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SOLVING ELASTODYNAMIC PROBLEMS OF 2D QUASICRYSTALS IN INHOMOGENEOUS MEDIA

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Abstract. Initial value problem for three dimensional (3D) elastodynamic system in two dimensional (2D) inhomogeneous quasicrystals is considered. An analytical method is studied for the solution of this problem. The system is written in terms of Fourier images of displacements with respect to lateral variables. The resulting problem is reduced to integral equations of the Volterra type. Finally, using Paley Wiener theorem it is shown that the solution of the initial value problem can be found by the inverse Fourier transform. A numerical example is considered for the comparison of the exact solution with the computed solution obtained by using the method.

Keywords: 2D quasicrystals; inhomogeneous media; elastodynamic system

MSC 2020: 35L52, 35Q86

1. Introduction

Shechtman [28] observed that certain Al-Mn alloys produced the unusual diffraction pattern of sharp spots which today are seen as the beginning of quasicrystal structures. This was the discovery of a new class of solids, which are defined as structures with long-range quasiperiodic order. These materials are named quasicrystals by Levine and Steinhardt [20] as a shortened form of quasiperiodic crystal. Periodic crystals can be described as an arrangement of atoms in a single type unit cell which is repeated periodically, whereas a quasicrystal (quasiperiodic crystal) is a structure that is ordered but not periodic. There are three main classes of quasicrystals: One dimensional (1D) quasicrystals that are given by quasiperiodic stacking of periodic planes, two dimensional (2D) quasicrystals that are quasiperiodic in two dimensions but periodic in the direction perpendicular to the aperiodic planes, and three dimensional (3D) quasicrystals that are quasiperiodic in all three dimensions.

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The physical properties of quasicrystals such as the structural, magnetic, thermal, piezoelectric properties are investigated in [10], [24], [30]. The theoretical and experimental results have proved the difference of their elastic behaviors compared with usual crystals [9], [30]. The quasiperiodicity of these materials in at least one direction results in the introduction of a new field, called phason field, which yields more complicated elastic equations.

In recent years, many authors studied general solutions of static elasticity in quasicrystals that plays an important role in solving boundary value problems. General solutions of static elasticity for 1D hexagonal quasicrystals and 2D decagonal, hexagonal, dodecagonal and octagonal quasicrystals are studied in [4], [14]–[16], [22], [25], [38], [39]. Plane elasticity of 3D quasicrystals and general solutions based on stress potential function is studied in [21]. Linear elasticity of cubic quasicrystals are considered in [13], [40]. The dynamic elasticity problems in quasicrystals have been investigated in [12], [31]. Fan and Mai in [12] have solved two-dimensional dynamic problems for 1D and 2D hexagonal quasicrystals. Wang extended the methods in [4] to derive the general solution of 3D dynamic problems for 1D hexagonal quasicrystals [31].

All these studies are considered in homogeneous quasicrystals. The study of wave propagation of elastic waves in inhomogeneous media has a great interest to characterize the properties of many important materials. However, the number of studies is not much. In [17], [18], [32], [33], wave propagation of elastic waves is studied in isotropic, inhomogeneous media. In [2], [3], [5], [6], [23], [26], [29], [35], [36], wave propagation of elastic waves is studied in anisotropic inhomogeneous media.

The dynamic equations of 3D elasticity for quasicrystals are more complicated if we compare with the equations which are considered in isotropic or anisotropic media. In addition to phonon displacement and stresses fields, there exist phason displacement and its stresses fields in quasicrystals. And these displacements have interaction. Thus, the study of elastodynamics in inhomogeneous quasicrystals are more scarce if compared with isotropic and anisotropic inhomogeneous media. The general solutions of the inhomogeneous linear elastic equations for pentagonal quasicrystals are obtained by minimizing the harmonic elastic energy which includes a nontrivial coupling between the phason and displacement variables in [7]. In [27] linear continuous inhomogeneous strains are considered in decagonal and octagonal quasicrystalline lattices. An analytical method of solving the initial value problem (IVP) for elastodynamic system in inhomogeneous 3D quasicrystals is developed in [34]. Existence, uniqueness and stability theorems for the theory of dynamic equations in inhomogeneous quasicrystals are stated in the paper. However, the IVP for 3D elastodynamic problems have not been studied in inhomogeneous 2D quasicrystals before.

This paper considers the IVP for 3D elastic problems of 2D vertically inhomogeneous quasicrystals. The method used in this paper is based on Fourier transformation and theory of integral equations following similar approach as in [1], [35], [36]. In Section 2, basic equations of elasticity theory for 2D quasicrystals and the formulation of the IVP are given. In Section 3, IVP is written in terms of Fourier images and this problem is reduced to a vector integral equation. In Section 4, properties of the vector integral equation are given and the construction of the unique solution of the IVP is described. In Section 5, the existence of the unique solution of the IVP for 3D elastic problems of 2D quasicrystals is proved. In the last section, an example of an IVP is considered. The problem is solved by the given method using Maple program and the computed solution is compared with the exact solution of the problem.

2. Equations of elastodynamic system for 2D quasicrystals

Quasicrystals are classified as 1D, 2D and 3D, according to the quasiperiodic directions; 2D quasicrystals in a 3D body have the atom arranged quasiperiodically in a plane and periodically in the orthogonal [37]. In this paper, we study 3D elastodynamic problems of 2D quasicrystals. In the class of 2D quasicrystals, there are five nonzero displacements, i.e., $u_1(x,t)$, $u_2(x,t)$, $u_3(x,t)$, $w_1(x,t)$, $w_2(x,t)$, where $x=(x_1,x_2,x_3)\in\mathbb{R}^3$, $t\geqslant 0$. In this section, basic relations of the elasticity theory of 2D quasicrystals [8], [9], [11] are presented.

Let $u_i(x,t)$, $i=1,2,3$, and $w_\beta(x,t)$, $\beta=1,2$, denote the phonon and phason displacements, respectively, where $x=(x_1,x_2,x_3)\in\mathbb{R}^3$, $t\geqslant 0$. The equations of motion are

$$\varrho\frac{\partial^2 u_i}{\partial t^2}=\sum_{j=1}^{3}\frac{\partial\sigma_{ij}}{\partial x_j}+f_i(x,t), \tag{2.1}$$

$$\varrho\frac{\partial^2 w_\beta}{\partial t^2}=\sum_{j=1}^{3}\frac{\partial H_{\beta j}}{\partial x_j}+g_\beta(x,t), \tag{2.2}$$

where $f_i(x,t)$ and $g_\beta(x,t)$ ($i=1,2,3$, $\beta=1,2$) are body forces for the phonon and phason displacements, respectively, and $\varrho>0$ is the density. Throughout the paper let ϱ depend on x_3 only, σ_{ij} is the phonon stress, and $H_{\beta j}$ is the phason stress. The stress components σ_{ij} and $H_{\beta j}$ are related to phonon strains E_{kl} and phason strains $F_{\alpha l}$ through the relations

$$\sigma_{ij}=\sum_{k,l=1}^{3}C_{ijkl}E_{kl}+\sum_{l=1}^{3}\sum_{\alpha=1}^{2}R_{ij\alpha l}F_{\alpha l},\quad i,j=1,2,3, \tag{2.3}$$

$$H_{\beta j}=\sum_{k,l=1}^{3}R_{kl\beta j}E_{kl}+\sum_{l=1}^{3}\sum_{\alpha=1}^{2}K_{\beta j\alpha l}F_{\alpha l},\quad j=1,2,3,\ \beta=1,2, \tag{2.4}$$

which are known as Generalized Hooke's law. In formulas (2.3), (2.4) the coefficients C_{ijkl} and K_{ijkl} are the constitutive tensors corresponding to the phanon and phason fields and R_{ijkl} are the constitutive tensors of the phanon-phason coupling.

The tensors C_{ijkl}, K_{ijkl} and R_{ijkl} have the following symmetries [9], [19]:

$$C_{ijkl} = C_{jikl} = C_{klij}, \quad K_{ijkl} = K_{klij}, \quad R_{ijkl} = R_{jikl}.$$

The problem is considered in vertically inhomogeneous quasicrystals so the elements of the tensors C_{ijkl}, K_{ijkl} and R_{ijkl} depend on x_3 and also these elements are considered as twice continuously differentiable functions.

According to the linear elastic theory of quasicrystals, the strain-displacement relations are given by the equations

$$E_{kl} = \frac{1}{2}\Big(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\Big), \quad F_{\alpha l} = \frac{\partial w_\alpha}{\partial x_l}, \quad k,l = 1,2,3, \ \alpha = 1,2. \tag{2.5}$$

In the present paper, IVP for 3D elastodynamic system in 2D quasicrystals (2.1), (2.2) with conditions

$$\begin{aligned} u_i(x,t)\big|_{t=0} &= 0, \quad \frac{\partial u_i(x,t)}{\partial t}\Big|_{t=0} = 0, \\ w_\beta(x,t)\big|_{t=0} &= 0, \quad \frac{\partial w_\beta(x,t)}{\partial t}\Big|_{t=0} = 0, \quad (i = 1,2,3, \ \beta = 1,2) \end{aligned} \tag{2.6}$$

is considered.

If phason and phonon displacements are defined as components of a vector function such as $\mathbf{V} = [u_1, u_2, u_3, w_1, w_2]^\top$ and $\mathbf{F} = [f_1, f_2, f_3, g_1, g_2]^\top$, then using equations (2.3), (2.4), (2.5), the IVP (2.1), (2.2), (2.6) can be written as

$$\frac{\partial^2 \mathbf{V}(x,t)}{\partial t^2} = P^1_{33}\frac{\partial^2 \mathbf{V}(x,t)}{\partial x_3^2} + \sum_{\substack{j,l=1 \\ j=l\neq 3}}^{3} P^1_{jl}\frac{\partial^2 \mathbf{V}(x,t)}{\partial x_j \partial x_l} + P^2_l \frac{\partial \mathbf{V}(x,t)}{\partial x_l} + \mathbf{F}(x,t), \tag{2.7}$$

$$\mathbf{V}(x,t)\big|_{t=0} = 0, \quad \frac{\partial \mathbf{V}(x,t)}{\partial t}\Big|_{t=0} = 0, \tag{2.8}$$

where

$$P^1_{33} = \frac{1}{\varrho}\begin{bmatrix} C_{1313} & C_{1323} & C_{1333} & R_{1313} & R_{1323} \\ C_{2313} & C_{2323} & C_{2333} & R_{2313} & R_{2323} \\ C_{3313} & C_{3323} & C_{3333} & R_{3313} & R_{3323} \\ R_{1313} & R_{2313} & R_{3313} & K_{1313} & K_{1323} \\ R_{1323} & R_{2323} & R_{3323} & K_{2313} & K_{2323} \end{bmatrix}, \tag{2.9}$$

and

$$P^1_{jl} = \frac{1}{\varrho}\begin{pmatrix} A_{jl} & B_{jl} \\ M_{jl} & N_{jl} \end{pmatrix}, \quad P^2_l = \frac{1}{\varrho}\begin{pmatrix} T_l & Y_l \\ R_l & Z_l \end{pmatrix}, \quad j,l = 1,2,3,$$

and the components of the matrices A_{jl}, B_{jl}, M_{jl} and N_{jl} are defined as

$$\begin{gathered}
A^{ik}_{jl} = \frac{1}{2}(C_{ijkl} + C_{ilkj}), \quad B^{i\alpha}_{jl} = \frac{1}{2}(R_{ij\alpha l} + R_{il\alpha j}), \\
M^{\beta k}_{jl} = \frac{1}{2}(R_{kl\beta j} + R_{kj\beta l}), \quad N^{\beta\alpha}_{jl} = \frac{1}{2}(K_{\beta j\alpha l} + K_{\beta l\alpha j}), \\
T^{ik}_{l} = \frac{\partial}{\partial x_3} C_{i3kl}, \quad Y^{i\alpha}_{l} = \frac{\partial}{\partial x_3} R_{i3\alpha l}, \\
R^{\beta k}_{l} = \frac{\partial}{\partial x_3} R_{kl\beta 3}, \quad Z^{\beta\alpha}_{l} = \frac{\partial}{\partial x_3} K_{\beta 3\alpha l}, \quad i,j,k,l = 1,2,3.
\end{gathered}$$

Let us consider that the matrix P^1_{33} is positive-definite for any nonzero $\mathbf{x} \in \mathbb{R}^3$, so the matrix P^1_{33} has to satisfy inequality

$$\mathbf{V}^\top P^1_{33} \mathbf{V} > 0$$

for arbitrary nonzero vectors $\mathbf{V} = (u_1, u_2, u_3, w_1, w_2)^\top \in \mathbb{R}^5$ and $\mathbf{x} \in \mathbb{R}^3$, where $\mathbf{V}^\top$ is the transpose of $\mathbf{V}$. Using (2.9) we find that

$$\mathbf{V}^\top P^1_{33} \mathbf{V} = \frac{1}{\varrho}\Big(\sum_{i,k=1}^{3} C_{i3k3} u_i u_k + \sum_{i=1}^{3}\sum_{\alpha=1}^{2} R_{i3\alpha 3} u_i w_\alpha + \sum_{\alpha,\beta=1}^{2} K_{\beta 3\alpha 3} w_\beta w_\alpha \Big).$$

If for any arbitrary nonzero vector $\mathbf{V} = (u_1, u_2, u_3, w_1, w_2)^\top \in \mathbb{R}^5$, the condition

$$\frac{1}{\varrho}\Big(\sum_{i,k=1}^{3} C_{i3k3} u_i u_k + \sum_{i=1}^{3}\sum_{\alpha=1}^{2} R_{i3\alpha 3} u_i w_\alpha + \sum_{\alpha,\beta=1}^{2} K_{\beta 3\alpha 3} w_\beta w_\alpha \Big) > 0 \tag{2.10}$$

holds, then the matrix P^1_{33} becomes positive definite for all $x \in \mathbb{R}^3$. This condition can be used to diagonalize the matrix P^1_{33}.

Notice that the matrix P^1_{33} is symmetric and under the condition (2.10), P^1_{33} is positive definite. Thus, there exists Λ such that

$$\Lambda^\top P^1_{33} \Lambda = D, \tag{2.11}$$

where $D = \operatorname{diag}(d_1, d_2, d_3, d_4, d_5)$.

Beside the assumption of the condition (2.10), the following assumptions and notations are used in the paper. Let ϱ_0, a, b and T be given positive numbers such that $\varrho(x_3) \geqslant \varrho_0 > 0$ and $0 < a \leqslant d_1, d_2, d_3, d_4, d_5 \leqslant b$, where $c = \sqrt{b/\varrho_0}$. And let the triangular domain $\Delta(T)$ used throughout the paper be defined as follows

$$\Delta(T) = \{(x_3, t)\colon\ t \in [0, T], \quad |x_3| \leqslant c(T - t)\}, \tag{2.12}$$

where T is a given positive number.

Let $\mathbf{V}(x,t) = \Lambda\widetilde{\mathbf{V}}(x,t)$. Plugging into the IVP (2.7), (2.8) and multiplying by $\Lambda^\top$ from the left-hand side, the IVP (2.7), (2.8) can be written as

$$\begin{aligned}
\frac{\partial^2\widetilde{\mathbf{V}}(x,t)}{\partial t^2} &= D\frac{\partial^2\widetilde{\mathbf{V}}}{\partial x_3^2} + \Big[2\Lambda^\top P_{33}^1\frac{\partial\Lambda}{\partial x_3} + \Lambda^\top P_3^2\Lambda\Big]\frac{\partial\widetilde{V}}{\partial x_3} \\
&+ \Big[\Lambda^\top P_{33}^1\frac{\partial^2\Lambda}{\partial x_3^2} + \Lambda^\top P_3^2\frac{\partial\Lambda}{\partial x_3}\Big]\widetilde{\mathbf{V}} \\
&+ \sum_{j,l=1}^{2}\Lambda^T P_{jl}^1\Lambda\frac{\partial^2\widetilde{\mathbf{V}}}{\partial x_j\partial x_l} + \Lambda^\top(P_{31}^1 + P_{13}^1)\frac{\partial}{\partial x_3}\Big[\Lambda\frac{\partial\widetilde{\mathbf{V}}}{\partial x_1}\Big] \\
&+ \sum_{l=1}^{2}\Lambda^\top P_l^2\Lambda\frac{\partial\widetilde{\mathbf{V}}}{\partial x_l} + \Lambda^\top(P_{32}^1 + P_{23}^1)\frac{\partial}{\partial x_3}\Big[\Lambda\frac{\partial\widetilde{\mathbf{V}}}{\partial x_2}\Big] + \widetilde{\mathbf{F}}(x,t),
\end{aligned} \tag{2.13}$$

$$\widetilde{\mathbf{V}}(x,t)\big|_{t=0} = 0, \quad \frac{\partial\widetilde{\mathbf{V}}(x,t)}{\partial t}\Big|_{t=0} = 0, \tag{2.14}$$

where the function $\widetilde{\mathbf{F}} = [\widetilde{F}_1,\widetilde{F}_2,\widetilde{F}_3,\widetilde{F}_4,\widetilde{F}_5]^\top$ is defined as $\widetilde{\mathbf{F}}(x,t) = \Lambda^\top\mathbf{F}(x,t)$.

3. Reduction to Volterra integral equation

Let the functions $\widetilde{\mathbf{V}}(x,t)$, $\widetilde{\mathbf{F}}(x,t)$ that are given in equations (2.13), (2.14) have Fourier transform with respect to x_1, x_2 denoted by $\widehat{\mathbf{V}}(\nu,x_3,t)$, $\widehat{\mathbf{F}}(\nu,x_3,t)$, respectively and let the Fourier image $\widehat{\mathbf{F}}(\nu,x_3,t)$ belong to the space $C(\mathbb{R}^2\times\Delta(T))$, $\nu = (\nu_1,\nu_2)$.

Applying the Fourier transform with respect to x_1, x_2

$$\widehat{\mathbf{V}}(\nu,x_3,t) = \mathcal{F}_{x_1,x_2}[\widetilde{\mathbf{V}}](\nu,x_3,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\widetilde{\mathbf{V}}(x,t)\mathrm{e}^{\mathrm{i}(\nu_1x_1+\nu_2x_2)}\,\mathrm{d}x_1\,\mathrm{d}x_2,$$

the dynamic equilibrium equations in 2D quasicrystals can be written in terms of the Fourier images

$$\begin{aligned}
\frac{\partial^2\widehat{\mathbf{V}}}{\partial t^2} &= D\frac{\partial^2\widehat{\mathbf{V}}}{\partial x_3^2} + \Big[2\Lambda^\top P_{33}^1\frac{\partial\Lambda}{\partial x_3} + \Lambda^\top P_3^2\Lambda\Big]\frac{\partial\widehat{\mathbf{V}}}{\partial x_3} + \Big[\Lambda^\top P_{33}^1\frac{\partial^2\Lambda}{\partial x_3^2} + \Lambda^\top P_3^2\frac{\partial\Lambda}{\partial x_3}\Big]\widehat{\mathbf{V}} \\
&- \sum_{j,l=1}^{2}\nu_j\nu_l\Lambda^\top P_{jl}^1\Lambda\widehat{\mathbf{V}} + \mathrm{i}\nu_1\Lambda^\top(P_{31}^1 + P_{13}^1)\frac{\partial\Lambda}{\partial x_3}\widehat{\mathbf{V}} \\
&+ \mathrm{i}\nu_1\Lambda^\top(P_{31}^1 + P_{13}^1)\Lambda\frac{\partial\widehat{\mathbf{V}}}{\partial x_3} + \mathrm{i}\nu_2\Lambda^\top(P_{32}^1 + P_{23}^1)\frac{\partial\Lambda}{\partial x_3}\widehat{\mathbf{V}} \\
&+ \mathrm{i}\nu_2\Lambda^\top(P_{32}^1 + P_{23}^1)\Lambda\frac{\partial\widehat{\mathbf{V}}}{\partial x_3} + \sum_{l=1}^{2}\nu_l\Lambda^\top P_l^2\Lambda\widehat{\mathbf{V}} + \widehat{\mathbf{F}}(\nu,x_3,t),
\end{aligned}$$

$$\widehat{\mathbf{V}}(\nu,x_3,t)\big|_{t=0} = 0, \quad \frac{\partial\widehat{\mathbf{V}}(\nu,x_3,t)}{\partial t}\Big|_{t=0} = 0.$$

Thus, each component $\widehat{V}_j(\nu, x_3, t)$, $j = 1, \dots, 5$, of $\widehat{\mathbf{V}}(\nu, x_3, t)$ satisfies that following IVP:

$$\begin{aligned}\frac{\partial^2 \widehat{V}_j}{\partial t^2} &= d_j \frac{\partial^2 \widehat{V}_j}{\partial x_3^2} + \sum_{k,m=1}^{5} \Big(2N^1_{jm} \frac{\partial \Lambda_{mk}}{\partial x_3} + N^2_{jm}\Lambda_{mk}\Big) \frac{\partial \widehat{V}_k}{\partial x_3} \\ &\quad + \sum_{k,m=1}^{5} \Big(N^1_{jm} \frac{\partial^2 \Lambda_{mk}}{\partial x_3^2} + N^2_{jm} \frac{\partial \Lambda_{mk}}{\partial x_3}\Big) \widehat{V}_k \\ &\quad + \sum_{k,m=1}^{5} N^3_{jm}\Lambda_{mk}\widehat{V}_k + \widehat{F}_j(\nu, x_3, t),\end{aligned} \tag{3.1}$$

$$\widehat{V}_j(\nu, x_3, t)\big|_{t=0} = 0, \qquad \frac{\partial \widehat{V}_j(\nu, x_3, t)}{\partial t}\Big|_{t=0} = 0, \tag{3.2}$$

where N^i_{jm} are components of the matrix $N^i(x_3)$, $i = 1, 2, 3$, defined as

$$\begin{aligned}N^1(x_3) &= \Lambda^\top P^1_{33}, \\ N^2(x_3) &= \Lambda^\top [P^2_3 + \mathrm{i}\nu_1(P^1_{31} + P^1_{13}) + \mathrm{i}\nu_2(P^1_{32} + P^1_{23})], \\ N^3(x_3) &= \Lambda^\top \Big[- \sum_{j,l=1}^{2} \nu_j \nu_l P^1_{jl} + \sum_{l=1}^{2} \nu_l P^2_l \Big].\end{aligned}$$

Let us consider the transform

$$y_j = \tau_j(x_3), \quad \tau_j(x_3) = \int_0^{x_3} c_j(\xi)\,\mathrm{d}\xi, \quad c_j^2(\xi) = \frac{1}{d_j(\xi)}, \quad x_3 = \tau_j^{-1}(y_j), \tag{3.3}$$

where d_j are the diagonal entries of D defined in (2.11).

$$U_j(\nu, y_j, t) = \widehat{V}_j(\nu, x_3, t)\big|_{x_3 = \tau_j^{-1}(y_j)}, \quad j = 1, 2, \dots, 5. \tag{3.4}$$

Using the given transformation and notations, the IVP (3.1), (3.2) becomes

$$\begin{aligned}\frac{\partial^2 U_j}{\partial t^2} &= \frac{\partial^2 U_j}{\partial y_j^2} - \widetilde{M}_j(y_j) \frac{\partial U_j}{\partial y_j} + \widehat{F}_j(\nu, x_3, t)\big|_{x_3 = \tau_j^{-1}(y_j)} \\ &\quad + \sum_{\substack{k,m=1 \\ k \neq j}}^{5} \Big(2N^1_{jm} \frac{\partial \Lambda_{mk}}{\partial x_3} + N^2_{jm}\Lambda_{mk}\Big) \frac{\partial \widehat{V}_k}{\partial x_3}\Big|_{x_3 = \tau_j^{-1}(y_j)} \\ &\quad + \sum_{k,m=1}^{5} \Big(N^1_{jm} \frac{\partial^2 \Lambda_{mk}}{\partial x_3^2} + N^2_{jm} \frac{\partial \Lambda_{mk}}{\partial x_3} + N^3_{jm}\Lambda_{mk}\Big) \widehat{V}_k\Big|_{x_3 = \tau_j^{-1}(y_j)},\end{aligned} \tag{3.5}$$

$$U_j(\nu, y_j, t)\big|_{t=0} = 0, \qquad \frac{\partial U_j(\nu, y_j, t)}{\partial t}\Big|_{t=0} = 0, \tag{3.6}$$

where

$$\widetilde{M}_j(y_j) = -\frac{\partial}{\partial y_j}\Big[\ln\sqrt{\frac{1}{d_j(\tau_j^{-1}(y_j))}}\Big] - \frac{1}{\sqrt{d_j(\tau_j^{-1}(y_j))}}\sum_{m=1}^{5}\Big(2N^1_{jm}\frac{\partial\Lambda_{mj}}{\partial x_3} + N^2_{jm}\Lambda_{mj}\Big)\Big|_{x_3=\tau_j^{-1}(y_j)}.$$

We seek a solution of problem (3.5), (3.6) in the form

$$U_j(\nu, y_j, t) = S_j(y_j)\overline{U}_j(\nu, y_j, t), \quad j = 1, 2, 3, \tag{3.7}$$

where the function $S_j(y_j)$ is defined by

$$S_j(y_j) = \exp\Big(\frac{1}{2}\int_0^{y_j}\widetilde{M}_j(\xi)\,\mathrm{d}\xi\Big).$$

With these new notations, equations (3.5) and (3.6) can be written as follows:

$$\begin{aligned}\frac{\partial^2\overline{U}_j}{\partial t^2} - \frac{\partial^2\overline{U}_j}{\partial y_j^2} &= \Big[\frac{1}{2}\frac{\partial\widetilde{M}_j}{\partial y_j} - \frac{1}{4}\widetilde{M}_j^2(y_j)\Big]\overline{U}_j + \frac{1}{S_j(y_j)}\widehat{F}_j\Big|_{x_3=\tau_j^{-1}(y_j)}\\ &+ \frac{1}{S_j(y_j)}\sum_{\substack{k,m=1\\k\neq j}}^{5} N^1_{jm}\frac{\partial}{\partial x_3}\Big(\frac{\partial\Lambda_{mk}}{\partial x_3}\widehat{V}_k\Big)\Big|_{x_3=\tau_j^{-1}(y_j)}\\ &+ \frac{1}{S_j(y_j)}\sum_{\substack{k,m=1\\k\neq j}}^{5}\Big(N^2_{jm}\frac{\partial}{\partial x_3}(\Lambda_{mk}\widehat{V}_k) + N^3_{jm}\Lambda_{mk}\widehat{V}_k\Big)\Big|_{x_3=\tau_j^{-1}(y_j)}\\ &+ \frac{1}{S_j(y_j)}\sum_{m=1}^{5}\Big(N^1_{jm}\frac{\partial^2\Lambda_{mj}}{\partial x_3^2} + N^2_{jm}\frac{\partial\Lambda_{mj}}{\partial x_3} + N^3_{jm}\Lambda_{mj}\Big)\widehat{V}_j\Big|_{x_3=\tau_j^{-1}(y_j)},\end{aligned} \tag{3.8}$$

$$\overline{U}_j(\nu, y_j, t)\big|_{t=0} = 0, \qquad \frac{\partial\overline{U}_j(\nu, y_j, t)}{\partial t}\Big|_{t=0} = 0. \tag{3.9}$$

Using transformation (3.3) and notations

$$\Lambda(x_3)\big|_{x_3=\tau_j^{-1}(y_j)} = \widetilde{\Lambda}(y_j), \quad c(x_3)\big|_{x_3=\tau_j^{-1}(y_j)} = \tilde{c}(y_j),$$
$$N^i(x_3)\big|_{x_3=\tau_j^{-1}(y_j)} = \widetilde{N}^i(y_j), \quad \widehat{F}_j(\nu, x_3, t)\big|_{x_3=\tau_j^{-1}(y_j)} = \overline{F}_j(\nu, y_j, t),$$

equation (3.8) can be written in the following form:

$$(3.10)\quad \frac{\partial^2\overline{U}_j}{\partial t^2}-\frac{\partial^2\overline{U}_j}{\partial y_j^2}=\Big[\frac{1}{2}\frac{\partial\widetilde{M_j}}{\partial y_j}-\frac{1}{4}\widetilde{M_j^2}(y_j)\Big]\overline{U}_j$$
$$+\frac{1}{S_j(y_j)\widetilde{\varrho}(y_j)}\sum_{\substack{k,m=1\\k\neq j}}^{5}\widetilde{N}^1_{jm}\Big(\frac{\partial^2\widetilde{\Lambda}_{mk}}{\partial y_j^2}\tilde{c}^2+\frac{\partial\widetilde{\Lambda}_{mk}}{\partial y_j}\frac{\partial\tilde{c}}{\partial y_j}\tilde{c}\Big)\widehat{V}_k$$
$$+\frac{1}{S_j(y_j)}\sum_{\substack{k,m=1\\k\neq j}}^{5}\frac{\partial}{\partial y_j}\Big(\widetilde{N}^1_{jm}\frac{\widetilde{\Lambda}_{mk}}{\partial y_j}\tilde{c}^2\widehat{V}_k\Big)-\frac{\partial}{\partial y_j}\Big(\widetilde{N}^1_{jm}\frac{\widetilde{\Lambda}_{mk}}{\partial y_j}\tilde{c}^2\Big)\widehat{V}_k$$
$$+\frac{1}{S_j(y_j)}\sum_{\substack{k,m=1\\k\neq j}}^{5}\frac{\partial}{\partial y_j}(\widetilde{N}^2_{jm}\tilde{c}\widetilde{\Lambda}_{mk}\widehat{V}_k)-\frac{\partial}{\partial y_j}(\widetilde{N}^2_{jm}\tilde{c})\widetilde{\Lambda}_{mk}\widehat{V}_k$$
$$+\frac{1}{S_j(y_j)}\sum_{m=1}^{5}\widetilde{N}^1_{jm}\Big(\frac{\partial^2\widetilde{\Lambda}_{mj}}{\partial y_j^2}\tilde{c}^2+\frac{\partial\widetilde{\Lambda}_{mj}}{\partial y_j}\frac{\partial\tilde{c}}{\partial y_j}\tilde{c}\Big)\overline{U}_j$$
$$+\frac{1}{S_j(y_j)}\sum_{m=1}^{5}\Big(\widetilde{N}^2_{jm}\frac{\partial\widetilde{\Lambda}_{mj}}{\partial y_j}\tilde{c}+\widetilde{N}^3_{jm}\widetilde{\Lambda}_{mj}\Big)\overline{U}_j$$
$$+\frac{1}{S_j(y_j)}\Big(\sum_{\substack{k,m=1\\k\neq j}}^{5}\widetilde{N}^3_{jm}\widetilde{\Lambda}_{mk}\widehat{V}_k+\overline{F}_j(\nu,y_j,t)\Big).$$

Using notations

$$\Psi_j(\nu,y_j,t)=\frac{1}{S_j(y_j)}\overline{F}_j(\nu,y_j,t),\quad L^1_j(y_j)=\frac{1}{2}\frac{\partial\widetilde{M_j}}{\partial y_j}-\frac{1}{4}\widetilde{M_j^2}(y_j),$$
$$L^2_j(y_j)=\frac{1}{S_j(y_j)}\sum_{m=1}^{5}\Big[\widetilde{N}^1_{jm}\Big(\frac{\partial^2\widetilde{\Lambda}_{mj}}{\partial y_j^2}\tilde{c}^2+\frac{\partial\widetilde{\Lambda}_{mj}}{\partial y_j}\frac{\partial\tilde{c}}{\partial y_j}\tilde{c}\Big)+\widetilde{N}^2_{jm}\frac{\partial\widetilde{\Lambda}_{mj}}{\partial y_j}\tilde{c}+\widetilde{N}^3_{jm}\widetilde{\Lambda}_{mj}\Big],$$
$$L^3_{jk}(y_j)=\frac{1}{S_j(y_j)}\sum_{m=1}^{5}\widetilde{N}^3_{jm}\widetilde{\Lambda}_{mk},\quad L^4_{jk}(y_j)=\frac{1}{S_j(y_j)}\sum_{m=1}^{5}\Big(\widetilde{N}^1_{jm}\frac{\widetilde{\Lambda}_{mk}}{\partial y_j}\tilde{c}^2\Big),$$
$$L^5_{jm}(y_j)=\frac{1}{S_j(y_j)}\widetilde{N}^2_{jm}\tilde{c},\quad L^6_{jk}(y_j)=\frac{1}{S_j(y_j)}\sum_{m=1}^{5}\widetilde{N}^1_{jm}\Big(\frac{\partial^2\widetilde{\Lambda}_{mk}}{\partial y_j^2}\tilde{c}^2+\frac{\partial\widetilde{\Lambda}_{mk}}{\partial y_j}\frac{\partial\tilde{c}}{\partial y_j}\tilde{c}\Big),$$

equation (3.10) can be written in the form

(3.11)
$$\frac{\partial^2\overline{U}_j}{\partial t^2}-\frac{\partial^2\overline{U}_j}{\partial y_j^2}=(L^1_j+L^2_j)\overline{U}_j+L^3_{jk}\widehat{V}_k+\sum_{\substack{k=1\\k\neq j}}^{5}\frac{\partial}{\partial y_j}(L^4_{jk}\widehat{V}_k)-\frac{\partial L^4_{jk}}{\partial y_j}\widehat{V}_k$$
$$+\sum_{\substack{k,m=1\\k\neq j}}^{5}\frac{\partial}{\partial y_j}(L^5_{jm}\widetilde{\Lambda}_{mk}\widehat{V}_k)-\frac{\partial L^5_{jm}}{\partial y_j}\widetilde{\Lambda}_{mk}\widehat{V}_k+L^6_{jk}\widehat{V}_k+\Psi_j(\nu,\xi,\eta).$$

Applying d'Alembert formula, it can be shown that the solution of (3.11) with initial conditions (3.9) can be found as

$$
\begin{aligned}
\text{(3.12)}\quad \overline{U}_j(\nu, y_j, t) = {} & \frac{1}{2}\int_0^t \int_{y_j-(t-\eta)}^{y_j+(t-\eta)} [L_j^1(\xi) + L_j^2(\xi)]\overline{U}_j(\nu, \xi, \eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\
& + \frac{1}{2}\int_0^t \int_{y_j-(t-\eta)}^{y_j+(t-\eta)} \sum_{\substack{k=1\\ j\neq k}}^{5} [L_{jk}^3(\xi) + L_{jk}^6(\xi)]\widehat{V}_k(\nu, \tau_j^{-1}(\xi), \eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\
& + \frac{1}{2}\int_0^t \int_{y_j-(t-\eta)}^{y_j+(t-\eta)} \Psi_j(\nu, \xi, \eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\
& - \frac{1}{2}\int_0^t \int_{y_j-(t-\eta)}^{y_j+(t-\eta)} \sum_{\substack{k=1\\ j\neq k}}^{5} \frac{\partial}{\partial \xi}[L_{jk}^4(\xi)]\widehat{V}_k(\nu, \tau_j^{-1}(\xi), \eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\
& - \frac{1}{2}\int_0^t \int_{y_j-(t-\eta)}^{y_j+(t-\eta)} \sum_{\substack{k,m=1\\ j\neq k}}^{5} \frac{\partial L_{jm}^5(\xi)}{\partial \xi}\widetilde{\Lambda}_{mk}\widehat{V}_k(\nu, \tau_j^{-1}(\xi), \eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\
& + \frac{1}{2}\int_0^t \sum_{\substack{k=1\\ j\neq k}}^{5} L_{jk}^4(\xi)\widehat{V}_k(\nu, \tau_j^{-1}(\xi), \eta)\Big|_{\xi=\tau_j(x_3)-(t-\eta)}^{\xi=\tau_j(x_3)+(t-\eta)}\,\mathrm{d}\eta \\
& + \frac{1}{2}\int_0^t \sum_{\substack{k,m=1\\ j\neq k}}^{5} L_{jm}^5(\xi)\widetilde{\Lambda}_{mk}(\xi)\widehat{V}_k(\nu, \tau_j^{-1}(\xi), \eta)\Big|_{\xi=\tau_j(x_3)-(t-\eta)}^{\xi=\tau_j(x_3)+(t-\eta)}\,\mathrm{d}\eta,
\end{aligned}
$$

and using the notations given in (3.4), (3.7) that are

$$
U_j(\nu, y_j, t) = S_j(y_j)\overline{U}_j(\nu, y_j, t), \quad \widehat{V}_j(\nu, x_3, t) = U_j(\nu, y_j, t)\big|_{y_j=\tau_j(x_3)},
$$

the equation (3.12) can be written as

$$
\begin{aligned}
\text{(3.13)}\quad & \widehat{V}_j(\nu, x_3, t) \\
& = \frac{S_j(\tau_j(x_3))}{2}\int_0^t \int_{\tau_j(x_3)-(t-\eta)}^{\tau_j(x_3)+(t-\eta)} [L_j^1(\xi) + L_j^2(\xi)]\frac{\widehat{V}_j(\nu, \xi, \eta)}{S_j(\xi)}\,\mathrm{d}\xi\,\mathrm{d}\eta \\
& \quad + \frac{S_j(\tau_j(x_3))}{2}\int_0^t \int_{\tau_j(x_3)-(t-\eta)}^{\tau_j(x_3)+(t-\eta)} \sum_{\substack{k=1\\ j\neq k}}^{5} [L_{jk}^3(\xi) + L_{jk}^6(\xi)]\widehat{V}_k(\nu, \tau_j^{-1}(\xi), \eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\
& \quad + \frac{S_j(\tau_j(x_3))}{2}\int_0^t \int_{\tau_j(x_3)-(t-\eta)}^{\tau_j(x_3)+(t-\eta)} \Psi_j(\nu, \xi, \eta)\,\mathrm{d}\xi\,\mathrm{d}\eta
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{S_j(\tau_j(x_3))}{2} \int_0^t \int_{\tau_j(x_3)-(t-\eta)}^{\tau_j(x_3)+(t-\eta)} \sum_{\substack{k=1\\ j\neq k}}^{5} \frac{\partial L_{jk}^4(\xi)}{\partial \xi} \widehat{V}_k(\nu, \tau_j^{-1}(\xi), \eta)\, \mathrm{d}\xi\, \mathrm{d}\eta \\
&- \frac{S_j(\tau_j(x_3))}{2} \int_0^t \int_{\tau_j(x_3)-(t-\eta)}^{\tau_j(x_3)+(t-\eta)} \sum_{\substack{k,m=1\\ j\neq k}}^{5} \frac{\partial L_{jm}^5(\xi)}{\partial \xi} \widetilde{\Lambda}_{mk} \widehat{V}_k(\nu, \tau_j^{-1}(\xi), \eta)\, \mathrm{d}\xi\, \mathrm{d}\eta \\
&+ \frac{S_j(\tau_j(x_3))}{2} \int_0^t \sum_{\substack{k=1\\ j\neq k}}^{5} L_{jk}^4(\xi) \widehat{V}_k(\nu, \tau_j^{-1}(\xi), \eta)\Big|_{\xi=\tau_j(x_3)-(t-\eta)}^{\xi=\tau_j(x_3)+(t-\eta)} \mathrm{d}\eta \\
&+ \frac{S_j(\tau_j(x_3))}{2} \int_0^t \sum_{\substack{k,m=1\\ j\neq k}}^{5} L_{jm}^5(\xi) \widetilde{\Lambda}_{mk}(\xi) \widehat{V}_k(\nu, \tau_j^{-1}(\xi), \eta)\Big|_{\xi=\tau_j(x_3)-(t-\eta)}^{\xi=\tau_j(x_3)+(t-\eta)} \mathrm{d}\eta.
\end{aligned}
$$

The function $\widehat{V}_j(\nu, x_3, t)$ given with equation (3.13) is the solution of the IVP (3.1), (3.2) that can be written in the form of integral equation as

$$
\widehat{\mathbf{V}}(\nu, x_3, t) = \mathbf{G}(\nu, x_3, t) + \int_0^t (\mathcal{K}\widehat{\mathbf{V}})(\nu, x_3, t, \eta)\, \mathrm{d}\eta, \tag{3.14}
$$

where the components $G_j(\nu, x_3, t)$, $j = 1, 2, \ldots, 5$, of $\mathbf{G} = (G_1, G_2, G_3, G_4, G_5)^\top$ are defined by

$$
G_j(\nu, x_3, t) = \frac{S_j(\tau(x_3))}{2} \int_0^t \int_{\tau(x_3)-(t-\eta)}^{\tau(x_3)+(t-\eta)} \Psi_j(\nu, \xi, \eta)\, \mathrm{d}\xi\, \mathrm{d}\eta, \tag{3.15}
$$

and $\mathbf{K} = (\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4, \mathcal{K}_5)^\top$ is the vector operator with components

$$
\begin{aligned}
(3.16)\quad &(\mathcal{K}_j \widehat{\mathbf{V}})(\nu, x_3, t, \eta) \\
&= \frac{S_j(\tau_j(x_3))}{2} \int_{\tau_j(x_3)-(t-\eta)}^{\tau_j(x_3)+(t-\eta)} [L_j^1(\xi) + L_j^2(\xi)] \frac{\widehat{V}_j(\nu, \xi, \eta)}{S_j(\xi)}\, \mathrm{d}\xi \\
&\quad + \frac{S_j(\tau_j(x_3))}{2} \int_{\tau_j(x_3)-(t-\eta)}^{\tau_j(x_3)+(t-\eta)} \sum_{\substack{k=1\\ j\neq k}}^{5} [L_{jk}^3(\xi) + L_{jk}^6(\xi)] \widehat{V}_k(\nu, \tau_j^{-1}(\xi), \eta)\, \mathrm{d}\xi \\
&\quad - \frac{S_j(\tau_j(x_3))}{2} \int_{\tau_j(x_3)-(t-\eta)}^{\tau_j(x_3)+(t-\eta)} \sum_{\substack{k=1\\ j\neq k}}^{5} \frac{\partial L_{jk}^4(\xi)}{\partial \xi} \widehat{V}_k(\nu, \tau_j^{-1}(\xi), \eta)\, \mathrm{d}\xi \\
&\quad - \frac{S_j(\tau_j(x_3))}{2} \int_{\tau_j(x_3)-(t-\eta)}^{\tau_j(x_3)+(t-\eta)} \sum_{\substack{k,m=1\\ j\neq k}}^{5} \frac{\partial L_{jm}^5(\xi)}{\partial \xi} \widetilde{\Lambda}_{mk} \widehat{V}_k(\nu, \tau_j^{-1}(\xi), \eta)\, \mathrm{d}\xi
\end{aligned}
$$

$$+\frac{S_j(\tau_j(x_3))}{2}\sum_{\substack{k=1\\ j\neq k}}^{5} L^4_{jk}(\xi)\widehat{V}_k(\nu,\tau_j^{-1}(\xi),\eta)\Big|_{\xi=\tau_j(x_3)-(t-\eta)}^{\xi=\tau_j(x_3)+(t-\eta)}$$
$$+\frac{S_j(\tau_j(x_3))}{2}\sum_{\substack{k,m=1\\ j\neq k}}^{5} L^5_{jm}(\xi)\widetilde{\Lambda}_{mk}(\xi)\widehat{V}_k(\nu,\tau_j^{-1}(\xi),\eta)\Big|_{\xi=\tau_j(x_3)-(t-\eta)}^{\xi=\tau_j(x_3)+(t-\eta)}.$$

4. Existence and uniqueness for the vector integral equation

We have shown that under the new notations and some assumptions, the system of equations (3.1)–(3.2) can be written in the form of operator integral equation (3.14). In this section, it will be shown that the integral equation (3.14) can be solved using the method of successive approximations and the unique solution that has continuous components for $(\nu, x_3, t) \in \mathbb{R}^2 \times \Delta(T)$ can be obtained.

To get the solution of the vector integral equation (3.14), the successive approximation method can be applied:

$$\begin{aligned}(4.1)\qquad &\widehat{\mathbf{V}}_i^{(0)}(\nu,x_3,t)=\mathbf{G}(\nu,x_3,t),\\ &\widehat{\mathbf{V}}_i^{(n)}(\nu,x_3,t)=\mathbf{G}(\nu,x_3,t)+\int_0^t\big(\mathcal{K}_i\widehat{\mathbf{V}}^{(n-1)}\big)(\nu,x_3,t,\eta)\,\mathrm{d}\eta,\quad n=1,2,\ldots\end{aligned}$$

Now our aim is to show that the series $\sum\limits_{n=0}^{\infty}\widehat{V}_i^{(n)}(\nu,x_3,t)$ uniformly converges to a vector function $\widehat{V}_i(\nu,x_3,t)$ that has continuous components and satisfies the integral equation.

Under the assumptions given in Section 2, it can be concluded that the components $G_i(\nu,x_3,t)$, $i=1,2,\ldots,5$, of $\mathbf{G}$ defined by (3.14) belong to the space $C(\mathbb{R}^2;\Delta(T))$.

Also, for the components $\mathcal{K}_i$ of $\mathbf{K}=(\mathcal{K}_1,\ldots,\mathcal{K}_5)^\top$ defined by (3.16) for $(x_3,t)\in\Delta(T)$, $\nu\in\mathbb{R}^2$, and any vector function $\widehat{\mathbf{V}}(\nu,x_3,t)$ with components $\widehat{V}_i(\nu,x_3,t)$ that belong to the space $C(\mathbb{R}^2\times\Delta(T))$, we can conclude that functions

$$\int_0^t(\mathcal{K}_i\widehat{\mathbf{V}})(\nu,x_3,t,\eta)\,\mathrm{d}\eta,\quad i=1,\ldots,5,$$

are continuous. Thus, the components of the vector function $\widehat{V}^{(n)}(\nu,x_3,t)$, $n=0,1,2,\ldots$ have components belonging to the space $C(\mathbb{R}^2\times\Delta(T))$.

Let $(x_3,t)\in\Delta(T)$, $|\nu|\leqslant\Omega$, for any positive number Ω and $\mathcal{A}$ be a positive number that depends on c,T,Ω and

$$\|\widehat{\mathbf{V}}\|(\nu,\eta)=\max_{i=1,\ldots,5}\ \max_{\xi\in[-c(T-\eta),c(T-\eta)]}|\widehat{V}_i(\nu,\xi,\eta)|.$$

Let Q be defined by

$$Q = \max_{j,k,l} \max_{\xi} \left\{ |L_j^l(\xi)|, \left| \frac{\partial L_j^4(\xi)}{\partial \xi} \right|, \sum_{m=1}^{5} |L_{jm}^5(\xi)\widetilde{\Lambda}_{mk}(\xi)|, \sum_{m=1}^{5} \left| \frac{\partial L_{jm}^5(\xi)}{\partial \xi} \widetilde{\Lambda}_{mk} \right|, |S_j(\xi)| \right\}$$
$$j, k = 1, 2, \ldots, 5, \; l = 1, 2, 3, 4, 6,$$

for any $(x_3, t) \in \Delta$, $\eta \in [0, t]$, $\xi \in [\tau_j(x_3) - (t - \eta), \tau_j(x_3) + (t - \eta)]$ and let

$$\|\widehat{V}_i(\nu, \tau^{-1}(\xi), \eta)\| \leqslant \|\widehat{V}\|(\nu, \eta).$$

Then from formulas (3.16) we obtain the following inequalities:

$$|(\mathcal{K}_i \widehat{\mathbf{V}})(\nu, x_3, t, \eta)| \leqslant \mathcal{A}_i(T, \Omega) \|\widehat{\mathbf{V}}\|(\nu, \eta), \quad i = 1, 2, \ldots, 5,$$

where $|\nu| < \Omega$ and

$$\mathcal{A}_i(T, \Omega) = 22TQ^2 + 2Q^2, \quad i = 1, 2, \ldots, 5,$$
$$\mathcal{A} = \max_{i=1,\ldots,5} \mathcal{A}_i(T, \Omega).$$

Thus, inequality

$$\left| \int_0^t (\mathcal{K}_i \widehat{\mathbf{V}})(\nu, x_3, t, \eta) \, d\eta \right| \leqslant \mathcal{A} \int_0^t \|\widehat{\mathbf{V}}\|(\nu, \eta) \, d\eta \tag{4.2}$$

is satisfied for any vector function $\widehat{\mathbf{V}}(\nu, x_3, t, \eta)$ with components which are continuous on $\mathbb{R}^2 \times \Delta(T)$.

Using (4.1) and inequality (4.2), we have

$$|\widehat{V}_i^{(n)}(\nu, x_3, t)| = \left| \int_0^t \left(\mathcal{K}_i \widehat{\mathbf{V}}^{(n-1)} \right)(\nu, x_3, t, \eta) \, d\eta \right|,$$

and thus,

$$|\widehat{V}_i^{(n)}(\nu, x_3, t)| \leqslant \mathcal{A} \int_0^t \|\widehat{\mathbf{V}}^{(n-1)}\|(\nu, \eta) \, d\eta.$$

Further, we choose a constant $\mathcal{A}_0$ such that $\|\widehat{V}_i^{(0)}\|(t) \leqslant \mathcal{A}_0$. Using the above inequality, we find that

$$\|\widehat{\mathbf{V}}^{(1)}\|(t) \leqslant \mathcal{A}_0 \mathcal{A} t, \; \|\widehat{\mathbf{V}}^{(2)}\|(t) \leqslant \mathcal{A}_0 \frac{(\mathcal{A}t)^2}{2!}, \ldots, \|\widehat{\mathbf{V}}^{(n)}\|(t) \leqslant \mathcal{A}_0 \frac{(\mathcal{A}t)^n}{n!}, \ldots.$$

Thus the series $\sum\limits_{n=0}^{\infty} \widehat{V}_i^{(n)}(\nu, x_3, t)$ is majorized for $(x_3, t) \in \Delta(T)$ by $\sum\limits_{n=0}^{\infty} \|\widehat{\mathbf{V}}^{(n)}\|(\nu, t)$ and the series $\sum\limits_{n=0}^{\infty} \|\widehat{\mathbf{V}}^{(n)}\|(\nu, t)$ is majorized by the numerical series

$$\sum_{n=0}^{\infty} \|\widehat{V}^{(n)}\|(\nu, t) \leqslant \sum_{n=0}^{\infty} \mathcal{A}_0 \frac{(\mathcal{A}T)^n}{n!} \max_{|\nu| \leqslant \Omega} \|\mathbf{G}\|(\nu, T), \quad n = 0, 1, 2, \ldots \tag{4.3}$$

Using inequality (4.3) and the Weierstrass first theorem, there exist functions $\widehat{V}_i(\nu, x_3, t) \in C(\mathbb{R}^2 \times \Delta(T))$ such that the series $\sum\limits_{n=0}^{\infty} \widehat{V}_i^{(n)}(\nu, x_3, t)$ converges uniformly and absolutely to $\widehat{V}_i(\nu, x_3, t)$ for $i = 1, 2, \dots, 5$.

Therefore, we get the following proposition.

Proposition 4.1. *Under the above assumptions and notations, the functions* $\widehat{V}_i^{(n)}(\nu, x_3, t)$ $(i = 1, \dots, 5,\ n = 0, 1, 2, \dots)$ *defined by* (4.1) *are continuous on* $\mathbb{R}^2 \times \Delta(T)$ *and there exist functions* $\widehat{V}_i(\nu, x_3, t) \in C(\mathbb{R}^2 \times \Delta(T))$ *such that the series* $\sum\limits_{n=0}^{\infty} \widehat{V}_i^{(n)}(\nu, x_3, t)$ *converges uniformly to* $\widehat{V}_i(\nu, x_3, t)$*, which are the components of the solution vector* $\widehat{\mathbf{V}}(\nu, x_3, t)$.

Proposition 4.2. $\widehat{\mathbf{V}}(\nu, x_3, t)$ *with the components* $\widehat{V}_i(\nu, x_3, t)$*,* $i = 1, 2, \dots, 5$*, is a solution of the vector integral equation* (3.14) *for* $(x_3, t) \in \Delta(T)$*,* $|\nu| \leqslant \Omega$.

P r o o f. To show that $\widehat{\mathbf{V}}(\nu, x_3, t)$ is a solution of (3.14) let us sum equation (4.1) with respect to n from 1 to N and add the vector function $\mathbf{G}(\nu, x_3, t)$ to both sides of the equation and get

$$\sum_{n=0}^{N} \widehat{\mathbf{V}}^{(n)}(\nu, x_3, t) = \mathbf{G}(\nu, x_3, t) + \int_0^t \sum_{n=0}^{N-1} (\mathcal{K}\widehat{\mathbf{V}}^{(n)})(\nu, x_3, t, \eta)\, \mathrm{d}\eta.$$

As N approaches to infinity, we find that $\widehat{\mathbf{V}}(\nu, x_3, t)$ satisfies (3.14) for $|\nu| \leqslant \Omega$, $(x_3, t) \in \Delta(T)$, $t \in [0, T]$. This means that $\widehat{\mathbf{V}}(\nu, x_3, t)$ is a solution of (3.14) for $|\nu| \leqslant \Omega$, $(x_3, t) \in \Delta(T)$, $t \in [0, T]$. $\square$

So we have the following theorem for existence of the solution of the vector integral equation.

Theorem 4.1. *Let* T *be a fixed positive number,* $\mathbf{K} = (\mathcal{K}_1, \mathcal{K}_2, \dots, \mathcal{K}_5)^\top$ *be the vector operator with the components defined by* (3.16)*. Then for any* $\mathbf{G} = (G_1, G_2, \dots, G_5)^\top$ *such that* $G_i = G_i(\nu, x_3, t) \in C(\mathbb{R}^2 \times \Delta(T)), i = 1, 2, \dots, 5$*, defined in* (3.15)*, there exists a unique solution* $\widehat{\mathbf{V}} = (\widehat{V}_1, \widehat{V}_2, \dots, \widehat{V}_5)$ *of the operator integral equation* (3.14) *such that* $\widehat{V}_i \in C(\mathbb{R}^2 \times \Delta(T)), \quad i = 1, 2, \dots, 5.$

P r o o f. Using Proposition 4.1 and 4.2, the existence of the solution for the integral equation can be shown. For the uniqueness of the solution let $\widehat{\mathbf{V}}(\nu, x_3, t)$ and $\widehat{\mathbf{V}}^*(\nu, x_3, t)$ be two solutions of (3.14) corresponding to the same vector function $\mathbf{G}(\nu, x_3, t)$ and let

$$\overline{\overline{\mathbf{V}}}(\nu, x_3, t) = \widehat{\mathbf{V}}(\nu, x_3, t) - \widehat{\mathbf{V}}^*(\nu, x_3, t)$$

for $|\nu| \leqslant \Omega$, $(x_3, t) \in \Delta(T)$. The vector function $\overline{\overline{\mathbf{V}}}(\nu, x_3, t)$ satisfies the following operator integral equality:

$$\overline{\overline{\mathbf{V}}}(\nu, x_3, t) = \int_0^t \Big(\mathbf{K}\overline{\overline{\mathbf{V}}}\Big)(\nu, x_3, t, \eta)\, \mathrm{d}\eta, \quad |\nu| \leqslant \Omega, \quad (x_3, t) \in \Delta(T). \tag{4.4}$$

Using the proposition above and (4.4) we have

$$\|\overline{\overline{\mathbf{V}}}\|(\nu, x_3, t) = \mathcal{A}\int_0^t \|\overline{\overline{\mathbf{V}}}\|(\nu, x_3, \eta)\, \mathrm{d}\eta, \quad |\nu| \leqslant \Omega, \quad t \in [0, T].$$

Applying Grownwall's lemma to (4.4), we find that

$$\|\overline{\overline{\mathbf{V}}}\|(\nu, x_3, t) = 0, \quad |\nu| \leqslant \Omega, \quad (x_3, t) \in \Delta(T).$$

From the continuity of $\overline{\overline{\mathbf{V}}}(\nu, x_3, t)$ we obtain

$$\overline{\overline{\mathbf{V}}}(\nu, x_3, t) \equiv 0, \quad |\nu| \leqslant \Omega, \quad (x_3, t) \in \Delta(T).$$

This means that $\widehat{\mathbf{V}}(\nu, x_3, t) \equiv \widehat{\mathbf{V}}^*(\nu, x_3, t)$ for $|\nu| \leqslant \Omega$, $(x_3, t) \in \Delta(T)$. Thus, the existence of a unique solution for the vector integral equation (3.14) is proved. $\square$

5. Existence and uniqueness of the IVP

In the previous section, existence of a unique solution for the vector integral equation (3.14) is proved. The solution $\widetilde{V}(x, t)$ of the IVP (2.13), (2.14) can be obtained by the inverse Fourier transform of $\widehat{\mathbf{V}}(\nu, x_3, t)$, that is, the solution of the vector integral equation (3.14). Since $\mathbf{V}(x, t) = \Lambda\widetilde{\mathbf{V}}(x, t)$, the solution of IVP (2.7), (2.8) can be obtained by multiplying the solution $\widetilde{\mathbf{V}}(x, t)$ of the IVP (2.13), (2.14) by Λ. Now we will show that the inverse Fourier transform with respect to $\nu = (\nu_1, \nu_2)$ can be applied to $\widehat{\mathbf{V}}(\nu, x_3, t)$ and the inverse Fourier transform of $\widehat{\mathbf{V}}(\nu, x_3, t)$ is the unique solution of (3.1), (3.2).

As an assumption given in Section 3, the Fourier image $\widehat{\mathbf{F}}(\nu, x_3, t)$ is considered in the space $C(\mathbb{R}^2 \times \Delta(T))$ and $\partial^{|\alpha|}\widehat{\mathbf{F}}(\nu, x_3, t)/(\partial\nu_1^{\alpha_1}\partial\nu_2^{\alpha_2})$ belongs to the class

$$C(\mathbb{R}^2 \times \Delta(T)) \cap C(\Delta(T); C_0(\mathbb{R}^2)),$$

where $C(\Delta(T); C_0(\mathbb{R}^2)$ is the class of continuous mappings of $(x_3, t) \in \Delta(T)$ into $C_0(\mathbb{R}^2)$, that is, the class of functions that are from $C(\mathbb{R}^2)$ with compact supports.

According to the assumptions and properties, $G_i(\nu, x_3, t)$, $i = 1, \ldots, 5$, defined by (3.15) satisfy

$$\frac{\partial^{|\alpha|}}{\partial\nu_1^{\alpha_1}\partial\nu_2^{\alpha_2}} G_i(\nu, x_3, t) \in C(\mathbb{R}^2 \times \Delta(T)) \cap C(\Delta(T); C_0(\mathbb{R}^2)).$$

For simplicity let us use following notation:

$$D_\nu^\alpha G_i(\nu, x_3, t) = \frac{\partial^{|\alpha|}}{\partial \nu_1^{\alpha_1} \partial \nu_2^{\alpha_2}} G_i(\nu, x_3, t), \quad i = 1, 2, \ldots, 5,$$

where $\alpha = (\alpha_1, \alpha_2)$, $\alpha_j \in \{0, 1, 2, \ldots\}$ and $|\alpha| = \alpha_1 + \alpha_2$. Thus, we have

$$D_\nu^\alpha G_i(\nu, x_3, t) \in C(\mathbb{R}^2 \times \Delta(T)) \cap C(\Delta(T); C_0(\mathbb{R}^2)).$$

Applying D_ν^α to the system of integral equations (3.14), we obtain

$$D_\nu^\alpha \widehat{\mathbf{V}}(\nu, x_3, t) = D_\nu^\alpha \mathbf{G}(\nu, x_3, t) + \int_0^t (\mathbf{K} D_\nu^\alpha \widehat{\mathbf{V}})(\nu, x_3, t, \eta) \, \mathrm{d}\eta. \tag{5.1}$$

Since this equation has the same form as (3.14), the solution which can be found by successive approximations satisfies $D_\nu^\alpha \widehat{\mathbf{V}}(\nu, x_3, t) \in C(\mathbb{R}^2 \times \Delta(T))$. Using equality (5.1), the result of the proposition we proved and Gronwall's lemma, we get the inequality

$$\|D_\nu^\alpha \widehat{\mathbf{V}}\|(\nu, t) \leqslant \mathrm{e}^{\mathcal{A}T} \max_{|\nu| \leqslant \Omega} \|D_\nu^\alpha \mathbf{G}\|(\nu, t), \quad t \in [0, T],$$

which together with equation (4.4) implies

$$D_\nu^\alpha \widehat{\mathbf{V}}(\nu, x_3, t) \in C(\Delta(T); C_0(\mathbb{R}^2)).$$

Hence, the components $\widehat{V}_j(\nu, x_3, t)$ of the solution $\widehat{\mathbf{V}}(\nu, x_3, t)$ are from the space

$$C(\mathbb{R}^2 \times \Delta(T)) \cap C(\Delta(T); C_0^\infty(\mathbb{R}^2)).$$

Using the Paley-Wiener theorem we find $(u_1, u_2, u_3, w_1, w_2)^\top = F_\nu^{-1}[\widehat{V}]$ is a unique solution of (2.13), (2.14) and so $\mathbf{V}(x, t)$ is a unique solution of (2.7), (2.8) such that each $\widetilde{V}_j(x, t)$, $j = 1, 2, \ldots, 5$, belongs to the class

$$C(\mathbb{R}^2 \times \Delta(T)) \cap C(\Delta(T); PW(\mathbb{R}^2)),$$

where $PW(\mathbb{R}^2)$ is the Paley-Wiener space [2].

So we have the following theorem for the existence of the unique solution of the IVP (2.1), (2.2), (2.6).

Theorem 5.1. *Consider the following assumptions are satisfied.*

(1) *c, T are positive numbers and $\Delta(T)$ is a triangular region defined in* (2.12).

(2) *The density of the medium $\varrho(x_3)$ and the elements of the tensors C_{ijkl}, K_{ijkl}, R_{ijkl} are twice continuously differentiable functions depending on x_3 over $[-cT, cT]$.*

(3) *For arbitrary nonzero vectors* $\mathbf{V} = (u_1, u_2, u_3, w_1, w_2)^\top \in \mathbb{R}^5$, *the condition*

$$\frac{1}{\varrho(x_3)}\Big(\sum_{i,k=1}^{3} C_{i3k3}u_iu_k + \sum_{i=1}^{3}\sum_{\alpha=1}^{2} R_{i3\alpha3}u_iw_\alpha + \sum_{\alpha,\beta=1}^{2} K_{\beta3\alpha3}w_\beta w_\alpha\Big) > 0$$

is satisfied for the positive definiteness of the matrix P^1_{33} *defined in* (2.9).

(4) *Consider* $\mathbf{F}(x,t) = [f_1, f_2, f_3, g_1, g_2]^\top$ *and the Fourier image* $\widehat{\mathbf{F}}(\nu, x_3, t)$ *of the function* $\widetilde{\mathbf{F}}(x,t) = \Lambda^\top \mathbf{F}(x,t)$ *is considered in the space* $C(\mathbb{R}^2 \times \Delta(T))$ *and* $\partial^{|\alpha|}\widehat{\mathbf{F}}(\nu, x_3, t)/(\partial\nu_1^{\alpha_1}\partial\nu_2^{\alpha_2})$ *belongs to the class*

$$C(\mathbb{R}^2 \times \Delta(T)) \cap C(\Delta(T); C_0(\mathbb{R}^2)),$$

where $\Lambda^\top$ *is the transpose of* Λ, *that is, the matrix diagonalizes matrix* P^1_{33} *defined in* (2.9).

Then a unique generalized solution of IVP (2.1), (2.2), (2.6) *exists where* $u_i(x,t)$, $w_\beta(x,t)$ $(i = 1,2,3,\ \beta = 1,2)$ *belong to the space*

$$C(\mathbb{R}^2 \times \Delta(T)) \cap C(\Delta(T); PW(\mathbb{R}^2)).$$

6. Computational example

As reviewed in the introduction, the problems studied in inhomogeneous quasicrystals are scarce. In the literature there is no computational example for inhomogeneous quasicrystals. In this section, a simple example for an IVP of elastodynamic system in vertically inhomoneous quasicrystal is considered. However, since there are no numerical examples and any other study in literature to validate the results of our method, the example is chosen simply for having an exact solution and do a comparison with the method explained in the paper.

Let us consider problem (2.1)–(2.2) with the initial condition (2.6), where

$$f_1(x,t) = \delta_N(x_1)\delta_N(x_2)(-5\mathrm{e}^{-t}(x_3+1)^2 - 6(t-1)(x_3+1)^2),$$
$$f_2(x,t) = f_3(x,t) = 0, \quad g_1(x,t) = g_2(x,t) = 0,$$

where N is a given positive number and the Fourier transform of the function $\delta_N(x_i)$, $i = 1,2$, is

$$F[\delta_N(x_i)] = e_N(\nu_i).$$

Here $e_N(\nu_i)$ is the function that is defined as

$$e_N(\nu_i) = \begin{cases} 1 & \text{for } |\nu_i| \leqslant N, \\ 0 & \text{for } |\nu_i| > N. \end{cases}$$

Also the components of the tensors are taken as

$$C_{1313} = C_{2323} = C_{3333} = K_{1313} = K_{2323} = (x_3 + 1)^2,$$

and all other components of the tensors are taken zero for simplification.

The problem has an exact solution that is a generalized function. To do a comparison, the problem is solved using the integral equation given in equation (3.14). An approximate solution of the problem is obtained solving integral equation (3.14) and using successive approximations that is written in Maple. The number of iterations of the method for successive approximations to calculate the approximate solution is 10. For nonsimple examples implementing the method needs more iterations and takes more time.

The components of the exact solution of the considered IVP are

$$u_1^e(x,t) = ((x_3+1)^2 \mathrm{e}^{-t} + (t-1)(x_3+1)^2)\delta_N(x_1)\delta_N(x_2),$$
$$u_2^e(x,t) = u_3^e(x,t) = 0, w_1^e(x,t) = w_2^e(x,t) = 0.$$

Let $\widehat{\mathbf{V}}^c(\nu, x_3, t)$, $\widehat{\mathbf{V}}^e(\nu, x_3, t)$ be the computed and the exact Fourier transforms of the solutions, respectively. The norm of the differences $\widehat{\mathbf{V}}^c(\nu, x_3, t) - \widehat{\mathbf{V}}^e(\nu, x_3, t)$ relative to max with respect to $(x_3, t) \in \Delta(T)$ and L_2 with respect to ν_1, ν_2 can be given by the equality

$$\|\widehat{\mathbf{V}}^c - \widehat{\mathbf{V}}^e\| = \max_{(x_3,t)\in\Delta(T)} \sqrt{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\widehat{\mathbf{V}}^c(\nu_1,\nu_2,x_3,t) - \widehat{\mathbf{V}}^e(\nu_1,\nu_2,x_3,t)|^2 \,\mathrm{d}\nu_1 \,\mathrm{d}\nu_2}.$$

The discrete analogue of this norm is given by

$$\|\widehat{\mathbf{V}}^c - \widehat{\mathbf{V}}^e\| = \max_{z_i,t_p} \sqrt{\sum_{j=-N}^{N}\sum_{k=-N}^{N} |\widehat{\mathbf{V}}^c(j\Delta, k\Delta, z_p, t_p) - \widehat{\mathbf{V}}^e(j\Delta, k\Delta, z_p, t_p)|^2 (\Delta)^2},$$

where (z_i, t_p) is the node of the discretization of the set $\Delta(T)$.

The discrete analogue form of the norm that is given by

$$\max_{z_i,t_p} \sqrt{\sum_{j=-N}^{N}\sum_{k=-N}^{N} |\widehat{\mathbf{V}}^c(j\Delta, k\Delta, z_p, t_p) - \widehat{\mathbf{V}}^e(j\Delta, k\Delta, z_p, t_p)|^2 (\Delta)^2}$$

approaches to

$$\max_{(x_3,t)\in(\Delta(T)} \sqrt{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\widehat{\mathbf{V}}^c(\nu_1,\nu_2,x_3,t) - \widehat{\mathbf{V}}^e(\nu_1,\nu_2,x_3,t)|^2 \,\mathrm{d}\nu_1 \,\mathrm{d}\nu_2}$$

as the steps of discretization tend to 0, cf. Table 1

Δ	$\|\widehat{\mathbf{V}}^c(\nu_1,\nu_2,x_3,t)-\widehat{\mathbf{V}}^e(\nu_1,\nu_2,x_3,t)\|$
1/10	$0.656.10^{-5}$
1/50	$0.116.10^{-5}$
1/100	$0.991.10^{-7}$

Table 1.

Moreover, according to the Parseval-Plancherel identity (theorem), we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\widehat{\mathbf{V}}^c(\nu_1,\nu_2,x_3,t)-\widehat{\mathbf{V}}^e(\nu_1,\nu_2,x_3,t)|^2\,\mathrm{d}\nu_1\,\mathrm{d}\nu_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\mathbf{V}^c(x_1,x_2,x_3,t)-\mathbf{V}^e(x_1,x_2,x_3,t)|^2\,\mathrm{d}x_1\,\mathrm{d}x_2.$$

Thus, according to this identity, we have

$$\|\mathbf{V}^c-\mathbf{V}^e\| = \max_{(x_3,t)\in(\Delta(T)}\sqrt{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\mathbf{V}^c(x_1,x_2,x_3,t)-\mathbf{V}^e(x_1,x_2,x_3,t)|^2\,\mathrm{d}x_1\,\mathrm{d}x_2},$$

which is equal (approximately) to

$$\max_{z_i,t_p}\sqrt{\sum_{j=-N}^{N}\sum_{k=-N}^{N}|\widehat{\mathbf{V}}^c(j\Delta,k\Delta,z_p,t_p)-\widehat{\mathbf{V}}^e(j\Delta,k\Delta,z_p,t_p)|^2(\Delta)^2}.$$

7. Conclusion

The initial value problem for 3D elastodynamic system in 2D vertically inhomogeneous quasicrystals is considered in the paper. Applying Fourier transform with respect to lateral variables, the IVP is written in terms of Fourier images. Using some other transforms, the IVP is written in the form of Volterra type vector integral equation. The existence and uniqueness theorems for the obtained integral equation are stated and proved using successive approximation method. Using the Paley-Wiener theorem, the existence of the inverse images for the solution of the IVP for the elastodynamic system is proved. Finally, the existence and uniqueness theorem for the original IVP is stated and an example is considered for the comparison of the exact solution with the solution obtained by using the method.

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