S^{-1}\alpha^{\mathfrak{r}+1}\beta^{\mathfrak{s}-1}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}}(b_1)))q((\alpha^{\mathfrak{r}-3}\beta^{\mathfrak{s}+2}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}-1}(c_{22})\alpha^{-2}\beta^{-2}\varphi^{-1}\psi^{-1}(x_{21}))\\
&\qquad\times S^{-1}\alpha^{\mathfrak{r}}\beta^{\mathfrak{s}}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}}(c_1))f(\beta^{-2}\psi^{-2}(x_{22}))\\
&\quad=\alpha(a)(\varphi^{-1}\psi^{-1}(b_{21})\varphi^{-1}\psi^{-1}(c_{21}))\otimes\overline{\alpha}(p)((\alpha^{\mathfrak{r}-2}\beta^{\mathfrak{s}+2}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}-1}(b_{22})\\
&\qquad\times((\alpha^{\mathfrak{r}-4}\beta^{\mathfrak{s}+2}\varphi^{\mathfrak{t}-1}\psi^{\mathfrak{u}-1}(c_{221})\alpha^{-3}\beta^{-2}\varphi^{-2}(x_{11}))S^{-1}\alpha^{\mathfrak{r}-1}\beta^{\mathfrak{s}}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}-1}(c_{12})))\\
&\qquad\times S^{-1}\alpha^{\mathfrak{r}+1}\beta^{\mathfrak{s}}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}}(b_1))q((\alpha^{\mathfrak{r}-3}\beta^{\mathfrak{s}+2}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}-2}(c_{222})\alpha^{-2}\beta^{-2}\varphi^{-1}\psi^{-1}(x_{12}))\\
&\qquad\times S^{-1}\alpha^{\mathfrak{r}}\beta^{\mathfrak{s}}\varphi^{\mathfrak{t}-1}\psi^{\mathfrak{u}}(c_{11}))f(\beta^{-1}\psi^{-1}(x_2)),
\end{aligned}$$

thus, $D(H)^{\{\mathfrak{r},\mathfrak{s},\mathfrak{t},\mathfrak{u}\}}$ is a BiHom-algebra.

Clearly $(D(H)^{\{\mathfrak{r},\mathfrak{s},\mathfrak{t},\mathfrak{u}\}},\overline{\Delta},\overline{\varepsilon},\overline{\varphi},\overline{\psi})$ is a BiHom-coalgebra. Now we show that $\overline{\Delta}$ and $\overline{\varepsilon}$ are all BiHom-algebra maps. For any $x,y\in H$, we have

$$\begin{aligned}
a_1\varphi^{-1}\psi^{-1}(b_{211})&\otimes\underline{p((\alpha^{\mathfrak{r}-3}\beta^{\mathfrak{s}+2}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}-1}(b_{22})\alpha^{-1}\beta^{-1}(-))S^{-1}\alpha^{\mathfrak{r}}\beta^{\mathfrak{s}}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}}(b_1))\star q}_{2}(x)\\
&\otimes a_2\varphi^{-1}\psi^{-1}(b_{212})\\
&\otimes\underline{p((\alpha^{\mathfrak{r}-3}\beta^{\mathfrak{s}+2}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}-1}(b_{22})\alpha^{-1}\beta^{-1}(-))S^{-1}\alpha^{\mathfrak{r}}\beta^{\mathfrak{s}}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}}(b_1))\star q}_{1}(y)\\
=a_1&\varphi^{-1}\psi^{-1}(b_{211})\otimes p((\alpha^{\mathfrak{r}-3}\beta^{\mathfrak{s}+2}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}-1}(b_{22})\alpha^{-1}\beta^{-1}\\
&\times(\alpha^{-2}\varphi^{-1}\psi^{-1}(y_1)\alpha^{-1}\beta^{-1}\varphi^{-2}(x_1)))S^{-1}\alpha^{\mathfrak{r}}\beta^{\mathfrak{s}}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}}(b_1))\\
&\times q(\alpha^{-1}\alpha^{-1}\psi^{-2}(y_2)\beta^{-2}\varphi^{-1}\psi^{-1}(x_2)),
\end{aligned}$$

and

$$\begin{aligned}
&a_1\varphi^{-1}\psi^{-1}(b_{121})\\
&\qquad\otimes\underline{p_2((\alpha^{\mathfrak{r}-3}\beta^{\mathfrak{s}+2}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}-1}(b_{122})\alpha^{-1}\beta^{-1}(-))S^{-1}\alpha^{\mathfrak{r}}\beta^{\mathfrak{s}}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}}(b_{11}))\star q_2}(x)\\
&\qquad\otimes a_2\varphi^{-1}\psi^{-1}(b_{221})\\
&\qquad\otimes\underline{p_1((\alpha^{\mathfrak{r}-3}\beta^{\mathfrak{s}+2}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}-1}(b_{222})\alpha^{-1}\beta^{-1}(-))S^{-1}\alpha^{\mathfrak{r}}\beta^{\mathfrak{s}}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}}(b_{21}))\star q_1}(y)\\
&=\ a_1\varphi^{-1}\psi^{-1}(b_{121})\otimes p(((\alpha^{\mathfrak{r}-4}\beta^{\mathfrak{s}+2}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}-2}(b_{222})\alpha^{-3}\beta^{-1}\varphi^{-1}\psi^{-1}(y_1))S^{-1}\alpha^{\mathfrak{r}-1}\\
&\qquad\times\beta^{\mathfrak{s}}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}-1}(b_{21}))((\alpha^{\mathfrak{r}-3}\beta^{\mathfrak{s}+1}\varphi^{\mathfrak{t}-1}\psi^{\mathfrak{u}-1}(b_{122})\alpha^{-2}\beta^{-2}\varphi^{-2}(y_1))S^{-1}\alpha^{\mathfrak{r}}\beta^{\mathfrak{s}-1}\\
&\qquad\times\varphi^{\mathfrak{t}-1}\psi^{\mathfrak{u}}(b_{11})))q(\alpha^{-1}\alpha^{-1}\psi^{-2}(y_2)\beta^{-2}\varphi^{-1}\psi^{-1}(x_2))\otimes a_2\varphi^{-1}\psi^{-1}(b_{221})\\
&=\ a_1\psi^{-1}(b_{21})\otimes p((\alpha^{\mathfrak{r}-3}\beta^{\mathfrak{s}+2}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}-3}(b_{2222})(\alpha^{-3}\beta^{-1}\varphi^{-1}\psi^{-1}(y_1)\\
&\qquad\times((S^{-1}\alpha^{\mathfrak{r}-3}\beta^{\mathfrak{s}-1}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}-3}(b_{2212})\alpha^{\mathfrak{r}-3}\beta^{\mathfrak{s}-1}\varphi^{\mathfrak{t}-2}\psi^{\mathfrak{u}-1}(b_{2211}))\\
&\qquad\times\alpha^{-2}\beta^{-3}\varphi^{-2}(x_1))))S^{-1}\alpha^{\mathfrak{r}}\beta^{\mathfrak{s}}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}-1}(b_1))\\
&\qquad\times q(\alpha^{-1}\alpha^{-1}\psi^{-2}(y_2)\beta^{-2}\varphi^{-1}\psi^{-1}(x_2))\otimes a_2\varphi^{-1}\psi^{-2}(b_{2221})
\end{aligned}$$

$$\begin{aligned}&\overset{(2.8)}{=} a_1\varphi^{-1}\psi^{-1}(b_{211})\otimes p((\alpha^{\mathfrak{r}-3}\beta^{\mathfrak{s}+2}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}-1}(b_{22})(\alpha^{-3}\beta^{-1}\varphi^{-1}\psi^{-1}(y_1)\alpha^{-2}\\ &\quad\times\beta^{-2}\varphi^{-2}(x_1)))S^{-1}\alpha^{\mathfrak{r}}\beta^{\mathfrak{s}}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}}(b_1))q(\alpha^{-1}\alpha^{-1}\psi^{-2}(y_2)\beta^{-2}\varphi^{-1}\psi^{-1}(x_2)).\end{aligned}$$

That means that $\overline{\Delta}$ preserves the multiplication. We also have $\overline{\varepsilon}((a\otimes p)(b\otimes q)) = \overline{\varepsilon}(a\otimes p)\overline{\varepsilon}(b\otimes q)$, hence, $D(H)^{\{\mathfrak{r},\mathfrak{s},\mathfrak{t},\mathfrak{u}\}}$ is a BiHom-bialgebra.

To prove that $S$ is the antipode, we compute as follows:

$$\begin{aligned}&\overline{S}(a_1\otimes p_2)(a_2\otimes p_1)\\
&\ = S\varphi^{-1}\psi^{-1}(a_{121})\varphi^{-1}\psi^{-1}(a_{221})\\
&\qquad\otimes p_2((S^{-1}\alpha^{\mathfrak{r}-1}\beta^{\mathfrak{s}}\varphi^{\mathfrak{t}+1}\psi^{\mathfrak{u}-2}(a_{122})S^{-1}\alpha^{-1}\beta^{-1}(-))\alpha^{\mathfrak{r}}\beta^{\mathfrak{s}}\varphi^{\mathfrak{t}-1}\psi^{\mathfrak{u}+1}(a_{11}))\\
&\qquad\times((\alpha^{\mathfrak{r}-3}\beta^{\mathfrak{s}+2}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}-1}(a_{222})\alpha^{-1}\beta^{-1}(-))S^{-1}\alpha^{\mathfrak{r}}\beta^{\mathfrak{s}}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}}(a_{21}))\star p_1\\
&\ = S\varphi^{-1}\psi^{-1}(a_{121})\varphi^{-1}\psi^{-1}(a_{221})\\
&\qquad\otimes p_2(((S^{-1}\alpha^{\mathfrak{r}-2}\beta^{\mathfrak{s}}\varphi^{\mathfrak{t}+1}\psi^{\mathfrak{u}-2}(a_{122})S^{-2}\alpha^{\mathfrak{r}-2}\beta^{\mathfrak{s}}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}}(a_{21}))S^{-1}\alpha^{-1}\beta^{-1}(-))\\
&\qquad\times(S^{-1}\alpha^{\mathfrak{r}-2}\beta^{\mathfrak{s}+1}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}-1}(a_{222})\alpha^{\mathfrak{r}}\beta^{\mathfrak{s}-1}\varphi^{\mathfrak{t}-1}\psi^{\mathfrak{u}+1}(a_{11})))\star p_1\\
&\overset{(2.5)}{=} S\psi^{-1}(a_{21})\varphi^{-1}\psi^{-2}(a_{2221})\\
&\qquad\otimes p_2((S^{-1}(S^{-1}\alpha^{\mathfrak{r}-3}\beta^{\mathfrak{s}+1}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}-2}(a_{2212})\alpha^{\mathfrak{r}-1}\beta^{\mathfrak{s}-1}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}-2}(a_{2211}))S^{-1}\alpha^{-1}\beta^{-1}(-))\\
&\qquad\times(S^{-1}\alpha^{\mathfrak{r}-2}\beta^{\mathfrak{s}+1}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}-2}(a_{2222})\alpha^{\mathfrak{r}}\beta^{\mathfrak{s}-1}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}+1}(a_1)))\star p_1\\
&\overset{(2.8)}{=} S\varphi^{-1}\psi^{-1}(a_{211})\varphi^{-1}\psi^{-1}(a_{212})\\
&\qquad\otimes p_2(S^{-1}\alpha^{-1}(-)(S^{-1}\alpha^{\mathfrak{r}-2}\beta^{\mathfrak{s}+1}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}}(a_{22})\alpha^{\mathfrak{r}}\beta^{\mathfrak{s}-1}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}+1}(a_1)))\star p_1\\
&\overset{(2.7)}{=} 1_H\varepsilon(a)\otimes p_2(S^{-1}(-))\star p_1 \overset{(2.8)}{=} 1_H\varepsilon(a)\otimes p\end{aligned}$$

and similarly we have $(a_1\otimes p_2)\overline{S}(a_2\otimes p_1) = 1_H\varepsilon(a)\otimes p$. Note that $\overline{S}$ commutes with the Hom-structure maps, then we get that $\overline{S}$ is the antipode of $D(H)^{\{\mathfrak{r},\mathfrak{s},\mathfrak{t},\mathfrak{u}\}}$.

At last, we show that $\overline{\mathrm{R}} = \sum(\alpha^{-\mathfrak{r}}\beta^{-\mathfrak{s}-1}\varphi^{-\mathfrak{t}-1}\psi^{-\mathfrak{u}}(e_i)\otimes\varepsilon)\otimes(1_H\otimes e^i)$ is a quasi-triangular structure in $D(H)^{\{\mathfrak{r},\mathfrak{s},\mathfrak{t},\mathfrak{u}\}}$.

For one thing, it is clear that $\overline{\mathrm{R}}$ is compatible with the Hom-structure maps of $D(H)^{\{\mathfrak{r},\mathfrak{s},\mathfrak{t},\mathfrak{u}\}}$. For another, for any $x, y\in H$, we have

$$\begin{aligned}&\sum((\alpha^{-\mathfrak{r}}\beta^{-\mathfrak{s}-1}\varphi^{-\mathfrak{t}-1}\psi^{-\mathfrak{u}}(e_i)\otimes\varepsilon)(\varphi^{-1}\psi(a_1)\otimes p_2(\varphi^{-1}\psi(-))))(x)\\
&\qquad\otimes((1_H\otimes e^i)(\varphi\psi^{-1}(a_2)\otimes p_1(\varphi\psi^{-1}(-))))(y)\\
&\quad = \sum\alpha^{-\mathfrak{r}}\beta^{-\mathfrak{s}-1}\varphi^{-\mathfrak{t}-1}\psi^{-\mathfrak{u}}(e_i)\varphi^{-1}\psi(a_1)\otimes p_2(\beta^{-1}\varphi^{-1}\psi(x))\otimes\beta\psi^{-2}(a_{221})\\
&\qquad\times e^i((\alpha^{\mathfrak{r}-3}\beta^{\mathfrak{s}+2}\varphi^{\mathfrak{t}+1}\psi^{\mathfrak{u}-1}(a_{222})\alpha^{-2}\beta^{-1}\varphi^{-1}(y_1))S^{-1}\alpha^{\mathfrak{r}}\beta^{\mathfrak{s}}\varphi^{\mathfrak{t}+1}\psi^{\mathfrak{u}-1}(a_{21}))\\
&\qquad\otimes p_1(\beta^{-1}\varphi\psi^{-2}(y_2))\\
&\quad\overset{(2.8)}{=} \sum\alpha^{-1}\beta(a_2)\alpha^{-\mathfrak{r}}\beta^{-\mathfrak{s}-2}\varphi^{-\mathfrak{t}-2}\psi^{-\mathfrak{u}}(y_1)\otimes p(\alpha^{-1}\beta^{-1}\varphi\psi^{-3}(y_2)\beta^{-2}\varphi^{-2}\psi(x))\\
&\qquad\otimes\beta(a_1)\otimes 1_{\Bbbk}\end{aligned}$$

and

$$\begin{aligned}
&\sum((\alpha^{-1}\beta(a_2)\otimes p_1(\alpha^{-1}\beta(-)))(\alpha^{-\mathfrak{r}}\beta^{-\mathfrak{s}-1}\varphi^{-\mathfrak{t}-1}\psi^{-\mathfrak{u}}(e_i)\otimes\varepsilon))(x)\\
&\qquad\otimes((\alpha^{-1}\beta(a_1)\otimes p_2(\alpha^{-1}\beta(-)))(1_H\otimes e^i))(y)\\
&\quad= \sum\alpha^{-1}\beta(a_2)\alpha^{-\mathfrak{r}}\beta^{-\mathfrak{s}-1}\varphi^{-\mathfrak{t}-2}\psi^{-\mathfrak{u}-1}(e_{i21})\\
&\qquad\otimes p_1((\alpha^{-4}\beta^2\varphi^{-1}\psi^{-1}(e_{i22})\alpha^{-3}(x))S^{-1}\alpha^{-1}\varphi^{-1}(e_{i1}))\otimes\beta(a_1)\otimes e^i(\beta^{-1}\psi^{-1}(y_2))\\
&\quad= \sum\alpha^{-1}\beta(a_2)\alpha^{-\mathfrak{r}}\beta^{-\mathfrak{s}-2}\varphi^{-\mathfrak{t}-2}\psi^{-\mathfrak{u}-1}(y_{21})\\
&\qquad\otimes p(((\alpha^{-5}\beta\varphi^{-1}\psi^{-2}(y_{22})\alpha^{-4}\psi^{-1}(x))S^{-1}\alpha^{-2}\beta^{-1}\varphi^{-2}(y_{12}))\alpha^{-2}\beta^{-3}(y_{11}))\\
&\qquad\otimes\beta(a_1)\otimes 1_{\Bbbk}\\
&\quad\overset{(2.8)}{=} \sum\alpha^{-1}\beta(a_2)\alpha^{-\mathfrak{r}}\beta^{-\mathfrak{s}-2}\varphi^{-\mathfrak{t}-2}\psi^{-\mathfrak{u}}(y_1)\otimes p(\alpha^{-1}\beta^{-1}\varphi\psi^{-3}(y_2)\beta^{-2}\varphi^{-2}\psi(x))\\
&\qquad\otimes\beta(a_1)\otimes 1_{\Bbbk},
\end{aligned}$$

hence, (Q2) holds. Further, we compute

$$\begin{aligned}
&\sum\alpha^{-1}\psi^{-1}(\alpha^{-\mathfrak{r}}\beta^{-\mathfrak{s}-1}\varphi^{-\mathfrak{t}-1}\psi^{-\mathfrak{u}}(e_i)\otimes\varepsilon)\beta^{-1}\varphi^{-1}(\alpha^{-\mathfrak{r}}\beta^{-\mathfrak{s}-1}\varphi^{-\mathfrak{t}-1}\psi^{-\mathfrak{u}}(o_i)\otimes\varepsilon)\\
&\qquad\otimes(1_H\otimes o^i)\otimes(1_H\otimes e^i)\\
&\quad=\sum\alpha^{-\mathfrak{r}-1}\beta^{-\mathfrak{s}-1}\varphi^{-\mathfrak{t}-1}\psi^{-\mathfrak{u}-1}(e_i)\alpha^{-\mathfrak{r}}\beta^{-\mathfrak{s}-2}\varphi^{-\mathfrak{t}-2}\psi^{-\mathfrak{u}}(o_i)\otimes(1_H\otimes o^i)\otimes(1_H\otimes e^i)\\
&\quad=\sum(\alpha^{-\mathfrak{r}}\beta^{-\mathfrak{s}-1}\varphi^{-\mathfrak{t}-1}\psi^{-\mathfrak{u}}(e_i)\otimes\varepsilon)\otimes(1_H\otimes {e^i}_2)\otimes(1_H\otimes {e^i}_1),
\end{aligned}$$

thus, (Q4) holds. Similarly one can prove (Q3). That completes the proof. □

**5.2. The representations of BiHom-type Drinfeld doubles.** For any $\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q},\mathfrak{r},\mathfrak{s},\mathfrak{t},\mathfrak{u}\in\mathbb{Z}$, we have the following results.

**Theorem 5.2.** *${}_{D(H)^{\{\mathfrak{r},\mathfrak{s},\mathfrak{t},\mathfrak{u}\}}}\mathcal{BM}$ is identified to ${}_H\mathcal{YD}^H(\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q})$.*

Proof. Define the functor $\mathscr{F}\colon {}_H\mathcal{YD}^H(\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q})\to {}_{D(H)^{\{\mathfrak{r},\mathfrak{s},\mathfrak{t},\mathfrak{u}\}}}\mathcal{BM}$ by

$$\mathscr{F}(U)=U\quad\text{as }\Bbbk\text{-space},\quad\mathscr{F}(f)=f$$

for any $(U,\alpha_U,\beta_U,\varphi_U,\psi_U)\in {}_H\mathcal{YD}^H(\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q})$, $f\in\mathrm{Mor}({}_H\mathcal{YD}^H(\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q}))$, and the $D(H)^{\{\mathfrak{r},\mathfrak{s},\mathfrak{t},\mathfrak{u}\}}$-action on $M$ is given by

$$(a\otimes p)\rightharpoondown u=p(\alpha^{-\mathfrak{m}}\beta^{-\mathfrak{n}+1}\varphi^{-\mathfrak{p}-1}\psi^{-\mathfrak{q}}(u_1))\alpha^{\mathfrak{r}+1}\beta^{\mathfrak{s}}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}+2}(a)\cdot\varphi_U^{-1}(u_0)$$

for any $u\in U$, $a\in H$, $p\in H^*$.

Now we check that $(U, \rightharpoondown, \alpha_U, \beta_U, \varphi_U, \psi_U)$ is a $D(H)^{\{\mathfrak{r},\mathfrak{s},\mathfrak{t},\mathfrak{u}\}}$-BiHom-module. We only prove the BiHom-associative law. Indeed, we have

$$
\begin{aligned}
&((a \otimes p)(b \otimes q)) \rightharpoondown u \\
&\quad = (p((\alpha^{\mathfrak{r}-3}\beta^{\mathfrak{s}+2}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}-1}(b_{22})\alpha^{-1}\beta^{-1}(-))S^{-1}\alpha^{\mathfrak{r}}\beta^{\mathfrak{s}}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}}(b_1)) \star q) \\
&\qquad \times (\alpha^{-\mathfrak{m}}\beta^{-\mathfrak{n}+1}\varphi^{-\mathfrak{p}-1}\psi^{-\mathfrak{q}}(u_1)) \\
&\qquad \times (\alpha^{\mathfrak{r}+1}\beta^{\mathfrak{s}}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}+2}(a)\alpha^{\mathfrak{r}+1}\beta^{\mathfrak{s}}\varphi^{\mathfrak{t}-1}\psi^{\mathfrak{u}+1}(b_{21})) \cdot \varphi_U^{-1}(u_0) \\
&\quad = p(\alpha^{-1}(-)) \\
&\qquad \times ((\alpha^{\mathfrak{r}-3}\beta^{\mathfrak{s}+2}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}-1}(b_{22})\alpha^{-\mathfrak{m}-2}\beta^{-\mathfrak{n}}\varphi^{-\mathfrak{p}-2}\psi^{-\mathfrak{q}}(u_{01}))S^{-1}\alpha^{\mathfrak{r}}\beta^{\mathfrak{s}}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}}(b_1)) \\
&\qquad \times \alpha^{\mathfrak{r}+2}\beta^{\mathfrak{s}}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}+2}(a)(\alpha^{\mathfrak{r}+1}\beta^{\mathfrak{s}}\varphi^{\mathfrak{t}-1}\psi^{\mathfrak{u}+1}(b_{21}) \cdot \beta_M^{-1}\varphi_M^{-1}(u_{00})) \\
&\qquad \times q(\alpha^{-\mathfrak{m}}\beta^{-\mathfrak{n}}\varphi^{-\mathfrak{p}-1}\psi^{-\mathfrak{q}}(u_1)) \\
&\quad = (\alpha(a) \otimes p(\alpha^{-1}(-))) \\
&\qquad \rightharpoondown (q(\alpha^{-\mathfrak{m}}\beta^{-\mathfrak{n}}\varphi^{-\mathfrak{p}-1}\psi^{-\mathfrak{q}}(u_1))\alpha^{\mathfrak{r}+1}\beta^{\mathfrak{s}}\varphi^{\mathfrak{t}}\psi^{\mathfrak{u}+2}(a) \cdot \beta_M^{-1}\varphi_M^{-1}(m_0)) \\
&\quad = (\alpha(a) \otimes p(\alpha^{-1}(-))) \rightharpoondown ((b \otimes q) \rightharpoondown \beta_U^{-1}(u)),
\end{aligned}
$$

as needed. Moreover, it is easy to obtain that $\mathscr{F}(f) \in \mathrm{Mor}({}_{D(H)^{\{\mathfrak{r},\mathfrak{s},\mathfrak{t},\mathfrak{u}\}}}\mathcal{BM})$. Hence, $\mathscr{F}$ is well-defined.

Conversely, define the functor $\mathscr{G}\colon {}_{D(H)^{\{\mathfrak{r},\mathfrak{s},\mathfrak{t},\mathfrak{u}\}}}\mathcal{BM} \to {}_H\mathcal{YD}^H(\mathfrak{m},\mathfrak{n},\mathfrak{p},\mathfrak{q})$ by

$$
\mathscr{G}(M) = M \quad \text{as } \Bbbk\text{-space}, \quad \mathscr{G}(f) = f
$$

for any $(M, \rightharpoondown, \alpha_M, \beta_M, \varphi_M, \psi_M) \in {}_{D(H)^{\{\mathfrak{r},\mathfrak{s},\mathfrak{t},\mathfrak{u}\}}}\mathcal{BM}$, $f \in \mathrm{Mor}({}_{D(H)^{\{\mathfrak{r},\mathfrak{s},\mathfrak{t},\mathfrak{u}\}}}\mathcal{BM})$, and the $H$-action, $H$-coaction on $M$ are given by

$$
\begin{aligned}
h \cdot m &= (\alpha^{-\mathfrak{r}-1}\beta^{-\mathfrak{s}}\varphi^{-\mathfrak{t}}\psi^{-\mathfrak{u}-2}(h) \otimes \varepsilon) \rightharpoondown m, \\
\varrho(m) &= \sum (1_H \otimes e^i) \rightharpoondown \beta_M^{-1}\varphi_M(m) \otimes \alpha^{\mathfrak{m}}\beta^{\mathfrak{n}}\varphi^{\mathfrak{p}}\psi^{\mathfrak{q}}(e_i)
\end{aligned}
$$

for any $m \in M$, $a \in H$. Next we show that $\mathscr{G}$ is well defined.

Firstly, $(M, \cdot, \alpha_M, \beta_M, \varphi_M, \psi_M)$ is an $H$-BiHom-module. Obviously we have $1_H \cdot m = \beta_M(m)$, and we also have

$$
\begin{aligned}
&\alpha(h) \cdot (g \cdot \beta_M^{-1}(m)) \\
&\quad = (\alpha^{-\mathfrak{r}}\beta^{-\mathfrak{s}}\varphi^{-\mathfrak{t}}\psi^{-\mathfrak{u}-2}(h) \otimes \varepsilon) \rightharpoondown ((\alpha^{-\mathfrak{r}-1}\beta^{-\mathfrak{s}}\varphi^{-\mathfrak{t}}\psi^{-\mathfrak{u}-2}(g) \otimes \varepsilon) \rightharpoondown \beta_M^{-1}(m)) \\
&\quad = (\alpha^{-\mathfrak{r}-1}\beta^{-\mathfrak{s}}\varphi^{-\mathfrak{t}}\psi^{-\mathfrak{u}-2}(h)\alpha^{-\mathfrak{r}-1}\beta^{-\mathfrak{s}}\varphi^{-\mathfrak{t}}\psi^{-\mathfrak{u}-2}(g) \otimes \varepsilon) \rightharpoondown m = (hg) \cdot m,
\end{aligned}
$$

which implies $M$ is an $H$-BiHom-module.

Secondly, it is easy to get $m_0\varepsilon(m_1) = \alpha_M(m)$. Moreover, for any $x \in H$, we obtain

$$\begin{aligned}
&\varphi_M(m_0) \otimes \Delta(m_1)\\
&\quad = \sum(1_H \otimes e^i(\varphi^{-1}(-))) \rightharpoondown \beta_M^{-1}\varphi_M^2(m) \otimes \alpha^{\mathfrak{m}}\beta^{\mathfrak{n}}\varphi^{\mathfrak{p}}\psi^{\mathfrak{q}}(e_{i1}) \otimes \alpha^{\mathfrak{m}}\beta^{\mathfrak{n}}\varphi^{\mathfrak{p}}\psi^{\mathfrak{q}}(e_{i2})\\
&\quad = \sum(1_H \otimes o^i(\alpha(-)) \star e^i(\beta\varphi^{-1}(-))) \rightharpoondown \beta_M^{-1}\varphi_M^2(m) \otimes \alpha^{\mathfrak{m}}\beta^{\mathfrak{n}}\varphi^{\mathfrak{p}}\psi^{\mathfrak{q}}(o_i)\\
&\qquad \otimes \alpha^{\mathfrak{m}}\beta^{\mathfrak{n}}\varphi^{\mathfrak{p}}\psi^{\mathfrak{q}+1}(e_i)\\
&\quad = \sum(1_H \otimes o^i) \rightharpoondown \beta_M^{-1}\varphi_M((1_H \otimes e^i) \rightharpoondown \beta_M^{-1}\varphi_M(m)) \otimes \alpha^{\mathfrak{m}}\beta^{\mathfrak{n}}\varphi^{\mathfrak{p}}\psi^{\mathfrak{q}}(o_i)\\
&\qquad \otimes \alpha^{\mathfrak{m}}\beta^{\mathfrak{n}}\varphi^{\mathfrak{p}}\psi^{\mathfrak{q}+1}(e_i)\\
&\quad = m_{00} \otimes m_{01} \otimes \psi(m_1),
\end{aligned}$$

which implies $M$ is an $H$-BiHom-comodule.

Finally, we have

$$\begin{aligned}
&\varrho(h \cdot m)\\
&= \sum((1_H \otimes e^i(\alpha(-)))(\alpha^{-\mathfrak{r}-1}\beta^{-\mathfrak{s}-1}\varphi^{-\mathfrak{t}+1}\psi^{-\mathfrak{u}-2}(h) \otimes \varepsilon))\\
&\quad \rightharpoondown \varphi_M(m) \otimes \alpha^{\mathfrak{m}}\beta^{\mathfrak{n}}\varphi^{\mathfrak{p}}\psi^{\mathfrak{q}}(e_i)\\
&= \sum(\alpha^{-\mathfrak{r}-1}\beta^{-\mathfrak{s}}\varphi^{-\mathfrak{t}}\psi^{-\mathfrak{u}-3}(h_{21}) \otimes e^i((\alpha^{-3}\beta\varphi\psi^{-3}(h_{22})\beta^{-1}(-))S^{-1}\beta^{-1}\varphi\psi^{-2}(h_1)) \star \varepsilon)\\
&\quad \rightharpoondown \varphi_M(m) \otimes \alpha^{\mathfrak{m}}\beta^{\mathfrak{n}}\varphi^{\mathfrak{p}}\psi^{\mathfrak{q}}(e_i)\\
&= \sum(\alpha^{-\mathfrak{r}-1}\beta^{-\mathfrak{s}}\varphi^{-\mathfrak{t}}\psi^{-\mathfrak{u}-3}(h_{21}) \otimes e^i(\beta^{-1}(-))) \rightharpoondown \varphi_M(m)\\
&\quad \otimes (\alpha^{\mathfrak{m}-3}\beta^{\mathfrak{n}+1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-3}(h_{22})\alpha^{\mathfrak{m}-1}\beta^{\mathfrak{n}}\varphi^{\mathfrak{p}}\psi^{\mathfrak{q}}(e_i))S^{-1}\alpha^{\mathfrak{m}}\beta^{\mathfrak{n}-1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-2}(h_1)\\
&= \sum(\alpha^{-\mathfrak{r}-1}\beta^{-\mathfrak{s}}\varphi^{-\mathfrak{t}}\psi^{-\mathfrak{u}-3}(h_{21}) \otimes \varepsilon) \rightharpoondown ((1_H \otimes e^i) \rightharpoondown \beta_M^{-1}\varphi_M(m_0))\\
&\quad \otimes (\alpha^{\mathfrak{m}-3}\beta^{\mathfrak{n}+1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-3}(h_{22})\alpha^{\mathfrak{m}-1}\beta^{\mathfrak{n}}\varphi^{\mathfrak{p}}\psi^{\mathfrak{q}}(e_i))S^{-1}\alpha^{\mathfrak{m}}\beta^{\mathfrak{n}-1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-2}(h_1)\\
&= \psi^{-1}(h_{21}) \cdot m_0 \otimes (\alpha^{\mathfrak{m}-3}\beta^{\mathfrak{n}+1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}-3}(h_{22})\alpha^{-1}(m_1))S^{-1}\alpha^{\mathfrak{m}}\beta^{\mathfrak{n}-1}\varphi^{\mathfrak{p}+1}\psi^{\mathfrak{q}+2}(h_1),
\end{aligned}$$

which implies (4.2), and hence $\mathscr{G}$ is well defined.

Notice that $\mathscr{F}$ and $\mathscr{G}$ are inverse with each other, the conclusion holds. □

**Corollary 5.3.** *For any* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{g}, \mathfrak{h}, \mathfrak{i}, \mathfrak{j}, \mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q}, \mathfrak{r}, \mathfrak{s}, \mathfrak{t}, \mathfrak{u} \in \mathbb{Z}$ *we have that* ${}_{D(H)^{\{\mathfrak{r},\mathfrak{s},\mathfrak{t},\mathfrak{u}\}}}\mathcal{BM}^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}}$ *is braided isomorphic to* $({}_H\mathcal{YD}^H(\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q}))^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}}$.

Proof. Clearly $\mathscr{F}$ is a strict monoidal functor. Note that ${}_{D(H)^{\{\mathfrak{r},\mathfrak{s},\mathfrak{t},\mathfrak{u}\}}}\mathcal{BM}^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}}$ is a braided category with the braiding **C** since

$$\overline{\mathrm{R}} = \sum(\alpha^{-\mathfrak{r}}\beta^{-\mathfrak{s}-1}\varphi^{-\mathfrak{t}-1}\psi^{-\mathfrak{u}}(e_i) \otimes \varepsilon) \otimes (1_H \otimes e^i)$$

is a quasitriangular structure in $D(H)^{\{\mathfrak{r},\mathfrak{s},\mathfrak{t},\mathfrak{u}\}}$, where **C** is derived by $\overline{\mathrm{R}}$ in the way of (3.1).

For any $U, V \in ({}_H\mathcal{YD}^H(\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q}))^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}}$, $u \in U$, $v \in V$, we have

$$\begin{aligned}
&\mathbf{C}_{\mathscr{F}(U),\mathscr{F}(V)}(u \otimes v)\\
&\quad = \sum (1_H \otimes e^i(\alpha^{-\mathfrak{a}}\beta^{-\mathfrak{b}}\varphi^{-\mathfrak{c}}\psi^{-\mathfrak{d}}(-))) \rightharpoonup \alpha_V^{\mathfrak{a}-\mathfrak{g}}\beta_V^{\mathfrak{b}-\mathfrak{h}-1}\varphi_V^{\mathfrak{c}-\mathfrak{i}+1}\psi_V^{\mathfrak{d}-\mathfrak{j}-1}(v)\\
&\qquad \otimes (\alpha^{\mathfrak{g}-\mathfrak{r}}\beta^{\mathfrak{h}-\mathfrak{s}-1}\varphi^{\mathfrak{i}-\mathfrak{t}-1}\psi^{\mathfrak{j}-\mathfrak{u}}(e_i) \otimes \varepsilon)\\
&\qquad \rightharpoonup \alpha_U^{-\mathfrak{a}+\mathfrak{g}}\beta_U^{-\mathfrak{b}+\mathfrak{h}-1}\varphi_U^{-\mathfrak{c}+\mathfrak{i}-1}\psi_U^{-\mathfrak{d}+\mathfrak{j}+1}(u)\\
&\quad = \sum e^i(\alpha^{-\mathfrak{m}-\mathfrak{g}}\beta^{-\mathfrak{n}-\mathfrak{h}}\varphi^{-\mathfrak{p}-\mathfrak{i}}\psi^{-\mathfrak{q}-\mathfrak{j}-1}(v_1))1_H \cdot \alpha_V^{\mathfrak{a}-\mathfrak{g}}\beta_V^{\mathfrak{b}-\mathfrak{h}-1}\varphi_V^{\mathfrak{c}-\mathfrak{i}}\psi_V^{\mathfrak{d}-\mathfrak{j}-1}(v_0)\\
&\qquad \otimes \varepsilon(u_1)\alpha^{\mathfrak{g}+1}\beta^{\mathfrak{h}-1}\varphi^{\mathfrak{i}-1}\psi^{\mathfrak{j}+2}(e_i)\\
&\quad = \alpha_V^{\mathfrak{a}-\mathfrak{g}}\beta_V^{\mathfrak{b}-\mathfrak{h}}\varphi_V^{\mathfrak{c}-\mathfrak{i}}\psi_V^{\mathfrak{d}-\mathfrak{j}-1}(v_0)\\
&\qquad \otimes \alpha^{-\mathfrak{m}+1}\beta^{-\mathfrak{n}-1}\varphi^{-\mathfrak{p}-1}\psi^{-\mathfrak{q}+1}(v_1) \cdot \alpha_U^{-\mathfrak{a}+\mathfrak{g}}\beta_U^{-\mathfrak{b}+\mathfrak{h}-1}\varphi_U^{-\mathfrak{c}+\mathfrak{i}-1}\psi_U^{-\mathfrak{d}+\mathfrak{j}+1}(u)\\
&\quad = \mathbf{t}_{U,V}(u \otimes v),
\end{aligned}$$

which implies the conclusion. □

**Corollary 5.4.** *The following statemant holds:* ${}_{D(H)^{\{\mathfrak{r},\mathfrak{s},\mathfrak{t},\mathfrak{u}\}}}\mathcal{BM}^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}}$ *is a full braided subcategory of* $\mathcal{Z}({}_H\mathcal{BM}^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}})$.

Proof. Directly from Theorem 4.8 and Corollary 5.3. □

**Corollary 5.5.** *For any*

$$\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{g}, \mathfrak{h}, \mathfrak{i}, \mathfrak{j}, \mathfrak{a}', \mathfrak{b}', \mathfrak{c}', \mathfrak{d}', \mathfrak{g}', \mathfrak{h}', \mathfrak{i}', \mathfrak{j}', \mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q}, \mathfrak{m}', \mathfrak{n}', \mathfrak{p}', \mathfrak{q}' \in \mathbb{Z},$$

*we have that* $({}_H\mathcal{YD}^H(\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q}))^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}}$ *is isomorphic to* $({}_H\mathcal{YD}^H(\mathfrak{m}', \mathfrak{n}', \mathfrak{p}', \mathfrak{q}'))^{\mathfrak{a}',\mathfrak{b}',\mathfrak{c}',\mathfrak{d}'}_{\mathfrak{g}',\mathfrak{h}',\mathfrak{i}',\mathfrak{j}'}$ *as braided categories.*

Proof. We have the following braided isomorphic

$$\begin{aligned}
({}_H\mathcal{YD}^H(\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q}))^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}} &\cong {}_{D(H)^{\{\mathfrak{r},\mathfrak{s},\mathfrak{t},\mathfrak{u}\}}}\mathcal{BM}^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}}, \quad \text{see Corollary 5.3 },\\
&\cong ({}_H\mathcal{YD}^H(\mathfrak{m}', \mathfrak{n}', \mathfrak{p}', \mathfrak{q}'))^{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}_{\mathfrak{g},\mathfrak{h},\mathfrak{i},\mathfrak{j}}, \quad \text{see Corollary 5.3},\\
&\cong ({}_H\mathcal{YD}^H(\mathfrak{m}', \mathfrak{n}', \mathfrak{p}', \mathfrak{q}'))^{\mathfrak{a}',\mathfrak{b}',\mathfrak{c}',\mathfrak{d}'}_{\mathfrak{g}',\mathfrak{h}',\mathfrak{i}',\mathfrak{j}'}, \quad \text{see Proposition 4.6}
\end{aligned}$$

as needed. □

### *References*

[1] *M. Aguiar, S. Mahajan*: Monoidal Functors, Species and Hopf Algebras. CRM Monograph Series 29. AMS, Providence, 2010. zbl MR doi

[2] *C. Balteanu, Z. Fiedorowicz, R. Schwänzl, R. Vogt*: Iterated monoidal categories. Adv. Math. *176* (2003), 277–349. zbl MR doi

[3] *S. Caenepeel, I. Goyvaerts*: Monoidal Hom-Hopf algebras. Commun. Algebra *39* (2011), 2216–2240. zbl MR doi

[4] *S. Caenepeel, D. Wang, Y. Yin*: Yetter-Drinfeld modules over weak bialgebras. Ann. Univ. Ferrara, Nuova Ser., Sez. VII *51* (2005), 69-98. zbl MR doi

[5] *Y. Chen, L. Zhang*: The category of Yetter-Drinfel'd Hom-modules and the quantum Hom-Yang-Baxter equation. J. Math. Phys. *55* (2014), Article ID 031702, 18 pages. zbl MR doi

[6] *X.-L. Fang, W. Liu*: Solutions of the BiHom-Yang-Baxter equations. Sb. Math. *209* (2018), 901–918; translation from Mat. Sb. *209* (2018), 128–145. zbl MR doi

[7] *S. Forcey, J. Siehler, E. S. Sowers*: Operads in iterated monoidal categories. J. Homotopy Relat. Struct. *2* (2007), 1–43. zbl MR

[8] *G. Graziani, A. Makhlouf, C. Menini, F. Panaite*: BiHom-associative algebras, BiHom-Lie algebras and BiHom-bialgebras. SIGMA, Symmetry Integrability Geom. Methods Appl. *11* (2015), Article ID 086, 34 pages. zbl MR doi

[9] *S. Guo, X. Zhang, S. Wang*: Braided monoidal categories and Doi-Hopf modules for monoidal Hom-Hopf algebras. Colloq. Math. *143* (2016), 79–103. zbl MR doi

[10] *S. Guo, X. Zhang, S. Wang*: The construction and deformation of BiHom-Novikov algebras. J. Geom. Phys. *132* (2018), 460–472. zbl MR doi

[11] *J. T. Hartwig, D. Larsson, S. D. Silvestrov*: Deformations of Lie algebras using $\sigma$-derivations. J. Algebra *295* (2006), 314–361. zbl MR doi

[12] *N. Hu*: $q$-Witt algebras, $q$-Lie algebras, $q$-holomorph structure and representations. Algebra Colloq. *6* (1999), 51–70. zbl MR

[13] *A. Joyal, S. Street*: Tortile Yang-Baxter operators in tensor categories. J. Pure Appl. Algebra *71* (1991), 43–51. zbl MR doi

[14] *C. Kassel*: Quantum Groups. Graduate Texts in Mathematics 155. Springer, New York, 1995. zbl MR doi

[15] *J. Li, L. Chen, B. Sun*: Bihom-Nijienhuis operators and T*-extensions of Bihom-Lie superalgebras. Hacet. J. Math. Stat. *48* (2019), 785–799. zbl MR doi

[16] *L. Liu, A. Makhlouf, C. Menini, F. Panaite*: $\{\sigma,\tau\}$-Rota-Baxter operators, infinitesimal Hom-bialgebras and the associative (bi)Hom-Yang-Baxter equation. Can. Math. Bull. *62* (2019), 355–372. zbl MR doi

[17] *D. Lu, X. Zhang*: Hom-L-R-smash biproduct and the category of Hom-Yetter-Drinfel'd-Long bimodules. J. Algebra Appl. *17* (2018), Article ID 1850133, 19 pages. zbl MR doi

[18] *S. Majid*: Representations, duals and quantum doubles of monoidal categories. Rend. Circ. Mat. Palermo, II. Ser., Suppl. *26* (1991), 197–206. zbl MR

[19] *S. Majid*: Quantum double for quasi-Hopf algebras. Lett. Math. Phys. *45* (1998), 1–9. zbl MR doi

[20] *A. Makhlouf, F. Panaite*: Yetter-Drinfeld modules for Hom-bialgebras. J. Math. Phys. *55* (2014), Article ID 013501, 17 pages. zbl MR doi

[21] *A. Makhlouf, F. Panaite*: Hom-L-R-smash products, Hom-diagonal crossed products and the Drinfeld double of a Hom-Hopf algebra. J. Algebra *441* (2015), 314–343. zbl MR doi

[22] *A. Makhlouf, S. D. Silvestrov*: Hom-algebra structures. J. Gen. Lie Theory Appl. *2* (2008), 51–64. zbl MR

[23] *A. Makhlouf, S. Silvestrov*: Hom-Lie admissible Hom-coalgebras and Hom-Hopf algebras. Generalized Lie Theory in Mathematics, Physics and Beyond. Springer, Berlin, 2009, pp. 189–206. zbl MR doi

[24] *A. Makhlouf, S. Silvestrov*: Hom-algebras and Hom-coalgebras. J. Algebra Appl. *9* (2010), 553–589. zbl MR doi

[25] *S. Montgomery*: Hopf algebras and their actions on rings. Regional Conference Series in Mathematics 82. AMS, Providence, 1993. zbl MR doi

[26] *A. Nenciu*: The center construction for weak Hopf algebras. Tsukaba J. Math. *26* (2002), 189–204. zbl MR doi

[27] *D. Yau*: The Hom-Yang-Baxter equation, Hom-Lie algebras, and quasi-triangular bialgebras. J. Phys. A, Math. Theor. *42* (2009), Article ID 165202, 12 pages. zbl MR doi

[28] *D. Yau*: Hom-quantum groups. I. Quasi-triangular Hom-bialgebras. J. Phys. A, Math. Theor. *45* (2012), Article ID 065203, 23 pages. zbl MR doi

[29] *X. Zhang, L. Dong*: Braided mixed datums and their applications on Hom-quantum groups. Glasg. Math. J. *60* (2018), 231–251. zbl MR doi

[30] *X. Zhang, S. Guo, S. Wang*: Drinfeld codoubles of Hom-Hopf algebras. Adv. Appl. Clifford Algebr. *29* (2019), Article ID 36, 26 pages. zbl MR doi

[31] *X. Zhang, D. Wang*: Cotwists of bicomonads and BiHom-bialgebras. Algebr. Represent. Theor. *23* (2020), 1355–1385. zbl MR doi

[32] *X. Zhang, W. Wang, X. Zhao*: Drinfeld twists for monoidal Hom-bialgebras. Colloq. Math. *156* (2019), 199–228. zbl MR doi

[33] *X. Zhao, X. Zhang*: Lazy 2-cocycles over monoidal Hom-Hopf algebras. Colloq. Math. *142* (2016), 61–81. zbl MR doi

[34] *H. Zhu, G. Liu, T. Yang*: Characterization of quasi-Yetter-Drinfeld modules. J. Algebra Appl. *19* (2020), Article ID 2050058, 16 pages. zbl MR doi

*Authors' addresses*: Xiaohui Zhang (corresponding author), School of Mathematical Sciences, Qufu Normal University, 57 Jingxuan West Road, Qufu, Shandong 273165, P. R. China, e-mail: `zxhhhhh@hotmail.com`; Wei Wang, Nanjing Research Institute of Electronic Engineering, 163 Xianlin Ave., Qixia District, Nanjing Jiangsu 210007, P. R. China, e-mail: `weiwang2012spring@yahoo.com`; Juzhen Chen, College of Science, Nanjing Forestry University, Nanjing Jiangsu 210037, P. R. China, e-mail: `juzhenchen@139.com`.