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TWO RESULTS OF n-EXANGULATED CATEGORIES

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Abstract. M. Herschend, Y. Liu, H. Nakaoka introduced n-exangulated categories, which are a simultaneous generalization of n-exact categories and $(n+2)$-angulated categories. This paper consists of two results on n-exangulated categories: (1) we give an equivalent characterization of axiom (EA2); (2) we provide a new way to construct a closed subfunctor of an n-exangulated category.

Keywords: n-exangulated category; homotopy cartesian square; half exact functor

MSC 2020: 18G80, 18E10

1. Introduction

The notion of extriangulated categories was introduced in [7], which can be viewed as a simultaneous generalization of exact categories and triangulated categories. The data of such a category is a triplet $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$, where $\mathscr{C}$ is an additive category, $\mathbb{E}\colon \mathscr{C}^{\rm op} \times \mathscr{C} \to \mathsf{Ab}$ is an additive bifunctor and $\mathfrak{s}$ assigns to each $\delta \in \mathbb{E}(C, A)$ a class of 3-term sequences with end terms A and C such that certain axioms hold. Recently, Herschend, Liu, Nakaoka in [1] introduced the notion of n-exangulated categories for any positive integer n. It should be noted that the case $n = 1$ corresponds to extriangulated categories. As typical examples we know that n-exact categories and $(n+2)$-angulated categories are n-exangulated categories, see [1], Propositions 4.34 and 4.5. However, there are some other examples of n-exangulated categories which are neither n-exact nor $(n+2)$-angulated, see [1], [2], [3], [6].

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Recall that homotopy cartesian squares in triangulated categories are the triangulated analogues of the pushout and pullback squares in abelian categories. It is well-known that the axiom (TR4) is equivalent to the homotopy cartesian axiom. Recently, Kong, Lin, Wang in [4] introduced the notion of homotopy cartesian squares and proposed a new shifted octahedron in extriangulated categories. Our first main result shows that Kong-Lin-Wang's result has a higher counterpart:

Theorem 1.1 (Theorem 3.4). *Let* $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ *be a pre-n-exangulated category. Then* $\mathscr{C}$ *satisfies* (EA2) *if and only if* $\mathscr{C}$ *satisfies* (EA2-1).

Relative theories are explicated by use of closed subfunctors in n-exangulated categories, see [1]. An additive subfunctor $\mathbb{F}$ of $\mathbb{E}$ is a closed subfunctor if $(\mathscr{C}, \mathbb{F}, \mathfrak{s}|_{\mathbb{F}})$ is an n-exangulated category, where $\mathfrak{s}|_{\mathbb{F}}$ is a restriction of $\mathfrak{s}$ to $\mathbb{F}$, see [1], Proposition 3.16. Our second main result provides a new way to construct a closed subfunctor of an n-exangulated category. This is a higher counterpart of Sakai's result.

Theorem 1.2 (Theorem 4.5). *Let* $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ *be an n-exangulated category and* $F\colon \mathscr{C} \to \mathscr{A}$ *a half exact functor to an abelian category* $\mathscr{A}$*. Then there exists a unique maximal closed subfunctor* $\mathbb{F}$ *of* $\mathbb{E}$ *such that* F *becomes a right exact functor from* $(\mathscr{C}, \mathbb{F}, \mathfrak{s}|_{\mathbb{F}})$ *to* $\mathscr{A}$.

This article is organized as follows. In Section 2, we review some elementary definitions and facts on n-exangulated categories. In Section 3, we prove our first main result. In Section 4, we prove our second main result.

2. Preliminaries

In this section, we briefly review basic concepts and results concerning n-exangulated categories.

Let $\mathscr{C}$ be an additive category and $\mathbb{E}\colon \mathscr{C}^{\rm op} \times \mathscr{C} \to \mathsf{Ab}$ (Ab is the category of abelian groups) an additive bifunctor. For any pair of objects $A, C \in \mathscr{C}$, an element $\delta \in \mathbb{E}(C, A)$ is called an $\mathbb{E}$-*extension* or simply an *extension*. We also write such δ as ${}_A\delta_C$ when we indicate A and C. The zero element ${}_A0_C = 0 \in \mathbb{E}(C, A)$ is called the *split* $\mathbb{E}$-*extension*. For any pair of $\mathbb{E}$-extensions ${}_A\delta_C$ and ${}_{A'}\delta'_{C'}$, let $\delta \oplus \delta' \in \mathbb{E}(C \oplus C', A \oplus A')$ be the element corresponding to $(\delta, 0, 0, \delta')$ through the natural isomorphism $\mathbb{E}(C \oplus C', A \oplus A') \simeq \mathbb{E}(C, A) \oplus \mathbb{E}(C, A') \oplus \mathbb{E}(C', A) \oplus \mathbb{E}(C', A')$.

For any $a \in \mathscr{C}(A, A')$ and $c \in \mathscr{C}(C', C)$, $\mathbb{E}(C, a)(\delta) \in \mathbb{E}(C, A')$ and $\mathbb{E}(c, A)(\delta) \in \mathbb{E}(C', A)$ are simply denoted by $a_*\delta$ and $c^*\delta$, respectively. Let ${}_A\delta_C$ and ${}_{A'}\delta'_{C'}$ be any pair of $\mathbb{E}$-extensions. A *morphism* $(a, c)\colon \delta \to \delta'$ of extensions is a pair of morphisms $a \in \mathscr{C}(A, A')$ and $c \in \mathscr{C}(C, C')$ in $\mathscr{C}$, satisfying the equality $a_*\delta = c^*\delta'$. Let $\mathscr{C}$ be an additive category as before, and let n be any positive integer.

Definition 2.1 ([1], Definition 2.7). Let $\mathbf{C}_{\mathscr{C}}$ be the category of complexes in $\mathscr{C}$. As its full subcategory, define $\mathbf{C}_{\mathscr{C}}^{n+2}$ to be the category of complexes in $\mathscr{C}$ whose components are zero in the degrees outside of $\{0,1,\ldots,n+1\}$. Namely, an object in $\mathbf{C}_{\mathscr{C}}^{n+2}$ is a complex $X_\bullet=\{X_i,d_i^X\}$ of the form

$$X_0 \xrightarrow{d_0^X} X_1 \xrightarrow{d_1^X} \cdots \xrightarrow{d_{n-1}^X} X_n \xrightarrow{d_n^X} X_{n+1}.$$

We write a morphism $f_\bullet\colon X_\bullet\to Y_\bullet$ simply $f_\bullet=(f_0,f_1,\ldots,f_{n+1})$, only indicating the terms of degrees $0,\ldots,n+1$.

Definition 2.2 ([1], Definition 2.11). By Yoneda lemma, any extension $\delta\in\mathbb{E}(C,A)$ induces natural transformations

$$\delta_\sharp\colon \mathscr{C}(-,C)\Rightarrow\mathbb{E}(-,A) \quad\text{and}\quad \delta^\sharp\colon \mathscr{C}(A,-)\Rightarrow\mathbb{E}(C,-).$$

For any $X\in\mathscr{C}$, these $(\delta_\sharp)_X$ and $\delta^\sharp_X$ are given as follows:
(1) $(\delta_\sharp)_X\colon \mathscr{C}(X,C)\to\mathbb{E}(X,A)\colon\ f\mapsto f^*\delta$.
(2) $\delta^\sharp_X\colon \mathscr{C}(A,X)\to\mathbb{E}(C,X)\colon\ g\mapsto g_*\delta$.
We simply denote $(\delta_\sharp)_X(f)$ and $\delta^\sharp_X(g)$ by $\delta_\sharp(f)$ and $\delta^\sharp(g)$, respectively.

Definition 2.3 ([1], Definition 2.9). Let $\mathscr{C},\mathbb{E},n$ be as before. Define a category $\text{Æ}:=\text{Æ}^{n+2}_{(\mathscr{C},\mathbb{E})}$ as follows:
(1) An object in $\text{Æ}^{n+2}_{(\mathscr{C},\mathbb{E})}$ is a pair $\langle X_\bullet,\delta\rangle$ of $X_\bullet\in\mathbf{C}_{\mathscr{C}}^{n+2}$ and $\delta\in\mathbb{E}(X_{n+1},X_0)$ satisfying

$$(d_0^X)_*\delta=0 \quad\text{and}\quad (d_n^X)^*\delta=0.$$

We call such a pair an $\mathbb{E}$-attached complex of length $n+2$. We also denote it by

$$X_0 \xrightarrow{d_0^X} X_1 \xrightarrow{d_1^X} \cdots \xrightarrow{d_{n-2}^X} X_{n-1} \xrightarrow{d_{n-1}^X} X_n \xrightarrow{d_n^X} X_{n+1} \overset{\delta}{\dashrightarrow}.$$

(2) For such pairs $\langle X_\bullet,\delta\rangle$ and $\langle Y_\bullet,\varrho\rangle$, a morphism $f_\bullet\colon\langle X_\bullet,\delta\rangle\to\langle Y_\bullet,\varrho\rangle$ is defined to be a morphism $f_\bullet\in\mathbf{C}_{\mathscr{C}}^{n+2}(X_\bullet,Y_\bullet)$ satisfying $(f_0)_*\delta=(f_{n+1})^*\varrho$.
We use the same composition and the identities as in $\mathbf{C}_{\mathscr{C}}^{n+2}$.

Definition 2.4 ([1], Definition 2.13). An *n-exangle* is a pair $\langle X_\bullet,\delta\rangle$ of $X_\bullet\in\mathbf{C}_{\mathscr{C}}^{n+2}$ and $\delta\in\mathbb{E}(X_{n+1},X_0)$ which satisfies the following conditions:
(1) The following sequence of functors $\mathscr{C}^{\mathrm{op}}\to\mathsf{Ab}$ is exact:

$$\mathscr{C}(-,X_0)\xrightarrow{\mathscr{C}(-,d_0^X)}\cdots\xrightarrow{\mathscr{C}(-,d_n^X)}\mathscr{C}(-,X_{n+1})\xrightarrow{\delta_\sharp}\mathbb{E}(-,X_0).$$

(2) The following sequence of functors $\mathscr{C}\to\mathsf{Ab}$ is exact:

$$\mathscr{C}(X_{n+1},-)\xrightarrow{\mathscr{C}(d_n^X,-)}\cdots\xrightarrow{\mathscr{C}(d_0^X,-)}\mathscr{C}(X_0,-)\xrightarrow{\delta^\sharp}\mathbb{E}(X_{n+1},-)\ .$$

In particular, any n-exangle is an object in Æ. A *morphism of n-exangles* simply means a morphism in Æ. Thus, n-exangles form a full subcategory of Æ.

Definition 2.5 ([1], Definition 2.22). Let $\mathfrak{s}$ be a correspondence which associates a homotopic equivalence class $\mathfrak{s}(\delta) = [{}_AX_{\bullet C}]$ to each extension $\delta = {}_A\delta_C$. Such $\mathfrak{s}$ is called a *realization* of $\mathbb{E}$ if it satisfies the following condition for any $\mathfrak{s}(\delta) = [X_\bullet]$ and any $\mathfrak{s}(\varrho) = [Y_\bullet]$.

(R0) For any morphism of extensions $(a, c)\colon \delta \to \varrho$, there exists a morphism $f_\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}(X_\bullet, Y_\bullet)$ of the form $f_\bullet = (a, f_1, \ldots, f_n, c)$. Such $f_\bullet$ is called a *lift* of (a, c).

In such a case, we simply say that "$X_\bullet$ realizes δ" whenever they satisfy $\mathfrak{s}(\delta) = [X_\bullet]$.

Moreover, a realization $\mathfrak{s}$ of $\mathbb{E}$ is said to be *exact* if it satisfies the following conditions:

(R1) For any $\mathfrak{s}(\delta) = [X_\bullet]$, the pair $\langle X_\bullet, \delta\rangle$ is an n-exangle.

(R2) For any $A \in \mathscr{C}$, the zero element ${}_A0_0 = 0 \in \mathbb{E}(0, A)$ satisfies

$$\mathfrak{s}({}_A0_0) = [A \xrightarrow{\mathrm{id_A}} A \to 0 \to \ldots \to 0 \to 0].$$

Dually, $\mathfrak{s}({}_00_A) = [0 \to 0 \to \ldots \to 0 \to A \xrightarrow{\mathrm{id_A}} A]$ holds for any $A \in \mathscr{C}$.

Note that the above condition (R1) does not depend on the representatives of the class $[X_\bullet]$.

Definition 2.6 ([1], Definition 2.23). Let $\mathfrak{s}$ be an exact realization of $\mathbb{E}$.

(1) An n-exangle $\langle X_\bullet, \delta\rangle$ is called an $\mathfrak{s}$-*distinguished* n-exangle if it satisfies $\mathfrak{s}(\delta) = [X_\bullet]$. We often simply say *distinguished n-exangle* when $\mathfrak{s}$ is clear from the context.

(2) An object $X_\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}$ is called an $\mathfrak{s}$-*conflation* or simply a *conflation* if it realizes an extension $\delta \in \mathbb{E}(X_{n+1}, X_0)$.

(3) A morphism f in $\mathscr{C}$ is called an $\mathfrak{s}$-*inflation* or simply an *inflation* if it admits a conflation $X_\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}$ satisfying $d_0^X = f$.

(4) A morphism g in $\mathscr{C}$ is called an $\mathfrak{s}$-*deflation* or simply a *deflation* if it admits a conflation $X_\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}$ satisfying $d_n^X = g$.

Definition 2.7 ([1], Definition 2.27). For a morphism $f_\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}(X_\bullet, Y_\bullet)$ satisfying $f_0 = \mathrm{id_A}$ for some $A = X_0 = Y_0$, its *mapping cone* $M^f_\bullet \in \mathbf{C}^{n+2}_{\mathscr{C}}$ is defined to be the complex

$$X_1 \xrightarrow{d_0^{M_f}} X_2 \oplus Y_1 \xrightarrow{d_1^{M_f}} X_3 \oplus Y_2 \xrightarrow{d_2^{M_f}} \ldots \xrightarrow{d_{n-1}^{M_f}} X_{n+1} \oplus Y_n \xrightarrow{d_n^{M_f}} Y_{n+1},$$

where

$$d_0^{M_f} = \begin{bmatrix} -d_1^X \\ f_1 \end{bmatrix}, \quad d_i^{M_f} = \begin{bmatrix} -d_{i+1}^X & 0 \\ f_{i+1} & d_i^Y \end{bmatrix} \quad (1 \leqslant i \leqslant n-1), \quad d_n^{M_f} = [\, f_{n+1} \quad d_n^Y \,].$$

The *mapping cocone* is defined dually, for morphisms $h_\bullet$ in $\mathbf{C}^{n+2}_{\mathscr{C}}$ satisfying $h_{n+1} = \mathrm{id}$.

Definition 2.8 ([1], Definition 2.32). A *pre-n-exangulated category* is a triplet $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ of additive category $\mathscr{C}$, additive bifunctor $\mathbb{E}\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathsf{Ab}$, and its exact realization $\mathfrak{s}$, satisfying the following condition.

(EA1) Let $A \xrightarrow{f} B \xrightarrow{g} C$ be any sequence of morphisms in $\mathscr{C}$. If both f and g are inflations, then so is $g \circ f$. Dually, if f and g are deflations, then so is $g \circ f$.

If the triplet $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ moreover satisfies the following conditions, then it is called an *n-exangulated category.*

(EA2) For $\varrho \in \mathbb{E}(D, A)$ and $c \in \mathscr{C}(C, D)$, let ${}_A\langle X_\bullet, c^*\varrho\rangle_C$ and ${}_A\langle Y_\bullet, \varrho\rangle_D$ be distinguished n-exangles. Then $(\mathrm{id_A}, \mathrm{c})$ has a *good lift* $f_\bullet$ in the sense that its mapping cone gives a distinguished n-exangle $\langle M^f_\bullet, (d^X_0)_*\varrho\rangle$.

(EA2$^{\mathrm{op}}$) Dual of (EA2).

Note that in the case when $n = 1$, a triplet $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is a 1-exangulated category if and only if it is an extriangulated category, see [1], Proposition 4.3.

Example 2.9. From [1], Propositions 4.34 and 4.5 we know that n-exact categories and $(n+2)$-angulated categories are n-exangulated categories. There are some other examples of n-exangulated categories which are neither n-exact nor $(n+2)$-angulated, see [1], [2], [3], [6].

3. An equivalent characterization of the axiom (EA2)

In this section we introduce the definition of homotopy cartesian squares in pre-n-exangulated categories and provide an equivalent statement of axiom (EA2).

Definition 3.1. The commutative diagram

$$\begin{array}{ccccccccc}
X_0 & \xrightarrow{f_0} & X_1 & \xrightarrow{f_1} & \cdots & \xrightarrow{f_{n-2}} & X_{n-1} & \xrightarrow{f_{n-1}} & X_n \\
\downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi_1} & & & & \downarrow{\scriptstyle \varphi_{n-1}} & & \downarrow{\scriptstyle \varphi_n} \\
Y_0 & \xrightarrow{g_0} & Y_1 & \xrightarrow{g_1} & \cdots & \xrightarrow{g_{n-2}} & Y_{n-1} & \xrightarrow{g_{n-1}} & Y_n
\end{array}$$

in a pre-n-exangulated category $\mathscr{C}$ is called *homotopy cartesian* if there exists an $\mathbb{E}$-extension $\delta \in \mathbb{E}(Y_n, X_0)$ such that the pair

$$\langle X_\bullet, \delta\rangle\colon\ X_0 \xrightarrow{\begin{pmatrix} -f_0 \\ \varphi_0 \end{pmatrix}} X_1 \oplus Y_0 \xrightarrow{\begin{pmatrix} -f_1 & 0 \\ \varphi_1 & g_0 \end{pmatrix}} X_2 \oplus Y_1 \xrightarrow{\begin{pmatrix} -f_2 & 0 \\ \varphi_2 & g_1 \end{pmatrix}} \cdots$$
$$\cdots \xrightarrow{\begin{pmatrix} -f_{n-1} & 0 \\ \varphi_{n-1} & g_{n-2} \end{pmatrix}} X_n \oplus Y_{n-1} \xrightarrow{(\varphi_n\ g_{n-1})} Y_n \overset{\delta}{\dashrightarrow} \tag{3.1}$$

forms a distinguished n-exangle, where δ is called a *differential.*

Remark 3.2. The distinguished n-exangle (3.1) in the definition of homotopy cartesian diagram can also be written as follows:

$$X_0 \xrightarrow{\begin{pmatrix} -f_0 \\ \varphi_0 \end{pmatrix}} X_1 \oplus Y_0 \xrightarrow{\begin{pmatrix} f_1 & 0 \\ \varphi_1 & g_0 \end{pmatrix}} X_2 \oplus Y_1 \xrightarrow{\begin{pmatrix} f_2 & 0 \\ -\varphi_2 & g_1 \end{pmatrix}} \cdots \tag{3.2}$$

$$\cdots \xrightarrow{\begin{pmatrix} f_{n-1} & 0 \\ (-1)^n \varphi_{n-1} & g_{n-2} \end{pmatrix}} X_n \oplus Y_{n-1} \xrightarrow{\begin{pmatrix} (-1)^{n+1}\varphi_n & g_{n-1} \end{pmatrix}} Y_n \overset{\delta}{\dashrightarrow}$$

In fact, the two distinguished n-exangles are isomorphic:

$$\begin{array}{ccccccc}
X_0 & \xrightarrow{\begin{pmatrix} -f_0 \\ \varphi_0 \end{pmatrix}} & X_1 \oplus Y_0 & \xrightarrow{\begin{pmatrix} -f_1 & 0 \\ \varphi_1 & g_0 \end{pmatrix}} & X_2 \oplus Y_1 & \xrightarrow{\begin{pmatrix} -f_2 & 0 \\ \varphi_2 & g_1 \end{pmatrix}} & \cdots \\
\| & & \| & & \downarrow{\scriptstyle \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}} & & \\
X_0 & \xrightarrow{\begin{pmatrix} -f_0 \\ \varphi_0 \end{pmatrix}} & X_1 \oplus Y_0 & \xrightarrow{\begin{pmatrix} f_1 & 0 \\ \varphi_1 & g_0 \end{pmatrix}} & X_2 \oplus Y_1 & \xrightarrow{\begin{pmatrix} f_2 & 0 \\ -\varphi_2 & g_1 \end{pmatrix}} & \cdots
\end{array}$$

$$\begin{array}{ccccc}
\cdots & \xrightarrow{\begin{pmatrix} -f_{n-1} & 0 \\ \varphi_{n-1} & g_{n-2} \end{pmatrix}} & X_n \oplus Y_{n-1} & \xrightarrow{\begin{pmatrix} \varphi_n & g_{n-1} \end{pmatrix}} & Y_n \overset{\delta}{\dashrightarrow} \\
 & & \downarrow{\scriptstyle \begin{pmatrix} (-1)^{n+1} & 0 \\ 0 & 1 \end{pmatrix}} & & \| \\
\cdots & \xrightarrow{\begin{pmatrix} f_{n-1} & 0 \\ (-1)^n \varphi_{n-1} & g_{n-2} \end{pmatrix}} & X_n \oplus Y_{n-1} & \xrightarrow{\begin{pmatrix} (-1)^{n+1}\varphi_n & g_{n-1} \end{pmatrix}} & Y_n \overset{\delta}{\dashrightarrow} .
\end{array}$$

Lemma 3.3 ([1], Proposition 3.6)**.** *Let $\mathscr{C}$ be an n-exangulated category, ${}_A\langle X_\bullet, \delta\rangle_C$ and ${}_B\langle Y_\bullet, \varrho\rangle_D$ be distinguished n-exangles. Suppose that we are given a commutative square*

$$\begin{array}{ccc}
X_0 & \xrightarrow{d_0^X} & X_1 \\
{\scriptstyle a}\downarrow & \circlearrowright & \downarrow{\scriptstyle b} \\
Y_0 & \xrightarrow[d_0^Y]{} & Y_1
\end{array}$$

in $\mathscr{C}$. Then the following holds.

(1) *There is a morphism $f_\bullet \colon \langle X_\bullet, \delta\rangle \to \langle Y_\bullet, \varrho\rangle$ which satisfies $f_0 = a$ and $f_1 = b$.*

(2) *If $X_0 = Y_0 = A$ and $a = 1_A$ for some $A \in \mathscr{C}$, then the above $f_\bullet$ can be taken to give a distinguished n-exangle $\langle M_\bullet^f, (d_0^X)_*\varrho\rangle$.*

Theorem 3.4. *Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be a pre-n-exangulated category. Then $\mathscr{C}$ satisfies* (EA2) *if and only if $\mathscr{C}$ satisfies* (EA2-1)*:*

Let $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-2}} X_{n-1} \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1} \overset{\delta}{\dashrightarrow}$ and $X_0 \xrightarrow{g_0} Y_1 \xrightarrow{g_1} \dots \xrightarrow{g_{n-2}} Y_{n-1} \xrightarrow{g_{n-1}} Y_n \xrightarrow{g_n} Y_{n+1} \overset{\delta'}{\dashrightarrow}$ be distinguished n-exangles, and $\varphi_1\colon X_1 \to Y_1$ be a morphism such that $\varphi_1 f_0 = g_0$. Then there exist morphisms $\varphi_i\colon X_i \to Y_i$ for $2 \leqslant i \leqslant n+1$, which give a morphism of distinguished n-exangles

$$\begin{array}{ccccccccccc}
X_0 & \xrightarrow{f_0} & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & \cdots \longrightarrow & X_n & \xrightarrow{f_n} & X_{n+1} & \overset{\delta}{\dashrightarrow} \\
\| & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_{n+1}} & \\
X_0 & \xrightarrow{g_0} & Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & \cdots \longrightarrow & Y_n & \xrightarrow{g_n} & Y_{n+1} & \overset{\delta'}{\dashrightarrow}
\end{array}$$

and moreover,

$$\begin{array}{ccccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & X_{n+1} \\
\downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_{n+1}} \\
Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & Y_{n+1}
\end{array}$$

is a homotopy cartesian diagram and $(f_0)_*\delta'$ is the differential.

P r o o f. (EA2) $\Rightarrow$ (EA2-1) We know that Lemma 3.3 is a consequence of (R0) and (EA2) by [1], and Lemma 3.3 $\Rightarrow$ (EA2-1) is clear.

(EA2-1) $\Rightarrow$ (EA2) Since $(\mathrm{id}, \varphi_{\mathrm{n}+1})\colon (\varphi_{\mathrm{n}+1})^*\delta' \to \delta'$ is a morphism of $\mathbb{E}$-extensions, by (R0), there exist morphisms $\varphi_i'\colon X_i \to Y_i$ for $1 \leqslant i \leqslant n$ such that the following diagram is commutative:

$$\begin{array}{ccccccccccc}
X_0 & \xrightarrow{f_0} & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & \cdots \longrightarrow & X_n & \xrightarrow{f_n} & X_{n+1} & \overset{(\varphi_{n+1})^*\delta'}{\dashrightarrow} \\
\| & & \downarrow{\scriptstyle \varphi_1'} & & \downarrow{\scriptstyle \varphi_2'} & & & \downarrow{\scriptstyle \varphi_n'} & & \downarrow{\scriptstyle \varphi_{n+1}} & \\
X_0 & \xrightarrow{g_0} & Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & \cdots \longrightarrow & Y_n & \xrightarrow{g_n} & Y_{n+1} & \overset{\delta'}{\dashrightarrow} .
\end{array}$$

By (EA2-1), there exist morphisms $\varphi_i''\colon X_i \to Y_i$ for $2 \leqslant i \leqslant n+1$, which give a morphism of distinguished n-exangles

$$\begin{array}{ccccccccccc}
X_0 & \xrightarrow{f_0} & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & \cdots \longrightarrow & X_n & \xrightarrow{f_n} & X_{n+1} & \overset{(\varphi_{n+1})^*\delta'}{\dashrightarrow} \\
\| & & \downarrow{\scriptstyle \varphi_1'} & & \downarrow{\scriptstyle \varphi_2''} & & & \downarrow{\scriptstyle \varphi_n''} & & \downarrow{\scriptstyle \varphi_{n+1}''} & \\
X_0 & \xrightarrow{g_0} & Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & \cdots \longrightarrow & Y_n & \xrightarrow{g_n} & Y_{n+1} & \overset{\delta'}{\dashrightarrow}
\end{array}$$

and moreover,

$$\begin{array}{ccccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & X_{n+1} \\
\downarrow{\scriptstyle \varphi_1'} & & \downarrow{\scriptstyle \varphi_2''} & & & & \downarrow{\scriptstyle \varphi_n''} & & \downarrow{\scriptstyle \varphi_{n+1}''} \\
Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & Y_{n+1}
\end{array}$$

is a homotopy cartesian diagram and $(f_0)_*\delta'$ is the differential.

Since $(\varphi_{n+1})^*\delta' = (\varphi''_{n+1})^*\delta'$, that is, $(\varphi_{n+1} - \varphi''_{n+1})^*\delta' = 0$, then by the exactness of

$$\mathscr{C}(X_{n+1}, Y_n) \xrightarrow{\mathscr{C}(X_{n+1}, g_n)} \mathscr{C}(X_{n+1}, Y_{n+1}) \xrightarrow{\delta'_\sharp} \mathbb{E}(X_{n+1}, X_0)\ ,$$

there is a morphism $h_{n+1}\colon X_{n+1} \to Y_n$ which gives $g_n h_{n+1} = \varphi_{n+1} - \varphi''_{n+1}$.

Since $g_n(\varphi'_n - \varphi''_n - h_{n+1}f_n) = g_n\varphi'_n - g_n\varphi''_n - g_n h_{n+1} f_n = g_n\varphi'_n - g_n\varphi''_n - \varphi_{n+1}f_n + \varphi''_{n+1}f_n = 0$, then by the exactness of

$$\mathscr{C}(X_n, Y_{n-1}) \xrightarrow{\mathscr{C}(X_n, g_{n-1})} \mathscr{C}(X_n, Y_n) \xrightarrow{\mathscr{C}(X_n, g_n)} \mathscr{C}(X_n, Y_{n+1})\ ,$$

there is a morphism $h_n\colon X_n \to Y_{n-1}$ which gives $g_{n-1}h_n = \varphi'_n - \varphi''_n - h_{n+1}f_n$.

Inductively, for $i = 2, 3, \ldots, n-1$, we obtain $h_i\colon X_i \to Y_{i-1}$ such that $g_{i-1}h_i = \varphi'_i - \varphi''_i - h_{i+1}f_i$. In particular, $g_1(\varphi'_1 - \varphi'_1 - h_2 f_1) = -g_1 h_2 f_1 = -\varphi'_2 f_1 + \varphi''_2 f_1 + h_3 f_2 f_1 = -g_1\varphi'_1 + g_1\varphi'_1 = 0$, and hence there exist $h_1\colon X_1 \to Y_0$ which gives $g_0 h_1 = -h_2 f_1$.

Set

$$\varphi_i = \begin{cases} \varphi'_i = \varphi''_i + h_{i+1}f_i + g_{i-1}h_i & \text{if } i = 2, 3, \ldots, n, \\ \varphi'_i + h_{i+1}f_i & \text{if } i = 1, \end{cases}$$

then $\varphi_1 f_0 = (\varphi'_1 + h_2 f_1)f_0 = \varphi'_1 f_0 = g_0$, $\varphi_2 f_1 = (\varphi''_2 + h_3 f_2 + g_1 h_2)f_1 = \varphi''_2 f_1 + g_1 h_2 f_1 = g_1\varphi'_1 + g_1 h_2 f_1 = g_1(\varphi'_1 + h_2 f_1) = g_1\varphi_1$. The commutative diagram

$$\begin{array}{ccccccc}
X_1 & \xrightarrow{\left(\begin{smallmatrix} -f_1 \\ \varphi'_1 \end{smallmatrix}\right)} & X_2 \oplus Y_1 & \xrightarrow{\left(\begin{smallmatrix} -f_2 & 0 \\ \varphi''_2 & g_1 \end{smallmatrix}\right)} & X_3 \oplus Y_2 & \xrightarrow{\left(\begin{smallmatrix} -f_3 & 0 \\ \varphi''_3 & g_2 \end{smallmatrix}\right)} & \cdots \\
\Big\| & & \Big\downarrow{\scriptstyle \left(\begin{smallmatrix} 1 & 0 \\ -h_2 & 1 \end{smallmatrix}\right)} & & \Big\downarrow{\scriptstyle \left(\begin{smallmatrix} 1 & 0 \\ -h_3 & 1 \end{smallmatrix}\right)} & & \\
X_1 & \xrightarrow{\left(\begin{smallmatrix} -f_1 \\ \varphi_1 \end{smallmatrix}\right)} & X_2 \oplus Y_1 & \xrightarrow{\left(\begin{smallmatrix} -f_2 & 0 \\ \varphi_2 & g_1 \end{smallmatrix}\right)} & X_3 \oplus Y_2 & \xrightarrow{\left(\begin{smallmatrix} -f_3 & 0 \\ \varphi_3 & g_2 \end{smallmatrix}\right)} & \cdots
\end{array}$$

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{\left(\begin{smallmatrix} -f_{n-1} & 0 \\ \varphi''_{n-1} & g_{n-2} \end{smallmatrix}\right)} & X_n \oplus Y_{n-1} & \xrightarrow{\left(\begin{smallmatrix} -f_n & 0 \\ \varphi''_n & g_{n-1} \end{smallmatrix}\right)} & X_{n+1} \oplus Y_n & \xrightarrow{\left(\begin{smallmatrix} \varphi''_{n+1} & g_n \end{smallmatrix}\right)} & Y_{n+1} \\
 & & \Big\downarrow{\scriptstyle \left(\begin{smallmatrix} 1 & 0 \\ -h_n & 1 \end{smallmatrix}\right)} & & \Big\downarrow{\scriptstyle \left(\begin{smallmatrix} 1 & 0 \\ -h_{n+1} & 1 \end{smallmatrix}\right)} & & \Big\| \\
\cdots & \xrightarrow{\left(\begin{smallmatrix} -f_{n-1} & 0 \\ \varphi_{n-1} & g_{n-2} \end{smallmatrix}\right)} & X_n \oplus Y_{n-1} & \xrightarrow{\left(\begin{smallmatrix} -f_n & 0 \\ \varphi_n & g_{n-1} \end{smallmatrix}\right)} & X_{n+1} \oplus Y_n & \xrightarrow{\left(\begin{smallmatrix} \varphi_{n+1} & g_n \end{smallmatrix}\right)} & Y_{n+1}
\end{array}$$

implies that

$$\begin{array}{l} X_1 \xrightarrow{\begin{pmatrix} -f_1 \\ \varphi_1 \end{pmatrix}} X_2 \oplus Y_1 \xrightarrow{\begin{pmatrix} -f_2 & 0 \\ \varphi_2 & g_1 \end{pmatrix}} X_3 \oplus Y_2 \xrightarrow{\begin{pmatrix} -f_3 & 0 \\ \varphi_3 & g_2 \end{pmatrix}} \cdots \\ \cdots \xrightarrow{\begin{pmatrix} -f_{n-1} & 0 \\ \varphi_{n-1} & g_{n-2} \end{pmatrix}} X_n \oplus Y_{n-1} \xrightarrow{\begin{pmatrix} -f_n & 0 \\ \varphi_n & g_{n-1} \end{pmatrix}} X_{n+1} \oplus Y_n \xrightarrow{(\varphi_{n+1}\ g_n)} Y_{n+1} \overset{(f_0)_*\delta'}{\dashrightarrow} \end{array}$$

is a distinguished n-exangle.

Consequently, $(\mathrm{id}_{X_0}, \varphi_{n+1})$ has a good lift $(\mathrm{id}_{X_0}, \varphi_1, \varphi_2, \ldots, \varphi_n, \varphi_{n+1})$. That is to say (EA2) holds. □

Corollary 3.5. *In Theorem* 3.4*, when* $n = 1$*, it is just the dual of Theorem* 3.3 *in* [4].

4. Closed subfunctors arising from half exact functors

In this section, we introduce methods of constructing closed subfunctors of an n-exangulated category $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ from half exact functors. Let us start with the following key lemma.

Lemma 4.1 ([1], Proposition 3.16)**.** *Let* $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ *be an* n*-exangulated category. For any additive subfunctor* $\mathbb{F} \subseteq \mathbb{E}$*, the following statements are equivalent.*
(1) $(\mathscr{C}, \mathbb{F}, \mathfrak{s}|_{\mathbb{F}})$ *is an* n*-exangulated category.*
(2) $\mathfrak{s}|_{\mathbb{F}}$*-inflations are closed under composition.*
(3) $\mathfrak{s}|_{\mathbb{F}}$*-deflations are closed under composition.*

We call an additive subfunctor $\mathbb{F} \subseteq \mathbb{E}$ *a closed subfunctor if it satisfies the above equivalent conditions. In this case, we call* $(\mathscr{C}, \mathbb{F}, \mathfrak{s}|_{\mathbb{F}})$ *a relative theory of* $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$*. For a relative theory* $(\mathscr{C}, \mathbb{F}, \mathfrak{s}|_{\mathbb{F}})$*, we briefly denote it by* $(\mathscr{C}, \mathbb{F})$*.*

Definition 4.2. Let $\mathscr{A}$ be an abelian category. An additive functor $F\colon \mathscr{C} \to \mathscr{A}$ is called a half exact functor if the sequence

$$FX_0 \xrightarrow{Ff_0} FX_1 \xrightarrow{Ff_1} \cdots \xrightarrow{Ff_{n-2}} FX_{n-1} \xrightarrow{Ff_{n-1}} FX_n \xrightarrow{Ff_n} FX_{n+1}$$

is exact for any distinguished n-exangle $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \ldots \xrightarrow{f_{n-2}} X_{n-1} \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1} \overset{\delta}{\dashrightarrow}$. Moreover, if Ff_n (or Ff_0) is an epimorphism (or monomorphism), we call F a right exact functor (or left exact functor).

Remark 4.3. If the category $\mathscr{C}$ is extriangulated, then Definition 4.2 coincides with the definition of half exact functor (homological functor) of extriangulated categories, cf. [5], [8].

Definition 4.4. Let $F\colon \mathscr{C} \to \mathscr{A}$ be a half exact functor. We define a subset $\mathbb{E}_R^F(X_{n+1}, X_0)$ of $\mathbb{E}(X_{n+1}, X_0)$ consisting of δ such that for any distinguished n-exangle $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-2}} X_{n-1} \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1} \overset{\delta}{\dashrightarrow}$, we have that Ff_n is an epimorphism in $\mathscr{A}$. Similarly, we define a subset $\mathbb{E}_L^F(X_{n+1}, X_0)$ of $\mathbb{E}(X_{n+1}, X_0)$ consisting of δ such that Ff_0 is a monomorphism in $\mathscr{A}$.

Note that the above definition is well-defined, that is, it does not depend on the choice of a distinguished n-exangle of δ. Moreover, $\mathbb{E}_R^F(X_{n+1}, X_0)$ defines the maximum closed subfunctor such that F becomes a right exact functor.

Theorem 4.5. *Let $F\colon \mathscr{C} \to \mathscr{A}$ be a half exact functor. Then the following statements hold.*

(1) *$\mathbb{E}_R^F$ is a closed subfunctor of $\mathbb{E}$, hence $(\mathscr{C}, \mathbb{E}_R^F)$ is a relative theory of $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$.*
(2) *F restricts to a right exact functor $F\colon (\mathscr{C}, \mathbb{E}_R^F) \to \mathscr{A}$.*
(3) *Let $(\mathscr{C}, \mathbb{F})$ be a relative theory of $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$. If F restricts to a right exact functor $F\colon (\mathscr{C}, \mathbb{F}) \to \mathscr{A}$, then we have $\mathbb{F} \subseteq \mathbb{E}_R^F$.*

Proof. (1) First of all, we claim that $\mathbb{E}_R^F$ is a subfunctor of $\mathbb{E}$. In fact, for any $\delta \in \mathbb{E}_R^F(X_{n+1}, X_0)$ and any morphism $a_0\colon X_0 \to Y_0$, by (EA2$^{\rm op}$), we obtain the following commutative diagram:

$$
\begin{array}{ccccccccccc}
X_0 & \xrightarrow{f_0} & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & \cdots \longrightarrow & X_n & \xrightarrow{f_n} & X_{n+1} & \overset{\delta}{\dashrightarrow} \\
\downarrow{\scriptstyle a_0} & & \downarrow{\scriptstyle a_1} & & \downarrow{\scriptstyle a_2} & & & \downarrow{\scriptstyle a_n} & & \| & \\
Y_0 & \xrightarrow{g_0} & Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & \cdots \longrightarrow & Y_n & \xrightarrow{g_n} & X_{n+1} & \overset{a_0\delta}{\dashrightarrow} .
\end{array}
$$

Applying F to the above diagram, we have the following commutative diagram with exact rows in $\mathscr{A}$:

$$
\begin{array}{cccccccccc}
FX_0 & \xrightarrow{Ff_0} & X_1 & \xrightarrow{Ff_1} & FX_2 & \xrightarrow{Ff_2} & \cdots \longrightarrow & FX_n & \xrightarrow{Ff_n} & FX_{n+1} \\
\downarrow{\scriptstyle Fa_0} & & \downarrow{\scriptstyle Fa_1} & & \downarrow{\scriptstyle Fa_2} & & & \downarrow{\scriptstyle Fa_n} & & \| \\
FY_0 & \xrightarrow{Fg_0} & FY_1 & \xrightarrow{Fg_1} & FY_2 & \xrightarrow{Fg_2} & \cdots \longrightarrow & FY_n & \xrightarrow{Fg_n} & FX_{n+1}.
\end{array}
$$

Since Ff_n is epimorphic, then Fg_n is also epimorphic. Consequently, we find that $a_0\delta \in \mathbb{E}_R^F(X_{n+1}, Y_0)$.

For any $\delta \in \mathbb{E}_R^F(X_{n+1}, X_0)$ and any morphism $a_{n+1} \colon Y_{n+1} \to X_{n+1}$, by (EA2), we obtain the commutative diagram

$$\begin{array}{ccccccccccc}
X_0 & \xrightarrow{g_0} & Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & \cdots \longrightarrow & Y_n & \xrightarrow{g_n} & Y_{n+1} & \overset{\delta a_{n+1}}{\dashrightarrow} \\
\| & & \downarrow{\scriptstyle a_1} & & \downarrow{\scriptstyle a_2} & & & \downarrow{\scriptstyle a_n} & & \downarrow{\scriptstyle a_{n+1}} & \\
X_0 & \xrightarrow{f_0} & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & \cdots \longrightarrow & X_n & \xrightarrow{f_n} & X_{n+1} & \overset{\delta}{\dashrightarrow}
\end{array}$$

such that

$$Y_1 \xrightarrow{\begin{pmatrix} -g_1 \\ a_1 \end{pmatrix}} Y_2 \oplus X_1 \xrightarrow{\begin{pmatrix} -g_2 & 0 \\ a_2 & f_1 \end{pmatrix}} Y_3 \oplus X_2 \xrightarrow{\begin{pmatrix} -g_3 & 0 \\ a_3 & f_2 \end{pmatrix}} \cdots$$

$$\cdots \xrightarrow{\begin{pmatrix} -g_{n-1} & 0 \\ a_{n-1} & f_{n-2} \end{pmatrix}} Y_n \oplus X_{n-1} \xrightarrow{\begin{pmatrix} -g_n & 0 \\ a_n & f_{n-1} \end{pmatrix}} Y_{n+1} \oplus X_n \xrightarrow{(\, a_{n+1} \ f_n \,)} X_{n+1} \overset{g_0\delta}{\dashrightarrow}$$

is a distinguished n-exangle. Note that $g_0\delta \in \mathbb{E}_R^F(X_{n+1}, Y_1)$ by the previous argument, we have the following exact sequence in it:

$$FY_1 \xrightarrow{F\begin{pmatrix} -g_1 \\ a_1 \end{pmatrix}} FY_2 \oplus FX_1 \xrightarrow{F\begin{pmatrix} -g_2 & 0 \\ a_2 & f_1 \end{pmatrix}} FY_3 \oplus FX_2 \xrightarrow{F\begin{pmatrix} -g_3 & 0 \\ a_3 & f_2 \end{pmatrix}} \cdots$$

$$\cdots \xrightarrow{F\begin{pmatrix} -g_{n-1} & 0 \\ a_{n-1} & f_{n-2} \end{pmatrix}} FY_n \oplus FX_{n-1} \xrightarrow{F\begin{pmatrix} -g_n & 0 \\ a_n & f_{n-1} \end{pmatrix}} FY_{n+1} \oplus FX_n \xrightarrow{F(\, a_{n+1} \ f_n \,)} FX_{n+1} \longrightarrow 0.$$

In order to prove that Fg_n is an epimorphism in $\mathscr{A}$, suppose that $k \colon FY_{n+1} \to M$ is a morphism in $\mathscr{A}$ satisfying $k \circ Fg_n = 0$. Let $(k, 0) \in \mathscr{C}(FY_{n+1} \oplus FX_n, M)$. Since $k \circ Fg_n = 0$, then $(k, 0)\begin{pmatrix} -Fg_n & 0 \\ Fa_n & Ff_{n-1} \end{pmatrix} = (-kFg_n, 0) = 0$. Thus, there exists a unique morphism $s \colon FX_{n+1} \to M$ such that $s \circ (Fa_{n+1}, Ff_n) = (k, 0)$. So $s \circ Fa_{n+1} = k$ and $s \circ Ff_n = 0$. That is to say, we have the following commutative diagram:

$$\begin{array}{ccccccc}
FY_n \oplus FX_{n-1} & \xrightarrow{\begin{pmatrix} -Fg_n & 0 \\ Fa_n & Ff_{n-1} \end{pmatrix}} & FY_{n+1} \oplus FX_n & \xrightarrow{(\, Fa_{n+1} \ Ff_n \,)} & FX_{n+1} & \longrightarrow & 0. \\
& \searrow{\scriptstyle 0} & \downarrow{\scriptstyle (k,0)} & \swarrow{\scriptstyle s} & & & \\
& & M & & & &
\end{array}$$

Note that Ff_n is an epimorphism, so we have $s = 0$. Then $k = s \circ Fa_{n+1} = 0$. This shows that Fg_n is an epimorphism in $\mathscr{A}$, hence we have $\delta a_{n+1} \in \mathbb{E}_R^F$. Thus, $\mathbb{E}_R^F$ is a subfunctor of $\mathbb{E}$.

Next, we need to show that $\mathbb{E}_R^F(X_{n+1}, X_0)$ is a subgroup of $\mathbb{E}(X_{n+1}, X_0)$. Note that $0 \in \mathbb{E}_R^F(X_{n+1}, X_0)$, we only need to show $\delta' - \delta \in \mathbb{E}_R^F(X_{n+1}, X_0)$ for any $\delta', \delta \in \mathbb{E}_R^F(X_{n+1}, X_0)$. For $(-\mathrm{id}, \mathrm{id}) \colon \mathrm{X}_0 \oplus \mathrm{X}_0 \to \mathrm{X}_0$ and $\begin{pmatrix} \mathrm{id} \\ \mathrm{id} \end{pmatrix} \colon X_{n+1} \oplus X_{n+1} \to$

X_{n+1}, we have $\delta' - \delta = (-\mathrm{id}, \mathrm{id})(\delta \oplus \delta')\begin{pmatrix}\mathrm{id}\\ \mathrm{id}\end{pmatrix}$; it is enough to show that $\delta \oplus \delta' \in \mathbb{E}_R^F(X_{n+1} \oplus X_{n+1}, X_0 \oplus X_0)$, and this follows from Proposition 3.2 in [1]. Thus, $\mathbb{E}_R^F$ is an additive subfunctor of $\mathbb{E}$.

Finally, let $X_n \xrightarrow{f_n} X_{n+1} \xrightarrow{g_n} Y_{n+1}$ be any sequence of morphisms in $\mathscr{C}$. Assume that f_n and g_n are $\mathfrak{s}|_{\mathbb{E}_R^F}$-deflations. By (EA1), we know that $g_n \circ f_n$ is an $\mathfrak{s}$-deflation. Thus, we assume

$$\mathfrak{s}(\delta) = [Z_0 \xrightarrow{h_0} Z_1 \xrightarrow{h_1} Z_2 \xrightarrow{h_2} \dots \xrightarrow{h_{n-2}} Z_{n-1} \xrightarrow{h_{n-1}} X_n \xrightarrow{g_n f_n} Y_{n+1}].$$

Note that Fg_n and Ff_n are epimorphic, then $F(g_n f_n)$ is an epimorphism, so $\delta \in \mathbb{E}_R^F$. This shows that $\mathfrak{s}|_{\mathbb{E}_R^F}$-deflations are closed under composition. Thus, $\mathbb{E}_R^F$ is a closed subfunctor of $\mathbb{E}$.

(2) and (3) follow immediately from the construction of $\mathbb{E}_R^F$. □

Remark 4.6. Dually, we obtain the statement with respect to $\mathbb{E}_L^F$ by applying the above proposition to a contravariant half exact functor $F\colon \mathscr{C} \to \mathscr{A}^{\mathrm{op}}$.

Example 4.7. Let $\mathscr{H} \subseteq \mathscr{C}$ be a full subcategory. Define subfunctors $\mathbb{E}_{\mathscr{H}}$ and $\mathbb{E}^{\mathscr{H}}$ of $\mathbb{E}$ by

$$\mathbb{E}_{\mathscr{H}}(X_{n+1}, X_0) = \{\delta \in \mathbb{E}(X_{n+1}, X_0)\colon (\delta_\sharp)_H = 0 \text{ for any } H \in \mathscr{H}\},$$
$$\mathbb{E}^{\mathscr{H}}(X_{n+1}, X_0) = \{\delta \in \mathbb{E}(X_{n+1}, X_0)\colon (\delta^\sharp)_H = 0 \text{ for any } H \in \mathscr{H}\}.$$

In Proposition 3.17 of [1], it is shown that $\mathbb{E}_{\mathscr{H}}$ and $\mathbb{E}^{\mathscr{H}}$ are closed subfunctors of $\mathbb{E}$. They are special cases of Theorem 4.5. Since the restricted Yoneda functors $Y_{\mathscr{H}}\colon \mathscr{C} \to \mathrm{Mod}\,\mathscr{H}$ and $Y^{\mathscr{H}}\colon \mathscr{C} \to \mathrm{Mod}\,\mathscr{H}^{\mathrm{op}}$ are half exact functors, moreover, $\mathbb{E}_{\mathscr{H}} = \mathbb{E}_R^{Y_{\mathscr{H}}}$ and $\mathbb{E}^{\mathscr{H}} = \mathbb{E}_L^{Y^{\mathscr{H}}}$ hold. In fact, for any distinguished n-exangle

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-2}} X_{n-1} \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1} \overset{\delta}{\dashrightarrow},$$

we have the following exact sequence:

$$\mathscr{C}(-, X_0)|_{\mathscr{H}} \xrightarrow{Y_{\mathscr{H}}(f_0)} \cdots \xrightarrow{Y_{\mathscr{H}}(f_{n-1})} \mathscr{C}(-, X_n)|_{\mathscr{H}} \xrightarrow{Y_{\mathscr{H}}(f_n)} \mathscr{C}(-, X_{n+1})|_{\mathscr{H}} \xrightarrow{\delta_\sharp} \mathbb{E}(-, X_0)|_{\mathscr{H}}.$$

Thus, we have $\mathbb{E}_R^{Y_{\mathscr{H}}}(X_{n+1}, X_0) = \mathbb{E}_{\mathscr{H}}(X_{n+1}, X_0)$.

Corollary 4.8. *In Theorem* 4.5*, when* $n = 1$*, it is just Proposition A in* [9].

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