Soufiane Abid; Khalid Atifi; El-Hassan Essoufi; Abderrahim Zafrar
Identification of source term in a nonlinear degenerate parabolic equation with memory

Applications of Mathematics, Vol. 69 (2024), No. 2, 209–232

Persistent URL: `http://dml.cz/dmlcz/152313`

IDENTIFICATION OF SOURCE TERM IN A NONLINEAR DEGENERATE PARABOLIC EQUATION WITH MEMORY

Soufiane Abid, Khalid Atifi, El-Hassan Essoufi, Settat,
Abderrahim Zafrar, Fez

Received February 25, 2023. Published online February 7, 2024.

Abstract. In this work, we consider an inverse backward problem for a nonlinear parabolic equation of the Burgers' type with a memory term from final data. To this aim, we first establish the well-posedness of the direct problem. On the basis of the optimal control framework, the existence and necessary condition of the minimizer for the cost functional are established. The global uniqueness and stability of the minimizer are deduced from the necessary condition. Numerical experiments demonstrate the effectiveness of this approach.

Keywords: inverse source problem; nonlinear parabolic equation; memory term; optimal control

MSC 2020: 35R30, 35K55, 49N45

1. Introduction

The nonlinear degenerate Burgers equations are a type of degenerate parabolic equations that describe nonlinear wave motions in a continuous medium (see [22], and [13]). These equations, first introduced by Johannes Martinus Burgers in 1948 (see [6]), were originally used to describe turbulent motion in fluids, but they can be also applied to other physical phenomena such as shock waves in plasmas (see [19], [5], [7]), gravity waves, and nonlinear deformations in solids (see [8], [4], [21]). Solving these nonlinear PDEs is challenging due to their degenerate nature. However, various numerical and analytical techniques have been developed to address these difficulties (see [15], [12], [14], [9], [3]). These types of equations are becoming increasingly prevalent in various fields of physics and engineering to describe nonlinear phenomena.

A parabolic equation with a memory term is a form of partial differential equation that describes the time evolution of a system based not only on its current state

DOI: 10.21136/AM.2024.0049-23

but also its past states. The memory term is typically represented by a convolution integral, which accounts for the cumulative effect of past states on the current state of the system. Systems that can be modelled using parabolic equations with memory terms include diffusion processes with time-dependent diffusivity, viscoelastic materials, and biological systems with memory. These types of equations are often difficult to solve analytically and numerical methods are frequently used to obtain approximate solutions.

The addition of a memory term to a nonlinear Burgers equation allows for the consideration of memory effects in the description of the temporal evolution of a system. This model is commonly used to model processes such as diffusion with time-dependent diffusivity, viscoelasticity of materials, and biological systems with memory. However, the fine time resolution required to capture memory effects can lead to stability issues when using numerical methods.

One solution to overcome the difficulties of numerically solving the Burgers equation with a memory term is to utilize deep learning-based schemes, such as deep neural networks. These have been shown to be highly effective in solving complex data processing problems, including prediction and classification tasks. They can be used to approximate numerical solutions of the Burgers equation with a memory term using simulated or experimental training data, and address the stability issues related to the fine temporal resolution required to capture memory effects. Deep neural networks can be trained to capture complex nonlinear relationships between input and output data, thus solving problems that are difficult to be solved by classical analytical or numerical methods.

To sum up the use of deep learning-based schemes, such as deep neural networks, can effectively overcome the difficulties associated with numerically solving the nonlinear Burgers equation with a memory term. These techniques are powerful and can be applied to solving complex problems in various fields.

The problem of identifying the parameters of such equations is of great importance for understanding how physical phenomena are generated and how they evolve in time and space. It is commonly used in many fields such as meteorology, oceanography, geophysics, plasma physics, fluid mechanics and civil engineering, to name but a few.

Solving this inverse problem also allows the development of prediction and control systems for physical phenomena, which has practical applications in many fields such as natural hazard management, environmental management and performance optimisation of industrial systems.

This explains the interest of many researchers in this subject, where we find many research papers in this matter.

An example of work on this topic is paper [2]. In this paper, the authors have considered a problem of identification of the source function in the Burgers type equation, given by

$$u_t(t,x) = \mu(t)u_{xx} + A(t)uu_x + B(t)u + C(t) + g(t)f(t,x). \tag{1.1}$$

Since the problem is studied in the Cauchy and boundary cases, they obtained sufficient conditions of existence and uniqueness of the solution.

Another example is papers [16] and [17], Leonenko et al. The study presented in these papers focuses on the use of classical statistical inference methods to analyze the solutions of the nonlinear diffusion equation, specifically in the context of the Burgers turbulence problem. The study aims to solve the parameter identification problem for Burgers flows using techniques such as parabolic rescaling, averaging and discretization. The solutions studied are long memory random sets and are generated from random initial data.

Let us also refer to [16], where the authors have considered the problems of parameter identification for a generalized Burgers equation. The identification scheme they used here is the quasilinearization method. Based on the development of the differentiability of the solution with respect to the unknown parameters, they have shown the existence and local convergence of the algorithm. They also presented some numerical examples that demonstrate the performance of the quasilinearization scheme.

In the present paper, we will study the inverse source problem in the following equation:

$$u_t(x,t) - (a(x)u_x)_x(x,t) + b(x)uu_x = \int_0^t k(x,s)u(x,s)\,\mathrm{d}s + f(x,t) \quad \forall\,(x,t) \in Q, \tag{1.2}$$

where $\Omega := (0,1)$, $Q := \Omega \times (0,T)$, $T > 0$ is a fixed moment of time, $k \in L^\infty(Q)$ and $f \in L^2(Q)$ is the source term, with the initial condition

$$u(x,0) = u_0(x) \quad \forall\,x \in \Omega, \tag{1.3}$$

where $u_0 \in L^2(\Omega)$. We consider the following homogeneous boundary conditions:

$$u(0,t) = u(1,t) = 0 \quad \forall\,t \in (0,T). \tag{1.4}$$

We suppose that $f(x,t) = h(x)R(x,t)$, where $h \in L^2(\Omega)$ and $R \in L^\infty(Q)$ is a known function. Furthermore, $a, b\colon [0,1] \mapsto \mathbb{R}$ are given functions with

$$a, b \in C^0([0,1]) \cap C^1((0,1]), \quad a, b > 0 \quad \text{in } (0,1], \quad b \leqslant C\sqrt{a}. \tag{1.5}$$

We will assume that

$$xa_x \leqslant Ka \quad \forall\,x \in [0,1] \tag{1.6}$$

for some $K \in [0,1)$; it will be said that we are in the weak degenerate case. We give an example of functions a and b as follows:

$$a(x) \equiv x^m, \quad b(x) \equiv x^\beta, \quad \text{with } 0 < m \leqslant 2\beta,\ m \in (0,1).$$

The remainder of this paper is briefly outlined as follows. Section 2 is devoted to the main results, we state the well-posedness of the direct problem in the first part. We show the existence and uniqueness of the solution using Gronwall's Lemma and Aubin-Lions' Lemma. The second part is reserved to the inverse source problem. We prove the Lipschitz continuity of the cost function and we provide the necessary optimality conditions in Section 3. Also, global uniqueness of the backward solution and its stability are provided in Section 4. In Section 5, we conclude the paper with some numerical experiments using neural network procedure.

2. Main results

2.1. Well-posedness of the direct problem. In this subsection, we will show the well-posedness of the direct problem. For this aim, we consider the following linear associated problem:

$$\begin{cases} u_t(x,t) - (a(x)u_x(x,t))_x + B(x,t)b(x,t)u_x(x,t) \\ \qquad = \displaystyle\int_0^t k(x,s)u(x,s)\,\mathrm{d}s + f(x,t), \quad (x,t) \in Q, \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{cases} \tag{2.1}$$

where $u_0 \in L^2(\Omega)$, $B \in L^\infty(Q)$ and $f \in L^\infty(Q)$.

We introduce the following weighted spaces:

$$\begin{aligned} H_a^1 &:= \{u \in L^2(\Omega)\colon\ u \text{ is absolutely continuous in } [0,1], \\ &\qquad\qquad \sqrt{a}u_x \in L^2(\Omega), u(0) = u(1) = 0\}, \\ H_a^2 &:= \{u \in H_a^1(\Omega)\colon\ au_x \in H^1(\Omega), u(0) = u(1) = 0\}. \end{aligned}$$

These spaces are endowed with the Hilbert norms

$$\|u\|_{H_a^1}^2 := \|u\|_2^2 + \|\sqrt{a}u_x\|_2^2, \quad \|u\|_{H_a^2}^2 := \|u\|_{H_a^1}^2 + \|(au_x)_x\|_2^2.$$

Proposition 2.1. *Assume that $u_0 \in L^2(\Omega)$ and $f \in L^2(Q)$, system* (2.1) *admits a unique solution u such that*

$$u \in W_T := L^2(0,T; H_a^1(\Omega)) \cap C([0,T]; L^2(\Omega)),$$

$$\sup_{t\in[0,T]} \|u(\cdot,t)\|_2^2 + \int_0^T \|\sqrt{a}u_x(\cdot,t)\|_2^2\,\mathrm{d}t \leqslant C_T(\|u_0\|_2^2 + \|f\|_{L^2(Q)}^2). \tag{2.2}$$

In addition, if $u_0 \in H^2_a(\Omega)$, one also has

$$
(2.3)\qquad \begin{aligned} &u \in \mathcal{U} := H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2_a(\Omega)) \cap C([0,T];H^1_a(\Omega)),\\ &\sup_{t\in[0,T]} \|u(\cdot,t)\|^2_{H^1_a(\Omega)} + \int_0^T (\|u_t(\cdot,t)\|_2^2 + \|(au_x)_x(\cdot,t)\|_2^2)\,\mathrm{d}t\\ &\qquad\qquad \leqslant C(\|u_0\|^2_{H^1_a(\Omega)} + \|f\|^2_{L^2(Q)}),\end{aligned}
$$

where the constant C depends on T, Ω, a, b and $\|B\|_{L^\infty(Q)}$.

Proof. The proof of this proposition is based on the results obtained in [1] and [11], Theorem 1.1.

First of all, we transform (2.1) into the following Cauchy problem:

$$
(2.4)\qquad \begin{cases} u'(t) + Au(t) = \displaystyle\int_0^t \varrho(x,s,u(s))\,\mathrm{d}s + f(t),\\ u(0) = u_0, \end{cases}
$$

where

$$
Au := -(au_x)_x + Bbu, \quad u \in D(A) := H^2_a(\Omega)
$$

and

$$
\varrho(x,s,u(s)) := k(x,s)u(x,s), \quad (x,s) \in Q.
$$

Now, we will check that (2.4) satisfies the assumptions of the above theorem. For this purpose, let $H^{-1}_a(\Omega)$ be the dual space of $H^1_a(\Omega)$ with respect to the pivot space $L^2(\Omega)$, endowed with the natural norm

$$
\|v\|_{H^{-1}_a(\Omega)} := \sup_{\|u\|_{H^1_a(\Omega)}=1} \langle v,u\rangle_{H^{-1}_a(\Omega),H^1_a(\Omega)}.
$$

For any $u \in H^1_a(\Omega)$ we have

$$
\langle Au,v\rangle_{H^{-1}_a(\Omega),H^1_a(\Omega)} = \int_0^1 au_xv_x\,\mathrm{d}x + \int_0^1 Bbuv\,\mathrm{d}x \quad \forall\, v \in H^1_a(\Omega)
$$

and

$$
\langle \varrho(x,s,u(s)),v\rangle_{H^{-1}_a(\Omega),H^1_a(\Omega)} = \int_0^1 k(x,s)uv\,\mathrm{d}x \quad \text{for a.e. } (x,s)\in Q \quad \forall\, v \in H^1_a(\Omega).
$$

Therefore, we can easily verify that the operators A and k satisfy the following properties:

(a) there exists a positive constant C such that $\|Au\|_{H^{-1}_a(\Omega)} \leqslant C\|u\|_{H^1_a(\Omega)}$ for all $u \in H^1_a(\Omega)$;

(b) there exists a positive constant C such that

$$
\|Au_1 - Au_2\|_{H^{-1}_a(\Omega)} \leqslant C\|u_1 - u_2\|_{H^1_a(\Omega)} \quad \text{for any } u_1,u_2 \in H^1_a(\Omega);
$$

(c) there exists $\lambda > 0$ and $\gamma > 0$ such that

$$\langle Au_1 - Au_2, u_1 - u_2\rangle_{H_a^{-1}(\Omega), H_a^1(\Omega)} + \lambda\|u_1 - u_2\|^2_{L^2(\Omega)} \geqslant \gamma\|u_1 - u_2\|^2_{H_a^1(\Omega)}$$

for any u_1 and u_2 in $H_a^1(\Omega)$;

(d) there exists a function $\beta\colon\ (0,T) \mapsto \mathbb{R}^+$ such that

$$\|\varrho(x,s,u_1) - \varrho(x,s,u_2)\|_{H_a^{-1}(\Omega)} \leqslant \beta(s)\|u_1 - u_2\|^2_{H_a^1(\Omega)} \quad \text{for a.e. } s \in (0,T)$$

for any $u_1, u_2 \in H_a^1(\Omega)$, β is explicitly given by

$$\beta(t) := \|k(x,s)\|_{L^\infty(\Omega)} \quad \text{for a.e. } s \in (0,T).$$

Furthermore, $k \in L^\infty(Q)$, $f \in L^2(Q)$, then all the assumptions of [11], Theorem 1.1 are verified. Consequently, problem (2.4) has a unique solution

$$u \in L^2(0,T,H_a^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$$

with $u_t \in L^2(0,T,H_a^{-1}(\Omega))$.

Moreover, by [18], Theorem 3.1, Chapter 1, we also have

$$u \in C(0,T;L^2(\Omega)),$$

then

$$u \in L^2(0,T,H_a^1(\Omega)) \cap C(0,T;L^2(\Omega))$$

and

$$\begin{aligned}
(2.5)\quad & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_2^2 + \|\sqrt{a}u_x\|_2^2 \\
&= -\langle bB(t)u_x(t);u(t)\rangle_2 + \left\langle \int_0^t k(x,s)u(x,s)\,\mathrm{d}s; u(t)\right\rangle_2 + \langle f(t);u(t)\rangle_2 \\
&\leqslant C\|\sqrt{a}u_x\|_2\|u\|_2 + \|f(t)\|_2\|u\|_2 + \left\| \int_0^t k(x,s)u(x,s)\,\mathrm{d}s\right\|_2 \|u\|_2 \\
&\leqslant \frac{1}{2}C\|\sqrt{a}u_x\|_2^2 + \frac{1}{2}C\|u\|_2^2 + \frac{1}{2}\|f(t)\|_2^2 + \|u\|_2^2 + \frac{1}{2}\left\| \int_0^t k(x,s)u(x,s)\,\mathrm{d}s\right\|_2^2 \\
&\leqslant \frac{1}{2}C\|\sqrt{a}u_x\|_2^2 + \frac{1}{2}C\|u\|_2^2 + \frac{1}{2}\|f(t)\|_2^2 + \|u\|_2^2 + \frac{1}{2}\|k\|^2_{L^\infty(Q)}T\int_0^t \|u(x,s)\|_2^2\,\mathrm{d}s \\
&\leqslant \frac{1}{2}C\|\sqrt{a}u_x\|_2^2 + C'\|u\|_2^2 + \frac{1}{2}\|f(t)\|_2^2 + \frac{1}{2}\|k\|^2_{L^\infty(Q)}T\int_0^t \|u(x,s)\|_2^2\,\mathrm{d}s.
\end{aligned}$$

We integrate the inequality on $[0,t]$ for all $t \in [0,T]$ and we get

$$\begin{aligned}(2.6)\qquad & \|u(t)\|_2^2 - \|u(0)\|_2^2 + 2\int_0^t \|\sqrt{a}u_x(s)\|_2^2\,\mathrm{d}s \\ & \leqslant C\int_0^t \|\sqrt{a}u_x(s)\|_2^2\,\mathrm{d}s + M\int_0^t \|u(s)\|_2^2\,\mathrm{d}s + \|f\|^2_{L^2(Q)},\end{aligned}$$

where

$$M = 2C' + \|k\|^2_{L^\infty(Q)}T^2.$$

Then

$$\begin{aligned}(2.7)\qquad & \|u(t)\|_2^2 + \int_0^t \|\sqrt{a}u_x(s)\|_2^2\,\mathrm{d}s \\ & \leqslant C\int_0^t \|\sqrt{a}u_x(s)\|_2^2\,\mathrm{d}s + M\int_0^t \|u(s)\|_2^2\,\mathrm{d}s + \|f\|^2_{L^2(Q)} + \|u_0\|_2^2.\end{aligned}$$

Let Ψ be defined as follows:

$$\Psi(t) := \|u(t)\|_2^2 + \int_0^t \|\sqrt{a}u_x(s)\|_2^2\,\mathrm{d}s.$$

From (2.7) we have

$$(2.8)\qquad \Psi(t) \leqslant \|u_0\|_2^2 + \|f\|^2_{L^2(Q)} + C''\int_0^t \Psi(s)\,\mathrm{d}s.$$

From Gronwall's Lemma we deduce at once a priori estimates

$$(2.9)\qquad \sup_{t\in[0,T]} \|u(t)\|_2^2 + \int_0^T \|\sqrt{a}u_x\|_2^2 \leqslant C_T(\|u_0\|_2^2 + \|f(t)\|^2_{L^2(Q)}).$$

□

Let us now turn to the following regularised problem:

$$(2.10)\qquad \begin{cases} u_t(x,t) - (a(x)u_x(x,t))_x + b(x,t)z(x,t)u_x(x,t) \\ \qquad = \displaystyle\int_0^t k(x,s)u(x,s)\,\mathrm{d}s + f(x,t), \quad (x,t)\in Q, \\ z(x,t) - \alpha^2(a(x)z_x(x,t))_x = u, \quad (x,t)\in Q, \\ u(x,t) = z(x,t) = 0, \quad (x,t)\in \partial\Omega\times[0,T], \\ u(x,0) = u_0(x), \quad x\in\Omega. \end{cases}$$

Recall that

$$\mathcal{U} := H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2_a(\Omega)) \cap C([0,T];H^1_a(\Omega)).$$

Theorem 2.1. *Let assumptions* (1.5) *and* (1.6) *be satisfied. Then for any* $u_0 \in H^1_a(\Omega)$ *and any* $f \in L^\infty(Q)$ *there exists a unique solution* (u_α, z_α) *to* (2.10) *with*

$$u_\alpha \in \mathcal{U}, \quad z_\alpha \in L^2(0,T;D(A^2)) \cap L^\infty(Q) \cap H^1(0,T;D(A)).$$

Furthermore, for some constants C *depending only on* T, a, b, $\|u_0\|_{H^1_a(\Omega)}$ *and* $\|f\|_{L^\infty(Q)}$, *one has:*

$$\sup_{t\in[0,T]} \|u_\alpha(t,\cdot)\|^2_{H^1_a(\Omega)} + \int_0^T [\|(u_\alpha)_t(t,\cdot)\|_2^2 + \|(a(u_\alpha)_x)_x(t,\cdot)\|_2^2]\,\mathrm{d}t \leqslant C, \tag{2.11}$$

$$\|u_\alpha\|_{L^\infty(Q)} \leqslant C, \tag{2.12}$$

$$\|z_\alpha\|^2_{L^\infty(L^2(\Omega))} + \alpha^2\|\sqrt{a}(z_\alpha)_x\|^2_{L^\infty(L^2(\Omega))} \leqslant C, \tag{2.13}$$

$$\alpha^2\|\sqrt{a}(z_\alpha)_x\|^2_{L^\infty(L^2(\Omega))} + \alpha^4\|(a(z_\alpha)_x)_x\|^2_{L^\infty(L^2(\Omega))} \leqslant C, \tag{2.14}$$

$$\|z_\alpha\|_{L^\infty(Q)} \leqslant C. \tag{2.15}$$

P r o o f. First, we prove the existence of a solution. For each $\bar{u} \in L^\infty(Q)$, let z be the unique solution of

$$\begin{cases} z - \alpha^2(a(x)z_x)_x = \bar{u}, & (x,t) \in Q, \\ z(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T). \end{cases} \tag{2.16}$$

From well-known results, one has $z \in L^2(0,T;D(A^2)) \cap L^\infty(Q)$ and

$$\begin{aligned} \|z\|^2_{L^\infty(L^2(\Omega))} + \alpha^2\|\sqrt{a}z_x\|^2_{L^\infty(L^2)} \leqslant C\|\bar{u}\|_{L^\infty(L^2)},\ \alpha^2\|\sqrt{a}z_x\|^2_{L^\infty(L^2)} \\ + \alpha^4\|(az_x)_x\|^2_{L^\infty(L^2)} \leqslant C\|\bar{u}\|_{L^\infty(L^2)},\ \|z\|_{L^\infty(Q)} \leqslant \|\bar{u}\|_{L^\infty(Q)}. \end{aligned} \tag{2.17}$$

In particular, the last estimate follows from the usual weak maximum principle.

Now, let us associate to z the unique solution of the following linear problem:

$$\begin{cases} u_t - (a(x)u_x)_x + b(x)zu_x = \displaystyle\int_0^t k(x,s)u(x,s)\,\mathrm{d}s + f, & (x,t) \in Q, \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ u(0,x) = u_0(x), \quad x \in \Omega. \end{cases} \tag{2.18}$$

It holds that $u \in \mathcal{U}$ and estimate (2.11) holds for u for some C depending on T, a, b, $\|u_0\|_{H^1_a}$, $\|k\|_{L^\infty(Q)}$ and $\|\bar{u}\|_{L^\infty(Q)}$. We will use the following result, whose proof is given below:

Lemma 2.1. *Let us assume that* $u_0 \in L^\infty(\Omega)$ *and* $f \in L^\infty(Q)$. *Then the solution to* (2.18) *satisfies*

$$\|u\|_{L^\infty(Q)} \leqslant M(T),$$

where $M(T) := \|u_0\|_{L^\infty(\Omega)} + T\|f\|_{L^\infty(Q)}$.

We define $\Lambda_\alpha\colon\ \bar u \in L^\infty(Q) \mapsto u \in L^\infty(Q)$. For every $R>0$, let us denote by B_R the closed ball of radius R centred at 0 in $L^\infty(Q)$. Given Lemma 2.1, we have $\Lambda_\alpha(B_R)\subset B_R$ if R is large enough.

Let us introduce the Banach space

$$V := \{v \in L^\infty(0,T;H^1_a(\Omega))\colon\ v_t \in L^2(Q)\}. \tag{2.19}$$

According to the classical Aubin-Lions results, the space V is compactly embedded in $L^\infty(Q)$, see [20]. Therefore, $\Lambda_\alpha\colon\ L^\infty(Q)\mapsto L^\infty(Q)$ is continuous and $\Lambda_\alpha(B_R)$ is precompact for any bounded set B_R. Hence, the hypotheses of Schauder's Theorem are satisfied and there exists at least one fixed point of Λ_α, which means that (2.10) has at least one solution (u_α, z_α) satisfying (2.11)–(2.15).

Now, we show the uniqueness of the solution. Consider another solution (u'_α, z'_α) to equation (2.10) verifying estimates (2.11)–(2.15), we put $w := u_\alpha - u'_\alpha$ and $v := z_\alpha - z'_\alpha$. Then

(2.20)
$$\begin{cases} w_t - (a(x)w_x)_x + b(x)z_\alpha w_x = -b(x)v(u'_\alpha)_x + \displaystyle\int_0^t k(x,s)w(x,s)\,\mathrm{d}s, & (x,t)\in Q,\\ v - \alpha^2(a(x)v_x)_x = w, \quad (x,t)\in Q,\\ w(x,t) = v(x,t) = 0, \quad (x,t)\in\partial\Omega\times(0,T),\\ w(x,0)=0, \quad x\in\Omega. \end{cases}$$

Since $u'_\alpha \in L^2(0,T;D(A))$, it is clear that we have $a(u'_\alpha)_x \in L^2(0,T;C(\Omega))$. Based on the first equation above, we can see that

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_2^2 + \|\sqrt{a}w_x\|_2^2 &\leqslant C\|z_\alpha\|_{L^\infty(\Omega)}\|\sqrt{a}w_x\|_2\|w\|_2 + \|v\|_2\|a(u'_\alpha)_x\|_{L^\infty(\Omega)}\|w\|_2 \\ &\quad + \|k\|_{L^\infty(Q)}\Big\|\int_0^t w(x,s)\,\mathrm{d}s\Big\|_2\|w\|_2 \\ &\leqslant \frac{1}{2}\|\sqrt{a}w_x\|_2^2 + \frac{C^2}{2}\|z_\alpha\|^2_{L^\infty(\Omega)}\|w\|_2^2 + \|a(u'_\alpha)_x\|_{L^\infty(\Omega)}\|v\|_2\|w\|_2 \\ &\quad + \frac{1}{2}\|w\|_2^2 + \frac{1}{2}\|k\|^2_{L^\infty}T\int_0^t \|w(x,s)\|_2^2\,\mathrm{d}s. \end{aligned} \tag{2.21}$$

Since $\|v\|_2 \leqslant \|w\|_2$, we get

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_2^2 + \|\sqrt{a}w_x\|_2^2 \\ &\quad\leqslant (C^2\|z_\alpha\|^2_{L^\infty(\Omega)} + 2\|a(u'_\alpha)_x\|_{L^\infty(\Omega)} + 1)\|w\|_2^2 + \|k\|_{L^\infty(Q)}T\int_0^t \|w(s)\|_2^2\,\mathrm{d}s. \end{aligned}$$

Integrating the last inequality, we get

$$\|w\|_2^2 + \int_0^t \|\sqrt{a} w_x(s)\|_2^2 \,\mathrm{d}s \leqslant (C^2 \|z_\alpha\|_{L^\infty(\Omega)}^2 + 2\|a(u'_\alpha)_x\|_{L^\infty} + 1 + \|k\|^2 T^2) \int_0^t \|w(s)\|_2^2 \,\mathrm{d}s.$$

Consequently, Gronwall's Lemma implies that $w \equiv 0$. By the second equation, we also get $v \equiv 0$ and therefore, uniqueness holds. □

P r o o f of Lemma 2.1. This is an easy consequence of the weak maximum principle for parabolic PDEs. For example, let us see that

$$u \leqslant M(t) := \|u_0\|_{L^\infty(\Omega)} + t\|f\|_{L^\infty(Q)} \quad \text{in } \Omega \times (0,t). \tag{2.22}$$

Let us introduce $\zeta := (M(t) - u)_-$ the negative part of $(M(t) - u)$. Then $\zeta(x,0) \equiv 0$ and $\zeta(0,t) \equiv \zeta(1,t) \equiv 0$. Multiplying the first equation in (2.18) by ζ and integrating over Ω, we see that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\zeta\|_2^2 + \int_0^1 |\sqrt{a}\zeta_x|^2 \,\mathrm{d}x + \int_0^1 b(x) z \zeta_x \zeta \,\mathrm{d}x = -\int_0^1 f\zeta \,\mathrm{d}x - \int_0^1 \|f\|_{L^\infty(Q)} \zeta \,\mathrm{d}x - \left\langle \int_0^t k(x,s)\zeta(x,s)\,\mathrm{d}s, \zeta \right\rangle_2,$$

whence

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\zeta\|_2^2 + \int_0^1 |\sqrt{a}\zeta_x|^2 \,\mathrm{d}x \leqslant C\|z\|_{L^\infty(Q)}^2 \int_0^1 |\zeta|^2 \,\mathrm{d}x + \|k\|_{L^\infty(Q)}^2 T \int_0^t \int_0^1 |\zeta|^2 \,\mathrm{d}x \,\mathrm{d}t.$$

After integrating, we get

$$\|\zeta\|_2^2 + \int_0^t \|\sqrt{a}\zeta_x\|_2^2 \,\mathrm{d}t \leqslant (C\|z\|_{L^\infty(Q)}^2 + \|k\|_{L^\infty(Q)}^2 T^2) \int_0^t \|\zeta\|_2^2 \,\mathrm{d}t.$$

In view of Gronwall's Lemma, we conclude that $\zeta \equiv 0$, that is, (2.22) holds. In a similar way, it can be proved that $u \geqslant -M(t)$ in $\Omega \times (0,t)$.

In the following result, we analyse the behaviour of the solution of (2.10) for a fixed f when $\alpha \to 0^+$.

Proposition 2.2. *Let the assumptions in Theorem* 2.1 *be satisfied and let us denote by* (u_α, z_α) *the unique solution to* (2.10) *for each* $\alpha > 0$. *Then*

$$z_\alpha \to u \quad \text{and} \quad u_\alpha \to u \quad \text{strongly in } L^2(0,T;H_a^1(\Omega)) \tag{2.23}$$

as $\alpha \to 0^+$, *where* u *is the unique solution to* (1.2)–(1.4).

P r o o f. From the estimates in Theorem 2.1 we see that, at least for a subsequence, one has

$$u_\alpha \to u \quad \text{and} \quad z_\alpha \to z \quad \text{weakly in } L^2(0,T;H_a^1(\Omega)), \tag{2.24}$$

$$(u_\alpha)' \to u' \quad \text{weakly in } L^2(0,T;L^2(\Omega)), \tag{2.25}$$

$$\sqrt{a}(u_\alpha)_x \to \sqrt{a}u_x \quad \text{weakly in } L^2(0,T;L^2(0,1)). \tag{2.26}$$

Let us consider the Banach space

$$W = \{w \in L^2(0,T;H_a^1(\Omega)) \colon\ w_t \in L^2(Q)\}$$

(endowed with its natural norm). From Aubin-Lions' Lemma, we know that W is compactly embedded in $L^2(Q)$ and consequently, we can assume that $u_\alpha \to u$ and $z_\alpha \to z$ strongly in $L^2(Q)$. It is clear that this suffices to conclude. □

2.2. Optimal control problem. Inverse source problem (ISP). In the present work, we are interested in the problem of identifying $h(x)$, where $h(x)$ is the spatial part of the source term $f(x,t)$ of problem (1.2), more precisely, we pose $f(x,t) = h(x)R(x,t)$. We will assume that h is sufficiently smooth and shall be kept independent of time t, with $R(x,t) \in L^\infty(Q)$. We suppose that there is a possibility to provide the additional temperature for the inverse heat problems at final time T:

$$u(x,T) = \widetilde{u}(x) \quad \forall\, x \in \Omega, \tag{2.27}$$

where the given observation data with noise $\widetilde{u}(x) \in L^2(\Omega)$ and satisfy the homogeneous Dirichlet boundary conditions. The source term identification problem will lead us to minimize the functional J given by

$$J(h) = \frac{1}{2}\|u(x,T,h) - \widetilde{u}(x)\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}\|h - h^b\|^2_{L^2(\Omega)}, \tag{2.28}$$

where $u(x,T,h)$ is the weak solution of (1.2) for a given coefficient $h(x)$, the positive real ε is called the regularization parameter and h^b is a priori (background) knowledge of the true state h^{exact} (the state to reconstruct). We consider that h^b partially known (example 20% of h^{exact} is known).

For the class of admissible set A_{ad} defined by

$$A_{\text{ad}} = \{h \in H^1(\Omega) \colon\ \|h\|^2_{H^1(\Omega)} \leqslant r\}, \quad r > 0,$$

the optimal control problem (P) is defined as follows:

$$(\mathrm{P})\colon\ \begin{cases} \text{Find } \tilde{h} \in A_{\text{ad}} \text{ such that} \\ J(\tilde{h}) = \min\limits_{h \in A_{\text{ad}}} J(h). \end{cases} \tag{2.29}$$

Theorem 2.2. *Let u be the weak solution of* (1.2)–(1.4). *The function*

$$\varphi\colon\ H^1(\Omega)\to L^2(0,T;H^1_a(\Omega))\cap C([0,T];L^2(\Omega))$$
$$h\mapsto u$$

is Lipschitz continuous, and the functional J is continuous in A_{ad}. Therefore, the problem (P) *has a unique solution $\tilde h$ in A_{ad}.*

Proof. Let $\delta h\in H^1(\Omega)$ be a small variation such that $h+\delta h\in A_{\rm ad}$. Consider $\delta z_\alpha = z_\alpha - z'_\alpha$ and $\delta u_\alpha = u^\delta_\alpha - u_\alpha$ with u_α and u^δ_α being respectively the weak solutions of (2.10) with the source terms h and $h^\delta = h+\delta h$. Then δu_α is the weak solution of the system

$$(2.30)\qquad \begin{cases} (\delta u_\alpha)_t - (a(x)(\delta u_\alpha)_x)_x + b(x)z_\alpha(\delta u_\alpha)_x \\ \qquad = -b(x)\delta z_\alpha(u^\delta_\alpha)_x + \displaystyle\int_0^t k(x,s)\delta u_\alpha(x,s)\,\mathrm{d}s + R\delta h, \quad (x,t)\in Q,\\ \delta z_\alpha - \alpha^2(a(x)(\delta z_\alpha)_x)_x = \delta u_\alpha, \quad (x,t)\in Q,\\ \delta u_\alpha(x,t)=\delta z_\alpha(x,t)=0, \quad (x,t)\in\partial\Omega\times(0,T),\\ \delta u_\alpha(x,0)=0, \quad x\in\Omega, \end{cases}$$

where

$$(2.31)\qquad \begin{cases} z_\alpha - \alpha^2(a(x)(z_\alpha)_x)_x = u_\alpha + \delta u_\alpha, \quad (x,t)\in Q,\\ z(x,t)=0, \quad (x,t)\in\partial\Omega\times(0,T) \end{cases}$$

and

$$(2.32)\qquad \begin{cases} z'_\alpha - \alpha^2(a(x)(z'_\alpha)_x)_x = u_\alpha, \quad (x,t)\in Q,\\ z'(x,t)=0, \quad (x,t)\in\partial\Omega\times(0,T). \end{cases}$$

Since $u^\delta_\alpha\in L^2(0,T;D(A))$, it is clear that we have $a(u^\delta_\alpha)_x\in L^2(0,T;C(\Omega))$. Based on the first equation above, we can see that

$$(2.33)\quad \begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\delta u_\alpha\|_2^2 + \|\sqrt{a}(\delta u_\alpha)_x\|_2^2\\ &\qquad\leqslant C\|z_\alpha\|_{L^\infty(\Omega)}\|\sqrt{a}(\delta u_\alpha)_x\|_2\|\delta u_\alpha\|_2 + \|\delta z_\alpha\|_2\|a(u_\alpha)_x\|_{L^\infty(\Omega)}\|\delta u_\alpha\|_2\\ &\qquad\quad + \|k\|_{L^\infty(Q)}\|\int_0^t \delta u_\alpha(x,s)ds\|_2\|\delta u_\alpha\|_2 + \|R\|_{L^\infty(\Omega)}\|\delta h\|_2\|\delta u_\alpha\|_2\\ &\qquad\leqslant \frac{1}{2}\|\sqrt{a}(\delta u_\alpha)_x\|_2^2 + \frac{C^2}{2}\|z_\alpha\|^2_{L^\infty(\Omega)}\|\delta u_\alpha\|_2^2 + \|a(u^\delta_\alpha)_x\|_{L^\infty(\Omega)}\|\delta z_\alpha\|_2\|\delta u_\alpha\|_2\\ &\qquad\quad + \frac{1}{2}\|\delta u_\alpha\|_2^2 + \frac{1}{2}\|k\|^2_{L^\infty(Q)}T\int_0^t\|\delta u_\alpha(x,s)\|_2^2\,\mathrm{d}s + \|R\|_{L^\infty(\Omega)}\|\delta h\|_2\|\delta u_\alpha\|_2. \end{aligned}$$

Since $\|\delta z_\alpha\|_2 \leqslant \|\delta u_\alpha\|_2$ and $2\|R\|_{L^\infty(\Omega)}\|\delta h\|_2\|\delta u_\alpha\|_2 \leqslant \|R\|^2_{L^\infty(\Omega)}\|\delta h\|_2^2 + \|\delta u_\alpha\|_2^2$, we get

(2.34)
$$\frac{\mathrm{d}}{\mathrm{d}t}\|\delta u_\alpha\|_2^2 + \|\sqrt{a}(\delta u_\alpha)_x\|_2^2 \leqslant (C^2\|z_\alpha\|^2_{L^\infty(\Omega)} + 2\|a(u^\delta_\alpha)_x\|_{L^\infty(\Omega)} + 2)\|\delta u_\alpha\|_2^2 + \|k\|_{L^\infty(Q)}T\int_0^t \|\delta u_\alpha(s)\|_2^2\,\mathrm{d}s + \|R\|^2_{L^\infty(\Omega)}\|\delta h\|_2^2.$$

Integrating the last inequality, we get

$$\|\delta u_\alpha(t)\|_2^2 + \int_0^t \|\sqrt{a}(\delta u_\alpha)_x(s)\|_2^2\,\mathrm{d}s \leqslant (C^2\|z_\alpha\|^2_{L^\infty(\Omega)} + 2\|a(u^\delta_\alpha)_x\|_{L^\infty} + 2 + \|k\|^2T^2) \times \int_0^t \|\delta u_\alpha(s)\|_2^2\,\mathrm{d}s + \|R\|^2_{L^\infty(\Omega)}\|\delta h\|_2^2.$$

Consequently, Gronwall's lemma implies that there exist two constants M_1 and M_2 such that

$$\sup_{t\in[0,T]} \|\delta u_\alpha\|_2^2 \leqslant M_1\|\delta h\|_2^2 \tag{2.35}$$

and

$$\int_0^T \|\sqrt{a}(\delta u_\alpha)_x(s)\|_2^2\,\mathrm{d}s \leqslant M_2\|\delta h\|_2^2 \tag{2.36}$$

by passing to the limits and knowing that

$$\delta u_\alpha \to \delta u \quad \text{and} \quad \delta z_\alpha \to \delta z \quad \text{weakly in } L^2(0,T;H^1_a(\Omega)), \tag{2.37}$$

$$(\delta u_\alpha)_t \to (\delta u)_t \quad \text{weakly in } \ L^2(0,T;L^2(\Omega)). \tag{2.38}$$

By Aubin-Lions' Lemma, the Hilbert space $Y := \{w \in L^2(0,T,H^2_a(\Omega)); w_t \in L^2(Q)\}$ is compactly embedded in $L^2(0,T;H^1_a(\Omega))$. Consequently,

$$\delta u_\alpha \to \delta u \quad \text{strongly in } L^2(0,T;H^1_a(\Omega)),$$

and then

$$\sqrt{a}(\delta u_\alpha)_x \to \sqrt{a}\delta u_x \quad \text{strongly in } L^2(0,T;L^2(0,1)). \tag{2.39}$$

This yields

$$\sup_{t\in[0,T]} \|\delta u\|_2^2 \leqslant M_1\|\delta h\|_2^2 \tag{2.40}$$

and

$$\int_0^T \|\sqrt{a}(\delta u)_x(s)\|_2^2\,\mathrm{d}s \leqslant M_2\|\delta h\|_2^2, \tag{2.41}$$

which gives the Lipschitz continuity of

$$\varphi\colon\ H^1(\Omega)\to L^2(0,T;H_a^1(\Omega))\cap C([0,T];L^2(\Omega))$$
$$h\mapsto u$$

and therefore the continuity of the functional J. □

3. Necessary optimality conditions

In this section, we give the necessary optimality conditions that must be satisfied by each optimal control h. First of all, we give the adjoint system associated with (1.2).

Theorem 3.1. *Let $\tilde h$ be the solution of the optimal control problem* (P). *Then there exists a set of functions $(u;\xi;h)$ satisfying*

$$\tag{3.1}\begin{cases} u_t(x,t)-(a(x)u_x)_x(x,t)+b(x)uu_x \\ \quad =\displaystyle\int_0^t k(x,s)u(x,s)\,\mathrm{d}s+f(x,t), \quad (x,t)\in Q, \\ u(x,t)=0, \quad (x,t)\in\partial\Omega\times(0,T), \\ u(x,0)=0, \quad x\in\Omega, \end{cases}$$

$$\tag{3.2}\begin{cases} \xi_t-(a(x)\xi_x)_x-\displaystyle\int_0^t k(x,s)\xi\,\mathrm{d}s+b(x)(u\xi)_x=(\theta-h)R, \quad (x,t)\in Q, \\ \xi(x,t)=0, \quad (x,t)\in\partial\Omega\times(0,T), \\ \xi(x,0)=0, \quad x\in\Omega, \end{cases}$$

$$\tag{3.3}\int_0^T\int_\Omega[u(x,T,h)-\widetilde{u}(x)]\xi(x,T)\,\mathrm{d}x\,\mathrm{d}t+\varepsilon\int_\Omega(h-h^b)(\theta-h)\,\mathrm{d}x\geqslant 0$$

for any $\theta\in A_{\rm ad}$.

Proof. For any $\theta\in A_{\rm ad}$ and $0\leqslant\delta\leqslant 1$, set

$$h_\delta=(1-\delta)h+\delta\theta\in A_{\rm ad}.$$

Therefore, there is a solution u_δ of equation (1.2) with the source term h_δ satisfying

$$\tag{3.4} J_\delta=J(h_\delta)=\frac{1}{2}\|u(x,T,h_\delta)-\widetilde{u}(x)\|^2_{L^2(\Omega)}+\frac{\varepsilon}{2}\|h_\delta-h^b\|^2_{L(\Omega)}.$$

Here, by taking the Fréchet derivative of J with the optimal solution $\tilde h$, we obtain

$$\tag{3.5}\frac{\mathrm{d}J_\delta}{\mathrm{d}\delta}\Big|_{\delta=0}=\int_\Omega[u(x,T,h)-\widetilde{u}(x)]\frac{\partial u}{\partial\delta}\Big|_{\delta=0}\,\mathrm{d}x+\varepsilon\int_\Omega(h-h^b)\cdot(\theta-h)\,\mathrm{d}x\geqslant 0.$$

By taking $\overline{u}_\delta = \partial u_\delta/\partial\delta$, $\overline{u}_\delta$ satisfies the system below with the source term h_δ:

$$(3.6)\qquad \begin{cases} (\overline{u}_\delta)_t - (a(x)(\overline{u}_\delta)_x)_x - \displaystyle\int_0^t k(x,s)\overline{u}_\delta \,\mathrm{d}s \\ \qquad +\dfrac{1}{2}b(x)((u+u_\delta)\overline{u}_\delta)_x = (\theta - h)R, \quad (x,t)\in Q, \\ \overline{u}_\delta(x,t) = 0, \quad (x,t)\in\partial\Omega\times(0,T), \\ \overline{u}_\delta(x,0) = 0, \quad x\in\Omega. \end{cases}$$

We take $\xi = \overline{u}_\delta|_{\delta=0}$. We find that ζ satisfies the following system:

$$(3.7)\qquad \begin{cases} \xi_t - (a(x)\xi_x)_x - \displaystyle\int_0^t k(x,s)\xi\,\mathrm{d}s + b(x)(u\xi)_x = (\theta-h)R, \quad (x,t)\in Q, \\ \xi(x,t) = 0, \quad (x,t)\in\partial\Omega\times(0,T), \\ \xi(x,0) = 0, \quad x\in\Omega. \end{cases}$$

The optimality condition then becomes

$$(3.8)\qquad \int_\Omega [u(x,T,h) - \widetilde{u}(x)]\xi(x,T)\,\mathrm{d}x + \varepsilon\int_\Omega (h-h^b)\cdot(\theta-h)\,\mathrm{d}x \geqslant 0.$$

□

R e m a r k 3.1. Let $L\xi \equiv \xi_t - (a(x)\xi_x)_x + \int_0^t k(x,s)\xi\,\mathrm{d}s + (u\xi)_x$ and suppose p is the solution of the following problem:

$$(3.9)\qquad \begin{cases} L^*p \equiv -p_t - (a(x)p_x)_x + \displaystyle\int_t^T k(x,s)p\,\mathrm{d}s + b(x)\xi p_x = 0, \quad (x,t)\in Q, \\ p(x,t) = 0, \quad (x,t)\in\partial\Omega\times(0,T), \\ p(x,T) = u(x,T) - \widetilde{u}, \quad x\in\Omega, \end{cases}$$

where L^* is the adjoint operator of L. From (3.8) and (3.9) we have

$$(3.10)\qquad \begin{aligned} 0 &= \int_0^T\int_\Omega \xi L^*p\,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_\Omega \xi(x,T)[u(x,T,h)-\widetilde{u}(x)]\,\mathrm{d}x + \int_0^T\int_\Omega pL\xi\,\mathrm{d}x\,\mathrm{d}t \\ 0 &= -\int_\Omega \xi(x,T)[u(x,T,h)-\widetilde{u}(x)]\,\mathrm{d}x + \int_0^T\int_\Omega pR(\theta-h)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Combining (3.10) and (3.8), one can easily obtain that

$$(3.11)\qquad \int_0^T\int_\Omega p(h-\theta)R\,\mathrm{d}x\,\mathrm{d}t + \varepsilon\int_\Omega (h-h^b)(\theta-h)\,\mathrm{d}x \geqslant 0.$$

Hence the theorem.

In comparison to the necessary condition (3.8), the new condition (3.11) does not permit to deduce the global uniqueness and stability of the minimizer. We therefore still use (3.8) in the following discussion.

4. Global uniqueness and stability

Theorem 4.1. *Suppose that $\widetilde{u}_1(x)$ and $\widetilde{u}_2(x)$ are two given functions in $L^2(\Omega)$. Let $h_1(x)$ and $h_2(x)$ be the minimizers of the optimal control problem* (P) *corresponding to $\widetilde{u}_1$ and $\widetilde{u}_2$, respectively. Then we have the estimate*

$$\|h_1(x) - h_2(x)\|_{L^2(\Omega)} \leqslant C\|\widetilde{u}_1(x) - \widetilde{u}_2(x)\|_{L^2(\Omega)},$$

where the constant C only depends on Ω and ε.

Proof. By taking $\theta = h_2$ when $h = h_1$ and taking $\theta = h_1$ when $h = h_2$ in (3.3), we have

$$\int_\Omega [u_1(x,T,h_1) - \widetilde{u}_1(x)]\xi_1(x,T)\,\mathrm{d}x + \varepsilon\int_\Omega (h_1 - h^b)\cdot(h_2 - h_1)\,\mathrm{d}x \geqslant 0 \tag{4.1}$$

and

$$\int_\Omega [u_2(x,T,h_2) - \widetilde{u}_2(x)]\xi_2(x,T)\,\mathrm{d}x + \varepsilon\int_\Omega (h_2 - h^b)\cdot(h_1 - h_2)\,\mathrm{d}x \geqslant 0, \tag{4.2}$$

where $\{u_i, \xi_i\}$ $(i = 1, 2)$ are solutions of systems (3.1)–(3.2) with $h = h_i$ $(i = 1, 2)$, respectively. We set

$$u_1 - u_2 = U, \quad \xi_1 + \xi_2 = \Xi,$$

where U and Ξ satisfy

$$\begin{cases} U_t(x,t) - (a(x)U_x)_x(x,t) + b(x)\dfrac{1}{2}(U(u_1+u_2))_x \\ \qquad = \displaystyle\int_0^t k(x,s)U(x,s)\,\mathrm{d}s + (h_1 - h_2)R(x,t), \quad (x,t) \in Q, \\ U(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ U(x,0) = 0, \quad x \in \Omega, \end{cases} \tag{4.3}$$

and

$$\begin{cases} \Xi_t - (a(x)\Xi_x)_x - \displaystyle\int_0^t k(x,s)\Xi\,\mathrm{d}s + b(x)(u_1\Xi)_x - b(x)(U\Xi)_x = 0, \quad (x,t) \in Q, \\ \Xi(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ \Xi(x,0) = 0, \quad x \in \Omega. \end{cases} \tag{4.4}$$

From the extremum principle we can see that (4.4) has only one null solution and therefore,

$$\xi_1(x,t) = -\xi_2(x,t). \tag{4.5}$$

Furthermore, ξ_1 solves the following equation:

(4.6)
$$\begin{cases} (\xi_1)_t - (a(x)(\xi_1)_x)_x - \int_0^t k(x,s)\xi_1\,\mathrm{d}s + b(x)(u_1\xi_1)_x = (h_2-h_1)R, & (x,t)\in Q,\\ \xi_1(x,t) = 0, \quad (x,t)\in\partial\Omega\times(0,T),\\ \xi_1(x,0)=0, \quad x\in\Omega. \end{cases}$$

From (4.3) and (4.6) we have

$$U(x,t) = -\xi_1(x,t). \tag{4.7}$$

From (4.1), (4.2), (4.5), and (4.7) we have

(4.8)
$$\begin{aligned} \varepsilon\int_0^1 |h_1-h_2|^2\,\mathrm{d}x &\leqslant \int_0^1 [u_1(x,T,h_1)-\widetilde{u}_1]\xi_1(x,T)\,\mathrm{d}x + \int_0^1 [u_2(x,T,h_2)-\widetilde{u}_2]\xi_2(x,T)\,\mathrm{d}x\\ &\leqslant \int_0^1 U(x,T)\xi_1(x,T)\,\mathrm{d}x + \int_0^1 (\widetilde{u}_2-\widetilde{u}_1)\xi_1(x,T)\,\mathrm{d}x\\ &\leqslant -\int_0^1 |\xi_1(x,T)|^2\,\mathrm{d}x + \frac{1}{2}\int_0^1 |\xi_1(x,T)|^2\,\mathrm{d}x + \frac{1}{2}\int_0^1 |\widetilde{u}_1-\widetilde{u}_2|^2\,\mathrm{d}x\\ &\leqslant -\frac{1}{2}\int_0^1 |\xi_1(x,T)|^2\,\mathrm{d}x + \frac{1}{2}\int_0^l |\widetilde{u}_1-\widetilde{u}_2|^2\,\mathrm{d}x. \end{aligned}$$

This yields

$$\|h_1-h_2\|_{L^2(\Omega)} \leqslant \sqrt{\frac{1}{2\varepsilon}}\|\widetilde{u}_1-\widetilde{u}_2\|_{L^2(\Omega)} \leqslant C\|\widetilde{u}_1-\widetilde{u}_2\|_{L^2(\Omega)}.$$

This completes the proof of Theorem 4.1. □

Remark 4.1. From Theorem 4.1 we can conclude that if there exists a constant η such that

$$\|\widetilde{u}_1-\widetilde{u}_2\|_{L^2(\Omega)} \leqslant \eta \quad\text{and}\quad \frac{\eta^2}{\varepsilon}\to 0,$$

then

$$\|h_1-h_2\|_{L^2(\Omega)} \to 0,$$

which is consistent with the existing results for the Tikhonov regularization method (TRM) with the L^2 norm of the gradient.

The minimization problem (1.2) can be formulated as

$$\min_{h\in\mathcal{U}} \mathcal{J}(u,h), \quad \mathcal{J}(u,h) = J(h) + \frac{1}{2}\|A(u) - f(x,t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|u(0,t)\|^2_{L^2(0,T)} + \frac{1}{2}\|u(1,t)\|^2_{L^2(0,T)} + \frac{1}{2}\|u(x,0) - u_0(x)\|^2_{L^2(\Omega)}, \tag{5.1}$$

where $A(u) = u_t(x,t) - (a(x)u_x(x,t))_x + b(x)u(x,t)u_x(x,t) - \int_0^t k(x,s)u(x,s)\,\mathrm{d}s$. We want to approximate $(u(t,x), h^{\text{exact}})$ with a deep neural network $(y(t,x;\theta_u), h(x;\theta_h))$, where θ_u, $\theta_h \in \mathbb{R}^k$ are the neural network's parameters. The goal is to find a set of parameters $\theta = (\theta_u, \theta_h)$ such that the function $(y(t,x;\theta_u), h(x;\theta_h))$ minimizes the error $\mathcal{J}(y)$. If the error $\mathcal{J}(y)$ is small, then $y(t,x;\theta_u)$ will closely satisfy the PDE differential operator, boundary conditions, and initial condition. Therefore, a θ_u which minimizes $\mathcal{J}$ produces a reduced-form model $y(t,x;\theta_u)$ which approximates the PDE solution $u(t,x)$.

Network architecture: We consider this new model which consists of two layers, the first is a simple version of LSTM cells, the second is a Network architecture of Taylor cell (5):

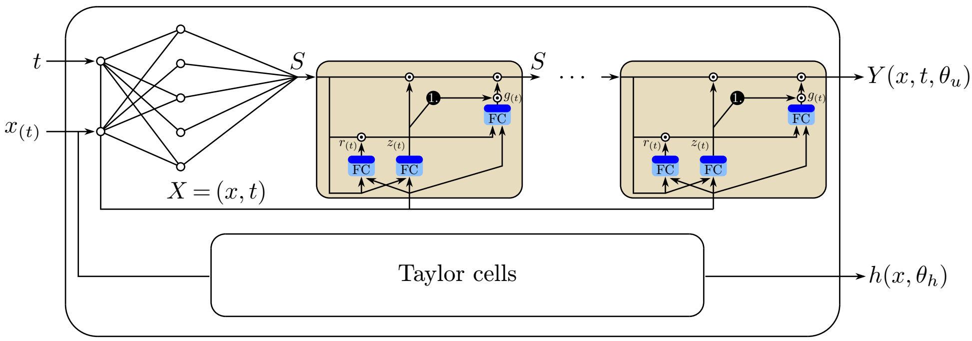

Figure 1. Neural network model.

We found the following network architecture:

$$\begin{aligned}
z_{(t)} &= \tanh(\theta^y_{xz} X_{(t)} + \theta^y_{hz} s_{(t-1)} + b_z),\\
r_{(t)} &= \tanh(\theta^y_{xr} X_{(t)} + \theta^y_{hr} s_{(t-1)} + b_r),\\
g_{(t)} &= \tanh(\theta^y_{xg} X_{(t)} + \theta^y_{hg} (r_{(t)} \otimes s_{(t-1)}) + b_g),\\
s_{(t)} &= z_{(t)} \otimes s_{(t-1)} + (1 - z_{(t)}) \otimes g_{(t)}.
\end{aligned}$$

The Taylor cell represented in Figure 1 gives the following network architecture:

$$h = \sum_{k=0}^{n} \theta_k^h x^k.$$

R e m a r k 5.1. We have $\theta_h = \theta^h$ and $\theta_u = \theta^y$.

The derivatives of $\widetilde{u}$ can be evaluated using automatic differentiation (see [10]), since it is parametrizing as a neural network.

Algorithm 1 (DL-ISP Algorithm)**.**

(1) Define boundary conditions.
(2) Define the architecture of neural networks by setting the number of layers, number of neurons in each layer and activation functions.
(3) Generate random training set D_M.
(4) Initialize the parameter set θ_0 and the learning rate α_0.
(5) Repeat until convergence criterion is satisfied.
    1. Randomly sample a mini-batch d_m of training examples from D_M.
    2. Compute the loss functional for the sampled mini-batch d_m:

$$\tilde{J}(\theta_n, d_m). \tag{5.2}$$

    3. Compute the gradient $\nabla_{\theta_n} \tilde{J}(\theta_n, d_m)$ for the sampled mini-batch d_m using backpropagation.
    4. Use the estimated gradient to take a descent step at d_m with learning rates to update θ_{n+1}:

$$\theta_{n+1} = \theta_n - \alpha_n \nabla_{\theta_n} \tilde{J}(\theta_n, d_m).$$

    The parameters are updated using the well-known ADAM algorithm with a decaying learning rate schedule.
(6) Save the model to be used for any $x \in \Omega$ and $t \in]0, T[$.

We implement the algorithm using TensorFlow, which is a software library for deep learning. TensorFlow has reverse mode automatic differentiation, which allows the calculation of derivatives for a broad range of functions. For example, TensorFlow can be used to calculate the gradient of the neural network 1 with respect to x or t, or θ. TensorFlow also allows for the training of models on graphics processing units (GPUs).

5.1. Numerical results. We now show some numerical experiments on the proposed methods for parameter identifications. For convenience, in all experiments, some basic parameters are taken as follows:

$$\varepsilon = 1, \quad T = 1, \quad a(x) = x, \quad b(x) = \sqrt{x}, \quad u_0(x) = x(1-x), \quad k(x,t) = 1.$$

It should be mentioned that in the following tests, two pictures are given in each test, the first one contains the curves of the functions (in blue) and of h numerically reconstructed by our algorithm (in red), the second picture contains the curve of the error between the two functions $h^{\rm exact}$ and h reconstructed.

Example 5.1. In the first example, we take

$$h^{\rm exact}(x) = \cos(\pi x) + 2.$$

Figures 2–5 show that the function h is recovered very well with err $\leqslant 3\%$.

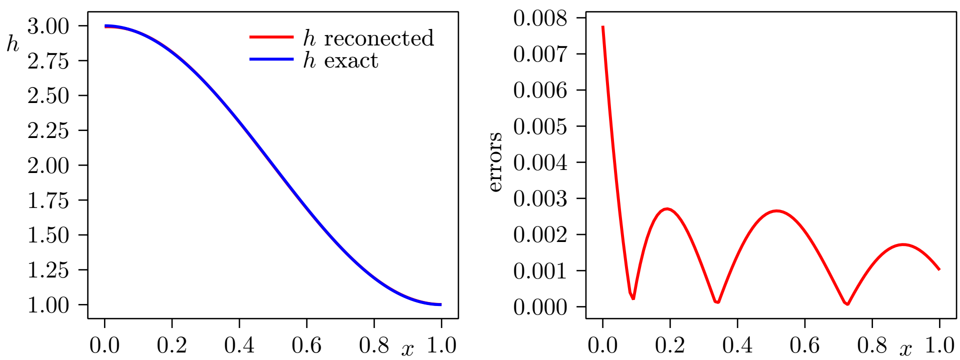

Figure 2. Test with err = 0%. This figure shows that we can rebuild h (left) by a small error (right).

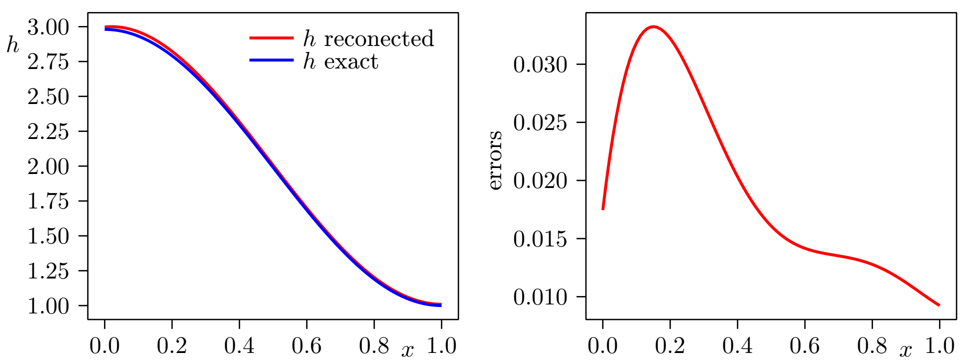

Figure 3. Test with err = 1%. This figure shows that we can rebuild h (left) by a small error (right).

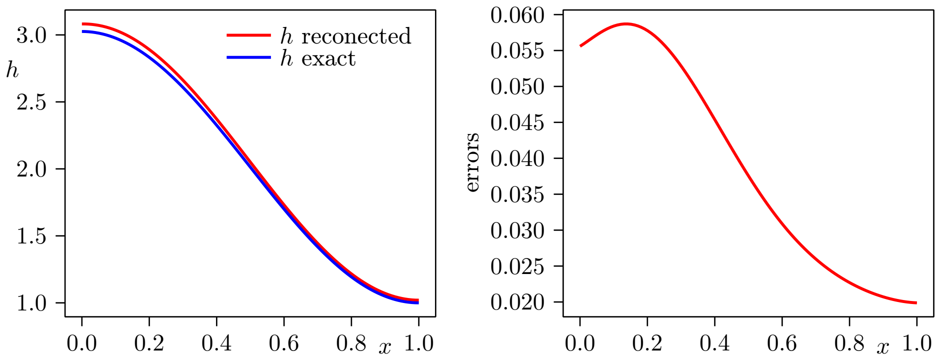

Figure 4. Test with err $= 2\%$. This figure shows that h begins to move away from h^{exact} (left) and that the error value increases (right).

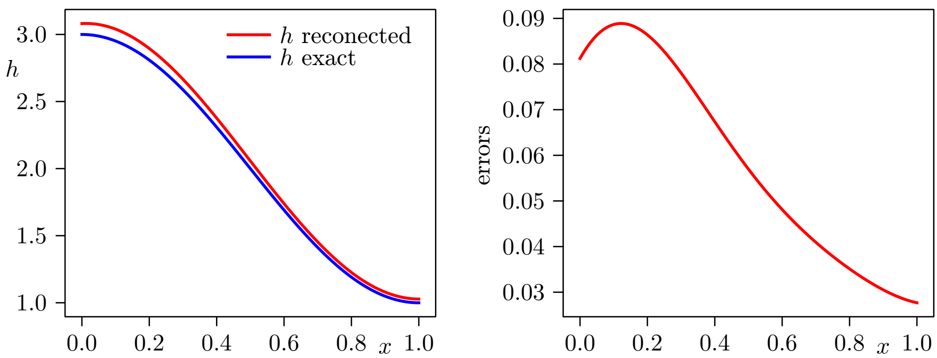

Figure 5. Test with err $= 3\%$. Function h continues to deviate from h^{exact} (left) and the error becomes noticeable (right).

In the remaining examples, we will rebuild h only for err $= 0\%$ (cf. Figures 6–9).

E x a m p l e 5.2. In this example, we take

$$h^{\text{exact}}(x) = \exp(x) + 1.$$

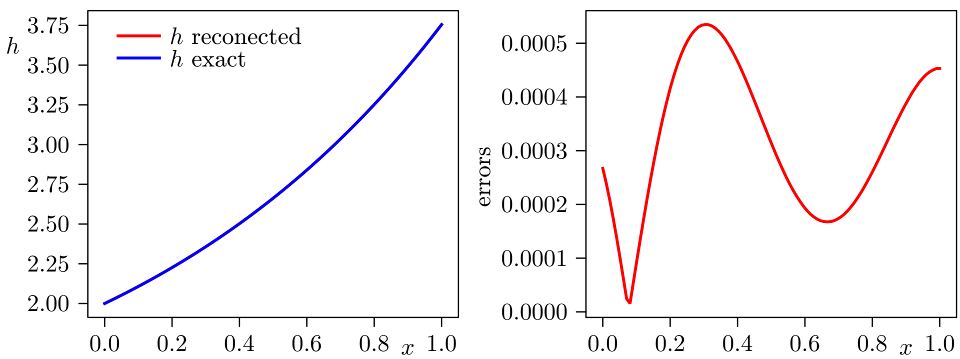

Figure 6. Test with err $= 0\%$. This figure shows that we can rebuild h (left) by a small error (right).

Example 5.3. In this example, we take

$$h^{\text{exact}}(x) = 2 - \cos(\pi x)\exp(x).$$

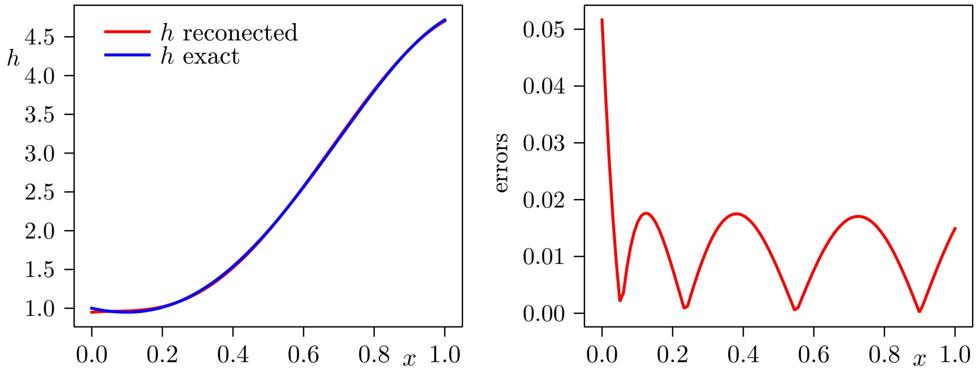

Figure 7. Test with err = 0%. This figure shows that we can rebuild h (left) by a small error (right).

Example 5.4. In this example, we take

$$h^{\text{exact}}(x) = \frac{\exp(\pi x)}{1 + x^2} - \ln(x) - 3.$$

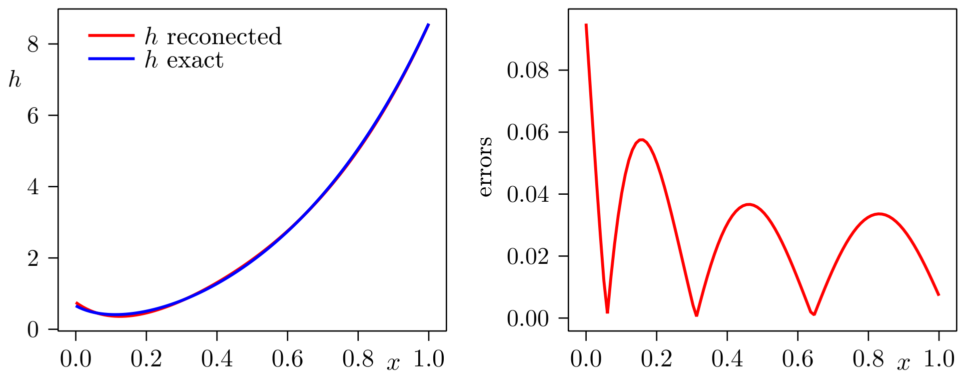

Figure 8. Test with err = 0%. This figure shows that we can rebuild h (left) by a small error (right).

Example 5.5. In this example, we take

$$h^{\text{exact}}(x) = 7 - \exp(x - 1) - \exp(1 - x).$$

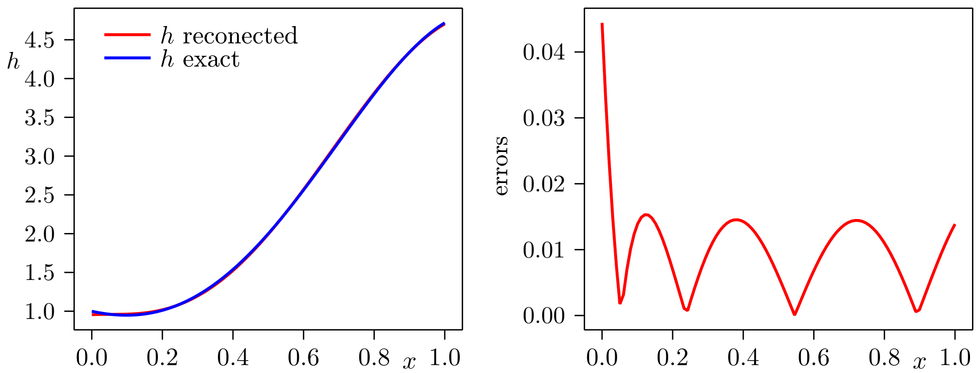

Figure 9. Test with err = 0%. This figure shows that we can rebuild h (left) by a small error (right).

6. Conclusion

In this paper, we have solved the inverse problem of recovering $h(x)$, the spatial part of the source coefficient in the following integro-differential heat equation with degenerate potential:

$$u_t(x,t) - (a(x)u_x(x,t))_x + b(x)u(x,t)u_x(x,t) = \int_0^t k(x,s)u(x,s)\,\mathrm{d}s + f(x,t),$$

where $f(x,t) = h(x)R(x,t)$, by minimizing the cost function J defined bellow. The existence, uniqueness and stability of the minimizer are proven, and the numerical results are obtained using a developed neural network model.

Further coming works will be subject to the same problem with both degenerate and singular terms.

References

[1] *B. Allal, G. Fragnelli*: Null controllability of degenerate parabolic equation with memory. Math. Methods Appl. Sci. *44* (2021), 9163–9190. zbl MR doi
[2] *Y. Y. Belov, K. V. Korshun*: An identification problem of source function in the Burgers-type equation. J. Sib. Fed. Univ., Math. Phys. *5* (2012), 497–506. (In Russian.) zbl
[3] *J. Biazar, H. Ghazvini*: Convergence of the homotopy perturbation method for partial differential equations. Nonlinear Anal., Real World Appl. *10* (2009), 2633–2640. zbl MR doi
[4] *M. A. Biot*: Mechanics of Incremental Deformations. John Wiley & Sons, New York, 1965. MR
[5] *D. Biskamp*: Collisionless shock waves in plasmas. Nuclear Fusion *13* (1973), Article ID 719. doi
[6] *J. M. Burgers*: A mathematical model illustrating the theory of turbulence. Advances in Applied Mechanics. Vol. 1. Academic Press, New York, 1948, pp. 171–199. zbl MR doi

[7] *B. Campanella, S. Legnaioli, S. Pagnotta, F. Poggialini, V. Palleschi*: Shock waves in laser-induced plasmas. Atoms *7* (2019), Article ID 57, 14 pages. doi
[8] *F. L. Carneiro, S. C. Ulhoa, J. W. Maluf, J. F. da Rocha-Neto*: Non-linear plane gravitational waves as space-time defects. Eur. Phys. J. C *81* (2021), Article ID 67, 9 pages. doi
[9] *I. Dağ, D. Irk, B. Saka*: A numerical solution of the Burgers' equation using cubic B-splines. Appl. Math. Comput. *163* (2005), 199–211. zbl MR doi
[10] *J.-P. Dussault*: La différentiation automatique et son utilisation en optimisation. RAIRO, Oper. Res. *42* (2008), 141–155. (In French.) zbl MR doi
[11] *M. Grasselli, A. Lorenzi*: Abstract nonlinear Volterra integrodifferential equations with nonsmooth kernels. Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat., IX. Ser., Rend. Lincei, Mat. Appl. *2* (1991), 43–53. zbl MR
[12] *M. Inc*: The approximate and exact solutions of the space- and time-fractional Burgers equations with initial conditions by variational iteration method. J. Math. Anal. Appl. *345* (2008), 476–484. zbl MR doi
[13] *A. Jeffrey, T. Taniuti*: Non-Linear Wave Propagation with Applications to Physics and Magnetohydrodynamics. Mathematics in Science and Engineering 9. Academic Press, New York, 1964. zbl MR doi
[14] *R. Jiwari*: A Haar wavelet quasilinearization approach for numerical simulation of Burgers' equation. Comput. Phys. Commun. *183* (2012), 2413–2423. zbl MR doi
[15] *R. Jiwari*: A hybrid numerical scheme for the numerical solution of the Burgers' equation. Comput. Phys. Commun. *188* (2015), 59–67. zbl MR doi
[16] *N. N. Leonenko, W. A. Woyczynski*: Parameter identification for singular random fields arising in Burgers' turbulence. J. Stat. Plann. Inference *80* (1999), 1–13. zbl MR doi
[17] *N. N. Leonenko, W. A. Woyczynski*: Parameter identification for stochastic Burgers' flows via parabolic rescaling. Probab. Math. Stat. *21* (2001), 1–55. zbl MR
[18] *J. L. Lions, E. Magenes*: Non-Homogeneous Boundary Value Problems and Applications. Vol. 1. Die Grundlehren der mathematischen Wissenschaften 181. Springer, Berlin, 1972. zbl MR doi
[19] *S. I. Popel, M. Y. Yu, V. N. Tsytovich*: Shock waves in plasmas containing variable-charge impurities. Phys. Plasmas *3* (1996), 4313–4315. doi
[20] *J. Simon*: Compact sets in the space $L^p(0,T;B)$. Ann. Mat. Pura Appl., IV. Ser. *146* (1986), 65–96. zbl MR doi
[21] *Z. Wang, J.-M. Vanden-Broeck, P. A. Milewski*: Two-dimensional flexural-gravity waves of finite amplitude in deep water. IMA J. Appl. Math. *78* (2013), 750–761. zbl MR doi
[22] *V. E. Zakharov, S. L. Musher, A. M. Rubenchik*: Hamiltonian approach to the description of non-linear plasma phenomena. Phys. Reports *129* (1985), 285–366. doi

Authors' addresses: *Soufiane Abid* (corresponding author), *Khalid Atifi*, *El-Hassan Essoufi*, Université Hassan Premier, Faculté des Sciences et Techniques, 50 Rue Ibnou Lhaytham B.P. 577, Settat 26002, Marocco, e-mail: s.abid@uhp.ac.ma, k.atifi.uhp@gmail.com, e.h.essoufi@gmail.com, *Abderrahim Zafrar*, Université Sidi Mohamed Ben Abdellah, Faculté des Sciences Dhar el Mahraz, Route d'Imouzzer, BP 2626, Fez 30000, Marocco, e-mail: zafrar.abd@gmail.com.