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Czechoslovak Mathematical Journal, Vol. 74 (2024), No. 1, 95–113

Persistent URL: http://dml.cz/dmlcz/152270

Czechoslovak Mathematical Journal, 74 (149) (2024), 95–113

HALL ALGEBRAS OF TWO EQUIVALENT EXTRIANGULATED CATEGORIES

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Received August 14, 2022. Published online November 29, 2023.

Abstract. For any positive integer n, let A_n be a linearly oriented quiver of type A with n vertices. It is well-known that the quotient of an exact category by projective-injectives is an extriangulated category. We show that there exists an extriangulated equivalence between the extriangulated categories $\mathcal{M}_{n+1}$ and $\mathcal{F}_n$, where $\mathcal{M}_{n+1}$ and $\mathcal{F}_n$ are the two extriangulated categories corresponding to the representation category of A_{n+1} and the morphism category of projective representations of A_n, respectively. As a by-product, the Hall algebras of $\mathcal{M}_{n+1}$ and $\mathcal{F}_n$ are isomorphic. As an application, we use the Hall algebra of $\mathcal{M}_{2n+1}$ to relate with the quantum cluster algebras of type A_{2n}.

Keywords: extriangulated category; extriangulated equivalence; Hall algebra; quantum cluster algebra

MSC 2020: 18E05, 18E10, 17B37

1. Introduction

In 1990, Ringel in [13] introduced the Hall algebra of a finite dimensional algebra over a finite field in his work of studying quantum groups. Hubery in [11] proved that the definition of the Hall algebra of module categories also applies to exact categories. Toën in [15] gave a construction of what he called *derived Hall algebras* for DG-enhanced triangulated categories satisfying certain finiteness conditions. Later, Xiao and Xu in [17] showed that Toën's definition also applies to any triangulated categories satisfying certain finiteness conditions.

Cluster algebras were introduced by Fomin and Zelevinsky in [8] and later the quantum cluster algebras were introduced by Berenstein and Zelevinsky in [3]. The

This work was partially supported by the National Natural Science Foundation of China (Grant Nos. 12301042, 12271257, 12271448).

DOI: 10.21136/CMJ.2023.0344-22

cluster multiplication theorem proved by Caldero and Keller (see [4]) showed the similarity between the multiplication in a cluster algebra and that in a dual Hall algebra. For a finite acyclic quiver Q, Ding, Xu and Zhang in [7] used the Hall algebra of the morphism category of projective representations of Q to realize the quantum cluster algebra with principal coefficients associated to Q. By introducing a bialgebra structure and an integration homomorphism on the Hall algebra of the morphism category, Fu, Peng and Zhang in [9] recovered the algebra homomorphism given in [7]. Recently, following the approach given in [9], Chen, Ding and Zhang in [6] obtained the quantum version of the cluster multiplication theorem by applying certain quotients of derived Hall subalgebras of Q.

Nakaoka and Palu in [12] introduced the notion of extriangulated categories, which is a unification of exact categories and triangulated categories. Recently, the Hall algebra of extriangulated categories has been defined in [16].

For any positive integer n, let $A_n := 1 \leftarrow 2 \leftarrow 3 \leftarrow \ldots \leftarrow n$ be the linearly oriented quiver of type A. Let k be a field and $\operatorname{mod} kA_n$ be the category of finite dimensional (left) kA_n-modules. For each vertex i of A_n, the indecomposable projective (or injective) kA_n-module corresponding to i is denoted by P_i (or I_i, respectively). Denote by $\mathscr{P}_{A_n}$ the subcategory of $\operatorname{mod} kA_n$ consisting of all projective kA_n-modules. Define $\mathcal{M}_{n+1}$ to be the quotient category $\operatorname{mod} kA_{n+1}/\langle P_{n+1}\rangle$. Denote by $C_2(\mathscr{P}_{A_n})$ the category of morphisms between projective kA_n-modules, and denote by $\mathcal{K}$ the subcategory consisting of all projective-injective objects in $C_2(\mathscr{P}_{A_n})$. Define $\mathcal{F}_n$ to be the quotient category $C_2(\mathscr{P}_{A_n})/\mathcal{K}$. We show that there exists an extriangulated equivalence between the extriangulated categories $\mathcal{M}_{n+1}$ and $\mathcal{F}_n$. As a by-product, the Hall algebras of $\mathcal{M}_{n+1}$ and $\mathcal{F}_n$ are isomorphic. As an application, we use the Hall algebra of $\mathcal{M}_{2n+1}$ to relate with the quantum cluster algebras of type A_{2n}.

In Section 2, we establish a functor from $\mathcal{M}_{n+1}$ to $\mathcal{F}_n$ and prove that it is an extriangulated equivalence. We consider the Hall algebras of the extriangulated categories $\mathcal{M}_{n+1}$ and $\mathcal{F}_n$ in Section 3. Section 4 is devoted to using the Hall algebra of $\mathcal{M}_{2n+1}$ to realize the cluster variables of quantum cluster algebras of type A_{2n}.

In this paper, let k be a field. Let $\mathscr{C}$ be an additive category, denote by $\operatorname{ind}(\mathscr{C})$ a complete set of indecomposable objects in $\mathscr{C}$. For an object $X \in \mathscr{C}$, we denote by $[X]$ and $\operatorname{Aut} X$ the isomorphism class and automorphism group of X, respectively. For several objects $X_1, \ldots, X_n$ in $\mathscr{C}$, we denote by $\langle X_i \colon 1 \leqslant i \leqslant n\rangle$ the subcategory of $\mathscr{C}$ which consists of direct summands of finite direct sums of these objects. Let $\mathcal{X}$ be a subcategory of $\mathscr{C}$, denote by $\mathcal{X}(M,N)$ the subgroup of $\operatorname{Hom}_{\mathscr{C}}(M,N)$ consisting of the morphisms which factor through an object of $\mathcal{X}$. The quotient category $\mathscr{C}/\mathcal{X}$ has the same objects as $\mathscr{C}$, but the morphism spaces are defined by $\operatorname{Hom}_{\mathscr{C}/\mathcal{X}}(M,N) := \operatorname{Hom}_{\mathscr{C}}(M,N)/\mathcal{X}(M,N)$ for any $M,N \in \mathscr{C}$. For any $f \in \operatorname{Hom}_{\mathscr{C}}(M,N)$, we denote by $\overline{f}$ the image of f in $\operatorname{Hom}_{\mathscr{C}/\mathcal{X}}(M,N)$. For a complex $X_\bullet$ in an abelian category, it ith homology group is denoted by $H^i(X_\bullet)$. For a finite set S, we denote by $|S|$ its cardinality. Vectors are always assumed to be column vectors.

2. The equivalence between two quotient categories

In this section, we recall some definitions and properties of morphism categories and extriangulated categories. Then we establish a functor from $\mathcal{M}_{n+1}$ to $\mathcal{F}_n$ and prove that it is an extriangulated equivalence.

2.1. Morphism categories. Let $\mathcal{E}$ be an exact category. Let $C_2(\mathcal{E})$ be the category whose objects are morphisms $M_0 \xrightarrow{f} M_1$ in $\mathcal{E}$, and every morphism from $M_0 \xrightarrow{f} M_1$ to $N_0 \xrightarrow{g} N_1$ is a pair (a, b) of morphisms in $\mathcal{E}$ such that the diagram

$$\begin{array}{ccc} M_0 & \xrightarrow{f} & M_1 \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ N_0 & \xrightarrow{g} & N_1 \end{array}$$

is commutative. Since we can view any object in $C_2(\mathcal{E})$ as a two-term complex, we also write $M_\bullet(f)$ or simply $M_\bullet$ as an object $M_0 \xrightarrow{f} M_1$ in $C_2(\mathcal{E})$. Let A be the path algebra of a finite acyclic quiver Q. Denote by $\mathscr{P} = \mathscr{P}_Q$ the subcategory consisting of all projective A-modules and by P_i the indecomposable projective module corresponding to the vertex i. In what follows, we consider the morphism category $C_2(\mathscr{P})$. For any object $P \in \mathscr{P}$, define two objects in $C_2(\mathscr{P})$

$$K_P := P \xrightarrow{1} P \quad \text{and} \quad Z_P := P \to 0. \tag{2.1}$$

For any A-module M, it has a minimal projective resolution[1]

$$0 \to \Omega_M \xrightarrow{i_M} P_M \to M \to 0. \tag{2.2}$$

Then we define an object in $C_2(\mathscr{P})$

$$C_M := \Omega_M \xrightarrow{i_M} P_M. \tag{2.3}$$

Since any two minimal projective resolutions of M are isomorphic, C_M is well defined up to isomorphisms. Hence, for any A-module M, we fix a minimal projective resolution as (2.2). Note that $C_2(\mathscr{P})$ is an exact category in the sense of the componentwise exactness. By [1], the objects C_P and K_P, where $P \in \text{ind}(\mathscr{P})$, provide a complete set of indecomposable projective objects in $C_2(\mathscr{P})$; and the objects Z_P and K_P

[1] The notations P_M and Ω_M will be used throughout the paper.

provide a complete set of indecomposable injective objects. Moreover, all K_P for $P \in \operatorname{ind}(\mathscr{P})$ give a complete set of indecomposable projective-injective objects. Hence,

$$\mathcal{F}_n = C_2(\mathscr{P}_{A_n})/\langle K_{P_i} \colon\ 1 \leqslant i \leqslant n\rangle.$$

Lemma 2.1 ([7], Proposition 2.4). *Any object $M_\bullet$ in $C_2(\mathscr{P})$ has a direct sum decomposition*

$$M_\bullet = K_P \oplus Z_Q \oplus C_M$$

for some projective A-modules P, Q and an A-module M. Moreover, the modules P, Q and M are uniquely determined up to isomorphisms.

Lemma 2.2. *Let $M_\bullet(f)$, $N_\bullet(g)$ be two objects in $C_2(\mathscr{P})$ and (a,b) be a morphism from $M_\bullet(f)$ to $N_\bullet(g)$. Then (a,b) factors through some projective-injective object K_P if and only if there exists a morphism $s\colon M_1 \to N_0$ such that $a = sf$ and $b = gs$.*

P r o o f. Suppose that (a,b) factors through some projective-injective object K_P, i.e., we have the commutative diagram

$$\begin{array}{ccc} M_0 & \xrightarrow{f} & M_1 \\ {\scriptstyle a_1}\downarrow & & \downarrow{\scriptstyle b_1} \\ P & \xrightarrow{1} & P \\ {\scriptstyle a_2}\downarrow & & \downarrow{\scriptstyle b_2} \\ N_0 & \xrightarrow{g} & N_1 \end{array}$$

with $a = a_2a_1$ and $b = b_2b_1$. Taking $s = a_2b_1$, we get that $sf = a_2b_1f = a_2a_1 = a$ and $gs = ga_2b_1 = b_2b_1 = b$ and finish the proof of necessity.

Suppose there exists a morphism $s\colon M_1 \to N_0$ such that $a = sf$ and $b = gs$. Then we have the commutative diagram

$$\begin{array}{ccc} M_0 & \xrightarrow{f} & M_1 \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle s} \\ N_0 & \xrightarrow{1} & N_0 \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle g} \\ N_0 & \xrightarrow{g} & N_1 \end{array}$$

such that $(1,g) \circ (a,s) = (a,gs) = (a,b)$, i.e., (a,b) factors through the projective-injective object K_P with $P = N_0$. Therefore, we completed the proof. $\square$

Consider the functor $F\colon \operatorname{mod} kA_{n+1} \to \mathcal{F}_n$ defined on indecomposable objects $M \in \operatorname{mod} kA_{n+1}$ by

$$F(M) = \begin{cases} C_M & \text{if } P_M \ncong P_{n+1}, \\ Z_{\Omega_M} & \text{otherwise.} \end{cases}$$

Given objects $M, N \in \operatorname{ind}(\operatorname{mod} kA_{n+1})$ and a morphism $f\colon M \to N$. Then there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_M & \xrightarrow{i_M} & P_M & \xrightarrow{\pi_M} & M & \longrightarrow & 0 \\ & & \downarrow a & & \downarrow b & & \downarrow f & & \\ 0 & \longrightarrow & \Omega_N & \xrightarrow{i_N} & P_N & \xrightarrow{\pi_N} & N & \longrightarrow & 0 \end{array} \tag{2.4}$$

and we put

$$F(f) = \begin{cases} \overline{(a,0)} & \text{if } P_N \cong P_{n+1}, \\ \overline{(a,b)} & \text{if } P_N \ncong P_{n+1}. \end{cases} \tag{2.5}$$

Suppose that there exist morphisms $a'\colon \Omega_M \to \Omega_N$, $b'\colon P_M \to P_N$ such that the diagram (2.4) is also commutative. Then it is easy to see that there exists a morphism $s\colon P_M \to \Omega_N$ such that $b - b' = i_N s$ and $a - a' = s i_M$. By Lemma 2.2, we know that $(a - a', b - b')$ as a morphism from C_M to C_N factors through some projective-injective object K_P, thus $\overline{(a,b)} = \overline{(a',b')}$. In what follows, we need to prove that the functor F is well-defined.

Lemma 2.3. *For any objects $M, N \in \operatorname{ind}(\operatorname{mod} kA_{n+1})$ and a morphism $f\colon M \to N$, $F(f)$ in (2.5) is well-defined.*

P r o o f. (i) If $P_M \ncong P_{n+1}$ and $P_N \ncong P_{n+1}$, as mentioned above, $F(f)$ is uniquely defined.

(ii) If $P_M \cong P_{n+1}$ and $P_N \ncong P_{n+1}$, in this case, $\operatorname{Hom}_{kA_{n+1}}(P_M, P_N) = 0$, i.e., $b = 0$ and thus $a = 0$ in (2.4). So $F(f) = 0$ is uniquely defined.

(iii) Suppose that the morphisms (a,b) and (a',b') make the diagram (2.4) commutative, then there exists a morphism $s\colon P_M \to \Omega_N$ such that $b - b' = i_N s$ and $a - a' = s i_M$. If $P_M \ncong P_{n+1}$ and $P_N \cong P_{n+1}$, applying Lemma 2.2 to the morphism $(a - a', 0)\colon C_M \to Z_{\Omega_N}$, we have that $\overline{(a,0)} = \overline{(a',0)}$. If $P_M \cong P_{n+1}$ and $P_N \cong P_{n+1}$, in this case $\operatorname{Hom}_{kA_{n+1}}(P_M, \Omega_N) = 0$, i.e., $s = 0$, so $a = a'$. Hence, $F(f) = \overline{(a,0)}$ is also uniquely defined. □

By the definition of F, it is clear that $F(\mathrm{Id}_M) = \mathrm{Id}_{F(M)}$ for $M \in \operatorname{ind}(\operatorname{mod} kA_{n+1})$. For any morphism $f\colon M \to N$, we denote the morphisms a and b like in (2.4) by a_f and b_f, respectively.

Lemma 2.4. *Let $M, N, L \in \operatorname{ind}(\operatorname{mod} kA_{n+1})$. For any $f \in \operatorname{Hom}_{kA_{n+1}}(M, N)$ and $g \in \operatorname{Hom}_{kA_{n+1}}(N, L)$, we have that $F(gf) = F(g)F(f)$.*

P r o o f. (i) If $P_L \ncong P_{n+1}$ and $P_N \cong P_{n+1}$, then $g = 0$, and thus $F(gf) = F(g)F(f) = 0$.

(ii) If $P_L \ncong P_{n+1}$ and $P_N \ncong P_{n+1}$, then $\overline{(a_{gf}, b_{gf})} = \overline{(a_g, b_g)} \circ \overline{(a_f, b_f)}$.

(iii) If $P_L \cong P_{n+1}$ and $P_N \cong P_{n+1}$, then $\overline{(a_{gf}, 0)} = \overline{(a_g, 0)} \circ \overline{(a_f, 0)}$.

(iv) If $P_L \cong P_{n+1}$ and $P_N \ncong P_{n+1}$, then $\overline{(a_{gf}, 0)} = \overline{(a_g, 0)} \circ \overline{(a_f, b_f)}$. □

Remark 2.5. In order to define a functor $F\colon \mathcal{C} \to \mathcal{D}$ between two Krull-Schmidt additive categories $\mathcal{C}$ and $\mathcal{D}$, assume that F has been well-defined on indecomposable objects and the morphisms between indecomposable objects, then for any objects $\bigoplus_{i=1}^{n} M_i$ with all M_i indecomposable, $F(M)$ is defined to be $\bigoplus_{i=1}^{n} F(M_i)$. For any morphism $f \in \operatorname{Hom}_{\mathcal{C}}\Big(\bigoplus_{i=1}^{n} M_i, \bigoplus_{j=1}^{m} N_j\Big)$ with all M_i, N_j indecomposable, write

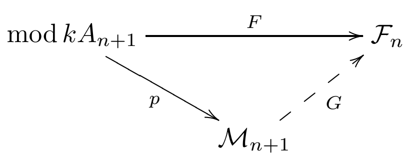
$$f = \begin{pmatrix} f_{11} & f_{12} & \dots & f_{1n} \\ f_{21} & f_{22} & \dots & f_{2n} \\ \vdots & \vdots & \dots & \vdots \\ f_{m1} & f_{m2} & \dots & f_{mn} \end{pmatrix},$$

then $F(f)$ is defined to be

$$\begin{pmatrix} F(f_{11}) & F(f_{12}) & \dots & F(f_{1n}) \\ F(f_{21}) & F(f_{22}) & \dots & F(f_{2n}) \\ \vdots & \vdots & \dots & \vdots \\ F(f_{m1}) & F(f_{m2}) & \dots & F(f_{mn}) \end{pmatrix}.$$

Moreover, assume that the map $F\colon \operatorname{Hom}_{\mathcal{C}}(M, N) \to \operatorname{Hom}_{\mathcal{D}}(F(M), F(N))$, $f \mapsto F(f)$ is bijective for any indecomposable objects M and N, then it is easy to see that the map $F\colon \operatorname{Hom}_{\mathcal{C}}\Big(\bigoplus_{i=1}^{n} M_i, \bigoplus_{j=1}^{m} N_j\Big) \to \operatorname{Hom}_{\mathcal{D}}\Big(\bigoplus_{i=1}^{n} F(M_i), \bigoplus_{j=1}^{m} F(N_j)\Big)$, $f \mapsto F(f)$ is also bijective. Hence, in what follows, to prove the functor F is fully faithful, we still check only for indecomposable objects.

Let $p\colon \operatorname{mod} kA_{n+1} \to \mathcal{M}_{n+1}$ be the projection functor. By definition, $\ker(F) = \langle P_{n+1} \rangle$. Hence, the functor F induces a functor $G\colon \mathcal{M}_{n+1} \to \mathcal{F}_n$ such that the diagram



is commutative.

Lemma 2.6. *The functor F is full.*

Proof. Let $M, N \in \operatorname{ind}(\operatorname{mod} kA_{n+1})$. Take any morphism

$$\overline{(a,c)} \in \operatorname{Hom}_{\mathcal{F}_n}(F(M), F(N))$$

with $(a,c) \in \operatorname{Hom}_{C_2(\mathscr{P}_{A_n})}(F(M), F(N))$.

If $P_N \cong P_{n+1}$, in this case, $c = 0$. Since $P_{n+1} = I_1$, the morphism a induces a commutative diagram as (2.4). Hence, by definition, we have $F(f) = \overline{(a,0)}$.

If $P_N \ncong P_{n+1}$ and $P_M \cong P_{n+1}$, in this case, $\operatorname{Hom}_{C_2(\mathscr{P}_{A_n})}(F(M), F(N)) = 0$, thus $(a,c) = (0,0)$ and $F(0) = \overline{(a,c)}$. If $P_N \ncong P_{n+1}$ and $P_M \ncong P_{n+1}$, there exists a unique morphism f such that we have the commutative diagram (2.4) with $b = c$. Hence, $F(f) = \overline{(a,c)}$. □

Lemma 2.7. *The functor G is faithful.*

Proof. Let $M, N \in \operatorname{ind}(\operatorname{mod} kA_{n+1})$. Take any $\overline{f} \in \operatorname{Hom}_{\mathcal{M}_{n+1}}(M, N)$ with $f \in \operatorname{Hom}_{\operatorname{mod} kA_{n+1}}(M, N)$. Then

$$G(\overline{f}) = F(f) = \overline{(a,c)} \in \operatorname{Hom}_{\mathcal{F}_n}(F(M), F(N)),$$

where $(a,c) \in \operatorname{Hom}_{C_2(\mathscr{P}_{A_n})}(F(M), F(N))$. By the definition of F, we have a commutative diagram as (2.4). Assume that $\overline{(a,c)} = 0$.

Case 1: $P_N \cong P_{n+1}$. In this case, $c = 0$. If $P_M \cong P_{n+1}$, $\overline{(a,0)} = 0$ means that $(a,0) \in \operatorname{Hom}_{C_2(\mathscr{P}_{A_n})}(Z_{\Omega_M}, Z_{\Omega_N})$ factors through some projective-injective object K_P, by Lemma 2.2 we obtain that $a = 0$. In the diagram (2.4), since $b \in \operatorname{End}(P_{n+1})$, we get that $b = 0$ or b is an isomorphism. For the former, we have that $f = 0$. For the latter, since $a = 0$, we get that $i_M = 0$ and thus $M \cong P_M \cong P_{n+1}$. So, $\overline{f} = 0$.

If $P_M \ncong P_{n+1}$, in this case also $c = 0$. Since $(a,0) \in \operatorname{Hom}_{C_2(\mathscr{P}_{A_n})}(C_M, Z_{\Omega_N})$ factors through some projective-injective object K_P, by Lemma 2.2 we obtain that there exists a morphism $s\colon P_M \to \Omega_N$ such that $a = si_M$. Hence, $bi_M = i_N a = i_N s i_M$ and then $(b - i_N s)i_M = 0$. So we obtain that there is a morphism $t\colon M \to P_N$ such that $b - i_N s = t\pi_M$ and thus $f\pi_M = \pi_N b = \pi_N(i_N s + t\pi_M) = \pi_N t \pi_M$. So $f = \pi_N t$, i.e., f factors through $P_N \cong P_{n+1}$. Hence, $\overline{f} = 0$.

Case 2: $P_N \ncong P_{n+1}$. If $P_M \cong P_{n+1}$, in this case, $a = c = 0$; since

$$\operatorname{Hom}_{kA_{n+1}}(P_M, P_N) = 0,$$

i.e., $b = 0$, we get that $f = 0$.

If $P_M \ncong P_{n+1}$, since $\overline{(a,c)} = \overline{(a,b)}$ factors through some projective-injective object K_P, by Lemma 2.2, there exists a morphism $s\colon P_M \to \Omega_N$ such that $b = i_N s$. Then we obtain that $f\pi_M = \pi_N b = \pi_N i_N s = 0$ and thus $f = 0$. □

In conclusion, we have the following proposition.

Proposition 2.8. *The functor* G *provides an equivalence*

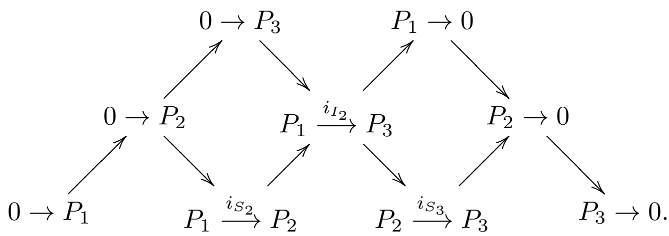
$$G\colon\ \mathrm{mod}\, kA_{n+1}/\langle P_{n+1}\rangle \xrightarrow{\ \simeq\ } C_2(\mathscr{P}_{A_n})/\langle K_{P_i}\colon\ 1\leqslant i\leqslant n\rangle$$

as additive categories.

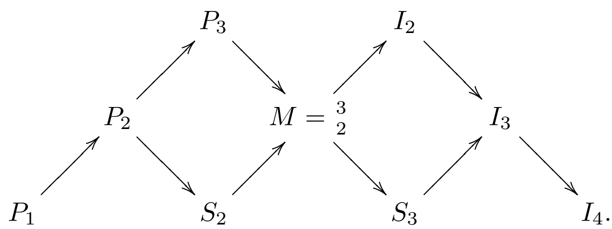
P r o o f. By Lemmas 2.6 and 2.7, it suffices to show that the functor G is dense. By Lemma 2.1, each object $M\in\mathcal{F}_n$ has a direct sum decomposition $M=Z_Q\oplus C_N$ for some projective module Q and $N\in\ \mathrm{mod}\, kA_n$. Observe that $F(N)=C_N$. Without loss of generality, we may assume that $Q\cong P_i$ for some $1\leqslant i\leqslant n$. Considering the minimal projective resolution $0\to P_i\to P_{n+1}\to I_{i+1}\to 0$, by the definition of F, we obtain that $F(I_{i+1})=Z_{P_i}$ and thus $G(N\oplus I_{i+1})=M$. □

We finish this section with a straightforward example illustrating Proposition 2.8.

Example 2.9. By Example 6.7 of [5], the Auslander-Reiten quiver of $\mathcal{F}_3$ is



It is easy to see that the Auslander-Reiten quiver of $\mathcal{M}_4$ is given by



For the morphism $f\colon\ S_2\to M$ in $\mathrm{mod}\, kA_4$, taking the minimal projective resolutions, we have the commutative diagram

(2.6)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_1 & \xrightarrow{i_{S_2}} & P_2 & \xrightarrow{\pi_{S_2}} & S_2 & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & P_1 & \xrightarrow{i_M} & P_3 & \xrightarrow{\pi_M} & M & \longrightarrow & 0
\end{array}$$

and $G(\overline{f})=\overline{(a,b)}\in \mathrm{Hom}_{\mathcal{F}_3}(C_{S_2},C_{I_2})$, since M viewed as a kA_3-module is just I_2 of A_3.

2.2. Extriangulated categories. According to [12], we recall that an extriangulated category is a triplet $(\mathscr{C}, \mathbb{E} = \mathbb{E}_{\mathscr{C}}, \mathfrak{s})$ satisfying the following conditions:

(1) $\mathbb{E}\colon \mathscr{C}^{\rm op} \times \mathscr{C} \to Ab$ is a biadditive functor.
(2) $\mathfrak{s}$ is an additive realization of $\mathbb{E}$.
(3) $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ satisfies some axioms denoted by (ET1)–(ET4), (ET3)$^{\rm op}$ and (ET4)$^{\rm op}$, see [12] for details.

Exact categories and triangulated categories are extriangulated categories, see [12], Example 2.13, Proposition 3.22. We use the following terminology:

▷ For any objects $A, C \in \mathscr{C}$, every element $\delta \in \mathbb{E}(C, A)$ is called an $\mathbb{E}$-*extension*. For any morphisms $a \in \mathscr{C}(A, A')$ and $c \in \mathscr{C}(C', C)$, we have $\mathbb{E}(C, a)(\delta) \in \mathbb{E}(C, A')$ and $\mathbb{E}(c, A)(\delta) \in \mathbb{E}(C', A)$. We simply denote them by $a_*\delta$ and $c^*\delta$, respectively.

▷ Two sequences of morphisms $A \xrightarrow{x} B \xrightarrow{y} C$ and $A \xrightarrow{x'} B' \xrightarrow{y'} C$ in $\mathscr{C}$ are said to be *equivalent* if there exists an isomorphism $b \in \mathscr{C}(B, B')$ such that $bx = x'$ and $y = y'b$. We denote the equivalence class of $A \xrightarrow{x} B \xrightarrow{y} C$ by $[A \xrightarrow{x} B \xrightarrow{y} C]$. The additive realization $\mathfrak{s}$ assigns to every extension $\delta \in \mathbb{E}(C, A)$ an equivalence class $[A \xrightarrow{x} B \xrightarrow{y} C]$. The sequence $A \xrightarrow{x} B \xrightarrow{y} C$ is called a *conflation*, x is called an *inflation* and y is called a *deflation*.

▷ An object P in $\mathscr{C}$ is called *projective* if for any conflation $A \xrightarrow{x} B \xrightarrow{y} C$ and any morphism $c\colon P \to C$, there exists a morphism $b\colon P \to B$ such that $yb = c$. We denote the full subcategory of projective objects in $\mathscr{C}$ by $\mathcal{P}(\mathscr{C})$. Dually, the *injective* objects are defined, and the full subcategory of injective objects in $\mathscr{C}$ is denoted by $\mathcal{I}(\mathscr{C})$. We say that $\mathscr{C}$ *has enough projective objects* if for any object $M \in \mathscr{C}$, there exists an $\mathbb{E}$-triangle $A \to P \to M \dashrightarrow$ satisfying $P \in \mathcal{P}(\mathscr{C})$. Dually, we define that $\mathscr{C}$ *has enough injective objects*.

Let $\mathfrak{I}$ be a full additive subcategory of $\mathscr{C}$ which is closed under isomorphisms. If $\mathfrak{I} \subseteq \mathcal{P}(\mathscr{C}) \cap \mathcal{I}(\mathscr{C})$, by Proposition 3.30 of [12], the ideal quotient $\mathscr{C}/\mathfrak{I}$ has the structure of an extriangulated category induced from that of $\mathscr{C}$. Explicitly, set $\overline{\mathscr{C}} = \mathscr{C}/\mathfrak{I}$, then $(\overline{\mathscr{C}}, \overline{\mathbb{E}}, \overline{\mathfrak{s}})$ is an extriangulated category given by

▷ $\overline{\mathbb{E}}(C, A) = \mathbb{E}(C, A)$ for any $A, C \in \mathscr{C}$.

▷ $\overline{\mathbb{E}}(\overline{c}, \overline{a}) = \mathbb{E}(c, a)$ for any morphisms $a \in \mathscr{C}(A, A'), c \in \mathscr{C}(c, c')$.

▷ For any $\overline{\mathbb{E}}$-extension $\delta \in \overline{\mathbb{E}}(C, A) = \mathbb{E}(C, A)$, define $\overline{\mathfrak{s}}(\delta) = \overline{\mathfrak{s}(\delta)} = [A \xrightarrow{\bar{x}} B \xrightarrow{\overline{y}} C]$ by $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$. Hence, $\mathcal{M}_{n+1}$ and $\mathcal{F}_n$ are two extriangulated categories.

According to [2], Definition 2.32, we recall the notion of extriangulated functors.

Definition 2.10. Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ and $(\mathscr{C}', \mathbb{E}', \mathfrak{s}')$ be two extriangulated categories. An additive covariant functor $\mathscr{F}\colon \mathscr{C} \to \mathscr{C}'$ is called an *extriangulated functor* if there exists a natural transformation

$$\Gamma = \{\Gamma_{(C,A)}\}_{(C,A)\in\mathscr{C}^{\rm op}\times\mathscr{C}}\colon \mathbb{E}(-,-) \Rightarrow \mathbb{E}'(\mathscr{F}^{\rm op}-, \mathscr{F}-)$$

such that $\mathfrak{s}(\delta) = [A \xrightarrow{f} B \xrightarrow{g} C]$ implies that

$$\mathfrak{s}'(\Gamma_{(C,A)}(\delta)) = [F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)].$$

An extriangulated functor is called an *extriangulated equivalence* if it gives an equivalence of categories.

Lemma 2.11. *The functor $G\colon \mathcal{M}_{n+1} \to \mathcal{F}_n$ preserves conflations.*

Proof. Take any conflation

$$\eta := M \xrightarrow{\bar{x}} N \xrightarrow{\overline{y}} Q$$

in $\mathcal{M}_{n+1}$, where

$$0 \to M \xrightarrow{x} N \xrightarrow{y} Q \to 0$$

is an exact sequence in $\operatorname{mod} kA_{n+1}$. By the Horseshoe Lemma, the minimal projective resolutions of M and Q yield the commutative diagram with exact rows and columns in $\operatorname{mod} kA_{n+1}$:

(2.7)
$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega_M & \xrightarrow{\binom{1}{0}} & \Omega_M \oplus \Omega_Q & \xrightarrow{(0\,1)} & \Omega_Q & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i_M} & & \downarrow & & \downarrow{\scriptstyle i_Q} & & \\
0 & \longrightarrow & P_M & \xrightarrow{\binom{1}{0}} & P_M \oplus P_Q & \xrightarrow{(0\,1)} & P_Q & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_M} & & \downarrow & & \downarrow{\scriptstyle \pi_Q} & & \\
0 & \longrightarrow & M & \xrightarrow{x} & N & \xrightarrow{y} & Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

It is well-known that the projective resolution

$$0 \to \Omega_M \oplus \Omega_Q \to P_M \oplus P_Q \to N \to 0$$

is isomorphic to a resolution of the form

$$0 \longrightarrow \Omega_N \oplus R \xrightarrow{i_N \oplus 1} P_N \oplus R \xrightarrow{(\pi_N 0)} N \longrightarrow 0$$

for some projective module $R \in \operatorname{mod} kA_{n+1}$. Since $\Omega_M \oplus \Omega_Q \cong \Omega_N \oplus R$, we have that $R \in \operatorname{mod} kA_n$.

Let us write $M = M_1 \oplus M_2$ so that P_{M_1} has no direct summand isomorphic to P_{n+1} and $P_{M_2} \cong P_{n+1}^i$ for some nonnegative integer i. Similarly, we assume that $N = N_1 \oplus N_2$ so that P_{N_1} has no direct summand isomorphic to P_{n+1} and $P_{N_2} \cong P_{n+1}^j$ for some nonnegative integer j and $Q = Q_1 \oplus Q_2$ so that P_{Q_1} has no direct summand isomorphic to P_{n+1} and $P_{Q_2} \cong P_{n+1}^k$ for some nonnegative integer k. Since $P_N \oplus R \cong P_M \oplus P_Q$, we have that $j = i + k$.

Let us modify the diagram (2.7) to get

(2.8)
$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega_{M_1} \oplus \Omega_{M_2} & \xrightarrow{a} & \Omega_{N_1} \oplus \Omega_{N_2} \oplus R & \xrightarrow{b} & \Omega_{Q_1} \oplus \Omega_{Q_2} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i_{M_1} \oplus i_{M_2}} & & \downarrow{\scriptstyle i_{N_1} \oplus i_{N_2} \oplus 1} & & \downarrow{\scriptstyle i_{Q_1} \oplus i_{Q_2}} & & \\
0 & \longrightarrow & P_{M_1} \oplus P_{n+1}^i & \xrightarrow{c} & P_{N_1} \oplus P_{n+1}^j \oplus R & \xrightarrow{d} & P_{Q_1} \oplus P_{n+1}^k & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_{M_1} \oplus \pi_{M_2}} & & \downarrow & & \downarrow{\scriptstyle \pi_{Q_1} \oplus \pi_{Q_2}} & & \\
0 & \longrightarrow & M_1 \oplus M_2 & \xrightarrow{\widetilde{x}} & N_1 \oplus N_2 & \xrightarrow{\widetilde{y}} & Q_1 \oplus Q_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

Set

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}, \quad c = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{pmatrix}, \quad b = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}, \quad d = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \end{pmatrix}.$$

Since $c_{32} \in \mathrm{Hom}_{kA_{n+1}}(P_{n+1}^i, R) = 0$, we get that $a_{32} = c_{32} i_{M_2} = 0$. Similarly, we have that $i_{N_1} a_{12} = c_{12} i_{M_2} = 0$ since $c_{12} \in \mathrm{Hom}_{kA_{n+1}}(P_{n+1}^i, P_{N_1}) = 0$. It follows that

$$\begin{pmatrix} i_{N_1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} a = (i_{M_1}\ 0) \begin{pmatrix} c_{11} \\ c_{31} \end{pmatrix}.$$

Note that $i_{Q_1} b_{12} = d_{12} i_{N_2} = 0$ since $d_{12} \in \mathrm{Hom}_{kA_{n+1}}(P_{n+1}^j, P_{Q_1}) = 0$. Hence, by the diagram (2.8), there is a commutative diagram

(2.9)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_{M_1} \oplus \Omega_{M_2} & \xrightarrow{a} & \Omega_{N_1} \oplus \Omega_{N_2} \oplus R & \xrightarrow{b} & \Omega_{Q_1} \oplus \Omega_{Q_2} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (i_{M_1} 0)} & & \downarrow{\scriptstyle \left(\begin{smallmatrix} i_{N_1} & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)} & & \downarrow{\scriptstyle (i_{Q_1} 0)} & & \\
 & & P_{M_1} & \xrightarrow[\left(\begin{smallmatrix} c_{11} \\ c_{31} \end{smallmatrix}\right)]{} & P_{N_1} \oplus R & \xrightarrow[(d_{11} d_{13})]{} & P_{Q_1} & &
\end{array}$$

in $\operatorname{mod} kA_n$. Set $S = P_{N_1} \oplus R$, $c' = \binom{c_{11}}{c_{31}}$, $d' = (d_{11}d_{13})$ and $d'' = (d_{21}d_{23})$. Consider the split exact sequence

$$0 \longrightarrow P_{M_1} \oplus P^i_{n+1} \xrightarrow{\begin{pmatrix} c' & 0 \\ c_{21} & c_{22} \end{pmatrix}} S \oplus P^j_{n+1} \xrightarrow{\begin{pmatrix} d' & 0 \\ d'' & d_{22} \end{pmatrix}} P_{Q_1} \oplus P^k_{n+1} \longrightarrow 0.$$

Observing that

$$\begin{pmatrix} d' & 0 \\ d'' & d_{22} \end{pmatrix} \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we obtain that d' is a retraction. Clearly, $\ker(d') \cong P_{M_1}$.

Similarly, we can check that c' is a section. Since $d'c' = 0$, there is a commutative diagram

$$\begin{array}{ccccccccc} & & P_{M_1} & \xrightarrow{c'} & P_{N_1} \oplus R & \xrightarrow{d'} & P_{Q_1} & & \\ & & {\scriptstyle s}\,\downarrow & & \| & & \| & & \\ 0 & \longrightarrow & P_{M_1} & \longrightarrow & P_{N_1} \oplus R & \xrightarrow[d']{} & P_{Q_1} & \longrightarrow & 0 \end{array}$$

in $\operatorname{mod} kA_n$. Since c' is a section, we obtain that s is a section. Thus, s is an isomorphism. So, we conclude that the second row in the diagram (2.9) is a short exact sequence in $\operatorname{mod} kA_n$. Hence, we obtain that $G(\eta)$ is isomorphic to

$$C_{M_1} \oplus Z_{\Omega_{M_2}} \xrightarrow{\overline{(a\,c')}} C_{N_1} \oplus Z_{\Omega_{N_2}} \oplus K_R \xrightarrow{\overline{(b\,d')}} C_{Q_1} \oplus Z_{\Omega_{Q_2}},$$

which is a conflation in $\mathcal{F}_n$. Therefore, $G(\eta)$ is a conflation in $\mathcal{F}_n$. □

Proposition 2.12. *The functor $G\colon \mathcal{M}_{n+1} \to \mathcal{F}_n$ is an extriangulated functor.*

Proof. By Lemma 2.11, we define

$$\Gamma_{X,Y}\colon \mathbb{E}_{\mathcal{M}_{n+1}}(Y,X) \to \mathbb{E}_{\mathcal{F}_n}(GY,GX); \quad \delta \mapsto \Gamma_{X,Y}(\delta)$$

such that

$$G(X) \xrightarrow{G(x)} G(L) \xrightarrow{G(y)} G(y) \overset{\Gamma_{X,Y}(\delta)}{\dashrightarrow}$$

is an $\mathbb{E}_{\mathcal{F}_n}$-triangle, where

$$X \xrightarrow{x} L \xrightarrow{y} Y \overset{\delta}{\dashrightarrow}$$

is an $\mathbb{E}_{\mathcal{M}_{n+1}}$-triangle. For $a \in \operatorname{Hom}_{\mathcal{M}_{n+1}}(X,X')$, consider the commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{x} & L & \xrightarrow{y} & Y & \overset{\delta}{\dashrightarrow} & \\ {\scriptstyle a}\downarrow & & \downarrow & & \| & & \\ X' & \longrightarrow & L' & \longrightarrow & Y & \overset{a_*\delta}{\dashrightarrow} & . \end{array} \tag{2.10}$$

Applying the functor G to the diagram (2.10), we have the commutative diagram

$$\begin{array}{ccccccc} G(X) & \xrightarrow{G(x)} & G(L) & \xrightarrow{G(y)} & G(Y) & \overset{\Gamma_{X,Y}(\delta)}{\dashrightarrow} & \\ \downarrow{\scriptstyle G(a)} & & \downarrow & & \| & & \\ G(X') & \longrightarrow & G(L') & \longrightarrow & G(Y) & \overset{\Gamma_{X',Y}(a_*\delta)}{\dashrightarrow} & . \end{array}$$

Hence, $G(a)_*\Gamma_{X,Y}(\delta) = \Gamma_{X',Y}(a_*\delta)$. Similarly, for any $c \in \mathrm{Hom}_{\mathcal{M}_{n+1}}(Y', Y)$, we have that $G(c)^*\Gamma_{X,Y}(\delta) = \Gamma_{X,Y'}(c^*\delta)$. Hence, $\Gamma\colon \mathbb{E}_{\mathcal{M}_{n+1}}(-,-) \Rightarrow \mathbb{E}_{\mathcal{F}_n}(G^{\mathrm{op}}-, G-)$ is a natural transformation and G is an extriangulated functor. $\square$

Combining Propositions 2.8 and 2.12, we obtain the following assertion.

Theorem 2.13. *The functor G is an extriangulated equivalence, i.e.,*

$$G\colon\ \mathrm{mod}\, kA_{n+1}/\langle P_{n+1}\rangle \xrightarrow{\backsimeq} C_2(\mathscr{P}_{A_n})/\langle K_{P_i}\colon\ 1 \leqslant i \leqslant n\rangle$$

as extriangulated categories.

3. Hall algebras

Recently, Gorsky, Nakaoka and Palu in [10] have defined the higher extensions $\mathbb{E}^n_{\mathscr{C}}(-,-)$ for any $n \in \mathbb{Z}$ in an extriangulated category $\mathscr{C}$. For any $n > 0$, they introduced two versions $\mathbb{E}^{-n}_{\mathrm{I}}$ and $\mathbb{E}^{-n}_{\mathrm{II}}$ of the higher negative extensions for the extriangulated categories having enough projective morphisms and enough injective morphisms, respectively. In general, the bifunctors $\mathbb{E}^{-n}_{\mathrm{I}}$ and $\mathbb{E}^{-n}_{\mathrm{II}}$ may not be isomorphic for $n > 0$. Let $\mathscr{C}$ has enough projective objects and enough injective objects and satisfies the following conditions:

(1) $\mathrm{Hom}(I, x)$ is a monomorphism for any injective object I and any inflation x;

(2) $\mathrm{Hom}(y, P)$ is a monomorphism for any projective object P and any deflation y.

Then $|\mathbb{E}^i_{\mathrm{I}}(X,Y)| = |\mathbb{E}^i_{\mathrm{II}}(X,Y)|$ for any objects $X, Y \in \mathscr{C}$ and any $i < 0$. In this case, we write $\mathbb{E}^i_{\mathscr{C}}(X,Y)$ as $\mathbb{E}^i_{\mathrm{I}}(X,Y)$ or $\mathbb{E}^i_{\mathrm{II}}(X,Y)$ in what follows. By [1], Proposition 3.2, the morphism category $C_2(\mathscr{P})$ has enough projective objects and enough injective objects, and it is hereditary. By [12], Proposition 3.24, $\{C_{P_i}\colon\ 1 \leqslant i \leqslant n\}$ is a complete set of indecomposable projective objects, and $\{Z_{P_i}\colon\ 1 \leqslant i \leqslant n\}$ is a complete set of indecomposable injective objects in $\mathcal{F}_n$. Thus, $\mathcal{F}_n$ has enough projective objects and enough injective objects and $\mathbb{E}^i_{\mathcal{F}_n}(X,Y) = 0$ for any $X, Y \in \mathcal{F}_n$ and $i > 1$. According to [10], Example 5.42 (iii), we get that $\mathbb{E}^i_{\mathcal{F}_n}(X,Y) = 0$ for any $X, Y \in \mathcal{F}_n$ and $i < -1$. Similarly, the extriangulated category $\mathcal{M}_{n+1}$ also satisfies these properties as $\mathcal{F}_n$. It is easy to see that the extriangulated categories $\mathcal{M}_{n+1}$ and $\mathcal{F}_N$ satisfy the conditions (1) and (2) above.

In what follows, let k be a finite field with q elements, and we set $v = \sqrt{q}$. Let $\mathscr{C}$ be an essentially small Krull-Schmidt additive k-linear category satisfying the conditions (1) and (2) above. We assume that $\mathscr{C}$ is left locally homologically finite (cf. [16]) and has enough projective objects and enough injective objects. We denote by $(X, L)_Y$ the set consisting of inflations $f\colon X \to L$ such that $\operatorname{cone}(f) \cong Y$ in $\mathscr{C}$. Dually, we define ${}_X(L, Y)$.

Definition 3.1. The Hall algebra $\mathcal{H}(\mathscr{C})$ of the extriangulated category $\mathscr{C}$ is a $\mathbb{Q}$-space with the basis $\{u_{[X]}\colon X \in \mathscr{C}\}$ and the multiplication defined by

$$u_{[X]} \diamond u_{[Y]} = \sum_{[L]} G_{XY}^L u_{[L]}$$

where

$$G_{XY}^L := \frac{|{}_Y(L,X)|}{|\mathrm{Aut}X|}\frac{\{L,X\}}{\{X,X\}} = \frac{|(Y,L)_X|}{|\mathrm{Aut}Y|}\frac{\{Y,L\}}{\{Y,Y\}} \tag{3.1}$$

and

$$\{M,N\} := \prod_{i>0} |\mathbb{E}_{\mathscr{C}}^{-i}(M,N)|^{(-1)^i}$$

for any $M, N \in \mathscr{C}$.

Remark 3.2. The Hall algebra defined here is the algebra opposite to that in Definition 6.2 of [16]. For any $X, Y, L \in \mathscr{C}$, by [16], Proposition 6.4 we have that

$$G_{XY}^L = \frac{|\mathbb{E}(X,Y)_L|}{|\mathrm{Hom}(X,Y)|} \cdot \frac{|\mathrm{Aut}L|}{|\mathrm{Aut}X| \cdot |\mathrm{Aut}Y|} \cdot \frac{\{L,L\}}{\{X,X\} \cdot \{Y,Y\}} \cdot \frac{1}{\{X,Y\}}, \tag{3.2}$$

where $\mathbb{E}(X,Y)_L$ denotes the set consisting of extensions $\delta \in \mathbb{E}(X,Y)$ such that $\mathfrak{s}(\delta) = [Y \xrightarrow{f} L \xrightarrow{g} X]$. Define the *dual Hall algebra* $\mathfrak{H}(\mathscr{C})$ to be the $\mathbb{Q}$-space with the basis $\{\nu_{[X]}\colon X \in \mathscr{C}\}$ and multiplication defined by

$$\nu_{[X]} \diamond \nu_{[Y]} = \sum_{[L]} \frac{|\mathbb{E}(X,Y)_L|}{|\mathrm{Hom}(X,Y)|} \cdot \frac{1}{\{X,Y\}} \nu_{[L]}.$$

By the formula (3.2), there is an algebra isomorphism

$$\varrho : \mathfrak{H}(\mathscr{C}) \to \mathcal{H}(\mathscr{C}) \tag{3.3}$$

defined on basis elements by

$$\varrho(\nu_{[X]}) = |\mathrm{Aut}X| \cdot \{X,X\} u_{[X]}.$$

Theorem 3.3. *There exists an isomorphism of algebras*

$$\varphi\colon \mathcal{H}(\mathcal{M}_{n+1}) \to \mathcal{H}(\mathcal{F}_n)$$

defined on basis elements by $u_{[X]} \mapsto u_{[G(X)]}$.

P r o o f. Since the functor G is an extriangulated equivalence, it is easy to get the following identities: $|\mathrm{Aut}(X)| = |\mathrm{Aut}(G(X))|$, $|_X(L,Y)| = |_{G(X)}(G(L),G(Y)|$ and $|(X,L)_Y| = |(G(X),G(L))_{G(Y)}|$. For any $X, Y \in \mathcal{M}_{n+1}$, take a conflation

$$P_1 \to P_0 \to Y$$

with P_0, P_1 projective in $\mathcal{M}_{n+1}$. By Corollary 5.14 of [10], we obtain that

$$\begin{aligned}\dim_k \mathbb{E}^{-1}_{\mathcal{M}_{n+1}}(X,Y) = {} & \dim_k \mathrm{Hom}_{\mathcal{M}_{n+1}}(X,Y) + \dim_k \mathrm{Hom}_{\mathcal{M}_{n+1}}(X,P_1)\\ & - \dim_k \mathrm{Hom}_{\mathcal{M}_{n+1}}(X,P_0) - \dim_k \mathrm{Hom}_{\underline{\mathcal{M}_{n+1}}}(X,Y),\end{aligned}$$

where $\underline{\mathcal{M}_{n+1}}$ is the quotient category of $\mathcal{M}_{n+1}$ by projectives. Since the functor G preserves projectives, it induces the equivalence between $\underline{\mathcal{M}_{n+1}}$ and $\underline{\mathcal{F}_n}$. Thus,

$$|\mathbb{E}^{-1}_{\mathcal{M}_{n+1}}(X,Y)| = |\mathbb{E}^{-1}_{\mathcal{F}_n}(G(X),G(Y))|.$$

Note that $\{M,N\} = |\mathbb{E}^{-1}_{\mathscr{C}}(M,N)|^{-1}$ for any $M, N \in \mathscr{C}$, where $\mathscr{C} = \mathcal{M}_{n+1}$ or $\mathcal{F}_n$. Thus, $\{M,N\} = \{G(M),G(N)\}$ for any $M, N \in \mathcal{M}_{n+1}$. Combining these identities, we easily obtain that φ is an isomorphism of algebras. □

Assume that the extriangulated category $\mathscr{C}$ is locally homologically finite, cf. [16], Section 7. For any $X, Y \in \mathscr{C}$, put

$$\langle X,Y\rangle_{\mathscr{C}} = \sum_{i\in\mathbb{Z}} (-1)^i \dim_k \mathbb{E}^i_{\mathscr{C}}(X,Y).$$

Since $\mathscr{C}$ is locally homologically finite, the sum above is finite, which descends to give a bilinear form

$$\langle -,-\rangle_{\mathscr{C}} \colon K_0(\mathscr{C}) \times K_0(\mathscr{C}) \to \mathbb{Z}$$

on the Grothendieck group $K_0(\mathscr{C})$ of $\mathscr{C}$ called the *Euler form* of $\mathscr{C}$. Note that for any $X, Y \in \mathscr{C}$,

$$\langle X,Y\rangle_{\mathscr{C}} = -\dim_k \mathbb{E}^{-1}_{\mathscr{C}}(X,Y) + \dim_k \mathrm{Hom}_{\mathscr{C}}(X,Y) - \dim_k \mathbb{E}^1_{\mathscr{C}}(X,Y)$$

if $\mathscr{C}$ is taken to be $\mathcal{M}_{n+1}$ or $\mathcal{F}_n$.

Define the *twisted dual Hall algebra* $\mathfrak{H}_q(\mathscr{C})$ to be the same space as $\mathfrak{H}(\mathscr{C})$, but with the multiplication defined by

$$\nu_{[X]} * \nu_{[Y]} = q^{\langle X,Y\rangle_{\mathscr{C}}} \nu_{[X]} \diamond \nu_{[Y]}$$

for any $X, Y \in \mathscr{C}$.

Lemma 3.4. *There exists an isomorphism of algebras*

$$\psi \colon \mathfrak{H}_q(\mathcal{M}_{n+1}) \to \mathfrak{H}_q(\mathcal{F}_n)$$

defined on basis elements by $\nu_{[X]} \mapsto \nu_{[G(X)]}$.

P r o o f. It is easy to see that $\langle X,Y\rangle_{\mathcal{M}_{n+1}}=\langle G(X),G(Y)\rangle_{\mathcal{F}_n}$ for any $X,Y\in\mathcal{M}_{n+1}$. Then by Theorem 3.3 and the isomorphism ϱ in (3.3), we finish the proof. $\square$

Let $K^b(\mathscr{P}_{A_n})$ be the homotopy category of bounded complexes over projective kA_n-modules. Define $K^{[-1,0]}(\mathscr{P}_{A_n})$ to be the full subcategory of $K^b(\mathscr{P}_{A_n})$ consisting of complexes situated in degree $0,-1$. Since $\operatorname{mod} kA_n$ is hereditary and has enough projectives, there exists a triangulated equivalence from $K^b(\mathscr{P}_{A_n})$ to the bounded derived category $D^b(A_n)$ of kA_n. When we view the objects in $D^b(A_n)$ as the direct sum of stalk complexes $M[i]$ with $M\in\operatorname{mod} kA_n, i\in\mathbb{Z}$, any object $M_\bullet\in K^{[-1,0]}(\mathscr{P}_{A_n})$ should be identified with $H^0(M_\bullet)\oplus H^{-1}(M_\bullet)[1]$. Clearly, $K^{[-1,0]}(\mathscr{P}_{A_n})$ is an extension-closed subcategory of $K^b(\mathscr{P}_{A_n})$. By Lemma 2.2, the category $\mathcal{F}_n$ is just $K^{[-1,0]}(\mathscr{P}_{A_n})$ if we view any object $M_\bullet$ in $\mathcal{F}_n$ as a complex with the (-1)st component and 0th component equal to M_0 and M_1, respectively. In what follows, for simplicity, we write $\langle -,-\rangle_{\operatorname{mod} kA_n}$ as $\langle -,-\rangle$. For any $M,N,X,Y\in\operatorname{mod} kA_n$, we set

$$_X\mathrm{Hom}_{kA_n}(M,N)_Y:=\{f\colon M\to N\colon \ker f\cong X \text{ and } \operatorname{coker} f\cong Y\}.$$

According to the characterization of the derived Hall algebra of $D^b(A_n)$ given by Toën, see [15] (see also the twisted version of [14], Proposition 2.1), we have the following assertion.

Proposition 3.5 ([15]). *The Hall algebra $\mathfrak{H}_q(D^b(A_n))$ is an associative unital algebra generated by the elements in $\{\nu_{M[i]}\colon M\in\operatorname{mod} kA_n,\ i\in\mathbb{Z}\}$ and the following relations*

$$\nu_{M[i]}*\nu_{N[i]}=q^{\langle M,N\rangle}\sum_{[L]}\frac{|\mathrm{Ext}^1_{kA_n}(M,N)_L|}{|\mathrm{Hom}_{kA_n}(M,N)|}\nu_{L[i]};\tag{3.4}$$

$$\nu_{M[i]}*\nu_{N[i+1]}=\nu_{M[i]\oplus N[i+1]};\tag{3.5}$$

$$\nu_{M[i+1]}*\nu_{N[i]}=q^{-\langle M,N\rangle}\sum_{[X],[Y]}|{}_X\mathrm{Hom}_{kA_n}(M,N)_Y|\nu_{Y[i]}*\nu_{X[i+1]};\tag{3.6}$$

$$\nu_{M[i]}*\nu_{N[j]}=q^{(-1)^{i-j}\langle M,N\rangle}\nu_{N[j]}*\nu_{M[i]},\quad i-j>1.\tag{3.7}$$

By abuse of notation, here we write $\nu_{M[i]}$ instead of $\nu_{[M[i]]}$ for $M\in\operatorname{mod} kA_n$ and $i\in\mathbb{Z}$.

Since $K^b(\mathscr{P}_{A_n})$ is equivalent to $D^b(A_n)$ as triangulated categories, the dual Hall algebra $\mathfrak{H}_q(K^b(\mathscr{P}_{A_n}))$ is isomorphic to $\mathfrak{H}_q(D^b(A_n))$. Since $\mathcal{F}_n$ is an extension-closed full subcategory of $K^b(\mathscr{P}_{A_n})$, the dual Hall algebra $\mathfrak{H}_q(\mathcal{F}_n)$ is a subalgebra of $\mathfrak{H}_q(K^b(\mathscr{P}_{A_n}))$. By Proposition 3.5, we give the characterization of $\mathfrak{H}_q(\mathcal{F}_n)$.

Proposition 3.6. *The dual Hall algebra $\mathfrak{H}_q(\mathcal{F}_n)$ is generated by the elements $\{\nu_M, \nu_{P[1]} \colon M \in \operatorname{mod} kA_n,\ P \in \mathscr{P}_{A_n}\}$ and the following relations:*

$$\nu_{P[1]} * \nu_{Q[1]} = \nu_{(P\oplus Q)[1]} = \nu_{Q[1]} * \nu_{P[1]}; \tag{3.8}$$

$$\nu_M * \nu_N = q^{\langle M,N\rangle} \sum_{[L]} \frac{|\mathrm{Ext}^1_{kA_n}(M,N)_L|}{|\mathrm{Hom}_{kA_n}(M,N)|} \nu_L; \tag{3.9}$$

$$\nu_M * \nu_{P[1]} = \nu_{M\oplus P[1]}; \tag{3.10}$$

$$\nu_{P[1]} * \nu_M = q^{-\langle P,M\rangle} \sum_{[F],[P']} |_{P'}\mathrm{Hom}_{kA_n}(P,M)_F| \nu_{F\oplus P'[1]} \tag{3.11}$$

for any $M, N \in \operatorname{mod} kA_n$ and $P, Q \in \mathscr{P}_{A_n}$.

4. Quantum cluster algebras

For an acyclic quiver Q with n vertices, define n_{ij} to be the number of arrows from i to j for any $1 \leqslant i, j \leqslant n$. Let $B_Q = (b_{ij})$ and $R_Q = (r_{ij})$ be the $n \times n$ matrices given by $b_{ij} = n_{ij} - n_{ji}$ and $r_{ij} = n_{ji}$ for any $1 \leqslant i, j \leqslant n$, respectively. It is easy to see that $B_{A_{2n}}$ is invertible. Take $\Lambda = -B^{-1}$, then (Λ, B) is a compatible pair, cf. [3], Definition 3.1. In what follows, we also denote by Λ the skew-symmetric bilinear form on $\mathbb{Z}^{2n}$ associated to the skew-symmetric matrix Λ.

Define the *quantum torus* $\mathcal{T}_\Lambda$ to be the $\mathbb{Z}[v, v^{-1}]$-algebra with a basis $\{X^\alpha \colon \alpha \in \mathbb{Z}^{2n}\}$ and the multiplication defined by

$$X^\alpha \star X^\beta = v^{\Lambda(\alpha,\beta)} X^{\alpha+\beta}. \tag{4.1}$$

Let $e_1, e_2, \ldots, e_{2n}$ be the standard basis of $\mathbb{Z}^{2n}$, set $x_i = X^{e_i}$. In what follows, for any $\alpha \in \mathbb{Z}^{2n}$, set $\alpha^* = (I_{2n} - R_{A_{2n}})\alpha$ and ${}^*\alpha = (I_{2n} - R^{\mathrm{tr}}_{A_{2n}})\alpha$, where I_{2n} is the identity matrix and $R^{\mathrm{tr}}_{A_{2n}}$ is the transpose of $R_{A_{2n}}$. For a module M, we use the corresponding lowercase boldface letter $\mathbf{m}$ to denote its dimension vector.

In order to relate this with the quantum torus $\mathcal{T}_\Lambda$, we need to use Λ to twist the multiplications of the Hall algebras. Define the Λ-*twisted dual Hall algebra* $\mathfrak{H}_\Lambda(\mathcal{F}_{2n})$ to be the same space as $\mathfrak{H}_q(\mathcal{F}_{2n})$, but with the multiplication defined by

$$\nu_{M\oplus P[1]} \star \nu_{N\oplus Q[1]} = v^{\Lambda((\mathbf{m}-\mathbf{p})^*, (\mathbf{n}-\mathbf{q})^*)} \nu_{M\oplus P[1]} * \nu_{N\oplus Q[1]}$$

for any $M, N \in \operatorname{mod} kA_{2n}$ and $P, Q \in \mathscr{P}_{A_{2n}}$.

Every object X of $\mathcal{M}_{2n+1}$ can be written as $M \oplus \bigoplus_{i=1}^{2n} a_i I_{i+1}$, where $M \in \operatorname{mod} kA_{2n+1}$ has no injective direct summands, any I_i is the indecomposable injective kA_{2n+1}-module corresponding to the vertex i, and $a_i \in \mathbb{Z}_{\geqslant 0}$; define the dimension vector $\mathbf{x}$ of X to be $\mathbf{m} - \sum_{i=1}^{2n} a_i \mathbf{p}_i$, where $\mathbf{m}$ is the dimension vector of M viewed as a kA_{2n}-module and any $\mathbf{p}_i$ is the dimension vector of the indecomposable projective kA_{2n}-module corresponding to the vertex i. Define the Λ-*twisted dual Hall algebra* $\mathfrak{H}_\Lambda(\mathcal{M}_{2n+1})$ to be the same space as $\mathfrak{H}_q(\mathcal{M}_{2n+1})$, but with the multiplication defined by

$$\nu_X \star \nu_Y = v^{\Lambda(\mathbf{x}^*, \mathbf{y}^*)} \nu_X * \nu_Y$$

for any $X, Y \in \mathcal{M}_{2n+1}$. Given a kA_{2n+1}-module M without injective direct summands, the kA_{2n}-module obtained by viewing M as a kA_{2n}-module is still denoted by M. It is easy to see that we have the following assertion.

Lemma 4.1. *There exists an isomorphism of algebras*

$$\psi_\Lambda \colon \mathfrak{H}_\Lambda(\mathcal{M}_{2n+1}) \to \mathfrak{H}_\Lambda(\mathcal{F}_n)$$

defined on basis elements by $\nu_X \mapsto \nu_{H^0(G(X)) \oplus H^{-1}(G(X))[1]}$. *Explicitly,* $\psi_\Lambda(\nu_M) = \nu_M$ *for any* $M \in \operatorname{mod} kA_{2n+1}$ *without injective direct summands and* $\nu_{I_{i+1}} = \nu_{P_i[1]}$ *for* $1 \leqslant i \leqslant 2n$.

Given a kA_{2n}-module M and a dimension vector $\mathbf{e} \in \mathbb{N}^{2n}$, we denote by $\mathrm{Gr}_{\mathbf{e}} M$ the set of all submodules V of M with $\mathbf{e}$ as the dimension vector. According to [9], we have the following statement.

Proposition 4.2. *There exists an algebra homomorphism* $\Psi \colon \mathfrak{H}_\Lambda(\mathcal{F}_{2n}) \to \mathcal{T}_\Lambda$ *such that*

$$\Psi(\nu_{M \oplus P[1]}) = \sum_{\mathbf{e}} v^{\langle \mathbf{p}-\mathbf{e}, \mathbf{m}-\mathbf{e} \rangle} |\mathrm{Gr}_{\mathbf{e}} M| X^{(\mathbf{p}-\mathbf{e})^* - {}^*(\mathbf{m}-\mathbf{e})}.$$

Combining Lemma 4.1 and Proposition 4.2, we obtain the following assertion.

Proposition 4.3. *There exists an algebra homomorphism* $\Phi \colon \mathfrak{H}_\Lambda(\mathcal{M}_{2n+1}) \to \mathcal{T}_\Lambda$ *such that*

$$\Phi(\nu_M) = \sum_{\mathbf{e}} v^{-\langle \mathbf{e}, \mathbf{m}-\mathbf{e} \rangle} |\mathrm{Gr}_{\mathbf{e}} M| X^{-\mathbf{e}^* - {}^*(\mathbf{m}-\mathbf{e})} \quad \text{and} \quad \Phi(I_{i+1}) = x_i$$

for any $M \in \operatorname{mod} kA_{2n+1}$ *without injective direct summands and* $1 \leqslant i \leqslant 2n$.

Acknowledgments. The authors would like to thank the anonymous reviewer for the careful reading and helpful comments.

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